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THE DISCRETE-TIME H_∞ CONTROL PROBLEM WITH MEASUREMENT FEEDBACK

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SUMMARY

This paper is concerned with the discrete-time H_∞ control problem with measurement feedback. We extend previous results by having weaker assumptions on the system parameters. We also show explicitly the structure of H_∞ controllers. Finally, we show that it is in certain cases possible, without loss of performance, to reduce the dynamical order of the controllers.

KEY WORDS H_∞ control Discrete-time systems Reduced order observers

1. INTRODUCTION

The H_∞ control problem has been studied extensively. First in continuous-time (see, for example, References 3, 4, 10 and 13) and later in discrete time (see, for example, References 1, 8, 6 and 14). For a more extensive reference list we refer to two recent books [2, 15].

The objective of this paper is to present a solution of the general discrete-time H_∞ control problem. One way to approach this problem is to transform the discrete-time H_∞ optimal control problem into an equivalent continuous-time H_∞ control problem via bilinear transformation. Then the continuous-time controllers that are solutions to the auxiliary problem can be obtained and transformed back to their discrete-time equivalent using inverse bilinear transformation. However, in our opinion it is more natural to solve this problem directly in discrete-time setting and in terms of the original system's performance. This approach leaves the possibility of directly observing the effect of certain physical parameters which might otherwise be blurred by the transformation to continuous-time. In view of this, and in accordance with earlier literature [1, 6, 8, 12], we take this direct approach in solving the discrete-time H_∞ optimal control problem.

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Compared to the existing literature, we solve this problem under weaker assumptions. All the existing literature on the discrete-time H_∞ control problem make the following assumptions on the system:

- The subsystem from the control input to the control output should be left invertible and should not have invariant zeros on the unit circle.
- The subsystem from the disturbance to the measurement output should be right invertible and should not have invariant zeros on the unit circle.

These conditions are the discrete-time analogue of what are called regular problems in continuous-time H_∞ control problems. In this paper, we remove the abovementioned left and right invertibility condition.

Moreover, we give a representation of one controller in a suitable form such that it becomes very transparent that this controller is a state *and disturbance* estimator in conjunction with a full-information feedback (i.e. a feedback of both state and disturbance). Such an interpretation was not available before and because of the involved formulas it was not very clear what kind of structure discrete-time H_∞ controllers should have.

Finally, a novel aspect of this paper is that we show that if certain states or disturbances are observed directly, then this yields the possibility of deriving a controller of lower MacMillan degree. This result again corresponds to those obtained in continuous-time case (see Reference 16).

The notation in this paper will be fairly standard. By $\mathbb{N}$ and $\mathbb{R}$ we denote the natural numbers and the real numbers, respectively. Moreover by σ we denote the shift

$$(\sigma x)(k) := x(k+1) \quad \forall k \in \mathbb{N}$$

$\operatorname{rank}_{\mathscr{K}}$ denotes the rank as a matrix with entries in the field $\mathscr{K}$. By $\mathbb{R}(z)$ we denote the field of real rational functions. Moreover, by $X^\dagger$ we denote the Moore–Penrose inverse of the matrix X. Finally, by $\rho(X)$ we denote the spectral radius of the matrix X.

2. PROBLEM FORMULATION AND MAIN RESULTS

We consider the following time-invariant system:

$$\Sigma: \begin{cases} \sigma x = Ax + Bu + Ew \\ y = C_1 x + D_{12} w \\ z = C_2 x + D_{21} u + D_{22} w \end{cases} \tag{1}$$

where for all $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, $y(k) \in \mathbb{R}^l$ is the measurement, $w(k) \in \mathbb{R}^q$ is the unknown disturbance and $z(k) \in \mathbb{R}^p$ is the output to be controlled. A, B, E, C_1, C_2, D_{12}, D_{21} and D_{22} are matrices of appropriate dimension.

If we apply a dynamic feedback law $u = Fy$ to Σ then the closed-loop system with zero initial conditions defines a convolution operator $\Sigma_{\mathrm{cl,F}}$ from w to y. We seek a feedback law $u = Fy$ which is internally stabilizing and which minimizes the $\mathscr{L}_2$-induced operator norm of $\Sigma_{\mathrm{cl,F}}$ over all internally stabilizing feedback laws. We will investigate dynamic feedback laws of the form:

$$\Sigma_{\mathrm{F}}: \begin{cases} \sigma p = Kp + Ly \\ u = Mp + Ny \end{cases} \tag{2}$$

We will say that the dynamic compensator Σ_{F}, given by (2), is internally stabilizing when applied to the system Σ, described by (1), if the following matrix is asymptotically stable:

$$\begin{pmatrix} A + BNC_1 & BM \\ LC_1 & K \end{pmatrix} \tag{3}$$

i.e. all its eigenvalues lie in the open unit disk. Denote by G_F the closed-loop transfer matrix. The $\mathscr{L}_2$-induced operator norm of the convolution operator $\Sigma_{cl,F}$ is equal to the H_∞ norm of the transfer matrix G_F and is given by:

$$\| G_F \|_\infty := \sup_{\theta \in [0, 2\pi]} \| G_F(e^{i\theta}) \|$$

$$= \sup_w \left\{ \frac{\| z \|_2}{\| w \|_2} \,\middle|\, w \in \mathscr{L}_2^l, w \neq 0 \right\}$$

where the $\mathscr{L}_2$-norm is given by

$$\| p \|_2 := \left(\sum_{k=0}^{\infty} p^T(k) p(k) \right)^{1/2}$$

and where $\| . \|$ denotes the largest singular value. We shall refer to the norm $\| G_F \|_\infty$ as the H_∞ norm of the closed-loop system.

In this paper we will derive necessary and sufficient conditions for the existence of a dynamic compensator Σ_F which is internally stabilizing and which is such that the closed-loop transfer matrix G_F satisfies $\| G_F \|_\infty < 1$. By scaling the plant we can thus, in principle, find the infimum of the H_∞ norm of the closed-loop system over all stabilizing controllers. This will involve a search procedure. Furthermore, if a stabilizing Σ_F exists which makes the H_∞ norm of the closed-loop system less than 1, then we derive an explicit formula for one particular F satisfying these requirements. We also give an alternative nonminimal representation for this controller whose structure makes clear that this controller is the interconnection of a current state and current disturbance estimator and a static full-information feedback. In Section 5 we show that in some cases we can reduce the dynamical order of the estimator and we will derive an explicit method to derive controllers of lower dynamical order.

In the formulation of our main result we will need the concept of invariant zero. Recall that z_0 is called an *invariant zero* of the system (A, B, C, D) if

$$\operatorname{rank}_{\mathbb{R}} \begin{pmatrix} z_0 I - A & -B \\ C & D \end{pmatrix} < \operatorname{rank}_{\mathbb{R}(z)} \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix}$$

We can now formulate one of our main results. This is an extension of References 1, 8 and 14.

Theorem 2.1

Consider the system (1). Assume that the systems (A, B, C_2, D_{21}) and (A, E, C_1, D_{12}) have no invariant zeros on the unit circle. The following statements are equivalent:

(i) There exists a dynamic compensator Σ_F of the form (2) such that the resulting closed-loop system is internally stable and the closed-loop transfer matrix G_F satisfies $\| G_F \|_\infty < 1$.

(ii) There exist symmetric matrices $P \geqslant 0$ and $Q \geqslant 0$ such that
   (a) We have

$$R > 0 \tag{4}$$

where

$$V := B^{\mathrm{T}}PB + D_{21}^{\mathrm{T}}D_{21}$$
$$R := I - D_{22}^{\mathrm{T}}D_{22} - E^{\mathrm{T}}PE + (E^{\mathrm{T}}PB + D_{22}^{\mathrm{T}}D_{21})V^{\dagger}(B^{\mathrm{T}}PE + D_{21}^{\mathrm{T}}D_{22})$$

(b) P satisfies the discrete algebraic Riccati equation:

$$P = A^{\mathrm{T}}PA + C_2^{\mathrm{T}}C_2 - \begin{pmatrix} B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2 \\ E^{\mathrm{T}}PA + D_{22}^{\mathrm{T}}C_2 \end{pmatrix}^{\mathrm{T}} G(P)^{\dagger} \begin{pmatrix} B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2 \\ E^{\mathrm{T}}PA + D_{22}^{\mathrm{T}}C_2 \end{pmatrix} \tag{5}$$

where

$$G(P) := \begin{pmatrix} D_{21}^{\mathrm{T}}D_{21} & D_{21}^{\mathrm{T}}D_{22} \\ D_{22}^{\mathrm{T}}D_{21} & D_{22}^{\mathrm{T}}D_{22} - I \end{pmatrix} + \begin{pmatrix} B^{\mathrm{T}} \\ E^{\mathrm{T}} \end{pmatrix} P (B \;\; E) \tag{6}$$

(c) For all $z \in \mathbb{C}$ with $|z| \geqslant 1$, we have

$$\operatorname{rank}_{\mathbb{R}} \begin{pmatrix} zI - A & -B & -E \\ B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2 & B^{\mathrm{T}}PB + D_{21}^{\mathrm{T}}D_{21} & B^{\mathrm{T}}PE + D_{21}^{\mathrm{T}}D_{22} \\ E^{\mathrm{T}}PA + D_{22}^{\mathrm{T}}C_2 & E^{\mathrm{T}}PB + D_{22}^{\mathrm{T}}D_{21} & E^{\mathrm{T}}PE + D_{22}^{\mathrm{T}}D_{22} - I \end{pmatrix}$$
$$= n + q + \operatorname{rank}_{\mathbb{R}(z)} C_2(zI - A)^{-1}B + D_{21}$$

(d) We have

$$S > 0 \tag{7}$$

where

$$W := D_{12}D_{12}^{\mathrm{T}} + C_1QC_1^{\mathrm{T}}$$
$$S := I - D_{22}D_{22}^{\mathrm{T}} - C_2QC_2^{\mathrm{T}} + (C_2QC_1^{\mathrm{T}} + D_{22}D_{12}^{\mathrm{T}})W^{\dagger}(C_1QC_2^{\mathrm{T}} + D_{12}D_{22}^{\mathrm{T}})$$

(e) Q satisfies the following discrete algebraic Riccati equation:

$$Q = AQA^{\mathrm{T}} + EE^{\mathrm{T}} - \begin{pmatrix} C_1QA^{\mathrm{T}} + D_{12}E^{\mathrm{T}} \\ C_2QA^{\mathrm{T}} + D_{22}E^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} H(Q)^{\dagger} \begin{pmatrix} C_1QA^{\mathrm{T}} + D_{12}E^{\mathrm{T}} \\ C_2QA^{\mathrm{T}} + D_{22}E^{\mathrm{T}} \end{pmatrix} \tag{8}$$

where

$$H(Q) := \begin{pmatrix} D_{12}D_{12}^{\mathrm{T}} & D_{12}D_{22}^{\mathrm{T}} \\ D_{22}D_{12}^{\mathrm{T}} & D_{22}D_{22}^{\mathrm{T}} - I \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} Q \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}^{\mathrm{T}} \tag{9}$$

(f) For all $z \in \mathbb{C}$ with $|z| \geqslant 1$, we have

$$\operatorname{rank}_{\mathbb{R}} \begin{pmatrix} zI - A & AQC_1^{\mathrm{T}} + ED_{12}^{\mathrm{T}} & AQC_2^{\mathrm{T}} + ED_{22}^{\mathrm{T}} \\ -C_1 & C_1QC_1^{\mathrm{T}} + D_{12}D_{12}^{\mathrm{T}} & C_1QC_2^{\mathrm{T}} + D_{12}D_{22}^{\mathrm{T}} \\ -C_2 & C_2QC_1^{\mathrm{T}} + D_{22}D_{12}^{\mathrm{T}} & C_2QC_2^{\mathrm{T}} + D_{22}D_{22}^{\mathrm{T}} - I \end{pmatrix}$$
$$= n + q + \operatorname{rank}_{\mathbb{R}(z)} C_1(zI - A)^{-1}E + D_{12}$$

(g) $\rho(PQ) < 1$. □

Remarks

(i) Necessary and sufficient conditions for the existence of an internally stablizing feedback compensator which makes the H_∞ norm of the closed-loop system less than some, *a priori* given, upper bound $\gamma > 0$ can be easily derived from Theorem 2.1 by scaling.

(ii) In this paper, we only investigate controllers of the form (2). This is not an essential restriction, since it can be shown that we cannot make the H_∞ norm less by allowing more general, possibly even nonlinear, causal feedbacks.

(iii) Conditions (b) is the standard Riccati equation used in discrete time H_∞ except that the inverse is replaced by a generalized inverse. Condition (c) is nothing else than the requirement that P must be a stabilizing solution of the Riccati equation. Conditions (b) and (c) uniquely determine, if it exists, the matrix P. In subsection 3.1 we show how to reduce these very general algebraic Riccati equations appearing in the above lemma to classical Riccati equations which can be solved using standard techniques. A similar comment can be made about conditions (d)–(f).

For the special cases of full-information and state feedback we can dispense with the second Riccati equation. Moreover, in these cases there always exist suitable static controllers. More specifically:

- Full information case: $C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}$, $\quad D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}$

  In this case we have $y_1 = x$ and $y_2 = w$, i.e. we know both the state and the disturbance of the system at time k. It is easy to check that $Q = 0$ satisfies conditions (d)–(f). Moreover this guarantees that the coupling condition (g) is automatically satisfied. Therefore there exists a stabilizing controller which yields a closed-loop system with the H_∞ norm strictly less than 1 if and only if there exists a positive semidefinite matrix P satisfying conditions (a)–(c). Moreover in that case we can find static output feedbacks $u = F_1 x + F_2 w$ with the desired properties. One particular choice for $F = (F_1, F_2)$ is given by:

$$F_1 := -V^\dagger(B^\mathrm{T}PA + D_{21}^\mathrm{T}C_2) + (I - V^\dagger V)F_0 \tag{10}$$

$$F_2 := -V^\dagger(B^\mathrm{T}PE + D_{21}^\mathrm{T}D_{22}) \tag{11}$$

  where F_0 is an arbitrary matrix such that $A + BF_1$ is stable.

- State feedback case: $C_1 = I$, $D_{12} = 0$

  In that case, it is easy to see that a necessary condition for the existence of a positive semidefinite matrix Q satisfying conditions (d)–(f) is that $\| D_{22} \| < 1$. In that case, it is easy to check that

$$Q = E(I - D_{22}D_{22}^\mathrm{T})^{-1}E^\mathrm{T}$$

  satisfies conditions (d)–(f). Condition (g) then reduces to

$$I - D_{22}D_{22}^\mathrm{T} - E^\mathrm{T}PE > 0 \tag{12}$$

  Moreover, condition (12) implies that condition (a) is automatically satisfied. Therefore there exists a stabilizing controller which yields a closed-loop system with the H_∞ norm strictly less than 1 if and only if there exists a positive semidefinite matrix P satisfying conditions (b), (c) and additionally condition (12).

  In that case we can find a static output feedback $u = Fx$ with the desired properties. One particular choice for F is given by:

$$F := -V^\dagger(B^\mathrm{T}PA + D_{21}^\mathrm{T}C_2 + [B^\mathrm{T}PE + D_{21}^\mathrm{T}D_{22}]\, R^{-1}[E^\mathrm{T}PA_x + D_{22}^\mathrm{T}C_x]) + (I - V^\dagger V)F_0$$

where F_0 is an arbitrary matrix such that $A + BF$ is stable (which can be shown to always exist) and

$$A_x := A - BV^{\dagger}[B^{T}PA + D_{21}^{T}C_2] \tag{13}$$

$$C_x := C_2 - D_{21}V^{\dagger}[B^{T}PA + D_{21}^{T}C_2] \tag{14}$$

3. THE PROOF OF THEOREM 2.1

The proof of Theorem 2.1 is divided into three parts. Each part establishes the proof for a certain part of the theorem. Every part is framed up as a subsection with a heading that represents a significant feature of its proof technique or its overall achievement. The rationale for dividing the proof into three parts is mainly due to the length and the complexity of the proof.

3.1. The existence of a solution to the algebraic Riccati equation

In this subsection we assume that part (i) of Theorem 2.1 is satisfied. We will show that the existence of P satisfying conditions (a)–(c) in (ii) of Theorem 2.1 is necessary. We begin with the following definition.

Definition 3.1

Let a system $\Sigma = (A, B, C, D)$ be given. The controllability subspace $\mathscr{R}^*(\Sigma)$ is the largest subspace $\mathscr{X}$ of $\mathbb{R}^n$ for which a mapping F exists such that

$$(A + BF)\mathscr{X} \subseteq \mathscr{X}$$
$$(C + DF)\mathscr{X} = \{0\}$$

and such that (A, B_1) is controllable where B_1 is an arbitrary matrix such that:

$$\operatorname{Im} B_1 = \mathscr{X} \cap B \operatorname{Ker} D \qquad \square$$

We next perform a basis transformation on the state and input spaces of Σ. We decompose the state-space $\mathscr{X} = \mathscr{R}^*(\Sigma_{ci}) \oplus \mathscr{X}_2$ where $\Sigma_{ci} = (A, B, C_2, D_2)$ and choose a basis adapted to this decomposition. We also decompose the controller input space $\mathscr{U} = \operatorname{Ker} V \oplus \mathscr{U}_2$ where V is as defined in Theorem 2.1. In the new bases, the matrices in the realization of Σ have a special form:

$$A = \begin{pmatrix} A_{11} - B_{12}F & A_{12} \\ -B_r F & A_r \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_r \end{pmatrix}, \quad E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$
$$C_1 = (\; C_{11} \quad C_{12} \;), \qquad D_{12} = D_{12} \tag{15}$$
$$C_2 = (\; -D_r F \quad C_r \;), \quad D_{21} = (\; 0 \quad D_r \;), \quad D_{22} = D_{22}$$

The above matrices have the following properties:

- (A_{11}, B_{11}) is controllable,
- (A_r, B_r, C_r, D_r) is left invertible,
- (A_r, B_r) is stabilizable if and only if (A, B) is stabilizable.

If part (i) of Theorem 2.1 holds, i.e., if the measurement feedback problem is solvable, then we also know that the full information H_∞ control problem is solvable. Let F_0 be such that

$A_{11} + B_{11}F_0$ is stable. Then it is easy to see that, after the preliminary feedback

$$u = \begin{pmatrix} F & 0 \\ F_0 & 0 \end{pmatrix} x + v \tag{16}$$

the subspace $\mathscr{R}^*(\Sigma_{\rm ci})$ does not affect the output to be controlled and the dynamics restricted to $\mathscr{R}^*(\Sigma_{\rm ci})$ is stable. Hence the achievable H_∞ norm using full-information feedback is completely determined by the following subsystem:

$$\Sigma_{\rm r}: \begin{cases} \sigma x_2 = A_{\rm r} x_2 + B_{\rm r} v_2 + E_2 w \\ z \quad = C_{\rm r} x_2 + D_{\rm r} v_2 + D_{22} w \end{cases} \tag{17}$$

However, for this subsystem the operator mapping the input v_2 to the output z is left invertible. Therefore we can apply the results from References 2 and 15 to obtain the following result:

Lemma 3.2

Consider the systems Σ and $\Sigma_{\rm r}$ defined by (1) and (17) respectively. Assume that the system (A, B, C_2, D_{21}) has no invariant zeros on the unit circle. Then also the system $(A_{\rm r}, B_{\rm r}, C_{\rm r}, D_{\rm r})$ has no invariant zeros on the unit circle. Moreover, the following statements are equivalent:

(i) There exists a full information feedback $u = F_1 x + F_2 w$ for the system Σ such that the resulting closed-loop system is internally stable and the closed-loop transfer matrix $G_{\rm F}$ satisfies $\| G_{\rm F} \|_\infty < 1$.

(ii) There exists a full information feedback $u = F_{1,\rm r} x + F_{2,\rm r} w$ for the system $\Sigma_{\rm r}$ such that the resulting closed-loop system is internally stable and the closed-loop transfer matrix $G_{\rm F,r}$ satisfies $\| G_{\rm F,r} \|_\infty < 1$.

(iii) There exists a symmetric matrix $P_r \geqslant 0$ such that

(a) We have

$$V_{\rm r} > 0, \qquad R_{\rm r} > 0$$

where

$$V_{\rm r} := B_{\rm r}^{\rm T} P_{\rm r} B_{\rm r} + D_{\rm r}^{\rm T} D_{\rm r}$$
$$R_{\rm r} := I - D_{22}^{\rm T} D_{22} - E_{\rm r}^{\rm T} P_{\rm r} E_{\rm r} + (E_{\rm r}^{\rm T} P_{\rm r} B_{\rm r} + D_{22}^{\rm T} D_{\rm r}) V_{\rm r}^{-1} (B_{\rm r}^{\rm T} P_{\rm r} E_{\rm r} + D_{\rm r}^{\rm T} D_{22})$$

(b) $P_{\rm r}$ satisfies the discrete algebraic Riccati equation:

$$P_{\rm r} = A_{\rm r}^{\rm T} P_{\rm r} A_{\rm r} + C_{\rm r}^{\rm T} C_{\rm r} - \begin{pmatrix} B_{\rm r}^{\rm T} P_{\rm r} A_{\rm r} + D_{\rm r}^{\rm T} C_{\rm r} \\ E_{\rm r}^{\rm T} P_{\rm r} A_{\rm r} + D_{22}^{\rm T} C_{\rm r} \end{pmatrix}^{\rm T} G_{\rm r}(P_{\rm r})^{-1} \begin{pmatrix} B_{\rm r}^{\rm T} P_{\rm r} A_{\rm r} + D_{\rm r}^{\rm T} C_{\rm r} \\ E_{\rm r}^{\rm T} P_{\rm r} A_{\rm r} + D_{22}^{\rm T} C_{\rm r} \end{pmatrix}$$

where

$$G_{\rm r}(P_{\rm r}) := \begin{pmatrix} D_{\rm r}^{\rm T} D_{\rm r} & D_{\rm r}^{\rm T} D_{22} \\ D_{22}^{\rm T} D_{\rm r} & D_{22}^{\rm T} D_{22} - I \end{pmatrix} + \begin{pmatrix} B_{\rm r}^{\rm T} \\ E_{\rm r}^{\rm T} \end{pmatrix} P_{\rm r} (B_{\rm r} \;\; E_{\rm r})$$

(c) The matrix $A_{\rm cl,P}$ is asymptotically stable where

$$A_{\rm cl,P} := A - (B_{\rm r} \;\; - E_{\rm r}) G_{\rm r}(P_{\rm r})^{-1} \begin{pmatrix} B_{\rm r}^{\rm T} P_{\rm r} A_{\rm r} + D_{\rm r}^{\rm T} C_{\rm r} \\ E_{\rm r}^{\rm T} P_{\rm r} A_{\rm r} + D_{22}^{\rm T} C_{\rm r} \end{pmatrix}$$ □

Proof. The implication (ii) ⇔ (iii) can be found in Reference 15. The implication (ii) ⇒ (i) can be easily checked using the arguments given before this lemma.

The implication (i) ⇒ (ii) can be derived in the following manner. First note that we can apply, without loss of generality the transformation (16). Suppose a stabilizing feedback, $v = F_{11}x_1 + F_{12}x_2 + F_2 w$ exists for the system $\mathscr{S}$ (after our preliminary transformation) which yields a closed-loop transfer matrix G_F satisfying $\| G_F \|_\infty < 1$. Then it is easy to check that the following dynamic compensator stabilizes Σ_r and yields the same closed-loop transfer matrix G_F:

$$\Sigma_F: \begin{cases} \sigma x_1 = (A_{11} + B_{12}F_0)x_1 + A_{12}x_2 + E_1 w \\ u \quad = \qquad F_{11}x_1 + F_{12}x_2 + F_2 w \end{cases}$$

However, Σ_r has a subsystem from v_2 to z which is left-invertible and hence, from Reference 14, we know that the existence of a suitable dynamic full-information feedback also guarantees the existence of a static full-information feedback. □

This lemma yields a solution P_r of a discrete-time Riccati equation for the reduced-order system. We can extend this matrix to the original state-space by setting it zero on $\mathscr{R}^*(\Sigma_{ci})$, i.e. if we define P by

$$P = \begin{pmatrix} 0 & 0 \\ 0 & P_r \end{pmatrix} \tag{18}$$

then P_r satisfies the conditions of Lemma 3.2 if and only if P satisfies the conditions of (a)–(c) of Theorem 2.1. The above can be combined to yield:

Lemma 3.3

Assume (A, B, C_2, D_{21}) has no invariant zeros on the unit circle. If part (i) of Theorem 2.1 is satisfied then there exists a symmetric matrix $P \geqslant 0$ satisfying (a)–(c) of part (ii) of Theorem 2.1.

We also need to know whether any solution P satisfying conditions (a)–(c) of Theorem 2.1 can be connected to a matrix P_r satisfying the conditions of Lemma 3.2. This is done in the following lemma:

Lemma 3.4

Let $P \geqslant 0$ be a matrix satisfying the conditions (a)–(c) of Theorem 2.1. Then

$$\operatorname{Ker} P \supseteq \mathscr{R}^*(\Sigma_{ci})$$

Hence, in our new bases, P will be of the form (18) for some matrix P_r. Moreover P_r satisfies the conditions in part (iii) of Lemma 3.2.

Proof. First note that condition (b) implies that

$$P \geqslant A_x^T P A_x + C_x^T C_x$$

where A_x and C_x are defined by (13) and (14) respectively. It is easily seen that this implies that Ker P is controlled invariant.

Secondly conditions (a) and (c) imply that

$$\operatorname{rank}(B^T P B + D_{21}^T D_{21}) = \operatorname{rank}_{\mathbb{R}(z)} C_2(zI - A)^{-1}B + D_{21}$$

These two properties, when combined with the decomposition of the state-space as introduced in the beginning of this section, yield the desired result. □

Using P_r, or equivalently P, we can also derive explicit formulas for static full-information compensators which achieve the desired objectives in parts (i) or (ii). This is outlined in the following lemma which is an extension of results in References 2 and 15.

Lemma 3.5

Let the systems Σ and Σ_r be defined by (1) and (17) respectively. Assume that a matrix $P_r \geqslant 0$ exists satisfying the conditions in part (iii) of Lemma 3.2. Moreover, define P by (18).

- A controller satisfying the conditions of part (ii) of Lemma 3.2 is described by:

$$F_{1,r} := -V^{-1}(B_r^T P_r A_r + D_r^T C_r)$$
$$F_{2,4} := -V^{-1}(B_r^T P_r E_r + D_r^T D_{22})$$

- A controller satisfying the conditions of part (i) of Lemma 3.2 is described by

$$F_1 = \begin{pmatrix} F & 0 \\ F_0 & F_{1,r} \end{pmatrix}$$

$$F_2 = (\,0 \quad F_{2,r})$$

where F and F_0 are the parameters of the preliminary feedback described before Lemma 3.2.

Alternatively, we can also describe a suitable controller for Σ in terms of the original system parameters of Σ:

$$F_1 := -V^\dagger(B^T PA + D_{21}^T C_2) + (I - V^\dagger V)\tilde{F}$$
$$F_2 := -V^\dagger(B^T PE + D_{21}^T D_{22})$$

where $\tilde{F}$ is an arbitrary matrix such that $A + BF_1$ is stable.

Proof. The first part of this lemma is a direct result of Reference 14. The second part of this lemma gives two controllers of which it can be easily shown that when applied to the reduced-order system they yield the same closed-loop transfer matrix as the controller given in the first part of this lemma when applied to the original system. Hence the closed-loop system has H_∞ norm strictly less than 1. Remains to check existence of a suitable $\tilde{F}$ to yield internal stability of the closed-loop system. This is shown by using the decomposition introduced in the beginning of this section together with stability of $A_r + B_r F_r$ and stabilizability of $(A_{11} + B_{12}F, B_{11})$.

In the next subsection we show that the part (i) of Theorem 2.1 also implies the remaining statements of the part (ii) of Theorem 2.1.

3.2. *A first system transformation*

In this subsection we assume that part (i) of Theorem 2.1 is satisfied and we show that part (ii) of Theorem 2.1 holds. A central component of the proof in this subsection is to transform the original system (1) into a new system. This transformation is designed in such a way that the problem of finding an internally stabilizing feedback which makes the H_∞ norm of the closed-loop system less than 1 for the original system would be equivalent to the problem of finding an internally stabilizing feedback which makes the H_∞ norm of the closed-loop system less than 1 for the new transformed system. Moreover, this new system has some very desirable properties which makes it much easier to work with. In particular, for this new system the disturbance decoupling problem with measurement feedback is solvable. We will perform the transformation in two steps. First we will perform a transformation related to the full-information H_∞ problem and next a transformation related to the filtering problem. We assume that we have a positive semidefinite matrix P satisfying conditions (a)–(c) of Theorem 2.1. We define the following system:

$$\Sigma_P: \begin{cases} \sigma x_P = A_P x_P + B u_P + E_P w_P \\ y_P = C_{1,P} x_P + D_{12,P} w_P \\ z_P = C_{2,P} x_P + D_{21,P} u_P + D_{22,P} w_P \end{cases} \tag{19}$$

where

$$\begin{aligned}
A_P &:= A + ER^{-1}(E^T P A_x + D_{22}^T C_x) \\
E_P &:= ER^{-1/2} \\
C_{1,P} &:= C_1 + D_{12} R^{-1}(E^T P A_x + D_{22}^T C_x) \\
C_{2,P} &:= (V^{1/2})^\dagger (B^T P A + D_{21}^T C_2 + [B^T P E + D_{21}^T D_{22}] R^{-1} [E^T P A_x + D_{22}^T C_x]) \\
D_{12,P} &:= D_{12} R^{-1/2} \\
D_{21,P} &:= V^{1/2} \\
D_{22,P} &:= (V^{1/2})^\dagger (B^T P E + D_{21}^T D_{22}) R^{-1/2}
\end{aligned}$$

where the matrix P satisfies parts (a)–(c) of Theorem 2.1 and the matrices A_x and C_x are defined by (13) and (14), respectively.

In order to continue, we need the system to be in the special basis as defined in the previous section. Using Lemma 3.4, we know that P is of the form (18) for some matrix P_r. We can then define the following system:

$$\Sigma_U: \begin{cases} \sigma x_U = A_U x_U + B_U u_U + E_U w \\ y_U = C_{1,U} x_U + D_{12,U} w \\ z_U = C_{2,U} x_U + D_{21,U} u_U + D_{22,U} w \end{cases} \tag{20}$$

where

$$\begin{aligned}
V_r &:= B_r^T P_r B_r + D_r^T D_r \\
A_U &:= A_r - B_r V_r^{-1}(B_r^T P_r A_r + D_r^T C_r) \\
B_U &:= B_r (0 \;\; V_r^{-1/2}) \\
E_U &:= E_2 - B_r V_r^{-1}(B_r^T P_r E_2 + D_r^T D_{22}) \\
C_{2,U} &:= C_r - D_r V_r^{-1}(B_r^T P_r A_r + D_r^T C_r) \\
C_{1,U} &:= -R^{-1/2}(E_2^T P_r A_u + D_{12}^T C_{2,U}) \\
D_{12,U} &:= R^{1/2} \\
D_{21,U} &:= D_r V^{-1/2} (0 \;\; V_r^{-1/2}) \\
D_{22,U} &:= D_{22} - D_r V_r^{-1}(B_r^T P_r E_2 + D_r^T D_{22})
\end{aligned}$$

where R is as defined in Theorem 2.1. We will show that Σ_U has a very nice property. In order to do this, we first recall the definition of the so-called inner systems. Moreover, some of the important properties of inner systems are also recalled in the following two lemmas.

Definition 3.6

A system is called *inner* if the system is internally stable, square (i.e. the number of inputs is equal to the number of outputs) and the transfer matrix of the system, denoted by G, satisfies:

$$G(z)G^{\mathrm{T}}(z^{-1}) = I \tag{21}$$

Lemma 3.7

Let the following square system be given:

$$\Sigma_{\mathrm{st}}: \begin{cases} \sigma x = Ax + Bu \\ z \;= Cx + Du \end{cases} \tag{22}$$

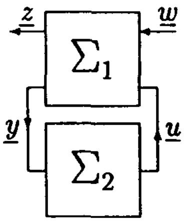
Assume that A is asymptotically stable. The system Σ_{st} is inner if there exists a matrix X satisfying:

(a) $X = A^{\mathrm{T}}XA + C^{\mathrm{T}}C$
(b) $D^{\mathrm{T}}C + B^{\mathrm{T}}XA = 0$
(c) $D^{\mathrm{T}}D + B^{\mathrm{T}}XB = I$

Proof. See References 6 and 15. □

Lemma 3.8

Suppose we have the following interconnection of two systems Σ_1 and Σ_2, both described by some state-space representation:



(23)

Assume Σ_1 is inner. Denote its transfer matrix from (w, u) to (z, y) by L. Moreover, assume that if we decompose L compatible with the sizes of w, u, z and y:

$$L\begin{pmatrix} w \\ u \end{pmatrix} := \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix} \tag{24}$$

we have $L_{21}^{-1} \in H_\infty$ and L_{22} is strictly proper. Then the following two statements are equivalent:

(i) The closed loop system (23) is internally stable and its closed-loop transfer matrix has H_∞ norm less than 1.
(ii) The system Σ_2 is internally stable and its transfer matrix has H_∞ norm less than 1.

Proof. See References 9 and 13. □

Now, we are ready to come back to the system Σ_U and establish some of its properties in the following lemma.

Lemma 3.9

The system Σ_U as defined by (20) is inner. Denote the transfer matrix of Σ_U by U. We decompose U compatible with the sizes of w, u_U, z_U and y_U:

$$U\begin{pmatrix} w \\ u_U \end{pmatrix} := \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} w \\ u_U \end{pmatrix} = \begin{pmatrix} z_U \\ y_U \end{pmatrix}$$

Then U_{21} is invertible and its inverse is in H_∞. Moreover U_{22} is strictly proper.

Proof. It can be easily checked that P_r satisfies the conditions (i)–(iii) of Lemma 3.7. Condition (i) of Lemma 3.7 turns out to be equal to the reduced-order discrete algebraic Riccati equation as given in Lemma 3.2. Conditions (ii) and (iii) follow by simply writing out the equations in terms of the system parameters of system (1).

The stable matrix $A_{cl,P}$, as defined in Lemma 3.2, can be written in the following form:

$$A_{cl,P} = A_U - E_U D_{12,U}^{-1} C_{1,U} \tag{25}$$

Next, we show that A_U is asymptotically stable. We know $P_r \geqslant 0$ and

$$P_r = A_U^T P_r A_U + (C_{1,U}^T \quad C_{2,U}^T)\begin{pmatrix} C_{1,U} \\ C_{2,U} \end{pmatrix} \tag{26}$$

It can be easily checked that $x \neq 0$, $A_U x = \lambda x$, $C_{1,U}x = 0$ and $C_{2,U}x = 0$ implies that $A_{cl,P}x = \lambda x$. Since $A_{cl,P}$ is stable we have Re $\lambda < 0$. Hence the realization (20) is detectable. By standard Lyapunov theory the existence of a positive semidefinite solution of (26) together with detectability guarantee asymptotic stability of A_U.

We can immediately write down a realization for U_{21}^{-1}:

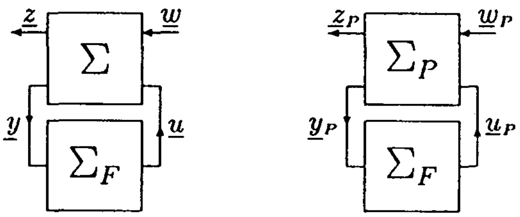
$$\Sigma_{U_{21}^{-1}}: \begin{cases} \sigma x_U = & A_{cl,P} x_U + E_U D_{12,U}^{-1} w \\ y_U = & -D_{12,U}^{-1} C_{1,U} x_U + D_{12,U}^{-1} w \end{cases}$$

Since $A_{cl,P}$ is stable we know that U_{21}^{-1} is an H_∞ function. Finally, the claim that U_{22} is strictly proper is trivial to check. □

We will now formulate our key lemma:

Lemma 3.10

Let P satisfy Theorem 2.1 part (ii) (a)–(c). Moreover, let Σ_F be an arbitrary linear time-invariant finite-dimensional compensator in the form (2). Consider the following two systems, where the system on the left is the interconnection of (1) and (2) and the system on the right is the interconnection of (19) and (2):

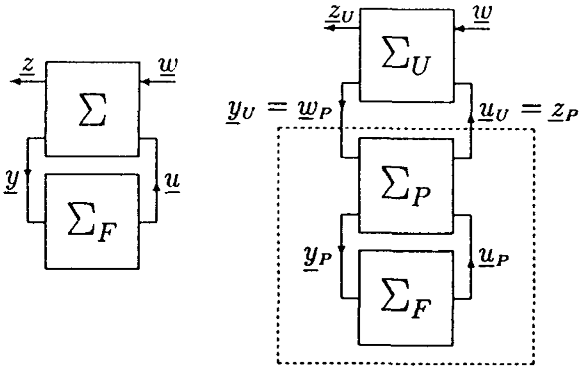


(27)

Then the following statements are equivalent:

(i) The system on the left is internally stable and its transfer matrix from w to z has H_∞ norm less than 1.
(ii) The system on the right is internally stable and its transfer matrix from w_P to z_P has H_∞ norm less than 1.

Proof. We investigate the following systems:



(28)

The system on the left is the same as the system on the left in (27) and the system on the right is described by the system (20) interconnected with the system on the right in (27). A realization for the system on the right is given by:

$$\sigma\begin{pmatrix} x_U - x_{2,P} \\ x_P \\ p \end{pmatrix} = \begin{pmatrix} A_{cl,P} & 0 & 0 \\ * & A+BNC_1 & BM \\ * & LC_1 & K \end{pmatrix}\begin{pmatrix} x_U - x_{2,P} \\ x_P \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ E+BND_{12} \\ LD_{12} \end{pmatrix} w$$

$$z_U = \begin{pmatrix} * & C_2+D_{21}NC_1 & D_{21}M \end{pmatrix}\begin{pmatrix} x_U - x_{2,P} \\ x_P \\ p \end{pmatrix} + (D_{22}+D_{21}ND_{12})w$$

where $A_{cl,P}$ is defined by (25). The asterisks denote matrices which are unimportant for this argument. The system on the right is internally stable if and only if the system described by the above set of equations is internally stable. If we also derive the system equations for the system on the left in (28) we immediately see that, since $A_{cl,P}$ is asymptotically stable, the system on the left is internally stable if and only if the system on the right is internally stable. Moreover, if we take zero initial conditions and both systems have the same input w then we have $z = z_U$, i.e. the input–output behaviour of both systems are equivalent. Hence the system on the left has H_∞ norm less than 1 if and only if the system on the right has H_∞ norm less than 1.

By Lemma 3.9 we may apply Lemma 3.8 to the system on the right in (28) and hence we find that the closed-loop system is internally stable and has H_∞ norm less than 1 if and only if the dashed system is internally stable and has H_∞ norm less than 1.

Since the dashed system is exactly the system on the right in (27) and the system on the left in (28) is exactly equal to the system on the left in (27) we have completed the proof. □

Using the previous lemma, we know that we only have to investigate the system Σ_P. This new system has a nice property which is outlined in the following lemma:

Lemma 3.11

There exists a matrix F_0 such that if we define:

$$F_{1,\mathrm{P}} = -D_{21,\mathrm{P}}^{\dagger}C_{2,\mathrm{P}} + (I - D_{21,\mathrm{P}}^{\dagger}D_{21,\mathrm{P}})F_0$$
$$F_{2,\mathrm{P}} = -D_{21,\mathrm{P}}^{\dagger}D_{22,\mathrm{P}}$$

then we have:

(i) $A_\mathrm{P} + BF_{1,\mathrm{P}}$ is stable
(ii) $C_{2,\mathrm{P}} + D_{21,\mathrm{P}}F_{1,\mathrm{P}} = 0$
(iii) $D_{22,\mathrm{P}} + D_{21,\mathrm{P}}F_{2,\mathrm{P}} = 0$

Proof. We first write everything in terms of the new basis introduced in the previous section. Hence the system parameters have the special form described by (15). Then it is easily checked that conditions (ii) and (iii) are always satisfied, independent of the specific choice for F_0. If we also write the matrix F_0 in the new basis,

$$F_0 = \begin{pmatrix} F_{0,11} & F_{0,12} \\ F_{0,21} & F_{0,22} \end{pmatrix}$$

then we have:

$$A_\mathrm{P} + BF_{1,\mathrm{P}} = \begin{pmatrix} A_{11} + B_{11}F_{0,11} & * \\ 0 & A_{\mathrm{cl,P}} \end{pmatrix}$$

where the asterisk denotes a matrix which is unimportant for our argument. According to Lemma 3.2, the matrix $A_{\mathrm{cl,P}}$ is asymptotically stable. Moreover, as noted in the previous section, (A_{11}, B_{11}) is controllable. Hence, any matrix F_0 such that $A_{11} + B_{11}F_{0,11}$ is stable satisfies the conditions of our lemma. Moreover, controllability guarantees the existence of such matrices F_0. □

Remark. The above lemma implies that the full-information feedback $u = F_{1,\mathrm{P}}x_\mathrm{P} + F_{2,\mathrm{P}}w_\mathrm{P}$ applied to Σ_P yields a stable closed-loop system for which the closed-loop H_∞ norm is equal to 0.

Next, we will look at the Riccati equation for the system Σ_P. It can be checked immediately that $X = 0$ satisfies (a)–(c) of Theorem 2.1 for the system Σ_P.

We dualize Σ_P. We know that (A, E, C_1, D_{12}) has no invariant zeros on the unit circle. It can be easily checked that this implies that $(A_\mathrm{P}, E, C_{1,\mathrm{P}}, D_{12})$ has no invariant zeros on the unit circle. Hence for the dual of Σ_P we know that $(A_\mathrm{P}^\mathrm{T}, C_{1,\mathrm{P}}^\mathrm{T}, E^\mathrm{T}, D_{21}^\mathrm{T})$ has no invariant zeros on the unit circle. If there exists an internally stabilizing feedback for the system Σ which makes the H_∞ norm of the closed-loop system less than 1 then the same feedback is internally stabilizing and makes the H_∞ norm of the closed-loop system less than 1 for the system Σ_P. If we dualize this feedback and apply it to the dual of Σ_P then it is again internally stabilizing and again it makes the H_∞ norm of the closed-loop system less than 1. We can now apply Lemma 3.3 which exactly guarantees the existence of a matrix $Y \geqslant 0$ satisfying the following conditions

(i) Y is such that $S_\mathrm{P} > 0$ where

$$W_\mathrm{P} := D_{12,\mathrm{P}}D_{12,\mathrm{P}}^\mathrm{T} + C_{1,\mathrm{P}}YC_{1,\mathrm{P}}^\mathrm{T}$$
$$S_\mathrm{P} := I - D_{22,\mathrm{P}}D_{22,\mathrm{P}}^\mathrm{T} - C_{2,\mathrm{P}}YC_{2,\mathrm{P}}^\mathrm{T} + (C_{2,\mathrm{P}}YC_{1,\mathrm{P}}^\mathrm{T} + D_{22,\mathrm{P}}D_{12,\mathrm{P}}^\mathrm{T})W_\mathrm{P}^{\dagger}(C_{1,\mathrm{P}}YC_{2,\mathrm{P}}^\mathrm{T} + D_{12,\mathrm{P}}D_{22,\mathrm{P}}^\mathrm{T})$$

(ii) Y satisfies the following discrete algebraic Riccati equation:

$$Y = A_P Y A_P^T + E_P E_P^T - \begin{pmatrix} C_{1,P} Y A_P^T + D_{12,P} E_P^T \\ C_{2,P} Y A_P^T + D_{22,P} E_P^T \end{pmatrix}^T H_P(Y)^\dagger \begin{pmatrix} C_{1,P} Y A_P^T + D_{12,P} E_P^T \\ C_{2,P} Y A_P^T + D_{22,P} E_P^T \end{pmatrix} \quad (29)$$

where

$$H_P(Y) := \begin{pmatrix} D_{12,P} D_{12,P}^T & D_{12,P} D_{22,P}^T \\ D_{22,P} D_{12,P}^T & D_{22,P} D_{22,P}^T - I \end{pmatrix} + \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix} Y \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix}^T \quad (30)$$

(iii) Y satisfies a stability condition: for all $z \in \mathbb{C}$ with $|z| \geqslant 1$, we have

$$\operatorname{rank}_{\mathbb{R}} \begin{pmatrix} zI - A & C_{1,P} Y A_P^T + D_{12,P} E_P^T & C_{2,P} Y A_P^T + D_{22,P} E_P^T \\ -C_{1,P} & C_{1,P} Y C_{1,P}^T + D_{12,P} D_{12,P}^T & C_{1,P} Y C_{2,P}^T + D_{12,P} D_{22,P}^T \\ -C_{2,P} & C_{2,P} Y C_{1,P}^T + D_{22,P} D_{12,P}^T & C_{2,P} Y C_{2,P}^T + D_{22,P} D_{22,P}^T - I \end{pmatrix}$$
$$= n + q + \operatorname{rank}_{\mathbb{R}(z)} C_1 (zI - A)^{-1} E + D_{12}$$

Note that Y satisfies the conditions (d)–(f) of Theorem 2.1 for the system Σ_P.

The following lemma relates the existence and the solution of the above conditions to the conditions in Theorem 2.1:

Lemma 3.12

There exists a matrix $Y \geqslant 0$ satisfying the above conditions if and only if there exist matrices $P \geqslant 0$ and $Q \geqslant 0$ satisfying the conditions in part (ii) of Theorem 2.1. Moreover, in that case we have:

$$Y = (I - QP)^{-1} Q$$

The above derivation yields the necessity part of Theorem 2.1:

Lemma 3.13

Let Σ, described by (1), be given with zero initial condition. Assume that (A, B, C_2, D_{21}) and (A, E, C_1, D_{12}) have no invariant zeros on the unit circle. If part (i) of Theorem 2.1 is satisfied then there exist matrices P and Q satisfying (a)–(f) of part (ii) of Theorem 2.1.

This completes the proof (i) $\Rightarrow$ (ii). In the next section we will prove the reverse implication. Moreover in case the desired compensator Σ_F exists we will derive an explicit formula for one choice for Σ_F which satisfies all requirements.

3.3. The transformation into a disturbance decoupling problem with measurement feedback

In this section we assume that there exist matrices P and Q satisfying part (ii) of Theorem 2.1 for the system (1) and we show that part (i) of Theorem 2.1 holds. First we transform our original system Σ into another system $\Sigma_{P,Y}$. We will show that a compensator is internally stabilizing and makes the H_∞ norm of the closed-loop system less than 1 for the system Σ if and only if the same compensator is internally stabilizing and makes the H_∞ norm of the closed-loop system less than 1 for our transformed system $\Sigma_{P,Y}$. Next we will show that $\Sigma_{P,Y}$

has a following very special property (see Reference 11):

> There exists an internally stabilizing compensator which makes the closed-loop transfer matrix equal to zero, i.e. w does not have any effect on the output of the system z. This property of $\Sigma_{\mathrm{P,Y}}$ has a special name: 'the Disturbance Decoupling Problem with Measurement feedback and internal Stability (DDPMS) is solvable'.

We know a matrix $Y := (I - QP)^{-1}Q$ exists satisfying the conditions as outlined in the previous section. Next, we define $\Sigma_{\mathrm{P,Y}}$. We start by transforming Σ into Σ_{P}. Then we apply the dual transformation on Σ_{P} to obtain $\Sigma_{\mathrm{P,Y}}$:

$$\Sigma_{\mathrm{P,Y}}: \begin{cases} \sigma x_{\mathrm{P,Y}} = A_{\mathrm{P,Y}} x_{\mathrm{P,Y}} + B_{\mathrm{P,Y}} u_{\mathrm{P,Y}} + E_{\mathrm{P,Y}} w_{\mathrm{P,Y}} \\ y_{\mathrm{P,Y}} = C_{1,\mathrm{P}} x_{\mathrm{P,Y}} + D_{12,\mathrm{P,Y}} w_{\mathrm{P,Y}} \\ z_{\mathrm{P,Y}} = C_{2,\mathrm{P,Y}} x_{\mathrm{P,Y}} + D_{21,\mathrm{P,Y}} u_{\mathrm{P,Y}} + D_{22,\mathrm{P,Y}} w_{\mathrm{P,Y}} \end{cases} \tag{31}$$

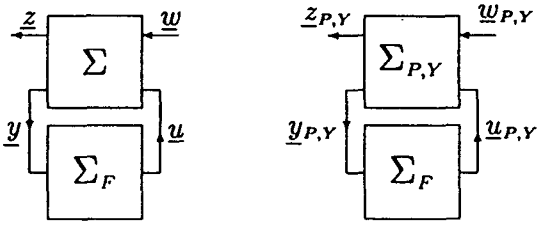
where

$$\begin{aligned}
A_y &:= A_{\mathrm{P}} - (A_{\mathrm{P}} Y C_{1,\mathrm{P}}^{\mathrm{T}} + E_{\mathrm{P}} D_{12,\mathrm{P}}^{\mathrm{T}}) W_{\mathrm{P}}^{\dagger} C_{1,\mathrm{P}} \\
E_y &:= E_{\mathrm{P}} - (A_{\mathrm{P}} Y C_{1,\mathrm{P}}^{\mathrm{T}} + E_{\mathrm{P}} D_{12,\mathrm{P}}^{\mathrm{T}}) W_{\mathrm{P}}^{\dagger} D_{12,\mathrm{P}} \\
A_{\mathrm{P,Y}} &:= A_{\mathrm{P}} + (A_y Y C_{2,\mathrm{P}}^{\mathrm{T}} + E_y D_{22,\mathrm{P}}^{\mathrm{T}}) S_{\mathrm{P}}^{-1} C_{2,\mathrm{P}} \\
C_{2,\mathrm{P,Y}} &:= S_{\mathrm{P}}^{-1/2} C_{2,\mathrm{P}} \\
B_{\mathrm{P,Y}} &:= B + (A_y Y C_{2,\mathrm{P}}^{\mathrm{T}} + E_y D_{22,\mathrm{P}}^{\mathrm{T}}) S_{\mathrm{P}}^{-1} D_{21,\mathrm{P}} \\
E_{\mathrm{P,Y}} &:= (A_{\mathrm{P}} Y C_{1,\mathrm{P}}^{\mathrm{T}} + E_{\mathrm{P}} D_{12,\mathrm{P}}^{\mathrm{T}} + [A_y Y C_{2,\mathrm{P}}^{\mathrm{T}} + E_y D_{22,\mathrm{P}}^{\mathrm{T}}] S_{\mathrm{P}}^{-1} [C_{2,\mathrm{P}} Y C_{1,\mathrm{P}}^{\mathrm{T}} + D_{22,\mathrm{P}} D_{12,\mathrm{P}}^{\mathrm{T}}]) \\
&\quad \times (W_{\mathrm{P}}^{1/2})^{\dagger} \\
D_{12,\mathrm{P,Y}} &:= W_{\mathrm{P}}^{1/2} \\
D_{21,\mathrm{P,Y}} &:= S_{\mathrm{P}}^{-1/2} D_{21,\mathrm{P}} \\
D_{22,\mathrm{P,Y}} &:= S_{\mathrm{P}}^{-1/2} (C_{2,\mathrm{P}} Y C_{1,\mathrm{P}}^{\mathrm{T}} + D_{22,\mathrm{P}} D_{12,\mathrm{P}}^{\mathrm{T}}) (W_{\mathrm{P}}^{1/2})^{\dagger}
\end{aligned}$$

When we first apply Lemma 3.10 on the transformation from Σ to Σ_{P} and then the dual of Lemma 3.10 on the transformation from Σ_{P} to $\Sigma_{\mathrm{P,Y}}$ we find:

Lemma 3.14

Let P satisfy Theorem 2.1 part (ii) (a)–(c). Moreover let an arbitrary linear time-invariant finite-dimensional compensator Σ_{F} be given, described by (2). Consider the following two systems, where the system on the left is the interconnection of (1) and (2) and the system on the right is the interconnection of (31) and (2):



The the following statements are equivalent:

(i) The system on the left is internally stable and its transfer matrix from w to z has H_∞ norm less than 1.
(ii) The system on the right is internally stable and its transfer matrix from $w_{\mathrm{P,Y}}$ to $z_{\mathrm{P,Y}}$ has H_∞ norm less than 1.

It remains to be shown that for $\Sigma_{P,Y}$ the (DDPMS) is solvable. We first need the following preliminary lemma.

Lemma 3.15

There exists a matrix K_0 such that if we define:

$$K_{1,P,Y} = -E_{P,Y}D_{12,P,Y}^{\dagger} + K_0(I - D_{12,P,Y}D_{12,P,Y}^{\dagger})$$
$$K_{2,P,Y} = -D_{12,P,Y}^{\dagger}$$

then we have:

(i) $A_{P,Y} + K_{1,P,Y}C_{1,P}$ is stable
(ii) $E_{P,Y} + K_{1,P,Y}D_{12,P,Y} = 0$
(iii) $D_{22,P,Y} + D_{22,P,Y}K_{2,P,Y}D_{12,P,Y} = 0$

Moreover, let $F_{1,P}$ and $F_{2,P}$ be as defined Lemma 3.11. If we define

$$F_{1,P,Y} := F_{1,P},$$
$$F_{2,P,Y} := -D_{21,P,Y}^{\dagger}D_{22,P,Y}$$

then we have:

(iv) $A_{P,Y} + B_{P,Y}F_{1,P,Y} = A_P + BF_{1,P}$ is stable
(v) $C_{2,P,Y} + D_{21,P,Y}F_{1,P} = 0$
(vi) $D_{22,P,Y} + D_{21,P,Y}F_{2,P,Y} = 0$

Proof. The construction of a suitable matrix K_0 satisfying conditions (i)–(iii) is dual to the derivation of a suitable F_0 satisfying the conditions of Lemma 3.5. Hence details are omitted. Conditions (iv)–(vi) can be checked via straightforward algebraic manipulations. □

Remark. The first part of the lemma is dual to Lemma 3.11 and shows that because of the dual transformation we can now observe the states of $\Sigma_{P,Y}$ perfectly. Surprisingly enough the property that Σ_P could be controlled perfectly is preserved: the second part of the lemma shows that also for $\Sigma_{P,Y}$ we can find a full-information feedback that stabilizes the system and yields a closed-loop system with H_∞ norm equal to 0.

Now we are ready to show the solvability of (DDPMS) for the system $\Sigma_{P,Y}$ in the following lemma.

Lemma 3.16

Let Σ_F be given by:

$$\Sigma_F: \begin{cases} \sigma p &= K_{P,Y}p + L_{P,Y}y_{P,Y} \\ u_{P,Y} &= M_{P,Y}p + N_{P,Y}y_{P,Y} \end{cases} \tag{32}$$

where

$$N_{P,Y} := -F_{2,P,Y}K_{2,P,Y}$$
$$M_{P,Y} := F_{1,P,Y} - N_{P,Y}C_{1,P}$$
$$L_{P,Y} := B_{P,Y}N_{P,Y} - K_{1,P,Y}$$
$$K_{P,Y} := A_{P,Y} + B_{P,Y}M_{P,Y} + K_{1,P,Y}C_{1,P}$$

The interconnection of Σ_F and $\Sigma_{P,Y}$ is internally stable and the closed-loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is zero.

Proof. We can write out the formulas for a state-space representation of the interconnection of $\Sigma_{P,Y}$ and Σ_F. We then apply the following basis transformation:

$$\begin{pmatrix} x_{P,Y} - p \\ p \end{pmatrix} = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{P,Y} \\ p \end{pmatrix}$$

After this transformation one immediately sees that the closed-loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is zero. Moreover the system matrix (3) after this transformation is given by:

$$\begin{pmatrix} A_{P,Y} + K_{1,P,Y}C_{1,P} & 0 \\ L_{P,Y}C_{1,P} & A_{P,Y} + B_{P,Y}F_{1,P,Y} \end{pmatrix}$$

Lemma 3.15 guarantees that this matrix is asymptotically stable. Hence Σ_F is internally stabilizing. □

We know Σ_F is internally stabilizing and the resulting closed-loop system has H_∞ norm less than 1 for the system $\Sigma_{P,Y}$. Hence, by applying Lemma 3.14, we find that Σ_F satisfies part (i) of Theorem 2.1. This completes the proof of (ii) ⇒ (i) of Theorem 2.1. We have already shown the reverse implication and hence the proof of Theorem 2.1 is completed.

4. CONTROLLER STRUCTURE

In the previous section, we found a controller for Σ which satisfies all requirements, but its structure is very cloudy. In this section we define a controller, which also achieves disturbance decoupling when applied to $\Sigma_{P,Y}$, but which has a very appealing structure.

We first need to construct a matrix with a desired stability property:

Lemma 4.1

There exists a matrix $\bar{K}_0$ such that

$$[I + \bar{K}_0(I - D_{12,P,Y}D^{\dagger}_{12,P,Y})C_{1,P}]\ (A_{P,Y} - E_{P,Y}D^{\dagger}_{12,P,Y})$$

is stable.

Proof. According to Lemma 3.15 there exists a matrix K_0 such that $A_1 + K_0C_1$ is stable where

$$A_1 = (A_{P,Y} - E_{P,Y}D^{\dagger}_{12,P,Y})$$
$$C_1 = (I - D_{12,P,Y}D^{\dagger}_{12,P,Y})C_{1,P}$$

Since, for discrete-time systems detectability of (C_1, A_1) implies that the pair (C_1A_1, A_1) is detectable there exists a matrix $\bar{K}_0$ such that $A_1 + \bar{K}_0C_1A_1$ is stable. This implies that $\bar{K}_0$ satisfies the conditions of the lemma. □

Remark. This lemma might look rather strange but it is essential. If we use one-step-ahead predictors then the estimator is stable if the filter gain K is such that $A + KC$ is stable. However, in this section we use current estimators where we also use the measurement $y(k)$

to estimate $x(k)$. In that case the estimator is stable if the filter gain is such that $(I+KC)A$ is stable. Intuitively the above lemma tells us that we can find a stable current estimator if we can find a stable one-step-ahead estimator.

Note that an optimal full-information feedback for $\Sigma_{P,Y}$ is given by:

$$u_{P,Y} = F_{1,P,Y}x_{P,Y} + F_{2,P,Y}w_{P,Y}$$

where we change $F_{2,P,Y}$ with respect to the previous section into:

$$F_{2,P,Y} := -D_{21,P,Y}^{\dagger}D_{22,P,Y} + (I - D_{21,P,Y}^{\dagger}D_{21,P,Y})F_0 YC_{1,P}^{T}(W_P)^{\dagger}$$

It can be shown, along the same lines as the proof of Lemma 3.16 that the following controller stabilizes $\Sigma_{P,Y}$ and achieves disturbance decoupling:

$$\begin{aligned}
\sigma p &= A_{P,Y}p + B_{P,Y}u_{P,Y} + E_{P,Y}\hat{w} - \bar{K}_0\Pi_1(\sigma y - C_{1,P}[A_{P,Y}p + B_{P,Y}u_{P,Y} + E_{P,Y}\hat{w}]) \\
\hat{w} &= D_{12,P,Y}^{\dagger}(y_{P,Y} - C_{1,P}p) \\
u_{P,Y} &= F_{1,P,Y}\hat{x} + F_{2,P,Y}\hat{w}
\end{aligned}$$

where

$$\Pi_1 := I - D_{12,P,Y}D_{12,P,Y}^{\dagger} = I - W_P W_P^{\dagger}$$

We are going to apply this controller to the system Σ. However, if we rewrite this controller in terms of the original system parameters it has a very special structure:

$$\begin{aligned}
\sigma\hat{x} &= A\hat{x} + Bu + E\hat{w} + \sigma K_1(y - \hat{y}) \\
\sigma\hat{w} &= R^{-1}(E^T PA_x + D_{22}^T C_x)[A\hat{x} + Bu + E\hat{w}] + \sigma K_2(y - \hat{y}) \\
\sigma\hat{y} &= C_1[A\hat{x} + Bu + E\hat{w}] + D_{12}R^{-1}(E^T PA_x + D_{22}^T C_x)[A\hat{x} + Bu + E\hat{w}] \\
u &= F_1\hat{x} + F_2\hat{w}
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= -\bar{K}_0\Pi_1 + YC_{1,P}^T W_P^{\dagger}(I + C_{1,P}\bar{K}_0\Pi_1) \\
K_2 &= D_{12,P}^T W_P^{\dagger}(I + C_{1,P}\bar{K}_0\Pi_1)
\end{aligned}$$

while F_1 and F_2 are defined by (10) and (11) respectively. We see that we have a full-information feedback:

$$u = F_1 x + F_2 w$$

where we replace the state x and the disturbance w by their respective estimates $\hat{x}$ and $\hat{w}$. For the state and the disturbance we have built estimators. If we write $s(k \mid k)$ for the estimate of the variable s at time k using measurements $y(0), \ldots y(k)$ and $s(k \mid k-1)$ for the estimate of the variable s at time k using measurements $y(0), \ldots y(k-1)$ then we can express the structure even clearer. We get the following form:

$$\Sigma_F \begin{cases}
x(k+1 \mid k+1) = x(k+1 \mid k) + K_1[y(k+1) - y(k+1) \mid k)] \\
w(k+1 \mid k+1) = w(k+1 \mid k) + K_2[y(k+1) - y(k+1 \mid k)] \\
x(k+1 \mid k) = Ax(k \mid k) + Bu(k) + Ew(k \mid k) \\
w(k+1 \mid k) = R^{-1}(E^T PA_x + D_{22}^T C_x)x(k+1 \mid k) \\
y(k+1) \mid k) = C_1 x(k+1 \mid k) + D_{12}w(k+1 \mid k) \\
u(k) = F_1 x(k \mid k) + F_2 w(k \mid k)
\end{cases}$$

Note that in the state feedback case we can identify a worst-case response for the disturbance w:

$$w(k) = R^{-1}(E^{\mathrm{T}} P A_x + D_{22}^{\mathrm{T}} C_x) x(k) \tag{33}$$

In the above controller we have to estimate $w(k+1 \mid k)$. Clearly past measurements do not tell us anything. However, this controller expects the worst-case response (33) and estimates this worst-case response.

5. REDUCED-ORDER ESTIMATOR-BASED CONTROLLER

In this section we show that for the singular H_∞ optimal control problem satisfying part (i) of Theorem 2.1 we can always find a solution which has dynamical order less than that of the plant and is of reduced-order observer-based structure. This result is analogous to those obtained in Reference 16 for continuous-time problems. Without loss of generality, we develop such a reduced-order observer-based controller for the system $\Sigma_{\mathrm{P,Y}}$ defined in the previous section. Consider the $\Sigma_{\mathrm{P,Y}}$ defined by (31). There exists a constant output prefeedback law $F_{\mathrm{pre}} y_{\mathrm{P,Y}}$ such that after applying this prefeedback law, namely setting

$$u_{\mathrm{P,Y}} \to F_{\mathrm{pre}} y_{\mathrm{P,Y}} + u_{\mathrm{P,Y}} \tag{34}$$

the direct feed-through term from $w_{\mathrm{P,Y}}$ from $z_{\mathrm{P,Y}}$ disappears. Hence without loss of generality, hereafter we assume that $D_{22,\mathrm{P,Y}} = 0$.

There exists an 'optimal' state feedback gain $F_{\mathrm{P,Y}}$ in the sense that

$$(C_{2,\mathrm{P,Y}} + D_{21,\mathrm{P}} F_{\mathrm{P,Y}})(sI - A_{\mathrm{P,Y}} - B_{\mathrm{P,Y}} F_{\mathrm{P,Y}})^{-1} E_{\mathrm{P,Y}} \equiv 0$$

with $A_{\mathrm{P,Y}} + B_{\mathrm{P,Y}} F_{\mathrm{P,Y}}$ stable. We need to construct an observer of low order. Without loss of generality but for simplicity of presentation, we assume that the matrices $C_{1,\mathrm{P}}$ and $D_{12,\mathrm{P,Y}}$ are already in the form

$$C_{1,\mathrm{P}} = \begin{pmatrix} 0 & C_{1,02} \\ I_{P-m_0} & 0 \end{pmatrix} \quad \text{and} \quad D_{12,\mathrm{P,Y}} = \begin{pmatrix} D_{12,0} \\ 0 \end{pmatrix} \tag{35}$$

where m_0 is the rank of $D_{12,\mathrm{P,Y}}$ and $D_{12,0}$ is of full rank. Then the given system $\Sigma_{\mathrm{P,Y}}$ can be written as,

$$\begin{cases} \sigma\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} w_{\mathrm{P,Y}} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u_{\mathrm{P,Y}} \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & C_{1,02} \\ I_{p-m_0} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_{12,0} \\ 0 \end{pmatrix} w_{\mathrm{P,Y}} \\ z_{\mathrm{P,Y}} = C_{2,\mathrm{P,Y}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_{21,\mathrm{P,Y}} U_{\mathrm{P,Y}} \end{cases} \tag{36}$$

where $(x_1', x_2')' = x_{\mathrm{P,Y}}$ and $(y_0', y_1')' = y_{\mathrm{P,Y}}$. We note that $y_1 \equiv x_1$. Thus, one needs to estimate only the state x_2 in the reduced-order estimator. Then following closely the procedure given in Reference 16, we first rewrite the state equation for x_1 in terms of the measured output y_1 and state x_2 as follows,

$$\sigma y_1 = A_{11} y_1 + A_{12} x_2 + E_1 w_{\mathrm{P,Y}} + B_1 u_{\mathrm{P,Y}} \tag{37}$$

where y_1 and $u_{P,Y}$ are known. Observation of x_2 is made via y_0 and

$$\tilde{y}_1 = A_{12}x_2 + E_1 w_{P,Y} = \sigma y_1 - A_{11}y_1 - B_1 u_{P,Y} \tag{38}$$

A reduced-order system for the estimation of state x_2 is given by

$$\begin{cases} \sigma x_2 = A_R x_2 + E_R w_{P,Y} + (A_{21} \ \ B_2)\begin{pmatrix} y_1 \\ u_{P,Y} \end{pmatrix} \\ y_R \ = C_R x_2 + D_R w_{P,Y} \end{cases} \tag{39}$$

where

$$A_R := A_{22}, \qquad E_R := E_2, \qquad C_R := \begin{pmatrix} C_{1,02} \\ A_{12} \end{pmatrix}, \qquad D_R := \begin{pmatrix} D_{12,0} \\ E_1 \end{pmatrix} \tag{40}$$

Based on (39), one can construct a reduced-order observer for x_2 as,

$$\sigma\hat{x}_2 = A_R\hat{x}_2 + (A_{21}B_2)\begin{pmatrix} y_1 \\ u_{P,Y} \end{pmatrix} + K_R[C_R\hat{x}_2 - y_R] \tag{41}$$

where K_R is the observer gain matrix which must be chosen such that $A_R + K_R C_R$ is asymptotically stable. Later, we will make a specific choice for K_R.

At this moment we have a reduced-order observer and an optimal state feedback. However, y_r contains a future measurement (the term σy_1 in (38). We apply a transformation to remove this term. We partition $K_R = (K_{R0}, K_{R1})$ compatible with the dimensions of the outputs $(y_0', \tilde{y}_1')'$, and at the same time define a new variable,

$$v := \hat{x}_2 + K_{R1}\tilde{y}_1$$

We then obtain the following reduced-order estimator-based controller,

$$\begin{cases} \sigma v \ = \ (A_R + K_R C_R)v + (B_2 + K_{R1}B_1)u_{P,Y} + G_R y_{P,Y} \\ \hat{x}_{P,Y} = \begin{pmatrix} 0 \\ I_{n-p+m_0} \end{pmatrix} v + \begin{pmatrix} 0 & I \\ 0 & -K_{R1} \end{pmatrix} y_{P,Y} \\ u_{P,Y} = F_{P,Y}\hat{x}_{P,Y} + F_{\text{pre}} y_{P,Y} \end{cases} \tag{42}$$

where

$$G_R = [-K_{R0}, A_{21} + K_{R1}A_{11} - (A_R + K_R C_R)K_{R1}]$$

and $F_{P,Y}$ is state feedback gain and F_{pre} is the output prefeedback gain.

Finally we need to choose K_R. This will be done such that the resulting controller achieves disturbance decoupling when applied to $\Sigma_{P,Y}$. We know that there exists an output injection such that:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} 0 & C_{1,02} \\ I_{p-m_0} & 0 \end{pmatrix} \tag{43}$$

is stable and

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} D_{12,0} \\ 0 \end{pmatrix} = 0 \tag{44}$$

Because the matrix in (43) is stable there exists a matrix L such that

$$A_{22} + K_{21}C_{1,02} + L(A_{12} + K_{11}C_{1,02})$$

is stable. Moreover (44) implies that

$$E_2 + K_{21}D_{12,0} + L(E_1 + K_{11}D_{12,0}) = 0$$

We then choose:

$$K_{\mathrm{R}} = (K_{\mathrm{R}0}\ \ K_{\mathrm{R}1}) = (K_{21} + LK_{11}\ \ L)$$

It is easy to check that the resulting controller is indeed stabilizing and achieves disturbance decoupling when applied to $\Sigma_{\mathrm{P,Y}}$.

Remark. It is interesting to point out that the state-space representation of the reduced-order estimator-based controller in (42) might not be minimal and hence the McMillan degree of this controller might be less than the dynamical order of its state-space representation (42). This is mainly due to the stable dynamics which become unobservable in the controlled output $z_{\mathrm{P,Y}}$ after the preliminary output feedback law (34).

A very interesting example is the state feedback case for $C_1 = I$ and $D_{12} = 0$. In this case, the preliminary output feedback F_{pre} in (34) can be chosen such that after this preliminary feedback $C_{2,\mathrm{P,Y}} = 0$ and $A_{\mathrm{P,Y}}$ is stable. Hence we can choose $F_{\mathrm{P,Y}} = 0$ but this implies that the reduced-order estimator-based controller (42) has McMillan degree equal to zero and it reduces to the static state feedback solution

$$u_{\mathrm{P,Y}} = F_{\mathrm{pre}}y$$

6. CONCLUSION

In this paper, we removed some standard assumptions on the system parameters. Moreover, we specified the structure of discrete time H_∞ controllers. Finally, we showed how to derive controllers of lower dynamical order without loss of performance. This is done by deriving reduced-order observers. Our results are obtained under the assumption that both systems (A, B, C_2, D_{21}) and (A, E, C_1, D_{12}) are free of invariant zeros on unit circle. A most trivial technique to handle invariant zeros on unit circle is to perturb the plant data such that the perturbed plant satisfies our assumptions. However, the resulting criteria for the existence of the solution to the H_∞ control problem for the perturbed plant are not *algebraic* in the nature. Hence the derivation of algebraic criteria directly in discrete domain for this case is an open problem.

Via the bilinear transform and our knowledge about the problems of invariant zeros on the imaginary axis for H_∞ control problems in continuous time (see References 5, 7 and 10), we know that in the case of invariant zeros on the unit circle several problems arise. These are mainly due to the fact that H_∞ controllers have a tendency of cancelling stable zeros of the system and will try to achieve this approximately if there are zeros on the unit circle. Hence we have poor stability margins. Moreover, the minimal achievable H_∞ norm may depend discontinuously on the system parameters if there are invariant zeros on the unit circle. Hence we also have numerical difficulties. The main problem in this respect is the nonuniqueness of (sub)optimal H_∞ controllers. Suppose we want to get closer and closer to the minimal achievable H_∞ norm. When can we avoid almost pole–zero cancellations near the unit circle? For this question, very little is known. However, there are examples where we can get very good stability margins even though there are zeros on the unit circle. Similarly there are examples where we always have bad stability margins near optimality. What is needed is a characterization of the achievable stability margin near optimality.

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