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LOOP TRANSFER RECOVERY DESIGN VIA NEW OBSERVER-BASED AND CSS ARCHITECTURE-BASED CONTROLLERS

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SUMMARY

Two new controller structures, namely the continuous-time current-type observer and current-type CSS (Chen–Saberi–Sannuti) architecture-based controllers, are considered in this paper for loop transfer recovery design for general non-strictly proper non-minimum phase systems. The proposed observer is structurally analogous to the current estimator of discrete-time systems, while the proposed CSS architecture falls into the category of the controller structures developed recently by Chen, Saberi and Sannuti.[2,3] The properties of these new structures are characterized. In particular, sets of necessary and sufficient conditions under which a target loop transfer function can be either exactly and/or asymptotically recovered by the abovementioned controllers are obtained. More importantly, the new current-type observer balances the observer structures for continuous-time and discrete-time linear systems.

KEY WORDS observer theory; CSS architecture; loop transfer recovery; robust control

1. INTRODUCTION

In classical as well as modern feedback control system design, many performance and robust stability objectives can be cast in terms of the maximum magnitude or maximum singular values of some particular closed-loop transfer functions, e.g., sensitivity and complementary sensitivity functions at certain points in a closed loop. A principal idea of 'loop shaping' is that such magnitude or singular value requirements on some closed-loop transfer functions can be directly determined by the corresponding singular values of certain related open-loop transfer functions. A prominent design methodology for multivariable systems which is based on such loop shaping concepts is LQG/LTR. Historically, LQG/LTR design philosophy involves two steps. The first step is to design a state feedback law that yields an open-loop transfer function which accommodates satisfactorily the given design specifications on the required sensitivity functions. Such an open-loop transfer function is called a target open-loop transfer function. The second step, called loop transfer recovery (LTR), involves the design of an output feedback control law such that the resulting open-loop transfer function would be either exactly or

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approximately the same as the target open-loop transfer function. In other words, the idea of LTR is to come up with a measurement feedback compensator to recover a specific open-loop transfer function prescribed in terms of a state feedback gain.

The topic of LTR has been the subject of a number of authors, major contributions coming from Athans,[1] Chen,[4] Doyle and Stein,[6,7] Goodman,[9] Kwakernaak,[10] Niemann *et al.*,[11,12] Saberi *et al.*[13,14] Saberi and Sannuti,[16] Søgaard-Andersen,[19] Stein and Athans,[21] and Zhang and Freudenberg.[22] During the last ten years the subject has attained a certain amount of maturity. More recently, Saberi, Chen and Sannuti have put together various aspects of LTR analysis and design in a book.[15] The problem of loop transfer recovery treated in Reference 15 is fairly general and complete. Our goals in this paper are to introduce two new controller architectures, namely, the current-type observer and CSS (Chen–Saberi–Sannuti) architecture-based controllers, for loop transfer recovery design. The new controller structures, to our knowledge, have not been studied before in any open literature.

The paper is organized as follows. Section 2 defines the LTR problem in precise terms. Section 3 deals with a current-type observer-based controller design and its properties, while in Section 4, we advocate using a new CSS architecture, namely, the current-type CSS architecture, for a controller. We next move on to show the advantages of using such a controller structure. Finally Section 5 draws the conclusions of our work.

Throughout this paper, A^{T} denotes the transpose of A and $\lambda(A)$ denotes the set of eigenvalues of A. $\mathbb{C}$ denotes the set of all complex numbers, $\mathbb{C}^- \coloneqq \{s \in \mathbb{C} | \mathrm{Re}(s) \leqslant -\delta\}$ for some desired $\delta > 0$* and $\mathbb{C}^+ \coloneqq \mathbb{C}/\mathbb{C}^-$. A matrix A is said to be stable if $\lambda(A) \subset \mathbb{C}^-$. Similarly, $\sigma_{\max}[A]$ and $\sigma_{\min}[A]$ respectively denote the maximum and minimum singular values of A. $\mathrm{Ker}(V)$ and $\mathrm{Im}(V)$ denote respectively the kernel and the image of V. Also, $C^{-1}\{\aleph\} \coloneqq \{x | Cx \in \aleph\}$, where C is a constant matrix and $\aleph$ is a vector space.

2. PROBLEM STATEMENT

In this section, we formulate in precise mathematical terms the problem of loop transfer recovery (LTR). Consider a linear finite-dimensional and time-invariant system characterized by

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. For the obvious reasons, we assume that Σ is stabilizable and detectable with respect to $\mathbb{C}^+$. In this paper, for simplicity, we concentrate on a case when plant uncertainties are modelled at the input point of a nominal plant model and hence the required loop transfer function is specified at the plant input point. However, our results can be generalized easily for the case when the required loop transfer function is specified at any arbitrary point. In fact, for the case when the required loop transfer function is specified at the plant output point,[10] our results can easily be dualized using the procedure given in Reference 15. Let F be a desired state feedback gain matrix such that (i) the closed-loop system is asymptotically stable, i.e., eigenvalues of $A - BF$ lie in $\mathbb{C}^-$, and (ii) the open-loop transfer function when the loop is broken at the input point of the given system meets some given

* Here we have strengthened the notion of stability in order to avoid pole–zero cancellations of closed-loop systems in bad locations, i.e., the neighbourhood of the imaginary axis.

frequency-dependent specifications. The state feedback control is

$$u = -Fx \tag{2}$$

and the loop transfer function evaluated when the loop is broken at the input point of the given system, the so-called desired *target loop* transfer function, is

$$L_t(s) = F\Phi B \tag{3}$$

where $\Phi = (sI - A)^{-1}$.

Arriving at an appropriate value for F is concerned with the issue of loop shaping which is an engineering art and often includes the use of linear quadratic regulator (LQR) design in which the cost matrices are used as free design parameters to generate the target loop transfer function $L_t(s)$ and thus the desired sensitivity and complementary sensitivity functions. The next step of design is to recover the target loop using only a measurement feedback controller. This is the problem of loop transfer recovery (LTR) and is the focus of this paper. To explain it clearly, consider the configuration of Figure 1 where $\mathscr{C}(s)$ and $P(s)$,

$$P(s) = C\Phi B + D \tag{4}$$

are respectively the transfer functions of a controller and of the given system. Given $P(s)$ and a desired target loop transfer function $L_t(s)$, one seeks then to design a proper $\mathscr{C}(s)$ such that $E(s)$,

$$E(s) \coloneqq L_t(s) - \mathscr{C}(s)P(s) \tag{5}$$

is either exactly or approximately equal to zero in the frequency region of interest while guaranteeing the internal stability of the resulting closed-loop system. Hereafter, we will call $E(s)$ the *recovery error*. To be precise, we say that exact LTR (ELTR) is achievable if there exists an internally stabilizing proper controller, $\mathscr{C}(s)$, such that the corresponding recovery error $E(s)$ is identically zero for all $s \in \mathbb{C}$, and asymptotic LTR (ALTR) is achievable if there exist a family of internally stabilizing proper controllers, $\mathscr{C}(s, \varepsilon)$, such that the corresponding recovery error

$$E(s, \varepsilon) \coloneqq L_t(s) - \mathscr{C}(s, \varepsilon)P(s) \to 0 \tag{6}$$

pointwise in $s \in \mathbb{C}$ as $\varepsilon \to 0$.

Regarding the structures of $\mathscr{C}(s)$ or $\mathscr{C}(s, \varepsilon)$, besides the Luenberger observer-based controller (see, for example, References 12 and 15) which is not very useful from the practical point of view because of the complicated relationships among its parameters, and the CSS architecture-based controllers,[2,3] the most commonly used ones for continuous LTR are the full and reduced order observer-based controllers. There are, however, three common structures for discrete LTR, namely, the full order (also called the prediction), the reduced order and the current-type observer- (or estimator-) based controllers (see for example, References 15 and 23). The main

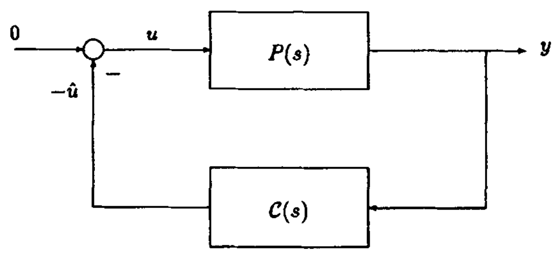

Figure 1. Plant with controller closed-loop configuration

purpose of this paper is to fill the gap in controller structures between continuous and discrete LTR by introducing a current-type observer-based controller for continuous-time systems. The new controller is structurally analogous to the current estimator-based controller of discrete-time systems. We will focus our attentions in this paper only on the properties of this new structure for LTR, although it has potential applications in H_2, H_∞ optimal control and many other control problems. As a byproduct, we also obtain a current-type CSS architecture-based controller. In order to proceed with our development, we recall from Reference 15 the following definitions of two geometric subspaces of linear systems.

Definition 2.1

Given a linear finite-dimensional and time-invariant system, Σ, characterized by the matrix quadruple (A, B, C, D), we define

1. $\mathcal{S}^-(\Sigma)$ as the minimal $(A-KC)$-invariant subspace of $\mathbb{R}^n$ containing in $\mathrm{Im}(B-KD)$ such that the eigenvalues of the map which is induced by $(A-KC)$ on the factor space $\mathbb{R}^n/\mathcal{S}^-$ are contained in $\mathbb{C}^-$ for some K.
2. $\mathcal{V}^+(\Sigma)$ as the maximal $(A-BF)$-invariant subspace of $\mathbb{R}^n$ contained in $\mathrm{Ker}(C-DF)$ such that the eigenvalues of $(A-BF)|\mathcal{V}^+$ are contained in $\mathbb{C}^+$ for some F.

3. CURRENT-TYPE OBSERVER-BASED CONTROLLER DESIGN

We introduce in this section a new observer-based controller. It is named as a current-type observer-based controller because it is structurally analogous to the current estimator-based controller of discrete-time systems (see, for example, References 8, 15 and 23). Without loss of generality but for simplicity of presentation, throughout this paper we assume that matrix D is in the form of,

$$D = \begin{bmatrix} D_0 \\ 0 \end{bmatrix} \tag{7}$$

where rank $(D_0) = $ rank $(D) = m_0$. Hence, we can rewrite the given system (1) as follows,

$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} D_0 \\ 0 \end{bmatrix} u \end{cases} \tag{8}$$

where C_0 and C_1 are matrices with appropriate dimensions. For the time being, we assume that $\dot{y}_1$, the *current value* of the measurement y_1, is available for feedback. We will utilize the following auxiliary measurement in the construction of our new observer,

$$\tilde{y} \coloneqq \begin{pmatrix} y_0 \\ y_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} C_0 x + D_0 u \\ C_1 x \\ C_1 A x + C_1 B u \end{pmatrix} = \begin{bmatrix} C_0 \\ C_1 \\ C_1 A \end{bmatrix} x + \begin{bmatrix} D_0 \\ 0 \\ C_1 B \end{bmatrix} u \tag{9}$$

For future use, we define

$$\tilde{C} \coloneqq \begin{bmatrix} C_0 \\ C_1 \\ C_1 A \end{bmatrix} \quad \text{and} \quad \tilde{D} \coloneqq \begin{bmatrix} D_0 \\ 0 \\ C_1 B \end{bmatrix} \tag{10}$$

Then the continuous-time current-type observer is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + K(\tilde{y} - \tilde{C}\hat{x} - \tilde{D}u) = (A - K\tilde{C})\hat{x} + (B - K\tilde{D})u + K\tilde{y} \tag{11}$$

where K is the observer gain and is the only free design parameter. Let $e \coloneqq x - \hat{x}$. We have

$$\dot{e} = (A - K\tilde{C})e \tag{12}$$

Obviously, $\lim_{t\to\infty} e(t) = 0$ provided that the observer gain K is chosen such that $A - K\tilde{C}$ is stable. Thus, $\hat{x}(t)$ is indeed an estimation of the state $x(t)$.

Now, let us get rid of $\dot{y}_1$. Let us partition the observer gain K as follows,

$$K = [K_0 \quad K_1 \quad \tilde{K}_1] \tag{13}$$

where K_0, K_1 and $\tilde{K}_1$ are of dimensions $n \times m_0$, $n \times (p - m_0)$ and $n \times (p - m_0)$, respectively. Also, let us define a new variable,

$$v \coloneqq \hat{x} - \tilde{K}_1 y_1 \tag{14}$$

It is simple to verify that

$$\dot{v} = \dot{\hat{x}} - \tilde{K}_1\dot{y}_1 = (A - K\tilde{C})v + (B - K\tilde{D})u + [K_0 \quad K_1 + (A - K\tilde{C})\tilde{K}_1]y \tag{15}$$

Finally, we obtain the continuous-time current-type observer-based controller,

$$\begin{cases} \dot{v} = (A - K\tilde{C})v + (B - K\tilde{D})u + [K_0 \;\; K_1 + (A - K\tilde{C})\tilde{K}_1]y \\ -u = -\hat{u} = F\hat{x} = Fv + [0 \;\; F\tilde{K}_1]y \end{cases} \tag{16}$$

where F is the desired state feedback gain matrix given in (2). The transfer function from y to $-u$ of this new observer-based controller is given by

$$\mathscr{C}_0(s) = F(sI - A + BF + K\tilde{C} - K\tilde{C}F)^{-1}[K_0 \quad K_1 + (A - K\tilde{C})\tilde{K}_1 - (B - K\tilde{D})F\tilde{K}_1] + [0 \quad F\tilde{K}_1] \tag{17}$$

A block diagram implementation of the current-type observer-based controller is depicted in Figure 2. Clearly, such a continuous-time current-type observer-based controller is physically realizable.

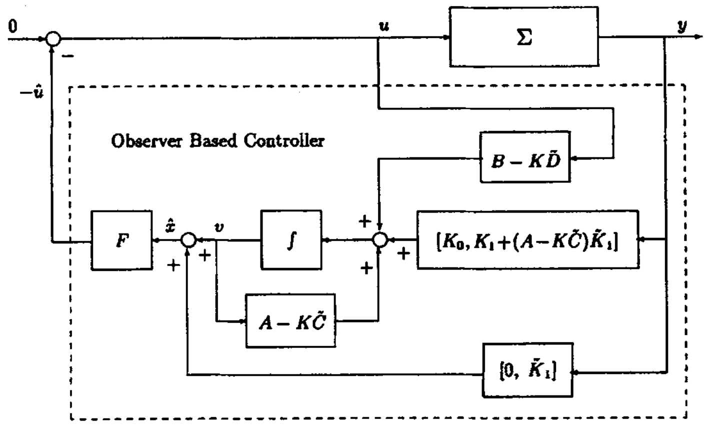

Figure 2. Plant with current-type observer-based controller

Remark 3.1

There is a fundamental difference between the continuous-time and discrete-time current-type observer-based controllers. In the discrete-time systems, one does not have to include the output y_1 in the auxiliary measurement $\tilde{y}$ (see, for example, References 8, 15 and 23). For the continuous-time systems, however, the resulting pair $(A, \tilde{C})$ will not be detectable, which implies that the stabilizing gain K does not exist, if y_1 is excluded from $\tilde{y}$.

We also note that by fixing $\tilde{K}_1 \equiv 0$, the new controller of (16) is reduced to the well-known conventional full order observer-based controller.

In what follows, we proceed to characterize the properties of this new controller structure. First, we have the following lemma.

Lemma 3.1

Consider a stabilizable and detectable system Σ characterized by the quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left-invertible. Let $L_t(s) = F\Phi B$ be a desired target loop transfer function. Then the recovery error $E_o(s)$ between the target loop transfer function $L_t(s)$ and that realized by the current-type observer based controller is given by

$$E_o(s) = M(s)[I + M(s)]^{-1}(I + F\Phi B) \tag{18}$$

where

$$M(s) = F(sI - A + K\tilde{C})^{-1}(B - K\tilde{D}) \tag{19}$$

Furthermore, for all $s \in \Omega$,

$$E_o(s) = 0 \quad \text{if and only if} \quad M(s) = 0$$

where Ω is the set of complex numbers for which $L_t(s)$ and $L_o(s) \coloneqq \mathscr{C}_o(s)P(s)$ are well defined.

Proof. It is simple to show that the transfer function of the current-type observer-based controller can be rewritten as

$$\mathscr{C}_o(s) = [I + M(s)]^{-1}(F(\Phi^{-1} + K\tilde{C})^{-1}[K_0 \;\; K_1 + (A - K\tilde{C})\tilde{K}_1] + [0 \;\; F\tilde{K}_1])$$

For future use, let us define

$$N(s) \coloneqq F(\Phi^{-1} + K\tilde{C})^{-1}[K_0 \;\; K_1 + (A - K)\tilde{\tilde{K}}_1] + [0 \;\; F\tilde{K}_1] \tag{20}$$

In view of the special form of matrix D in (7), we have

$$\begin{aligned}
N(s)&P(s) \\
&= F(\Phi^{-1} + K\tilde{C})^{-1}[K_0C_0 + K_1C_1 + (A - K\tilde{C})\tilde{K}_1C_1]\Phi B + F\tilde{K}_1C_1\Phi B + F(\Phi^{-1} + K\tilde{C})^{-1}K_0D_0 \\
&= F(\Phi^{-1} + K\tilde{C})^{-1}\{[K_0C_0 + K_1C_1 + (A - K\tilde{C})\tilde{K}_1C_1 + (\Phi^{-1} + K\tilde{C})\tilde{K}_1C_1]\Phi B + K_0D_0\} \\
&= F(\Phi^{-1} + K\tilde{C})^{-1}[(K_0C_0 + K_1C_1 + \tilde{K}_1C_1A)\Phi B + \tilde{K}_1C_1B + K_0D_0] \\
&= F(\Phi^{-1} + K\tilde{C})^{-1}K\tilde{C}\Phi B + F(\Phi^{-1} + K\tilde{C})^{-1}K\tilde{D} \\
&= F[I - (\Phi^{-1} + K\tilde{C})^{-1}\Phi^{-1}]\Phi B + F(\Phi^{-1} + K\tilde{C})^{-1}K\tilde{D} \\
&= F\Phi B - M(s)
\end{aligned} \tag{21}$$

Now it is straightforward to show that

$$E_o(s) = L_t(s) - \mathscr{C}_o(s)P(s) = L_t(s) - [I + M(s)]^{-1}N(s)P(s) = M(s)[I + M(s)]^{-1}(I + F\Phi B)$$

Using (18), it is trivial to verify that $E_o(s) = 0$ if and only if $M(s) = 0$. This completes the proof of Lemma 3.1. □

Equations (18) and (19) present a clear perspective to study the basic mechanism of LTR via the proposed controller. In fact, they facilitate the study of $E_o(s)$ in terms of the study of $M(s)$. Thus Lemma 3.1 and the expression for $M(s)$ as given by (19) form a basis for our study. As we will see shortly, the conditions under which a desired target loop transfer function can be either exactly and/or asymptotically recovered by a current-type observer-based controller turn out to be the constraints on the finite and infinite zero structures of the auxiliary system $\tilde{\Sigma}$ characterized by the matrix quadruple $(A, B, \tilde{C}, \tilde{D})$. Hence, it is important for us to investigate the properties of $\tilde{\Sigma}$. We have the following proposition.

Proposition of 3.1

1. $\tilde{\Sigma}$ is of (non)-minimum phase if and only if Σ is of (non)-minimum phase. In fact, the invariant zeros of $\tilde{\Sigma}$ and Σ are the same.
2. $\tilde{\Sigma}$ is stabilizable and detectable if and only if Σ is stabilizable and detectable.
3. Orders of infinite zeros of $\tilde{\Sigma}$ are reduced by one from those of Σ.
4. $\mathcal{V}^+(\tilde{\Sigma}) = \mathcal{V}^+(\Sigma)$.
5. $\mathcal{S}^-(\tilde{\Sigma}) = \mathcal{S}^-(\Sigma) \cap C^{-1}\{\mathrm{Im}(D)\}$.
6. $\mathcal{S}^-(\tilde{\Sigma}) = \{0\}$ if and only if Σ is left-invertible and of minimum phase with no infinite zeros of order higher than one.

Proof. See Appendix. □

Remark 3.2

Here we note that the auxiliary system $\tilde{\Sigma}$ associated with the continuous-time current-type observer-based controller, always has the same invariant zeros as those of the original plant Σ. The auxiliary system corresponding to the discrete-time current estimator-based controller, however, always has extra invariant zeros at 0, the origin of the complex plane (see, for example, Reference 15).

Now, we are ready to present the necessary and sufficient conditions for ELTR and ALTR. The following is the main result of this section.

Theorem 3.1

Consider a stabilizable and detectable system Σ characterized by the quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left-invertible. Let $L_t(s) = F\Phi B$ be a desired target loop transfer function. Then we have the following:

1. $L_t(s)$ is exactly recoverable by a current-type observer-based controller if and only if
$$S^-(\Sigma) \cap C^{-1}\{\mathrm{Im}(D)\} \subseteq \mathrm{Ker}(F) \tag{22}$$
2. $L_t(s)$ is asymptotically recoverable by a current-type observer-based controller if and only if
$$\mathcal{V}^+(\Sigma) \subseteq \mathrm{Ker}(F) \tag{23}$$

Proof. In view of Lemma 3.1, it is sufficient to prove the above theorem by deriving the conditions under which $M(s)$ can be made identically zero or arbitrarily small. Let us consider an auxiliary system characterized by

$$\Sigma_{\text{aux}}: \begin{cases} \dot{x} = A^{\mathrm{T}}x + \tilde{C}^{\mathrm{T}}u + F^{\mathrm{T}}w \\ z = B^{\mathrm{T}}x + \tilde{D}^{\mathrm{T}}u \end{cases} \tag{24}$$

Then, with the state feedback law, $u = -K^{\mathrm{T}}x$, the closed-loop transfer function from w to z, denoted by $T_{zw}(s)$, is simply $T_{zw}(s) = M^{\mathrm{T}}(s)$. Hence, the problem of finding an observer gain matrix K such that $A - K\tilde{C}$ is stable and $M(s) = 0$ is equivalent to the well-known disturbance decoupling problem with internal stability (DDPS) for Σ_{aux} of (24). Similarly, the problem of finding a family of observer gains $K(\varepsilon)$ such that as $\varepsilon \to 0$, $A - K(\varepsilon)\tilde{C}$ is stable and the corresponding

$$M(s, \varepsilon) \coloneqq F[sI - A + K(\varepsilon)\tilde{C}]^{-1}[B - K(\varepsilon)\tilde{D}] \to 0$$

pointwise in $s \in \mathbb{C}$, is equivalent to the well-known almost disturbance decoupling problem with internal stability (ADDPS) for Σ_{aux}. Thus, the results of Theorem 3.1 follow from Proposition 3.1 and the well-known results of the DDPS and ADDPS (see, for example, References 5 and 20). □

Remark 3.3

By utilizing the auxiliary system Σ_{aux} of (24), one can construct a current-type observer-based controller that achieves ELTR or ALTR using any existing design methodologies. Currently, there are three design procedures available for LTR. They are the H_2-optimization, the H_∞-optimization, and the asymptotic time-scale and eigenstructure assignment (ATEA) based design algorithms (see, for example, Reference 15). We refer the interested readers to Reference 15 for detail.

Remark 3.4

In the case when a target loop transfer function is both exactly and asymptotically recoverable, it is in general desirable to recover it exactly because (i) compensators that achieve ELTR do not have high-gain problems and (ii) the overall achieved loop preserves all properties of the given target loop. On the other hand, it was pointed out by one of our referees that, in some cases, compensators achieving ALTR may add some additional roll-off at high frequencies to the achieved loop. However, the price one has to pay for this additional roll-off may be very high as it is well-known that the ALTR is always associated with high gains.

The following are some interesting corollaries.

Corollary 3.1

Consider a stabilizable and detectable system Σ characterized by the quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left-invertible. Then

1. any arbitrarily given target loop transfer function is exactly recoverable by the current-type observer-based controller if and only if Σ is left-invertible and of minimum phase with no infinite zeros of order higher than one;

2. any arbitrarily given target loop transfer function is asymptotically recoverable by the current-type observer-based controller if and only if Σ is left-invertible and of minimum phase.

Proof. Observing that $\mathcal{S}^-(\Sigma) \cap C^{-1}\{\mathrm{Im}(D)\} = \{0\}$ if and only if Σ is left-invertible and of minimum phase with no infinite zeros of order higher than one, and $\mathcal{V}^+ = \{0\}$ if and only if Σ is left-invertible and of minimum phase, the results follow from Theorem 3.1. □

Corollary 3.2

Consider a stabilizable and detectable system Σ characterized by the quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left-invertible. Then

1. if a target loop transfer function is exactly recoverable by any arbitrarily structured output feedback controller, it can be exactly recovered by a current-type observer-based controller;
2. if a target loop transfer function is asymptotically recoverable by any arbitrarily structured output feedback controller, it can be asymptotically recovered by a current-type observer-based controller.

Proof. It follows from Theorem 3.1 and the results in Chapter 10 of Reference 15. □

The following remarks are in order.

Remark 3.5

We recall from References 4 and 15 that the condition under which a desired target loop transfer function $L_t(s) = F\Phi B$ can be exactly recovered by a conventional full order observer-based controller is $\mathcal{S}^-(\Sigma) \subseteq \mathrm{Ker}(F)$. Clearly, this is stronger than the condition given in (22) (see also Example 3.1). Also, from the analysis of References 4 and 15, we know that the high-gain feature of LTR design is merely associated with the infinite zero structure of the given system. Since orders of infinite zeros of $\tilde{\Sigma}$ are reduced by one from those of Σ (see item 3 of Proposition 3.1), the current-type observer-based controller in general requires smaller gain than the conventional full order observer-based controller for the same degree of recovery.

Remark 3.6

In this paper, we are mainly focusing our attentions on the cases where a target loop transfer function is either exactly or asymptotically recoverable. For the non-recoverable case, one can exactly follow the procedures in Reference 15 to design a current observer-based controller that shapes the recovery error over certain band of frequencies or achieves recovery in a given control subspace.

We illustrate our results in the following example.

Example 3.1

Let us consider an example in the seminal work of Doyle and Stein.[6] The given plant Σ is characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = [2 \quad 1]x + 0.u$$

It is simple to verify that Σ is invertible and of minimum phase with one invariant zero at $s = -2$ and one infinite zero of order one. The desired target loop transfer function is characterized by a state feedback gain

$$F = [50 \quad 10]$$

It is well understood in LTR literature that such a target loop transfer function cannot be exactly recovered using the conventional full order observer-based controller. Nevertheless, it can be exactly recovered by a current-type observer-based controller (see Corollary 3.1). From (10), we obtain

$$\tilde{C} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \quad \text{and} \quad \tilde{D} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then it is easy to see that a current-type observer-based controller with

$$\tilde{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

achieves exact loop transfer recovery (note that $B - \tilde{K}\tilde{D} \equiv 0$ and hence $M(s) \equiv 0$).

4. CURRENT-TYPE CSS ARCHITECTURE-BASED CONTROLLER DESIGN

It turned that the observer-based controller architecture is not the best structure for LTR. Recently, Chen *et al.*[2,3] focused on a number of theoretical and numerical studies of the LTR design concept and its potential practicability when an arbitrary structure is used for controller. The investigations of References 2 and 3 lead to an important conclusion that the dynamic structure on so-called architecture of a controller plays a predominant role in dictating the controller gain and thus the controller bandwidth required to achieve a certain degree of recovery. Based on the study of loop transfer recovery mechanism, Chen *et al.*[2,3] developed an architecture that can be called a *CSS architecture*[15] for controller. There have been two CSS architecture-based controllers available till now, namely the full and reduced order ones which structurally correspond to the full and reduced order observer-based controllers, respectively. Following the footsteps of References 2 and 3, we propose in this section a new current-type CSS architecture-based controller.

The motivation behind the CSS architecture is very simple. Let us first study the physical interpretation of the recovery matrix $M(s)$. To do so, one can view the current-type observer-based controller as a device having two inputs, (i) the plant input u and (ii) the plant output y, and one output, $-\hat{u}$. Then it is simple to show that

$$-\hat{u}(s) = M(s)u(s) + N(s)y(s) \tag{25}$$

where $N(s)$ is as given in (20). It is clear that $M(s)$ is the transfer function or the link from the plant input point u to the controller output point $-\hat{u}$. Thus, whenever LTR is achieved, the controller output does not entail any feedback from the plant input. Based on this observation, we are inspired to remove the abovementioned link right from the start of design. Once the link is removed, or what is now called CSS architecture, we embark on a new design philosophy which is outside the realm of observer theory and hence the separation principle is no longer valid. Without the blessing of the separation principle, one has to prove that the design objectives of closed-loop stability and recovering the target loop shape can be simultaneously achieved. We intend to do exactly this in the following.

The time-domain equations of a current-type CSS architecture-based controller are given by

$$\begin{cases} \dot{v} = (A - K\tilde{C})v + [K_0\ K_1 + (A - K\tilde{C})\tilde{K}_1]y \\ -u = -\hat{u} = Fv + [0\ F\tilde{K}_1]y \end{cases} \tag{26}$$

where F again is the desired state feedback gain, $\tilde{C}$ and $\tilde{D}$ are as defined in (10), and

$$K = [K_0\ K_1\ \tilde{K}_1]$$

is the new controller gain matrix, the only free design parameter in (26). The transfer function of such a controller is

$$\mathscr{C}_c(s) = F(sI - A + K\tilde{C})^{-1}[K_0\ K_1 + (A - K\tilde{C})\tilde{K}_1] + [0\ F\tilde{K}_1] \tag{27}$$

A block diagram implementation of a current-type CSS architecture-based controller is depicted in Figure 3.

We impose right from the beginning of design that the controller gain K is chosen such that $A - K\tilde{C}$ is stable, i.e., the controller of (26) is open-loop stable. Also, another point to be emphasized is this. In the case of current-type observer-based controller, the separation principle is valid and hence once the matrix $A - K\tilde{C}$ is designed to be stable, the closed-loop-stability of the plant Σ together with the controller $\mathscr{C}_o(s)$ is guaranteed. This is not so in the case of CSS architecture-based controllers. So we intend to design the current-type CSS architecture gain matrix K to meet the following goals:

1. The current-type CSS architecture-based controller is open-loop asymptotically stable, i.e., $\lambda(A - K\tilde{C}) \subset \mathbb{C}^-$.
2. The closed-loop system comprising the given system Σ and the current-type CSS architecture-based controller (26) is asymptotically stable, i.e., $\lambda(A_{cl}) \subset \mathbb{C}^-$, where
$$A_{cl} \coloneqq \begin{bmatrix} A - K\tilde{C} - HDF & HC - HDF[0\ \tilde{K}_1]C \\ -BF & A - BF[0\ \tilde{K}_1]C \end{bmatrix} \tag{28}$$
and where $H \coloneqq [K_0\ K_1 + (A - K\tilde{C})\tilde{K}_1]$.
3. To achieve exact or asymptotic loop transfer recovery, i.e., the attained loop transfer function $L_c(s) \coloneqq \mathscr{C}_c(s)P(s)$ is either exactly or approximately equal to the target loop transfer function $L_t(s)$.

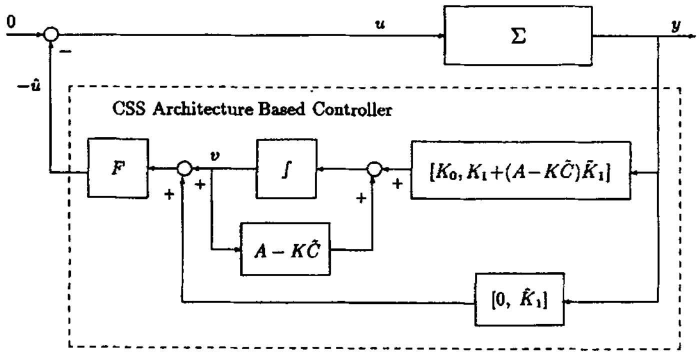

Figure 3. Plant with current-type CSS architecture-based controller

At first, by means of the following lemma, we characterize the recovery error between the target and attained loop transfer function via the current-type CSS architecture-based controller.

Lemma 4.1

Consider a stabilizable and detectable system Σ characterized by the quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left-invertible. Let $L_t(s) = F\Phi B$ be a desired target loop transfer function. Then the recovery error $E_c(s)$ between the target loop transfer function $L_t(s)$ and that realized by the current-type CSS architecture based controller is given by

$$E_c(s) = L_t(s) - L_c(s) = M(s) \tag{29}$$

where

$$M(s) = F(sI - A + K\tilde{C})^{-1}(B - K\tilde{D}) \tag{30}$$

Proof. In view of (20), (21) and (27) we have

$$\mathscr{C}_c(s) = L_t(s) - \mathscr{C}_c(s)P(s) = M(s)$$

This proves Lemma 4.1. □

Lemma 4.1 establishes a powerful interconnection between the properties of the current-type CSS architecture-based and the current-type observer-based controllers. Next, the following theorem reveals that if a desired target loop transfer function $L_t(s)$ is exactly (or respectively asymptotically) recoverable by a current-type observer-based controller, it is also exactly (or respectively asymptotically) recoverable by a current-type CSS architecture-based controller.

Theorem 4.1

Consider a stabilizable and detectable system Σ characterized by the quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left-invertible. Let $L_t(s)$ be a desired target loop transfer function. Then we have the following:

1. $L_t(s)$ can be exactly recovered via a current-type CSS architecture-based controller if and only if

$$\mathscr{S}^-(\Sigma) \cap C^{-1}\{\mathrm{Im}(D)\} \subseteq \mathrm{Ker}(F) \tag{31}$$

2. $L_t(s)$ can be asymptotically recovered via a current-type CSS architecture-based controller if and only if

$$\mathscr{V}^+(\Sigma) \subseteq \mathrm{Ker}(F) \tag{32}$$

Proof. In view of the results of Lemma 4.1 and the proof of Theorem 3.1, in order to prove Theorem 4.1 for each case considered, all one requires is to prove that the resulting closed-loop system under the current-type CSS architecture-based controller is asymptotically stable. First, let us define a new variable,

$$w \coloneqq x - v - \bar{K}_1 y_1 \tag{33}$$

We have

$$\begin{aligned}
\dot{w} &= \dot{x} - \dot{v} - \tilde{K}_1\dot{y}_1 \\
&= Ax + Bu - (A - K\tilde{C})v - K_0y_0 - K_1y_1 - (A - K\tilde{C})\tilde{K}_1y_1 - \tilde{K}_1C_1Ax - \tilde{K}_1C_1Bu \\
&= Ax + Bu - (A - K\tilde{C})v - K_0C_0x - K_1C_1x - \tilde{K}_1C_1Ax - K_0D_0u - \tilde{K}_1C_1Bu - (A - K\tilde{C})y_1 \\
&= (A - K\tilde{C})(x - v - \tilde{K}_1y_1) + (B - K\tilde{D})u \\
&= (A - K\tilde{C})w + (B - K\tilde{D})u
\end{aligned} \tag{34}$$

and

$$-u = Fv + [0 \quad F\tilde{K}_1]y = Fx - Fw \tag{35}$$

Then it is simple to verify that the dynamic matrix of the closed-loop system with a current-type CSS architecture-based controller is given by

$$A_{\rm cl} \coloneqq \begin{bmatrix} A - K\tilde{C} + (B - K\tilde{D})F & -(B - K\tilde{D})F \\ BF & A - BF \end{bmatrix} \tag{36}$$

The rest of our proof follows along the same lines of reasoning given in References 4 and 15. □

Remark 4.1

Again, one can use any design methodologies mentioned in Remark 3.3 to construct the current-type CSS architecture-based controller.

The above development culminating in Theorem 4.1 shows clearly that a current-type CSS architecture-based controller can do whatever the corresponding current-type observer-based controller or any arbitrarily structured controllers can do. Our next task is to show the benefits of CSS architecture-based controllers. Let us note that the task leading to Theorem 4.1 pursued only the capabilities of CSS architecture-based controllers. Theorem 4.1, in the case of asymptotic recovery, merely examines whether the recovery error $E_c(s)$ can be asymptotically rendered zero. It does not take into account the rate at which the recovery error tends to zero. A crucial question that arises here is how does the compromised level of recovery when the CSS architecture-based controller is used, compared with that when the observer-based controller is used. To seek an answer to such a question, one needs to study the rate at which different recovery errors tend to zero. Theorem 4.2 presents the result of such a study.

Theorem 4.2

Consider a stabilizable and detectable system Σ characterized by the quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left-invertible. Let the given target loop transfer function $L_t(s)$ be asymptotically recoverable. Also, let the same gain matrix K be used for both current-type observer-based and CSS architecture-based controllers and be such that $\sigma_{\max}[M(j\omega)]$ is small (say, $\ll 1$ but nonzero) for all ω. Furthermore, assume that

$$\sigma_{\min}[L_t(j\omega)] = \sigma_{\min}[F(j\omega I - A)^{-1}B] \gg 1 \quad \text{for all} \quad \omega \in \Omega_c \tag{37}$$

for some frequency region of interest, Ω_c. Then for all $\omega \in \Omega_c$, the error between the target loop transfer function and the one attained by the current-type CSS architecture-based controller, is always less than the corresponding one attained by the current-type observer-based controller. More specifically, we have

$$\sigma_{\max}[E_o(j\omega)] \geqslant \sigma_{\max}[E_c(j\omega)] \quad \text{for all} \quad \omega \in \Omega_c \tag{38}$$

Proof. It follows from the similar arguments as in Theorem 7 of Reference 2. □

Remark 4.2

It is well known (see, for example, Reference 7) that in order to have a good command following and disturbance rejection properties, the target loop transfer function $L_t(j\omega)$ has to be large and consequently, the minimum singular value $\sigma_{\min}[L_t(j\omega)]$ should be large in the appropriate frequency region. Thus the condition (37) is always satisfied in all practical situations.

We illustrate the above result in the following example.

Example 4.1

Consider a system[3] characterized by

which is square and invertible and one invariant zero at $s = 0{\cdot}3$. Since the given system is of non-minimum phase, not all the target loop transfer functions are recoverable. The geometric subspace $\mathcal{V}^+(\Sigma)$ for this example is given by

$$A = \begin{bmatrix} -25 & -25 & 1 & -25 \\ 0 & 0 & 0 & 1 \\ -6 & 1 & 0{\cdot}3 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$V^+(\Sigma) = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

Let the target loop transfer function, $L_t(s) = F\Phi B$, be specified by the following gain matrix,

$$F = \begin{bmatrix} -13 & 50 & 0 & 10 \\ 11 & 250 & 0 & 50 \end{bmatrix}$$

It is simple to see that the specified target loop transfer function is asymptotically recoverable since $\mathcal{V}^+(\Sigma) \subseteq \text{Ker}(F)$. From (10), we obtain

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -25 & -25 & 1 & -25 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \tilde{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Using the H_2-optimization-based algorithm (see, for example, Reference 15), we obtain a gain matrix,

$$K = \begin{bmatrix} 0{\cdot}985\,490\,66 & -0{\cdot}025\,349\,23 & 0{\cdot}998\,348\,55 & -0{\cdot}029\,878\,38 \\ -0{\cdot}025\,349\,23 & 1{\cdot}367\,509\,84 & -0{\cdot}003\,400\,46 & 0{\cdot}883\,195\,93 \\ 6{\cdot}742\,081\,27 & 21{\cdot}629\,345\,95 & 0{\cdot}580\,565\,03 & 14{\cdot}187\,561\,98 \\ -0{\cdot}029\,878\,38 & 0{\cdot}883\,195\,93 & -0{\cdot}238\,550\,59 & 95{\cdot}134\,419\,70 \end{bmatrix}$$

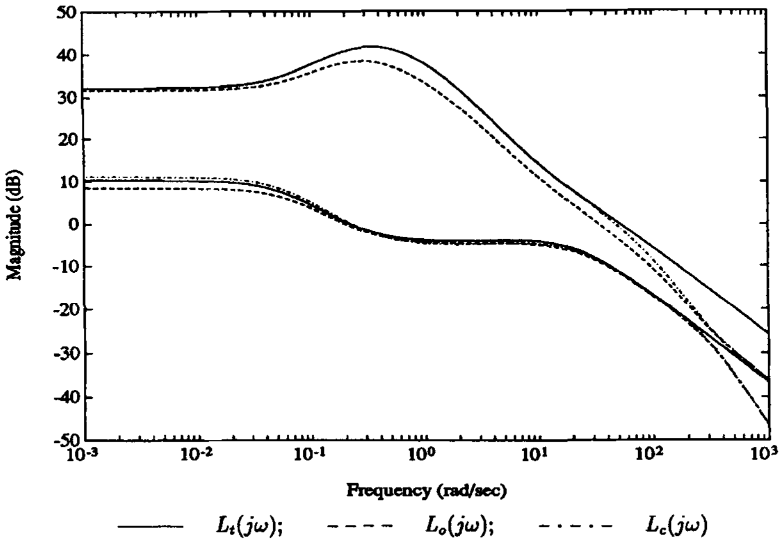

Figure 4. The maximum and minimum singular values of $L_t(j\omega)$, $L_o(j\omega)$ and $L_c(j\omega)$

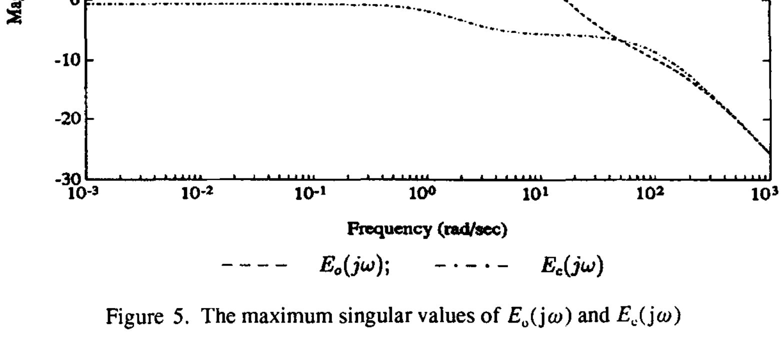

Figure 5. The maximum singular values of $E_o(j\omega)$ and $E_c(j\omega)$

which places $\lambda(A-K\bar{C})$ at $\{-103{\cdot}0974, -1{\cdot}0297, -1{\cdot}4373, -0{\cdot}2937\}$. Let the above K be used for both current-type observer- and CSS architecture-based controllers. The plots of the maximum and minimum singular values of the target loop transfer function as well as the two recovered loop transfer functions, one for the current-type observer-based controller and another for the current-type CSS architecture-based controller, are given in Figure 4. Also, the maximum singular values of the recovery errors $E_0(j\omega)$ and $E_c(j\omega)$ are plotted in Figure 5. Both figures clearly show that the CSS architecture-based controller is much better than the observer-based controller for LTR. It is worth noting that the maximal values of $\sigma_{max}[E_o(j\omega)]$ and $\sigma_{max}[E_c(j\omega)]$ in the frequency range shown in the plots are 45·1537 and 0·9177, respectively.

5. CONCLUDING REMARKS

In this paper, we have introduced two new controller structures, namely the current-type observer- and current-type CSS architecture-based controllers, for LTR design. It turned out that these controllers have the capacity to do whatever any arbitrarily structured controllers can do. As expected, the CSS architecture-based, however, requires much smaller gain than the observer-based one for the same degree of recovery. It is worth noting that the new current-type observer has balanced the observer structures for continuous-time and discrete-time systems and it has potential applications in H_2 and H_∞ optimal control. This will be the subject of our future research.

APPENDIX: PROOF OF PROPOSITION 3.1

In what follows we first recall from References 17 and 18 the special co-ordinate basis (SCB) for a linear time-invariant system. Such a co-ordinate basis has a distinct feature of explicitly displaying the finite and infinite zero structures of a given system as well as other system geometric properties.

Consider a system Σ characterized by the quadruple (A, B, C, D) as in (1). It is simple to verify that there exist nonsingular transformations U and V such that

$$UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \tag{39}$$

where m_0 is the rank of matrix D. Hence hereafter, without loss of generality, it is assumed that the matrix D has the form given on the right-hand side of (39). One can now rewrite the system of (1) as,

$$\begin{cases} \dot{x} = Ax + [B_0 \;\; B_1] \begin{pmatrix} u_0 \\ \tilde{u}_1 \end{pmatrix} \\ \begin{pmatrix} y_0 \\ \tilde{y}_1 \end{pmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ \tilde{u}_1 \end{pmatrix} \end{cases} \tag{40}$$

where the matrices B_0, B_1, C_0 and C_1 have appropriate dimensions. We have the following theorem.

Theorem A.1 (SCB)

For any given system Σ characterized by (A, B, C, D), there exist

1. nonnegative integers n_a^-, n_a^+, n_b, n_c, n_d, $m_d \leq m - m_0$ and q_i, $i = 1, \ldots, m_d$, and
2. nonsingular state, output and input transformations Γ_1, Γ_2 and Γ_3 which take the given Σ into a special co-ordinate basis that displays explicitly both the finite and infinite zero structures of Σ.

The special co-ordinate basis is described by the following set of equations:

$$x = \Gamma_1 \tilde{x}, \quad y = \Gamma_2 \tilde{y}, \quad u = \Gamma_3 \tilde{u}$$
$$\tilde{x} = [x_b^{\mathrm{T}}, x_a^{\mathrm{T}}, x_c^{\mathrm{T}}, x_d^{\mathrm{T}}]^{\mathrm{T}}, \quad x_a = [(x_a^-)^{\mathrm{T}}, (x_a^+)^{\mathrm{T}}]^{\mathrm{T}}$$
$$\tilde{x}_d = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \ldots, x_{m_d}^{\mathrm{T}}]^{\mathrm{T}}$$
$$\tilde{y} = [y_0^{\mathrm{T}}, y_d^{\mathrm{T}}, y_b^{\mathrm{T}}]^{\mathrm{T}}, \quad y_d = [y_1, y_2, \ldots, y_{m_d}]^{\mathrm{T}}$$
$$\tilde{u} = [u_0^{\mathrm{T}}, u_d^{\mathrm{T}}, u_c^{\mathrm{T}}]^{\mathrm{T}}, \quad u_d = [u_1, u_2, \ldots, u_{m_d}]^{\mathrm{T}}$$

and

$$\dot{x}_b = A_{bb} x_b + B_{0b} y_0 + L_{bd} y_d, \quad y_b = C_b x_b \tag{41}$$
$$\dot{x}_a^- = A_{aa}^- x_a^- + B_{0a}^- y_0 + L_{ad}^- y_d + L_{ab}^- y_b \tag{42}$$
$$\dot{x}_a^+ = A_{aa}^+ x_a^+ + B_{0a}^- y_0 + L_{ad}^+ y_d + L_{ab}^+ y_b \tag{43}$$
$$\dot{x}_c = A_{cc} x_c + B_{0c} y_0 + L_{cb} y_b + L_{cd} y_d + B_c [E_{ca}^- x_a^- + E_{ca}^+ x_a^+] + B_c u_c \tag{44}$$
$$y_0 = C_{0a}^- x_a^- + C_{0a}^+ x_a^+ + C_{0b} x_b + C_{0c} x_c + C_{0d} x_d + u_0 \tag{45}$$

and for each $i = 1, \ldots, m_d$,

$$\dot{x}_i = A_{q_i} x_i + L_{i0} y_0 + L_{id} y_d + B_{q_i} \left[u_i + E_{ia} x_a + E_{ib} x_b + E_{ic} x_c + \sum_{j=1}^{m_d} E_{ij} x_j \right] \tag{46}$$

$$y_i = C_{q_i} x_i, \quad y_d = C_d x_d \tag{47}$$

Here the states x_a^-, x_a^+, x_b, x_c and x_d are respectively of dimensions n_a^-, n_a^+, n_b, n_c and $n_d = \Sigma_{i=1} q_i$, while x_i is of dimension q_i for each $i = 1, \ldots, m_d$. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the following form:

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i - 1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1, 0, \ldots, 0] \tag{48}$$

Furthermore, we have $\lambda(A_{aa}^-) \in \mathbb{C}^-$, $\lambda(A_{aa}^+) \in \mathbb{C}^+$, the pair (A_{cc}, B_c) is controllable and the pair (A_{bb}, C_b) is observable.

Proof. See Sannuti and Saberi,[18] and Saberi and Sannuti.[17] □

The following are some important properties of the special co-ordinate basis.

Property A1

1. Invariant zeros of Σ are the eigenvalues of A_{aa}. Moreover, the invariant zeros which are in $\mathbb{C}^-$ and $\mathbb{C}^+$ are respectively the eigenvalues of A_{aa}^- and A_{aa}^+.
2. The given system Σ is right-invertible if and only if x_b and hence y_b are nonexistent, left-invertible if and only if x_c and hence u_c are nonexistent, invertible if and only if both x_b and x_c are nonexistent.
3. Let $\bar{x}_0 = m_0$. Let $\bar{q}_j$ be an integer such that exactly $\bar{q}_j$ elements of q_i, $i = 1, \ldots, m_d$, are equal to j. Also, let k be an integer such that $\bar{q}_j = 0$ for all $j > k$. Then Σ has $j\bar{q}_j$ number of infinite zeros of order j, for $j = 0, \ldots, k$.
4. $x_a^+ \oplus x_c \oplus x_d$ spans $\mathcal{S}^-(\Sigma)$ and $x_a^+ \oplus x_c$ spans $\mathcal{V}^+(\Sigma)$.

Now we are ready to prove Proposition 3.1. Without loss of generality but for simplicity of presentation, we assume that the given system Σ is in the form of the special co-ordinate basis given in Theorem A.1. Let us partition $x_d = [(x_{d1})^{\mathrm{T}}, (x_{d0})^{\mathrm{T}}, (x_{d2})^{\mathrm{T}}]^{\mathrm{T}}$, where x_{d1} is the part of the output associated with the infinite zeros of order one, x_{d0} is the rest of the output associated with the infinite zeros of order higher than one and x_{d2} consists of the state variables corresponding to the rest of the infinite zeros. Now, by an appropriate permutation transformation of the state variables, we can partition the given system as follows,

$$x = [(x_{d1})^{\mathrm{T}}, (x_{d0})^{\mathrm{T}}, (x_b)^{\mathrm{T}}, (x_a^-)^{\mathrm{T}}, (x_a^+)^{\mathrm{T}}, (x_c)^{\mathrm{T}}, (x_{d2})^{\mathrm{T}}]^{\mathrm{T}}$$

and

$$A - B_0C_0 = \begin{bmatrix} E_{d11} & E_{d10} & E_{b1} & E_{a1}^- & E_{a1}^+ & E_{c1} & E_{d12} \\ L_{d01} & L_{d00} & 0 & 0 & 0 & 0 & C_{d2} \\ L_{bd1} & L_{bd0} & A_{bb} & 0 & 0 & 0 & 0 \\ L_{ad1}^- & L_{ad0}^- & L_{ab}^-C_b & A_{aa}^- & 0 & 0 & 0 \\ L_{ad1}^+ & L_{ad0}^+ & L_{ab}^+C_b & 0 & A_{aa}^+ & 0 & 0 \\ L_{cd1} & L_{cd0} & L_{cb}C_b & B_cE_{ca}^- & B_cE_{ca}^+ & A_{cc} & 0 \\ A_{d21} & A_{d20} & B_{d2}E_{b2} & B_{d2}E_{a2}^- & B_{d2}E_{a2}^+ & B_{d2}E_{c2} & A_{d22}+E_{d22} \end{bmatrix}$$

$$B = [B_0 \ B_1] = \begin{bmatrix} B_{d01} & I & 0 & 0 \\ B_{d00} & 0 & 0 & 0 \\ B_{b0} & 0 & 0 & 0 \\ B_{a0}^- & 0 & 0 & 0 \\ B_{a0}^+ & 0 & 0 & 0 \\ B_{c0} & 0 & 0 & B_c \\ B_{d02} & 0 & B_{d2} & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & C_{0b} & C_{0a}^- & C_{0a}^+ & C_{0c} & C_{0d2} \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_b & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let us define

$$\hat{C} := \begin{bmatrix} C_0 \\ C_1(A - B_0C_0) \\ C_1 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{D}} := \begin{bmatrix} \mathbf{I}_{m_0} & 0 \\ 0 & \mathbf{C}_1\mathbf{B}_1 \\ 0 & 0 \end{bmatrix}$$

We note that $\hat{C} = \Gamma\tilde{C}$ and $\hat{D} = \Gamma\tilde{D}$, where Γ is a nonsingular matrix,

$$\hat{\Gamma} = \begin{bmatrix} I & 0 & 0 \\ -C_1B_0 & 0 & I \\ 0 & I & 0 \end{bmatrix}$$

Thus, establishing the required properties for a system characterized by $(A, B, \tilde{C}, \tilde{D})$, is equivalent to doing the same for a system characterized by the quadruple $(A, B, \hat{C}, \hat{D})$. We next rewrite $\hat{C}$ and $\hat{D}$ in the form,

$$\tilde{C} = \begin{bmatrix} 0 & 0 & C_{0b} & C_{0a}^- & C_{0a}^+ & C_{0c} & C_{0d2} \\ E_{d11} & E_{d10} & E_{b1} & E_{a1}^- & E_{a1}^+ & E_{c1} & E_{d12} \\ L_{d01} & L_{d00} & 0 & 0 & 0 & 0 & C_{d2} \\ C_bL_{bd11} & C_bL_{bd10} & C_bA_{bb} & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_b & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} I_{m_0} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is trivial then to verify that the system characterized by $(A, B, \hat{C}, \hat{D})$ has the same finite and infinite zero structures, and the invertibility properties as the system characterized by $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1)$ does, where

$$\tilde{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ L_{d01} & L_{d00} & 0 & 0 & 0 & 0 & C_{d2} \\ L_{bd1} & L_{bd0} & A_{bb} & 0 & 0 & 0 & 0 \\ L^-_{ad1} & L^-_{ad0} & L^-_{ab}C_b & A^-_{aa} & 0 & 0 & 0 \\ L^+_{ad1} & L^+_{ad0} & L^+_{ab}C_b & 0 & A^+_{aa} & 0 & 0 \\ L_{cd1} & L_{cd0} & L_{cb}C_b & B_cE^-_{ca} & B_cE^+_{ca} & A_{cc} & 0 \\ A_{d21} & A_{d20} & B_{d2}E_{b2} & B_{d2}E^-_{a2} & B_{d2}E^+_{a2} & B_{d2}E_{c2} & A_{d22}+B_{d2}E_{d22} \end{bmatrix}. \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_{d2} & 0 \end{bmatrix}$$

and

$$\tilde{C}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & C_{d2} \\ 0 & 0 & C_bA_{bb} & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_b & 0 & 0 & 0 & 0 \end{bmatrix}$$

Next, we define a dual system,

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u}, \quad \bar{y} = \bar{C}\bar{x}$$

where

$$\bar{A} = \tilde{A}_1^{\mathrm{T}}, \quad \bar{B} = \tilde{C}_1^{\mathrm{T}}, \quad \bar{C} = \tilde{B}_1^{\mathrm{T}}$$

$$\bar{x} = [(\bar{x}_0)^{\mathrm{T}}, (\bar{x}_a^-)^{\mathrm{T}}, (\bar{x}_a^+)^{\mathrm{T}}, (\bar{x}_b)^{\mathrm{T}}, (\bar{x}_d)^{\mathrm{T}}]^{\mathrm{T}}, \quad \bar{u} = [(\bar{u}_d)^{\mathrm{T}}, (\bar{u}_c)^{\mathrm{T}}]^{\mathrm{T}}, \quad \bar{y} = [(\bar{y}_d)^{\mathrm{T}}, (\bar{y}_b)^{\mathrm{T}}]^{\mathrm{T}}$$

The dynamic equations of the above dual system are givenby

$$\dot{\bar{x}}_0 = \bar{A}_{00}\bar{x}_0 + \bar{A}_{10}\bar{x}_a^- + \bar{A}_{20}\bar{x}_a^+ + \bar{A}_{b0}\bar{x}_b + \bar{A}_{d0}\bar{x}_d + \bar{K}_c\bar{u}_c$$

$$\dot{\bar{x}}_a^- = \bar{A}^-_{aa}\bar{x}_a^- + \bar{L}^-_{ab}\bar{y}_b + \bar{L}^-_{ad}\bar{y}_d$$

$$\dot{\bar{x}}_a^+ = \bar{A}^+_{aa}\bar{x}_a^+ + \bar{L}^+_{ab}\bar{y}_b + \bar{L}^+_{ad}\bar{y}_d$$

$$\dot{\bar{x}}_b = \bar{A}_{bb}\bar{x}_b + \bar{L}_{bd}\bar{y}_d, \quad \bar{y}_b = \bar{C}_b\bar{x}_b$$

$$\dot{\bar{x}}_d = \bar{A}_d\bar{x}_d + \bar{L}_d\bar{y}_d + \bar{B}_d[\bar{u}_d + \bar{E}_0\bar{x}_0], \quad \bar{y}_d = \bar{C}_d\bar{x}_d$$

where

$$\bar{A}_{00} = \begin{bmatrix} 0 & L^{\mathrm{T}}_{d01} & L^{\mathrm{T}}_{bd0} \\ 0 & L^{\mathrm{T}}_{d00} & L^{\mathrm{T}}_{bd0} \\ 0 & 0 & A^{\mathrm{T}}_{bb} \end{bmatrix}, \quad \bar{K}_c \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ A^{\mathrm{T}}_{bb}C^{\mathrm{T}}_b & 0 & 0 & C^{\mathrm{T}}_b \end{bmatrix}$$

$$\bar{A}_{10} = [L^-_{ad1}, L^-_{ad0}, L^-_{ab}C_b]^{\mathrm{T}}, \quad \bar{A}_{20} = [L^+_{ad1}, L^+_{ad0}, L^+_{ab}C_b]^{\mathrm{T}}$$

$$\bar{A}_{b0} = [L_{cd1}, L_{cd0}, L_{cb}C_b]^{\mathrm{T}}, \quad \bar{A}_{d0} = [A_{d21}, A_{d20}, B_{d2}E_{b2}]^{\mathrm{T}}$$

$$\bar{A}^-_{aa} = (A^-_{aa})^{\mathrm{T}}, \quad \bar{L}^-_{ab} = (E^-_{ca})^{\mathrm{T}}, \quad \bar{L}^-_{ad} = (E^-_{a2})^{\mathrm{T}}, \quad \bar{A}^+_{aa} = (A^+_{aa})^{\mathrm{T}}$$

$$\bar{L}^+_{ab} = (E^+_{ca})^{\mathrm{T}}, \quad \bar{L}^+_{ad} = (E^+_{a2})^{\mathrm{T}}, \quad \bar{A}_{bb} = (A_{cc})^{\mathrm{T}}, \quad \bar{L}_{bd} = (E_{c2})^{\mathrm{T}}$$

$$\bar{E}_0 = [0, I, 0], \quad \bar{A}_d = (A_{dd2})^{\mathrm{T}}, \quad \bar{C}_d = (B_{d2})^{\mathrm{T}}, \quad \bar{C}_b = (B_c)^{\mathrm{T}}, \quad \bar{L}_d = (E_{d22})^{\mathrm{T}}$$

Next, we shall perform some transformations among the state variables in order to bring the new system characterized by $(\bar{A}, \bar{B}, \bar{C})$ into the form of the SCB. It follows from the results in Appendix A.2 of Reference 18 that there exists a nonsingular transformation T such that

$$[(\bar{x}_0)^{\mathrm{T}}, (\bar{x}_a^-)^{\mathrm{T}}, (\bar{x}_a^+)^{\mathrm{T}}, (\bar{x}_b)^{\mathrm{T}}, (\bar{x}_d)^{\mathrm{T}}]^{\mathrm{T}} = T[(\bar{\bar{x}}_0)^{\mathrm{T}}, (\bar{x}_a^-)^{\mathrm{T}}, (\bar{x}_a^+)^{\mathrm{T}}, (\bar{x}_b)^{\mathrm{T}}, (\bar{x}_d)^{\mathrm{T}}]^{\mathrm{T}}$$

and

$$\begin{aligned}
\dot{\bar{\bar{x}}}_0 &= \bar{A}_{00}\bar{\bar{x}}_0 + \bar{K}_c\bar{u}_c + \bar{A}_{10}\bar{x}_a^- + \bar{A}_{20}\bar{x}_a^+ + \tilde{\bar{L}}_{b0}\bar{y}_b + \tilde{\bar{L}}_{d0}\bar{y}_d \\
\dot{\bar{x}}_a^- &= \bar{A}_{aa}^-\bar{x}_a^- + \bar{L}_{ab}^-\bar{y}_b + \bar{L}_{ad}^-\bar{y}_d \\
\dot{\bar{x}}_a^+ &= \bar{A}_{aa}^+\bar{x}_a^+ + \bar{L}_{ab}^+\bar{y}_b + \bar{L}_{ad}^+\bar{y}_d \\
\dot{\bar{x}}_b &= \bar{A}_{bb}\bar{x}_b + \bar{L}_{bd}\bar{y}_d, \quad \bar{y}_b = \bar{C}_b\bar{c}_b \\
\dot{\bar{x}}_d &= \bar{A}_d\bar{x}_d + \bar{L}_d\bar{y}_d + \bar{B}_d[\bar{u}_d + \bar{E}_0\bar{\bar{x}}_0 + \tilde{\bar{E}}_b\bar{x}_b + \tilde{\bar{E}}_d\bar{x}_d], \quad \bar{y}_d = \bar{C}_d\bar{x}_d
\end{aligned}$$

We note that the above system is not in the standard form of the SCB since $\bar{A}_{10}$ and $\bar{A}_{20}$ are not in the range space of $\bar{K}_c$. Noting that the pair $(\bar{A}_{00}, \bar{K}_c)$ is completely controllable, it follows from Lemma 1.3.1 of Reference 15 that there exists another nonsingular transformation S such that

$$[(\bar{\bar{x}}_0)^{\mathrm{T}}, (\bar{x}_a^-)^{\mathrm{T}}, (\bar{x}_a^+)^{\mathrm{T}}, (\bar{x}_b)^{\mathrm{T}}, (\bar{x}_d)^{\mathrm{T}}]^{\mathrm{T}} = S[(\bar{x}_c^-)^{\mathrm{T}}, (\bar{x}_a^-)^{\mathrm{T}}, (\bar{x}_a^+)^{\mathrm{T}}, (\bar{x}_b)^{\mathrm{T}}, (\bar{x}_d)^{\mathrm{T}}]^{\mathrm{T}}$$

and

$$\begin{aligned}
\dot{\bar{x}}_c &= \bar{A}_{cc}\bar{x}_c + \bar{B}_c[\bar{E}_{ca}^-\bar{x}_a^- + \bar{E}_{ca}^+\bar{x}_a^+] + \tilde{\bar{L}}_{b0}\bar{y}_b + \tilde{\bar{L}}_{d0}\bar{y}_d \\
\dot{\bar{x}}_a^- &= \bar{A}_{aa}^-\bar{x}_a^- + \bar{L}_{ab}^-\bar{y}_b + \bar{L}_{ad}\bar{y}_d \\
\dot{\bar{x}}_a^+ &= \bar{A}_{aa}^+\bar{x}_a^+ + \bar{L}_{ab}^+\bar{y}_b + \bar{L}_{ad}^+\bar{y}_d \\
\dot{\bar{x}}_b &= \bar{A}_{bb}\bar{x}_b + \bar{L}_{bd}\bar{y}_d, \quad \bar{y}_b = \bar{C}_b\bar{x}_b \\
\dot{\bar{x}}_d &= \bar{A}_d\bar{x}_d + \bar{L}_d\bar{y}_d + \bar{B}_d[\bar{u}_d + \bar{E}_c\bar{x}_c + \bar{E}_a^-\bar{x}_a^- + \bar{E}_a^+\bar{x}_a^+ + \bar{E}_b\bar{x}_b + \bar{E}_d\bar{x}_d], \quad \bar{y}_d = \bar{C}_d\bar{x}_d
\end{aligned}$$

where $(\bar{A}_{cc}, \bar{B}_c)$ is a controllable pair, and $\bar{E}_{ca}^-, \ldots, \bar{E}_a^+$ are some constant matrices with appropriate dimensions. It is now trivial to see that the above system is in the standard form of the SCB. Hence all the properties listed in Proposition 3.1 can be verified easily by the properties of the SCB (see Property A.1) and by some simple algebra.

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