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Brief Paper

Simultaneous Finite- and Infinite-zero Assignments of Linear Systems*

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Key Words—Invariant zeros; infinite zeros; zero placement; linear system theory.

Abstract—A simultaneous finite- and infinite-zero assignment problem via sensor selection for linear multivariable systems is proposed. By sensor selection we mean an appropriate choice of the output matrix C. Here, by utilizing the well-known Burnovsky canonical form for a linear system characterized by the matrix pair (A, B), we obtain an explicit construction algorithm that generates a non-empty set $\mathfrak{C}$ of output matrices such that for any member C of this set, the corresponding system characterized by the triple (A, B, C) has the prescribed finite- and infinite-zero structures. Two examples are also given to illustrate our results.

1. *Introduction and problem statement*

The problem of finite-zero (invariant-zero or transmission-zero) assignment of linear multivariable systems has attracted considerable attention from many researchers during the last two decades (see e.g. Emami-Naeini and van Dooren, 1982; Karcanias *et al.*, 1988; Kouvaritakis and MacFarlane, 1976; Patel, 1978; Vardulakis, 1980). Recently, Syrmos and Lewis (1993) re-examined this problem using semistate descriptions, while Syrmos (1993) re-solved it by squaring down a system from the outputs to the inputs. The results reported by these authors are valid for a certain special class of systems, i.e. the resulting systems are always of uniform rank with relative degree one. Also, in what is called a squaring down approach, similar but general results were reported earlier by Sannuti and Saberi (1987). Their results are more general in the sense that the systems considered by them are allowed to have any fixed infinite-zero structure (see e.g. Theorem 4.1 of Sannuti and Saberi, 1987).

Next, it is important to point out that all the results reported in the literature so far, including the ones mentioned above, deal solely with the *finite-zero* assignment problem. That is, the *infinite-zero* structure of the resulting system is either always fixed or of not much concern. To date, to the best of our knowledge, there are no published methods for dealing with simultaneous *finite-* and *infinite-zero* assignment. In this regard we emphasize a point the recent research has brought out, namely that the finite- and infinite-zero structures of a given system play dominant roles in a number of control theoretical problems such as H_2 or H_∞ optimal control, disturbance decoupling, loop transfer recovery, and flexible eigenstructure assignment. The problem of finite- and infinite-zero assignment and its solution thus have a number of important consequences for several other control theoretical problems. In view of this, in this paper, we propose for the first time an explicit method that assigns simultaneously both the finite- and infinite-zero structures in linear multivariable systems. In particular, given a matrix pair (A, B) we develop an explicit construction algorithm that generates a non-empty set $\mathfrak{C}$ of output matrices such that for any member C of this set, the corresponding system characterized by the matrix triple (A, B, C) has the prescribed finite- and infinite-zero structures. We note that selecting an output matrix C corresponds to selecting a set of sensors that define the measured output.

It is useful to compare in detail our method with the squaring down method of Sannuti and Saberi (1987). The latter method starts with a system characterized by a given matrix triple (A, B, C), and then designs pre- and post-compensators such that the compensated system has some additional finite (invariant) zeros at chosen locations while keeping it infinite zero structure fixed. On the other hand, in this paper, we start with a given matrix pair (A, B) while the choice of the matrix C is completely free. Accordingly, we choose a matrix C such that the corresponding system characterized by the matrix triple (A, B, C) has the prescribed finite- and infinite-zero structures. The physical applicability of our method is quite different from the squaring down method of Sannuti and Saberi (1987).

We first recall the definitions of invariant zeros and infinite zeros of a linear multivariable system. Consider a system Σ characterized by

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. Let $H(s) := C(sI_n - A)^{-1}B$ be the transfer function of Σ. Then a complex scalar z is said to be an invariant zero of the system Σ or $H(s)$ if

$$\operatorname{rank}\begin{bmatrix} zI_n - A & -B \\ C & 0 \end{bmatrix} < n + \text{normal rank}\,\{H(s)\}, \tag{2}$$

where the normal rank of $H(s)$ is defined as the rank of $H(s)$ over the field of rational functions. We refer interested readers to Saberi *et al.* (1993) for a complete study of invariant zeros and the associated algebraic and geometric multiplicities as well as zero directions. The infinite-zero structure of $H(s)$ can be defined from the point of view of Smith–McMillan theory. To define the zero structure of $H(s)$ at infinity, one can use the familiar Smith–McMillan description of the zero structure at finite frequencies. Namely, a rational matrix $H(s)$ possesses an infinite zero of order q when $H(1/z)$ has a finite zero of precisely that order at $z = 0$ (see e.g. Commault and Dion, 1982; Hung and MacFarlane, 1981; Pugh and Ratcliffe, 1979; Rosenbrock, 1970; Verghese, 1978). The number of zeros at infinity together with their orders indeed defines an infinite-zero structure. Note that the invariant zeros and the orders of the infinite zeros of Σ are also respectively related to the structural-invariant index lists $\mathscr{I}_1$ and $\mathscr{I}_4$ of Morse (1973). As is well known, both these structures of linear systems play extremely important roles in modern control theory.

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In this paper, we consider a simultaneous finite- and infinite-zero assignment problem from the system

$$\dot{x} = Ax + Bu, \tag{3}$$

with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Without loss of generality, we assume throughout this paper that B is of maximal rank. Our goal, as mentioned earlier, is to find an output matrix C such that the resulting system (A, B, C) has the desired finite- and infinite-zero structures. As will be seen shortly, our construction method in fact yields a set $\mathscr{C}$ of output matrices such that for all elements of $\mathscr{C}$, the corresponding system are square-invertible with the same desired invariant zeros and the same desired infinite-zero structure.

This paper is organized as follows. In Section 2, we recall some background material that is used in the derivation of our results. Section 3 gives our main results, while Section 4 draws some concluding remarks.

Throughout the paper, A' denotes the transpose of A, 0 denotes a scalar zero or a zero matrix with appropriate dimension while I_k denotes the identity matrix of dimension $k \times k$. With a slight abuse of notation, I_k with $k \le 0$ is treated as an empty matrix. Also, ★ denotes some constant matrix with appropriate dimension that is not of much interest in the given context. A set $\mathscr{W}$ of complex scalars is said to be self-conjugate if for any $w \in \mathscr{W}$, its complex conjugate $\bar{w} \in \mathscr{W}$.

2. *Preliminaries*

In this section, we recall the well-known Brunovsky canonical form of a constant pair (A, B), and the special coordinate basis for a linear time-invariant system (A, B, C) from Sannuti and Saberi (1987). The former plays a central role in the development of our construction algorithms, while the latter is the key to the proofs of our results.

2.1. *The Brunovsky canonical form.* The Brunovsky canonical form for a linear system or simply for a constant matrix pair (A, B) can be summarized in the following theorem.

Theorem 2.1. (*The Brunovsky canonical form.*) Consider a linear system (3) or a pair (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. There exist nonsingular state and input transformations $T_s \in \mathbb{R}^{n\times n}$ and $T_i \in \mathbb{R}^{m\times m}$ such that

$$\tilde{A} := T_s^{-1}AT_s = \begin{bmatrix} A_o & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & I_{k_1-1} & \dots & 0 & 0 \\ \star & \star & \star & \dots & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & I_{k_m-1} \\ \star & \star & \star & \dots & \star & \star \end{bmatrix},$$
$$\tilde{B} := T_s^{-1}BT_i = \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \vdots & 1 \end{bmatrix}, \tag{4}$$

where $k_i > 0$, $i = 1, \dots, m$, A_o is of dimension $n_o := n - \sum_{i=1}^{m} k_i$, and its eigenvalues are the uncontrollable modes of the pair (A, B). The set of integers, $\mathscr{C} := \{n_o, k_1, \dots, k_m\}$, is called the controllability index of the pair (A, B).

Proof. This is well known. We refer to Saberi (1985) for a construction algorithm for T_s and T_i. Also, an m-file that realizes such a canonical form can be found in a commercially available software package, Linear Systems Toolbox (Lin *et al.*, 1991). ■

2.2. *A special coordinate basis.* In this subsection we recall the special coordinate basis for a linear time-invariant system from Sannuti and Saberi (1987). Such a coordinate basis has a distinct feature of explicitly displaying the finite- and infinite-zero structures of a given system. We have the following theorem.

Theorem 2.2. (*The special coordinate basis.*) Consider a linear time-invariant system Σ of (1). Then there exist nonsingular state, input and output transforms $\Gamma_s \in \mathbb{R}^{n\times n}$, $\Gamma_o \in \mathbb{R}^{p\times p}$ and $\Gamma_i \in \mathbb{R}^{m\times m}$ such that

$$\tilde{A} := \Gamma_s^{-1}A\Gamma_s = \begin{bmatrix} A_{aa} & L_{ab}C_b & 0 & L_{ad}C_d \\ 0 & A_{bb} & 0 & L_{bd}C_d \\ B_cE_{ca} & B_cE_{cb} & A_{cc} & L_{cd}C_d \\ B_dE_{da} & B_dE_{db} & B_dE_{dc} & A_{dd} \end{bmatrix}, \tag{5}$$

$$\tilde{B} := \Gamma_s^{-1}B\Gamma_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix},$$
$$\tilde{C} := \Gamma_o^{-1}C\Gamma_s = \begin{bmatrix} 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix} \tag{6}$$

where the pair (A_{bb}, C_b) is completely observable and in fact (A_{bb}', C_b') is in the Brunovsky canonical form, (A_{cc}, B_c) is completely controllable and in the Brunovsky canonical form as well, and the triple (A_{dd}, B_d, C_d) has the special form

$$C_d = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \tag{7}$$

$$A_{dd} = \begin{bmatrix} 0 & I_{q_1-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_{q_{m_d}-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} + L_{dd}C_d + B_dE_{dd},$$
$$B_d = \begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix} \tag{8}$$

for some $q_i > 0$, $i = 1, \dots, m_d$, and some appropriate dimensional matrices L_{dd} and E_{dd}.

Proof. See Sannuti and Saberi (1987). Again, an m-file that realizes such a special coordinate basis is also reported in the Linear Systems Toolbox of Lin *et al.* (1991). ■

In what follows, we state some important properties of the special coordinate basis that are pertinent to our present work.

Property 2.1.

(1) The invariant zeros of the given system Σ of are given by the eigenvaues of A_{aa}, which correspond to the structural-invariant index list $\mathscr{I}_1$ of Morse (1973).
(2) The infinite-zero structure of Σ is fully specified by the set of integers, $\mathcal{Q} := \{q_1, \dots, q_{m_d}\}$, which corresponds to the structural-invariant index list $\mathscr{I}_4$ of Morse (1973).
(3) The controllability index of (A_{bb}', C_b') corresponds to the structural-invariant index list $\mathscr{I}_3$ of Morse (1973). Moreover, the system Σ is right-invertible if and only if the $\mathscr{I}_3$ list is an empty set.
(4) The controllability index of (A_{cc}, B_c) corresponds to the structural-invariant index list $\mathscr{I}_2$ of Morse (1973). Moreover, the system Σ is left-invertible if and only if the $\mathscr{I}_2$ list is an empty set.
(5) The system is invertible if and only if both the $\mathscr{I}_2$ and $\mathscr{I}_3$ lists of Morse (1973) are empty sets.

Next, we introduce the following theorem, which is mainly due to Chen *et al.* (1992) and which is crucial to the development of our main results in this paper.

Theorem 2.3. Consider an invertible linear system characterized by a matrix triple (A, B, C). It follows from Theorem

2.2 that there exist appropriate nonsingular transformations Γ_s, Γ_o and Γ_i such that

$$\Gamma_s^{-1}A\Gamma_s = \begin{bmatrix} A_{aa} & L_{ad}C_d \\ B_dE_{da} & A_{dd} \end{bmatrix},$$
$$\Gamma_s^{-1}B\Gamma_i = \begin{bmatrix} 0 \\ B_d \end{bmatrix}, \quad \Gamma_o^{-1}C\Gamma_s = [0 \quad C_d], \tag{9}$$

where $\lambda(A_{aa})$ corresponds to the invariant zeros of (A, B, C), while the triple (A_{dd}, B_d, C_d) is in the form of (7) and (8). Let

$$\hat{B} := \Gamma_s \begin{bmatrix} K_a \\ B_d \end{bmatrix} \Gamma_i^{-1} \tag{10}$$

Then the new triple $(A, \hat{B}, C)$ is also invertible and has the same infinite-zero structure as the given system (A, B, C). Moreover, the invariant zeros of $(A, \hat{B}, C)$ are given by the eigenvalues of the matrix $A_{aa} - K_aE_{da}$.

Proof. The theorem follows from the proofs of Theorems 2.1 and 3.1 of Chen *et al.* (1992). ∎

3. *Simultaneous finite- and infinite-zero assignment*

We present in this section the main results of this paper. We begin with the simultaneous finite- and infinite-zero assignment problem for single-input single-output (SISO) systems because the solution to this problem is relatively simple and intuitive. It is helpful in understanding the derivation of the result for multiple-input multiple-output (MIMO) systems given later in Section 3.2. Moreover, the set of output matrices obtained for the SISO case is complete.

3.1. *Simultaneous zero assignment for SISO systems.* We consider in this subsection the finite- and infinite-zero assignment problem for the system (3) with $m = 1$. Note that for this case, the controllability index of the system is simply given by $\mathscr{C} := \{n_o, k_1\}$, where n_o and k_1 are respectively the dimensions of the uncontrollable and controllable subspaces of the given system. We first have the following theorem. The proof of this theorem is constructive and it gives an explicit expression of a set $\mathfrak{C}$ of output matrices for each of whose elements the corresponding system has the prescribed finite- and infinite-zero structures.

Theorem 3.1. (*The SISO case.*) Consider a system characterized by $(A, \underline{b})$ with $A \in \mathbb{R}^{n\times n}$ and $\underline{b} \in \mathbb{R}^{n\times 1}$. Let $\mathscr{C} := \{n_o, k_1\}$ be the controllability indices of $(A, \underline{b})$. Also, let $\{u_1, \ldots, u_{n_o}\}$ be the uncontrollable modes of $(A, \underline{b})$. Then for any given integer q_1, $0 < q_1 \le k_1$ and a set of self-conjugate scalars, $\{z_1, \ldots, z_{k_1-q_1}\}$, there exists a non-empty set of output matrices $\mathfrak{C} \subset \mathbb{R}^{1\times n}$ such that for any $\underline{c} \in \mathfrak{C}$ the resulting system $(A, \underline{b}, \underline{c})$ has $n_o + k_1 - q_1$ invariant zeros at $\{u_1, \ldots, u_{n_o}, z_1, \ldots, z_{k_1-q_1}\}$ and has an infinite-zero structure $\mathcal{Q} = \{q_1\}$, i.e. the relative degree of $(A, \underline{b}, \underline{c})$ is equal to q_1.

Proof. It follows from Theorem 2.1 that there exist nonsingular state and input transformations T_s and T_i such that $(A, \underline{b})$ can be transformed into the Brunovsky canonical form (4). Next, we rewrite (4) as

$$\tilde{A} = \begin{bmatrix} A_o & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_1-q_1-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{q_1-1} \\ \star & \star & \star & \star & \star \end{bmatrix}, \quad \tilde{\underline{b}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{11}$$

Let

$$a(s) := s^{k_1-q_1} + a_1 s^{k_1-q_1-1} + \cdots + a_{k_1-q_1} \tag{12}$$

be a polynomial having roots at $z_1, \ldots, z_{k_1-q_1}$. Also, let us define

$$\underline{a} := [a_{k_1-q_1-1} \quad \ldots \quad a_1].$$

Then the desired set of output matrices $\mathfrak{C}$ is given by

$$\mathfrak{C} := \{\underline{c} \in \mathbb{R}^{1\times n} \mid \underline{c} = \alpha[\underline{d} \quad a_{k_1-q_1} \quad \underline{a} \quad 1 \quad 0]T_s^{-1}, \; 0 \ne \alpha \in \mathbb{R}, \underline{d} \in \mathbb{R}^{1\times n_o}\}. \tag{13}$$

In what follows, we shall proceed to prove that the resulting system with any $\underline{c} \in \mathfrak{C}$ has all the properties stated in Theorem 3.1. Let us define

$$\hat{A}_{aa} := \begin{bmatrix} 0 & I_{k_1-q_1-1} \\ 0 & 0 \end{bmatrix}', \quad \hat{E}_{da} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}', \quad \hat{K}_a := [a_{k_1-q_1} \quad \underline{a}]', \tag{14}$$

$$\hat{A}_{dd} := \begin{bmatrix} 0 & I_{q_1-1} \\ \star & \star \end{bmatrix}', \quad \hat{C}_d := \begin{bmatrix} 0 \\ 1 \end{bmatrix}', \quad \hat{B}_d := [1 \quad 0]'. \tag{15}$$

It is simple to see that the pair $(\hat{A}_{aa}, \hat{E}_{da})$ is completely observable and

$$\hat{A}_{aa}^c := \hat{A}_{aa} - \hat{K}_a\hat{E}_{da}$$

has eigenvalues $z_1, \ldots, z_{k_1-q_1}$. Also, it is straightforward to verify that the system $(A, \hat{\underline{b}}, \hat{\underline{c}})$, where

$$\hat{A} := \tilde{A}' = \begin{bmatrix} A_o' & 0 & \star\cdot\hat{C}_d \\ 0 & \hat{A}_{aa} & \star\cdot\hat{C}_d \\ 0 & \hat{B}_d\hat{E}_{da} & \hat{A}_{dd} \end{bmatrix}, \quad \hat{\underline{b}} := \begin{bmatrix} \underline{d}' \\ \hat{K}_a \\ \hat{B}_d \end{bmatrix}, \tag{16}$$

$$\hat{\underline{c}} := \tilde{\underline{b}}' = [0 \quad 0 \quad \hat{C}_d]. \tag{17}$$

Then it follows from Appendix B of Chen *et al.* (1992) (see also Theorem 2.3) that there exists a nonsingular state transformation T such that

$$T^{-1}\hat{A}T = \begin{bmatrix} A_o' & \underline{d}'\hat{E}_{da} & \star\cdot\hat{C}_d \\ 0 & \hat{A}_{aa}^c & \star\cdot\hat{C}_d \\ 0 & \hat{B}_d\hat{E}_{da} & \hat{A}_{dd}^\star \end{bmatrix}, \quad T^{-1}\hat{\underline{b}} = \begin{bmatrix} 0 \\ 0 \\ \hat{B}_d \end{bmatrix} \tag{18}$$

and

$$\hat{\underline{c}}T = [0 \quad 0 \quad \hat{C}_d], \tag{19}$$

where

$$\hat{A}_{dd}^\star = \hat{A}_{dd} + \hat{B}_d \cdot \star. \tag{20}$$

Note that $(T^{-1}\hat{A}T, T^{-1}\hat{\underline{b}}, \hat{\underline{c}}T)$ is now in the form of the special coordinate basis of Theorem 2.2. Thus it follows from the properties of this special coordinate basis that $(T^{-1}\hat{A}T, T^{-1}\hat{\underline{b}}, \hat{\underline{c}}T)$, or equivalently $(A, \underline{b}, \underline{c})$, has an infinite-zero structure $\mathcal{Q} = \{q_1\}$ and has invariant (finite) zeros at

$$\lambda(A_o') \cup \lambda(\hat{A}_{aa}^c) = \{u_1, \ldots, u_{n_o}, z_1, \ldots, z_{k_1-q_1}\}. \tag{21}$$

This completes the proof of Theorem 3.1. ∎

The following corollary shows that $\mathfrak{C}$ of (13) is complete.

Corollary 3.1. The set of output matrices $\mathfrak{C}$ in (3.3) is complete, i.e. any output matrix $\underline{c}$ for which the resulting system $(A, \underline{b}, \underline{c})$ has all properties listed in Theorem 3.1 is a member of $\mathfrak{C}$.

Proof. Let $\underline{c}$ be such that the resulting system $(A, \underline{b}, \underline{c})$ has invariant zeros at $\{u_1, \ldots, u_{n_o}\}$ and $\{z_1, \ldots, z_{k_1-q_1}\}$ and has relative degree $q_1 \le k_1$. It is obvious that $\underline{c}$ can be written in conformity with (11) as

$$\underline{c} = [\underline{d} \quad e \quad \underline{h} \quad g \quad \underline{z}]T_s^{-1}, \tag{22}$$

where $\underline{d} \in \mathbb{R}^{1\times n_o}$, $e \in \mathbb{R}$, $\underline{h} \in \mathbb{R}^{1\times(k_1-q_1-1)}$, $g \in \mathbb{R}$ and $\underline{z} \in \mathbb{R}^{1\times(q_1-1)}$. Note that

$$[A^{q_1}\underline{b} \quad \ldots \quad A\underline{b} \quad \underline{b}] = T_s[\tilde{A}^{q_1}\tilde{\underline{b}} \quad \ldots \quad \tilde{A}\tilde{\underline{b}} \quad \tilde{\underline{b}}]T_s^{-1}$$
$$= T_s \begin{bmatrix} 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \star & \ldots & 1 & 0 \\ \star & \ldots & \star & 1 \end{bmatrix} T_s^{-1}. \tag{23}$$

Then it is simple to verify that the fact that $(A, \underline{b}, \underline{c})$ has a relative degree q_1, i.e.

$$\underline{c}\underline{b} = \underline{c}A\underline{b} = \cdots = \underline{c}A^{q_1-1}\underline{b} = 0 \tag{24}$$

and $\underline{c}A^{q_1}\underline{b} \ne 0$, implies $\underline{z} = 0$ and $g \ne 0$. Thus we have

$$\underline{c} = \alpha[\underline{d}/\alpha \quad e/\alpha \quad \underline{h}/\alpha \quad 1 \quad 0]T_s^{-1},$$

where $\alpha = 1/g$. Following the same procedure as in (14)–(20), it can be shown that the invariant zeros of $(A, \underline{b}, \underline{c})$ are given by $\lambda(A_o) = \{u_1, \ldots, u_{n_o}\}$ and

$$\lambda(\hat{A}_{aa} - [e/\alpha \quad \underline{h}/\alpha]'\hat{E}_{da}) = \{z_1, \ldots, z_{k_1-q_1}\}. \tag{25}$$

Since $(\hat{A}_{aa}, \hat{E}_{da})$ is a single-output system, $[e/\alpha \;\; \underline{h}/\alpha]'$ is determined uniquely by the closed-loop eigenvalues $\{z_1, \ldots, z_{k_1-q_1}\}$. Hence

$$[\underline{e}/\alpha \quad \underline{h}/\alpha] = [a_{k_1-q_1} \quad \underline{a}] \tag{26}$$

and $\underline{c} \in \mathfrak{C}$. ■

Remark 3.1. We should like to point out here that in Theorem 3.1, we are able to assign an infinite zero or relative degree of any arbitrary order between 1 and k_1 for the resulting system $(A, \underline{b}, \underline{c})$. Obviously, this is much more widely applicable in most practical situations. The result of Syrmos (1993), for example, can only generate a $\underline{c}$ such that the relative degree of the resulting system is equal to one, i.e. $q_1 = 1$.

We illustrate the above result in the following example.

Example 3.1. Consider a system characterized by

$$\dot{x} = Ax + \underline{b}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u. \tag{27}$$

It is simple to see that the pair $(A, \underline{b})$ is already in the form of the Brunovsky canonical form with a controllability index $\mathscr{C} = \{0, 3\}$. Then it follows from Theorem 3.1 that one is free to choose output matrices such that the resulting systems have

(1) infinite-zero structure $\mathcal{Q} = \{3\}$ with no invariant zero;
(2) $\mathcal{Q} = \{2\}$ with one invariant zero;
(3) $\mathcal{Q} = \{1\}$ with two invariant zeros.

The systems with the following output matrices respectively have such properties:

$$\underline{c}_1 = \alpha[1 \quad 0 \quad 0],$$
$$\underline{c}_2 = \alpha[a_1 \quad 1 \quad 0],$$
$$\underline{c}_3 = \alpha[a_2 \quad a_1 \quad 1],$$

where $0 \neq \alpha \in \mathbb{R}$. This can easily be verified by computing the corresponding transfer functions. We have

$$H_1(s) := \underline{c}_1(sI_3 - A)^{-1}\underline{b} = \frac{\alpha}{s(s^2 - 1)},$$

$$H_2(s) := \underline{c}_2(sI_3 - A)^{-1}\underline{b} = \frac{\alpha(s + a_1)}{s(s^2 - 1)},$$

$$H_3(s) := \underline{c}_3(sI_3 - A)^{-1}\underline{b} = \frac{\alpha(s^2 + a_1 s + a_2)}{s(s^2 - 1)}.$$

Note that only $\underline{c}_3$ or the system $H_3(s)$ can be obtained from Syrmos' approach.

3.2. *Simultaneous zero assignment for MIMO systems.* Next, we proceed to solve the simultaneous finite- and infinite-zero assignment problem for MIMO systems. As in the SISO case, we shall first state our result in a theorem and then give a constructive proof that generates an explicit expression for a non-empty set $\mathfrak{C}$ of output matrices such that for any $C \in \mathfrak{C}$, the resulting system (A, B, C) is square-invertible and has the chosen finite- and infinite-zero structures. A construction procedure is also summarized as an easy-to-follow algorithm.

Theorem 3.2. (The MIMO case.) Consider a system characterized by (A, b), with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, with B being of full rank. Let $\mathscr{C} := \{n_o, k_1, \ldots, k_m\}$ be the controllability index of (A, B). Also, let $\{u_1, \ldots, u_{n_o}\}$ be the uncontrollable modes of (A, B). Then, for any given set of integers, $\mathcal{Q} := \{q_1, \ldots, q_m\}$, with $0 < q_i \leq k_i$, $i = 1, \ldots, m$, and a set of self-conjugate scalars $\{z_1, \ldots, z_l\}$, where $l := \sum_{i=1}^{m} (k_i - q_i)$, there exists a non-empty set $\mathfrak{C} \subseteq \mathbb{R}^{m \times n}$ of output matrices such that for any $C \in \mathfrak{C}$, the corresponding system (A, B, C) is square-invertible with $n_o + l$ invariant zeros at $\{u_1, \ldots, u_{n_o}, z_1, \ldots, z_l\}$ and has an infinite-zero structure $\mathcal{Q} = \{q_1, \ldots, q_m\}$.

Proof. Again, it follows from Theorem 2.1 that there exist nonsingular transformation T_s and T_i such that the pair (A, B) can be transformed into the Brunovsky canonical form (4). Next, we rewrite $\tilde{A}$ and $\tilde{B}$ as

$$\tilde{A} = \begin{bmatrix} A_o & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_1-q_1-1} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{q_1} & \ldots & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \ldots & \star & \star & \star & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & I_{k_m-q_m-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & I_{q_m-1} \\ \star & \star & \star & \star & \star & \ldots & \star & \star & \star & \star \end{bmatrix}, \tag{28}$$

$$\tilde{B} = \begin{bmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 1 \end{bmatrix},$$

and we define

$$\tilde{A}_{aa} := \begin{bmatrix} 0 & I_{k_1-q_1-1} & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & I_{k_m-q_m-1} \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \quad \tilde{L}_{ad} := \begin{bmatrix} 0 & \ldots & 0 \\ 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \\ 0 & \ldots & 1 \end{bmatrix}. \tag{29}$$

Note that $(\tilde{A}_{aa}, \tilde{L}_{ad})$ is completely controllable, and in fact is in the Brunovsky canonical form. Let us also define

$$\mathbf{F}_a := \{\tilde{F}_a \in \mathbb{R}^{m \times l} \mid \lambda(\tilde{A}_{aa} - \tilde{L}_{ad}\tilde{F}_a) = \{z_1, \ldots, z_l\}\}. \tag{30}$$

Then, we partition any $\tilde{F}_a \in \mathbf{F}_a$ in conformity with (29) as

$$\tilde{F}_a = \begin{bmatrix} F_{11}^0 & F_{11}^1 & \ldots & F_{1m}^0 & F_{1m}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{m1}^0 & F_{m1}^1 & \ldots & F_{mm}^0 & F_{mm}^1 \end{bmatrix}, \tag{31}$$

and define a corresponding $m \times n$ matrix, in conformity with (28),

$$\tilde{C} := \begin{bmatrix} D_1 & F_{11}^0 & F_{11}^1 & 1 & 0 & \ldots & F_{1m}^0 & F_{1m}^1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ D_m & F_{m1}^0 & F_{m1}^1 & 0 & 0 & \ldots & F_{mm}^0 & F_{mm}^1 & 1 & 0 \end{bmatrix}, \tag{32}$$

where

$$\tilde{D} = \begin{bmatrix} D_1 \\ \vdots \\ D_m \end{bmatrix} \quad (33)$$

is any arbitrary matrix of dimension $m \times n_o$. Then the desired set $\mathfrak{C}$ of matrices is given by

$$\mathfrak{C} := \{C \in \mathbb{R}^{n\times m} \mid C = \Gamma\tilde{C}T_s^{-1}, \text{ with } \tilde{D} \in \mathbb{R}^{m\times n_o}, \tilde{F}_a \in \mathbf{F}_a, \Gamma \in \mathbb{R}^{m\times m} \text{ and } \det(\Gamma) \neq 0\}. \quad (34)$$

We now proceed to prove the properties of the resulting system (A, B, C) for $C \in \mathfrak{C}$. We note that the finite- and infinite zero-structures of (A, B, C) are equivalent to those of $(\tilde{A}, \tilde{B}, \tilde{C})$ because they are related by some nonsingular transformations T_s, T_i and Γ. Noting the structure of $(\tilde{A}, \tilde{B}, \tilde{C})$, it is simple to see that there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P^{-1}\tilde{A}P = \begin{bmatrix} A_o & 0 & 0 \\ 0 & \tilde{A}_{aa} & \tilde{L}_{ad}\tilde{C}_d \\ 0 & \tilde{B}_d\cdot\star & \tilde{A}_{dd} \end{bmatrix}, \quad P^{-1}\tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \tilde{B}_d \end{bmatrix} \quad (35)$$

and

$$\tilde{C}P = [\tilde{D} \quad \tilde{F}_a \quad \tilde{C}_d], \quad (36)$$

where

$$\tilde{A}_{dd} := \begin{bmatrix} 0 & I_{q_1-1} & \cdots & 0 & 0 \\ \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{q_m-1} \\ \star & \star & \cdots & \star & \star \end{bmatrix}, \quad \tilde{B}_d := \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}, \quad (37)$$

$$\tilde{C}_d := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 1 & 0 \end{bmatrix}. \quad (38)$$

Again, as was done in the SISO case, by taking the dual result to Theorem 2.3, it can be shown that $(\tilde{A}, \tilde{B}, \tilde{C})$, or equivalently the system (A, B, C), has an infinite-zero structure $\mathcal{Q} = \{q_1, \ldots, q_m\}$ and has invariant zeros at

$$\lambda(A_o) \cup \lambda(\tilde{A}_{aa} - \tilde{L}_{ad}\tilde{F}_a) = \{u_1, \ldots, u_{n_o}, z_1, \ldots, z_l\}. \quad (39)$$

This completes the proof of Theorem 3.2. ■

The following remarks are in order.

Remark 3.2.

(1) The uncontrollable modes of (A, b) are automatically included in the set of invariant zeros of (A, B, C) for any C such that (A, B, C) is square-invertible. Hence the invariant zeros at $u_1, \ldots, u_{n_o}$, in both Theorems 3.1 and 3.2, cannot be re-assigned. However, they can be excluded from the invariant zeros of a left-invertible or non-invertible system.

(2) In order to have a square-invertible system (A, B, C), the set $\mathcal{Q}$ must not have zero elements. However, if one wishes to have a non-invertible system then the elements of $\mathcal{Q}$ might be set to zero.

(3) By selecting $\mathcal{Q} = \{1, \ldots, 1\}$ in Theorem 3.2, we recover the result of Syrmos (1993). In this case we obtain a set of uniform-rank systems with relative degree of one. Obviously, our result in Theorem 3.2 is much more general than that of Syrmos.

(4) Unfortunately, it can be shown by some examples that $\mathfrak{C}$ of (34) is not necessarily complete for $m > 1$. That is there exists an output matrix C such that the resulting system (A, B, C) has all the properties listed in Theorem 3.2, but $C \notin \mathfrak{C}$.

Remark 3.3. Note that the procedure for the construction of the desired set $\mathfrak{C}$ of output matrices is buried in the proof of Theorem 3.2. We should like to summarize as follows an easy-to-follow step-by-step algorithm that generates this $\mathfrak{C}$.

(1) Given (A, B), compute nonsingular transformations T_s and T_i such that $(T_s^{-1}AT_s, T_s^{-1}BT_i)$ is in Brunovsky canonical form and obtain the controllability index $\{n_o, k_1, \ldots, k_m\}$.

(2) Specify a desired infinite-zero structure for the resulting systems in a set of integers $\{q_1, \ldots, q_m\}$ with $0 < q_i \leq k_i$, $i = 1, \ldots, m$.

(3) Specify a set of desired invariant zeros $\{z_1, \ldots, z_l\}$, where $l = \sum_{i=1}^{m} (k_i - q_i)$, which must be self-conjugate.

(4) Define $(\tilde{A}_{aa}, \tilde{L}_{ad})$ as in (29) and compute the set $\mathbf{F}_a$ as in (30).

(5) Finally, compute the desired set of output matrices $\mathfrak{C}$ as in (34).

We illustrate Theorem 3.2 by the following example.

Example 3.2. Consider a system characterized by

$$\dot{x} = Ax + Bu = \begin{bmatrix} -2 & -5 & -4 & -4 \\ 2 & 3 & 3 & 3 \\ -2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 & 0 \\ 1 & 1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} u. \quad (40)$$

Using the software package of Lin *et al.* (1991), we have found that

$$T_s = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_i = I_2, \quad (41)$$

and the Brunovsky canonical form of (A, B) as

$$\tilde{A} = T_s^{-1}AT_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 1 \end{bmatrix}, \quad \tilde{B} = T_s^{-1}B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (42)$$

with a controllability index $\mathscr{C} = \{0, 2, 2\}$. Employing the procedure as in the proof of Theorem 3.2, we obtain the following set of output matrices:

$$\mathfrak{C}_1 = \left\{\Gamma\begin{bmatrix} a_1 & 1 & a_2 & 0 \\ a_3 & 0 & a_4 & 1 \end{bmatrix}T_s^{-1} \mid a_1 + a_4 = a_1a_4 - a_2a_3 = 2, \Gamma \in \mathbb{R}^{2\times 2} \text{ with } \det(\Gamma) \neq 0\right\},$$

such that for any $C \in \mathfrak{C}_1$, the resulting system (A, B, C) has an infinite-zero structure $\mathcal{Q} = \{1, 1\}$ and two invariant zeros at $-1 \pm j1$. The following is another set of output matrices that we obtain:

$$\mathbb{C}_2 = \left\{\Gamma\begin{bmatrix} 1 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \end{bmatrix}T_s^{-1} \mid a \in \mathbb{R}, \Gamma \in \mathbb{R}^{2\times 2} \text{ with } \det(\Gamma) \neq 0\right\}.$$

It is simple to verify that for any $C \in \mathbb{C}_2$, the corresponding system (A, B, C) has an infinite-zero structure $\mathcal{Q} = \{1, 2\}$ and one invariant zero at -1.

4. *Concluding remarks*

We have proposed an explicit method that assigns simultaneously both the finite- and infinite-zero structures in linear multivariable systems by an appropriate selection of the output matrix. Selecting an output matrix corresponds to selecting a set of sensors that define the measured output. The computations involved in our method are rather simple. The required computations for the transformation matrices of the Brunovsky canonical form can easily be done by an m-file brunovsk.m in a commercially available software package, Linear Systems Toolbox (Lin *et al.*, 1991). We should like to note that our current method deals only with the assignment of finite and infinite zeros, i.e., in terms of the terminology of Morse (1973), only with the lists $\mathcal{I}_1$ and $\mathcal{I}_4$. In other words, the systems that result from our current method always have $\mathcal{I}_2$ and $\mathcal{I}_3$ as empty lists. We believe that our method can be extended to yield systems having all desired structural invariant index lists $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$. This is the subject of our future research.

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