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A simple algorithm for the stable/unstable decomposition of a linear discrete-time system

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A simple algorithm that decomposes the dynamics of a linear discrete-time system into stable modes, unstable modes and those on the unit circle, is considered here. The relationship between such a decomposition and the stable/unstable decomposition of continuous-time systems is also established.

1. Introduction

In many control problems, it is often useful to separate the dynamics of the given systems into the stable and unstable parts (see for example Hsu and Hou 1991). It is also desirable in many applications to decompose the stable and unstable zero dynamics of the given systems (see for example Chen *et al*. 1992, 1993 and Saberi *et al*. 1993). In general, the stable/unstable decomposition of a linear continuous-time system is rather easy and can be done using the well-known and numerically well-behaved Schur decomposition technique. An m-file that realizes such a decomposition has been reported by Lin *et al*. (1991). In principle, one can utilize the same technique to obtain a stable/unstable decomposition for discrete-time systems. However, due to the ordering of eigenstructures in Schur decomposition, the procedures involved are rather complicated. In this note we present a simple algorithm that computes a non-singular transformation T such that the dynamic matrix, say A, of a linear discrete-time system is decomposed as

$$T^{-1}AT = \begin{bmatrix} A_{\odot} & 0 & 0 \\ 0 & A_{\circ} & 0 \\ 0 & 0 & A_{\otimes} \end{bmatrix} \tag{1.1}$$

where the eigenvalues of $A_{\odot}$, $A_{\circ}$ and $A_{\otimes}$ are respectively inside, on and outside the unit circle of the complex plane. More importantly, we will also show the relationship between the stable/unstable decomposition of discrete-time systems and that of continuous-time systems.

2. Main results

We give our main results in this section. We will convert the stable/unstable decomposition of a linear discrete-time system into an equivalent problem for an auxiliary continuous-time system. First, let us recall the following lemma. The proof of this lemma is very straightforward.

Lemma 2.1: *Consider a real square matrix X. Assume that X does not have eigenvalues at* -1 *and let* $\widetilde{X} := (X + I)^{-1}(X - \mathrm{I})$. *Then we have*

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(1) *X has all its eigenvalues inside the unit circle if and only if $\widetilde{X}$ has all its eigenvalues in the open left half-plane*;

(2) *X has all its eigenvalues on the unit circle if and only if $\widetilde{X}$ has all its eigenvalues on the imaginary axis*;

(3) *X has all its eigenvalues outside the unit circle if and only if $\widetilde{X}$ has all its eigenvalues in the open right-half plane*.

We are ready to present our results.

Proposition 2.1: *Consider a real square matrix A and assume that it does not have eigenvalues at -1. Let $\widetilde{A} := (A+I)^{-1}(A-I)$. Also, let T be a non-singular transformation such that*

$$T^{-1}\widetilde{A}T = \begin{bmatrix} \widetilde{A}_- & 0 & 0 \\ 0 & \widetilde{A}_0 & 0 \\ 0 & 0 & \widetilde{A}_+ \end{bmatrix} \tag{2.1}$$

where $\widetilde{A}_-$, $\widetilde{A}_0$ and $\widetilde{A}_+$ have their eigenvalues in the open left half-plane, on the imaginary axis and in the open right half-plane, respectively. Then

$$T^{-1}AT = \begin{bmatrix} A_{\odot} & 0 & 0 \\ 0 & A_{\circ} & 0 \\ 0 & 0 & A_{\otimes} \end{bmatrix} \tag{2.2}$$

where the eigenvalues of $A_{\odot}$, $A_{\circ}$ and $A_{\otimes}$ are respectively inside, on and outside the unit circle of the complex plane. □

Proof: First note that $\widetilde{A} = (A+I)^{-1}(A-I)$ implies that $A = (I+\widetilde{A})(I-\widetilde{A})^{-1}$. Then we have

$$\begin{aligned} T^{-1}AT &= T^{-1}(I+\widetilde{A})TT^{-1}(I-\widetilde{A})^{-1}T \\ &= (I+T^{-1}\widetilde{A}T)(I-T^{-1}\widetilde{A}T)^{-1} \\ &= \begin{bmatrix} I+\widetilde{A}_- & 0 & 0 \\ 0 & I+\widetilde{A}_0 & 0 \\ 0 & 0 & I+\widetilde{A}_+ \end{bmatrix} \begin{bmatrix} I-\widetilde{A}_- & 0 & 0 \\ 0 & I-\widetilde{A}_0 & 0 \\ 0 & 0 & I-\widetilde{A}_+ \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (I+\widetilde{A}_-)(I-\widetilde{A}_-)^{-1} & 0 & 0 \\ 0 & (I+\widetilde{A}_0)(I-\widetilde{A}_0)^{-1} & 0 \\ 0 & 0 & (I+\widetilde{A}_+)(I-\widetilde{A}_+)^{-1} \end{bmatrix} \end{aligned}$$

In view of Lemma 2.1, the result follows. □

The following remarks deal with the cases when the matrix A has eigenvalues at -1.

Remark 2.1: If A has eigenvalues at -1 but does not have eigenvalues at 1, then the non-singular transformation of the stable/unstable decomposition, T, can be obtained using the same procedure as in Proposition 2.1 by re-defining $\widetilde{A}$ as $\widetilde{A} := (I-A)^{-1}(I+A)$. □

Remark 2.2: If A has eigenvalues at both 1 and -1, then the following procedure should be used to determine T. First, let $\lambda(A)$ be the set of the eigenvalues of A and then do the following.

Step 1. If $\max\{|\lambda|: \lambda \in \lambda(A)\} \leqslant 1$, set $T_\alpha = I$, $A_* = A$ and go to Step 3. Otherwise, determine

$$\alpha := \min\{|\lambda|: \lambda \in \lambda(A) \text{ and } |\lambda| > 1\} \tag{2.3}$$

and define

$$\widetilde{A}_\alpha := [2A/(\alpha + 1) + I]^{-1}[2A/(\alpha + 1) - I] \tag{2.4}$$

It is simple to show that $2A/(\alpha + 1)$ does not have any eigenvalues on the unit circle and hence $\widetilde{A}_\alpha$ has no eigenvalues on the imaginary axis.

Step 2. Utilizing the result of Proposition 2.1, find a non-singular transformation T_α such that

$$T_\alpha^{-1}\widetilde{A}_\alpha T_\alpha = \begin{bmatrix} \widetilde{A}_{\alpha-} & 0 \\ 0 & \widetilde{A}_{\alpha+} \end{bmatrix} \tag{2.5}$$

where the eigenvalues of $\widetilde{A}_{\alpha-}$ and $\widetilde{A}_{\alpha+}$ are respectively in the open left and the open right half complex plane. It is simple to verify that

$$T_\alpha^{-1} A T_\alpha = \begin{bmatrix} A_* & 0 \\ 0 & A_\otimes \end{bmatrix} \tag{2.6}$$

where the eigenvalues of $A_\otimes$ are outside the unit circle and the eigenvalues of A_* are either on or inside the unit circle of the complex plane.

Step 3. If $\min\{|\lambda|: \lambda \in \lambda(A)\} \geqslant 1$, set $T_\beta = I$ and go to Step 5. Otherwise, compute

$$\beta := \max\{|\lambda|: \lambda \in \lambda(A) \text{ and } |\lambda| < 1\} \tag{2.7}$$

and define

$$\widetilde{A}_{*\beta} := [2A_*/(\beta + 1) + I]^{-1}[2A_*/(\beta + 1) - I] \tag{2.8}$$

Again, it is easy to see that $2A_*/(\beta + 1)$ does not have any eigenvalues on the unit circle and hence $\widetilde{A}_{*\beta}$ has no eigenvalues on the imaginary axis.

Step 4. Next, utilizing the result of Proposition 2.1, find a non-singular transformation $T_{*\beta}$ such that

$$T_{*\beta}^{-1}\, \widetilde{A}_{*\beta} T_{*\beta} = \begin{bmatrix} \widetilde{A}_{*\beta-} & 0 \\ 0 & \widetilde{A}_{*\beta+} \end{bmatrix} \tag{2.9}$$

where $\widetilde{A}_{*\beta-}$ and $\widetilde{A}_{*\beta+}$ have their eigenvalues in the open left and the open right half-plane, respectively.

Step 5. Finally, it is straightforward to verify that the non-singular transformation T

$$T := T_\alpha \begin{bmatrix} T_{*\beta} & 0 \\ 0 & I \end{bmatrix} \tag{2.10}$$

has the following property

$$T^{-1}AT = \begin{bmatrix} A_{\odot} & 0 & 0 \\ 0 & A_{\bigcirc} & 0 \\ 0 & 0 & A_{\otimes} \end{bmatrix} \tag{2.11}$$

where $A_{\odot}$, $A_{\bigcirc}$ and $A_{\otimes}$ have their eigenvalues inside, on and outside the unit circle of the complex plane, respectively. □

Remark 2.3: In general, the algorithm given in Remark 2.2 is numerically well behaved. It may have some numerical difficulties when matrix A has eigenvalues very 'close' to the unit circle. However, these difficulties can be overcome by grouping these 'awkward' eigenvalues with those on the unit circle and this can be done by slightly modifying the procedure of Remark 2.2. We would like to note that this is a very common problem. In fact, a similar situation also arises in the continuous-time case when the given dynamic matrix has eigenvalues very 'close' to the imaginary axis. □

We illustrate the above results in the following examples.

Example 2.1: Consider a system with dynamic matrix

$$A = \begin{bmatrix} 34{\cdot}2 & 1{\cdot}8 & 8{\cdot}2 & -16{\cdot}6 \\ -64{\cdot}4 & -1{\cdot}6 & -16{\cdot}4 & 33{\cdot}2 \\ 4{\cdot}2 & -0{\cdot}2 & 2{\cdot}2 & -2{\cdot}6 \\ 61{\cdot}6 & 0{\cdot}4 & 14{\cdot}6 & -32{\cdot}8 \end{bmatrix} \tag{2.12}$$

which has eigenvalues at 2, 0, j and $-j$. First, let us compute

$$\widetilde{A} = \begin{bmatrix} -5{\cdot}7333333 & -35{\cdot}2666667 & -18{\cdot}4 & -31{\cdot}8 \\ 12{\cdot}1333333 & 70{\cdot}8666667 & 36{\cdot}8 & 63{\cdot}6 \\ -1{\cdot}4000000 & -6{\cdot}6000000 & -3{\cdot}4 & -5{\cdot}8 \\ -13{\cdot}5333333 & -70{\cdot}4666667 & -37{\cdot}2 & -62{\cdot}4 \end{bmatrix}$$

Using the software package on Lin *et al*. (1991), we obtain

$$T = \begin{bmatrix} -0{\cdot}3364018 & -0{\cdot}3402378 & -0{\cdot}2105630 & 0{\cdot}3746343 \\ 0{\cdot}6728035 & 0{\cdot}6804755 & 0{\cdot}4211261 & -0{\cdot}6556101 \\ -0{\cdot}0708214 & -0{\cdot}0686780 & -0{\cdot}6259673 & 0{\cdot}0000000 \\ -0{\cdot}6550982 & -0{\cdot}6453485 & 0{\cdot}6216760 & 0{\cdot}6556101 \end{bmatrix} \tag{2.13}$$

and

$$T^{-1}AT = \left[\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0{\cdot}7844393 & 64{\cdot}8404082 & 0 \\ 0 & -0{\cdot}0249126 & -0{\cdot}7844393 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right] \tag{2.14}$$

This verifies the result of Proposition 2.1. □

Example 2.2: Consider another system with dynamic matrix

$$A = \begin{bmatrix} 1{\cdot}60 & 6{\cdot}00 & 7{\cdot}40 & 6{\cdot}20 & 2{\cdot}20 \\ -2{\cdot}38 & -4{\cdot}45 & -7{\cdot}27 & -6{\cdot}76 & -2{\cdot}66 \\ 4{\cdot}42 & 9{\cdot}05 & 13{\cdot}43 & 11{\cdot}84 & 4{\cdot}94 \\ -2{\cdot}06 & -4{\cdot}65 & -6{\cdot}49 & -5{\cdot}62 & -2{\cdot}42 \\ -2{\cdot}28 & -8{\cdot}20 & -10{\cdot}62 & -9{\cdot}06 & -2{\cdot}96 \end{bmatrix} \tag{2.15}$$

The eigenvalues of A are at 1·5, 1, −1, 0 and 0·5. Hence, we would have to use the procedure of Remark 2.2. First we have $\alpha = 1{\cdot}5$ and

$$\widetilde{A}_\alpha = \begin{bmatrix} -0{\cdot}1688312 & 21{\cdot}8181818 & 27{\cdot}0337662 & 26{\cdot}1610390 & 5{\cdot}6311688 \\ -1{\cdot}1948052 & -11{\cdot}3636364 & -14{\cdot}4010390 & -14{\cdot}4664935 & -3{\cdot}4348052 \\ 2{\cdot}3838384 & 20{\cdot}4646465 & 26{\cdot}2832323 & 26{\cdot}6602020 & 6{\cdot}5438384 \\ -1{\cdot}2640693 & -13{\cdot}5151515 & -17{\cdot}4271861 & -17{\cdot}9399134 & -4{\cdot}1440693 \\ -0{\cdot}8196248 & -25{\cdot}2929293 & -31{\cdot}6760750 & -30{\cdot}7354690 & -7{\cdot}2596248 \end{bmatrix}$$

Using the package of Lin *et al*. (1991), we find

$$T_\alpha = \begin{bmatrix} 0{\cdot}5260730 & -0{\cdot}4759932 & -0{\cdot}4694273 & 0{\cdot}4255322 & -0{\cdot}1761533 \\ -0{\cdot}2455007 & -0{\cdot}3201876 & 0{\cdot}6942234 & 0{\cdot}3754696 & 0{\cdot}4712102 \\ 0{\cdot}4559299 & 0{\cdot}5946341 & 0{\cdot}1322330 & -0{\cdot}2002505 & -0{\cdot}7178249 \\ -0{\cdot}3156438 & -0{\cdot}4116698 & -0{\cdot}2446311 & -0{\cdot}6758453 & 0{\cdot}2598262 \\ -0{\cdot}5962160 & 0{\cdot}3845110 & -0{\cdot}4694273 & 0{\cdot}4255322 & 0{\cdot}4051527 \end{bmatrix}$$

and

$$T_\alpha^{-1} A T_\alpha = \begin{bmatrix} A_* & 0 \\ 0 & A_\otimes \end{bmatrix}$$

$$= \left[\begin{array}{cccc|c} -1 & 0{\cdot}0211936 & 2{\cdot}9630208 & -5{\cdot}2095969 & 0 \\ 0 & 0 & -1{\cdot}0909336 & -1{\cdot}2820860 & 0 \\ 0 & 0 & 0{\cdot}5 & 0{\cdot}2075349 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1{\cdot}5 \end{array}\right]$$

Next, we have $\beta = 0{\cdot}5$ and

$$\widetilde{A}_{*\beta} = \begin{bmatrix} 7 & -0{\cdot}1695489 & -14{\cdot}3704733 & 19{\cdot}4414739 \\ 0 & -1 & -1{\cdot}7454937 & -1{\cdot}2582407 \\ 0 & 0 & -0{\cdot}2 & 0{\cdot}1423096 \\ 0 & 0 & 0 & 0{\cdot}1428571 \end{bmatrix}$$

Again, using the software package of Lin *et al*. (1991), we get

$$T_{*\beta} = \begin{bmatrix} 0{\cdot}0211889 & 0{\cdot}8933552 & -1 & 0 \\ 0{\cdot}9997755 & -0{\cdot}0189334 & 0 & 0{\cdot}8483480 \\ 0 & 0{\cdot}4489521 & 0 & -0{\cdot}2029648 \\ 0 & 0{\cdot}0000000 & 0 & -0{\cdot}4889897 \end{bmatrix}$$

$$T = \begin{bmatrix} -0{\cdot}4647394 & 0{\cdot}2682318 & -0{\cdot}5260730 & -0{\cdot}5166115 & -0{\cdot}1761533 \\ -0{\cdot}3253176 & 0{\cdot}0984160 & 0{\cdot}2455007 & -0{\cdot}5961342 & 0{\cdot}4712102 \\ 0{\cdot}6041612 & 0{\cdot}4554152 & -0{\cdot}4559299 & 0{\cdot}5755384 & -0{\cdot}7178249 \\ -0{\cdot}4182655 & -0{\cdot}3840153 & 0{\cdot}3156438 & 0{\cdot}0308937 & 0{\cdot}2598262 \\ 0{\cdot}3717915 & -0{\cdot}7506632 & 0{\cdot}5962160 & 0{\cdot}2133955 & 0{\cdot}4051527 \end{bmatrix} \tag{2.16}$$

and

$$T^{-1}AT = \left[\begin{array}{cc|cc|c} 0 & -0\cdot4804181 & 0 & 0 & 0 \\ 0 & 0\cdot5 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & -1\cdot9640299 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1\cdot5 \end{array}\right] \tag{2.17}$$

□

3. Conclusions

We have presented in this short paper a simple algorithm for the stable/unstable decomposition of linear discrete-time systems. We have also shown the relationship between such a decomposition and that of continuous-time systems.

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