INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL, VOL. 4, 503–519 (1994)

A NON-RECURSIVE METHOD FOR SOLVING THE GENERAL DISCRETE-TIME RICCATI EQUATIONS RELATED TO THE H_∞ CONTROL PROBLEM

BEN M. CHEN

Department of Electrical Engineering, National University of Singapore, Kent Ridge Crescent, Singapore 0511

ALI SABERI

School of Electrical Engineering and Computer Science, Washington State University, Pullman, Washington 99164, U.S.A.

AND

YACOV SHAMASH

College of Engineering and Applied Sciences, State University of New York at Stony Brook, Stony Brook, NY 11794, U.S.A.

SUMMARY

In this paper we propose a nonrecursive method for solving the general discrete-time algebraic Riccati equation related to the H_∞ control problem (H_∞-DARE). We have achieved this by casting the problem of solving a given H_∞-DARE to the problem of solving an auxiliary continuous-time algebraic Riccati equation associated with the H_∞ control problem (H_∞-CARE) for which the well known nonrecursive methods of solving are available. The advantages of our approach are: it reduces the computation involved in the recursive algorithms while giving much more accurate solutions, and it readily provides the properties of the general H_∞-DARE.

KEY WORDS Algebraic Riccati equations H_∞ control Robust control

1. INTRODUCTION

The discrete-time algebraic Riccati equation (DARE) has been investigated extensively in the literature (see, for example References 3, 7, 8, 11–13). Here, most of the work was based on the discrete-time algebraic Riccati equation appearing in linear quadratic control problem (hereafter we will refer to such a DARE as the H_2-DARE). Also, recently the problem of H_∞ control and that of differential games for discrete-time systems, have been studied by a number of researchers including References 2, 5, 10, 15. This work gives rise to a different kind of algebraic Riccati equation (hereafter we call it an H_∞-DARE). Analysing and solving such an H_∞-DARE are very difficult, primarily because of an indefinite nonlinear term and because we cannot *a priori* guarantee the existence of solutions. In a recent paper,[16] Stoorvogel and Weeren, for the first time to our knowledge, have derived a recursive method to find a

CCC 1049–8923/94/040503–17

stabilizing solution, if it exists, to this general H_∞-DARE. Their method proceeds as follows:

(1) solve a Lyapunov equation to determine an initial condition for an H_2-DARE;
(2) utilize the recursive algorithm of Reference 3 to solve this H_2-DARE and use the solution as an initial condition for the H_∞-DARE;
(3) compute the solution for the H_∞-DARE using another recursive algorithm.

A main drawback of the above procedure, however, is that it is in general computationally very expensive, as it involves two recursive steps. In this paper, we propose a nonrecursive method for solving this H_∞-DARE. We cast the problem of solving a given H_∞-DARE to the problem of solving a continuous-time algebraic Riccati equation associated with the H_∞ control problem (H_∞-CARE), which can be obtained easily from the given data of H_∞-DARE. Hence, one can utilize the well-known nonrecursive methods to solve this auxiliary problem to obtain the solution to the H_∞-DARE. The advantages of our approach over the recursive method are threefold: (a) it reduces the computation involved while giving much more accurate solutions, (b) it brings a clear intuition to the conditions associated with the H_∞-DARE, and (c) some of the properties of the H_∞-DARE follow readily from the continuous-time counterpart.

The outline of this paper is as follows. In Section 2, we introduce the detailed problem statement while in Section 3 we give our main results, namely a nonrecursive method for solving the H_∞-DAREs. The proofs of the main results are given in Section 4, and Section 5 contains a numerical example that illustrates our procedure. Finally, we make the concluding remarks in Section 6.

Throughout this paper A' denotes the transpose of A, and I denotes an identity matrix of appropriate dimension.

2. PROBLEM STATEMENT

In this paper we propose a nonrecursive procedure that generates symmetric positive semidefinite matrices, P, such that

$$V(P) := B'PB + D_1'D_1 > 0 \tag{1}$$

$$R(P) := \gamma^2 I - D_2'D_2 - E'PE + (E'PB + D_2'D_1)V(P)^{-1}(B'PE + D_1'D_2) > 0 \tag{2}$$

and such that the following discrete-time algebraic Riccati equation (DARE) is satisfied:

$$P = A'PA + C'C - \begin{bmatrix} B'PA + D_1'C \\ E'PA + D_2'C \end{bmatrix}' G(P)^{-1} \begin{bmatrix} B'PA + D_1'C \\ E'PA + D_2'C \end{bmatrix} \tag{3}$$

where

$$G(P) := \begin{bmatrix} D_1'D_1 & D_1'D_2 \\ D_2'D_1 & D_2'D_2 - \gamma^2 I \end{bmatrix} + \begin{bmatrix} B' \\ E' \end{bmatrix} P [B \ \ E] \tag{4}$$

The conditions (1) and (2) guarantee that the matrix $G(P)$ is invertible. In this paper we are particularly interested in solutions P of (1), (2) and (3) such that all the eigenvalues of the matrix A_{cl} are inside the unit circle, where

$$A_{cl} := A - [B \ \ E]\, G(P)^{-1} \begin{bmatrix} B'PA + D_1'C \\ E'PA + D_2'C \end{bmatrix} \tag{5}$$

The interest for this particular Riccati equation stems from the discrete-time H_∞ control theory.[15] Also, it is simple to see that by letting $E = 0$ and $D_2 = 0$, (1), (2) and (3) reduce to the well-known Riccati equation from linear quadratic control theory. We first recall that the relation between the above Riccati equation and the discrete-time H_∞ γ-suboptimal full information feedback control problem (see, for example, References 5 and 15). Let us define a system $\Sigma_{\rm FI}$ by

$$\Sigma_{\rm FI}: \begin{cases} x(k+1) = A\; x(k) + B\; u(k) + E\; w(k) \\ y(k) = \begin{pmatrix} I \\ 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ I \end{pmatrix} w(k) \\ z(k) = C\; x(k) + D_1\; u(k) + D_2\; w(k) \end{cases} \tag{6}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ the disturbance input, $z \in \mathbb{R}^p$ the controlled output and $y \in \mathbb{R}^{n+q}$ the measurement. The following lemma is due to Stoorvogel.[15]

Lemma 2.1

Consider a given system (6). Assume that (A, B, C, D_1) is left invertible and has no invariant zeros on the unit circle. Then the following two statements are equivalent:

1. There exists a static feedback $u = K_1 x + K_2 w$, which stabilizes $\Sigma_{\rm FI}$ and makes the H_∞ norm of the closed-loop transfer function from w to z less than γ.
2. There exists a symmetric positive semidefinite solution P to (1), (2) and (3) such that matrix $A_{\rm cl}$ of (5) has all its eigenvalues inside the unit circle.

For the discrete-time H_∞ γ-suboptimal full state feedback control problem, one of the conditions associated with the DARE of (3), is slightly different from those for the full information case. Namely, condition (2) should be replaced by

$$D_2' D_2 + E' P E < \gamma^2 I \tag{7}$$

To be more specific, let us consider the following system $\Sigma_{\rm SF}$,

$$\Sigma_{\rm SF}: \begin{cases} x(k+1) = A\; x(k) + B\; u(k) + E\; w(k) \\ y(k) = x(k) \\ z(k) = C\, x(k) + D_1\; u(k) + D_2\; w(k) \end{cases} \tag{8}$$

where x, u, w and z are the same as the full information case. The following result is recalled from Stoorvogel.[14]

Lemma 2.2

Consider a given system (8). Assume that (A, B, C, D_1) is left invertible and has no invariant zeros on the unit circle. Then the following two statements are equivalent:

1. There exists a static state feedback $u = Kx$, which stabilizes $\Sigma_{\rm SF}$ and makes the H_∞ norm of the closed-loop transfer function from w to z less than γ.
2. There exists a symmetric $P \geqslant 0$ such that (1), (7) and (3) are satisfied and such that matrix $A_{\rm cl}$ of (5) has all its eigenvalues inside the unit circle.

To differentiate the H_∞ γ-suboptimal control problems for the full information feedback case and the full state feedback case, we introduce the following definitions.

Definition 2.1

The DARE of (3) with conditions (1) and (2) are referred to as the H_∞-DARE for the full information problem. Moreover, a symmetric positive semidefinite P is said to be the stabilizing solution of H_∞-DARE for the full information problem if it satisfies (1), (2) and (3), and is such that the matrix A_{cl} has all its eigenvalues inside the unit circle.

Definition 2.2

The DARE of (3) with conditions (1) and (7) are referred to as the H_∞-DARE for the full state feedback problem. Moreover, a symmetric positive semidefinite P is said to be the stabilizing solution of H_∞-DARE for the full state feedback problem if it satisfies (1), (3) and (7), and is such that the matrix A_{cl} has all its eigenvalues inside the unit circle.

The following remark connects the H_∞-DARE to that given in Reference 2, which appears in a different form.

Remark 2.1

Assume that that $D_2 = 0$ and $D_1'[C \; D_1] = [0 \; I]$, it is simple to verify that (1) is automatically satisfied, (2) and (7) can be rewritten as

$$R(P) = I - \gamma^{-2}E'(I + PBB')^{-1}PE > 0 \tag{9}$$

and

$$E'PE < \gamma^2 I \tag{10}$$

respectively, and (3) can be rewritten as,

$$P = C'C + A'P[I + (BB' - \gamma^{-2}EE')P]^{-1}A \tag{11}$$

Also, A_{cl} as defined in (5) is equivalent to

$$A_{cl} = [I + (BB' - \gamma^{-2}EE')P]^{-1}A \tag{12}$$

Remark 2.2

We should point out that if matrix S defined as

$$S := \begin{pmatrix} D_1'D_1 & D_1'D_2 \\ D_2'D_1 & D_2'D_2 - \gamma^2 I \end{pmatrix}$$

is invertible, then the problem of finding the stabilizing solution to the H_∞-DARE for the full information problem or for the full state feedback problem basically can be reduced to that of solving a generalized eigenvalue problem (see Reference 16). However, if matrix S is not invertible, such a technique is not applicable.

It is shown in Reference 16 that a stabilizing solution to the H_∞-DARE for the full information problem or H_∞-DARE for the full state feedback problem, if it exists, is unique. Also, the existence of the stabilizing solution to the H_∞-DARE for the full information problem is necessary, but not sufficient, for the existence of the stabilizing solution to the H_∞-DARE for the full state feedback problem. Moreover, the stabilizing solution to the

H_∞-DARE for the full state feedback problem is also a stabilizing solution to the H_∞-DARE for the full information problem since condition (7) implies (3).

The problem of obtaining the stabilizing solution to the H_∞-DARE for the full state feedback can be easily converted to the problem of finding a stabilizing solution to H_∞-DARE for the full information problem. This can be done as follows. Given an H_∞-DARE for the full state feedback problem, we form and solve the H_∞-DARE for the full information problem.

(1) If the H_∞-DARE for the full information problem does not have a stabilizing solution, then the given H_∞-DARE for the full state feedback problem does not have a stabilizing solution.
(2) Suppose that the H_∞-DARE for the full information problem has a stabilizing solution, say P. If P satisfies (7), then P is the stabilizing solution to the given H_∞-DARE for the full state feedback problem. Otherwise, the given H_∞-DARE for the full state feedback problem does not have a stabilizing solution.

As such, in this paper, without loss of generality but for simplicity of presentation, we will focus only on the H_∞-DARE for the full information problem. Our goal is to derive a nonrecursive (instead of recursive) method for solving the H_∞-DARE for the full information problem.

3. MAIN RESULTS

In this section, we provide a nonrecursive method for computing the stabilizing solution to the H_∞-DARE for the full information problem, i.e., (1), (2) and (3). We first define an auxiliary H_∞-CARE from the given system data and we connect the stabilizing solution for the given H_∞-DARE to the stabilizing solution for the auxiliary H_∞-CARE, for which nonrecursive methods of obtaining solutions are available.

We assume that matrix A in the given H_∞-DARE has no eigenvalues at -1. This is without loss of generality since in the discrete-time H_∞ control problem associated with the given H_∞-DARE, one can always apply a pre-feedback law to relocate the eigenvalues of A that are at -1, provided that (A, B) is stabilizable. In what follows, we define an auxiliary H_∞-CARE,

$$0 = \tilde{P}\tilde{A} + \tilde{A}'\tilde{P} + \tilde{C}'\tilde{C} - \begin{bmatrix} \tilde{B}'\tilde{P} + \tilde{D}_1'\tilde{C} \\ \tilde{E}'\tilde{P} + \tilde{D}_2'\tilde{C} \end{bmatrix}' \tilde{G}^{-1} \begin{bmatrix} \tilde{B}'\tilde{P} + \tilde{D}_1'\tilde{C} \\ \tilde{E}'\tilde{P} + \tilde{D}_2'\tilde{C} \end{bmatrix} \tag{13}$$

with associated condition

$$\tilde{D}_2'(I - \tilde{D}_1(\tilde{D}_1'\tilde{D}_1)^{-1}\tilde{D}_1')\tilde{D}_2 < \gamma^2 I \tag{14}$$

where

$$\left.\begin{aligned} \tilde{A} &:= (A+I)^{-1}(A-I) \\ \tilde{B} &:= 2(A+I)^{-2}B \\ \tilde{E} &:= 2(A+I)^{-2}E \\ \tilde{C} &:= C \\ \tilde{D}_1 &:= D_1 - C(A+I)^{-1}B \\ \tilde{D}_2 &:= D_2 - C(A+I)^{-1}E \end{aligned}\right\} \tag{15}$$

and

$$\tilde{G} := \begin{pmatrix} \tilde{D}_1'\tilde{D}_1 & \tilde{D}_1'\tilde{D}_2 \\ \tilde{D}_2'\tilde{D}_1 & \tilde{D}_1'\tilde{D}_2 - \gamma^2 I \end{pmatrix} \tag{16}$$

If matrix $\tilde{D}_1$ is injective, then condition (14) implies $\tilde{G}$ in (16) is invertible. Again, we are particularly interested in solution $\tilde{P}$ of (13) such that the eigenvalues of $\tilde{A}_{cl}$ are in the open left half plane, where

$$\tilde{A}_{cl} := \tilde{A} - [\tilde{B}\ \tilde{E}]\tilde{G}^{-1}\begin{bmatrix} \tilde{B}'\tilde{P} + \tilde{D}_1'\tilde{C} \\ \tilde{E}'\tilde{P} + \tilde{D}_2'\tilde{C} \end{bmatrix} \tag{17}$$

We note that under the conditions that $\tilde{D}_1$ is injective, $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}_1)$ has no invariant zeros on the jω axis, and (14), the above H_∞-CARE (13) is related to the continuous-time H_∞ γ-suboptimal full information feedback control problem for the following system,

$$\tilde{\Sigma}_{FI}: \begin{cases} \dot{x} = \tilde{A}\ x + \tilde{B}\ u + \tilde{E}\ w \\ y = \begin{pmatrix} I \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} w \\ z = \tilde{C}\ x + \tilde{D}_1\ u + \tilde{D}_2\ w \end{cases} \tag{18}$$

The following lemma is recalled from Reference 14.

Lemma 3.1

Consider a given system (18). Assume that $\tilde{D}_1$ is injective and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}_1)$ has no invariant zeros on the jω axis. Then the following two statements are equivalent:

1. There exists a static feedback law $u = \tilde{K}_1 x + \tilde{K}_2 w$, which stabilizes $\tilde{\Sigma}_{FI}$ and makes the H_∞ norm of the closed-loop transfer function from w to z less than γ.
2. Condition (14) holds and there exists a symmetric $\tilde{P} \geqslant 0$ such that (13) is satisfied and such that the matrix $\tilde{A}_{cl}$ of (17) has all its eigenvalues in the open left half plane.

Now, we are ready to present our main results.

Theorem 3.1

Assume that A has no eigenvalues at -1. Then the following two statements are equivalent:

1. (A, B) is stabilizable and (A, B, C, D_1) is left invertible with no invariant zeros on the unit circle. Moreover, there exists a symmetric positive semidefinite matrix P such that (1), (2) and (3) are satisfied along with the matrix A_{cl} of (5) having all its eigenvalues inside the unit circle.
2. $(\tilde{A}, \tilde{B})$ is stabilizable, $\tilde{D}_1$ is injective and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}_1)$ has no invariant zeros on the jω axis, and (14) holds. Moreover, there exists a symmetric positive semidefinite solution $\tilde{P}$ of the H_∞-CARE (13) such that the eigenvalues of $\tilde{A}_{cl}$, where $\tilde{A}_{cl}$ is as in (17), are in the open left half complex plane.

Furthermore, P and $\tilde{P}$ are related by $P = 2(A' + I)^{-1}\tilde{P}(A + I)^{-1}$.

Remark 3.1

We should point out that the left invertibility of (A, B, C, D_1) is a necessary condition for the existence of the stabilizing solution to the H_∞-DARE for the full information problem (see Reference 16). Moreover, following the proof of Theorem 3.1 in the next section and the properties of the continous-time algebraic Riccati equation, it is easy to show that the condition that (A, B, C, D_1) has no invariant zeros on the unit circle is also necessary for the existence of the stabilizing solution to the H_∞-DARE for the full information problem.

Remark 3.2

From Theorem 3.1, a noniterative method of obtaining the stabilizing solution P to the H_∞-DARE for the full information problem can be established as follows:

1. Obtain the auxiliary H_∞-CARE.
2. Obtain the stabilizing solution $\tilde{P}$ to the H_∞-CARE using some well-known noniterative methods. For clarity, we recall in the following a so-called Schur method (see Reference 9): define a Hamiltonian matrix

$$\mathbf{H} = \begin{bmatrix} \tilde{A} - [\tilde{B}\ \tilde{E}]\tilde{G}^{-1}[\tilde{D}_1\ \tilde{D}_2]'\tilde{C} & -[\tilde{B}\ \tilde{E}]\tilde{G}^{-1}[\tilde{B}\ \tilde{E}]' \\ -\tilde{C}'\{I - [\tilde{D}_1\ \tilde{D}_2]\tilde{G}^{-1}[\tilde{D}_1\ \tilde{D}_2]'\}\tilde{C} & -\{\tilde{A} - [\tilde{B}\ \tilde{E}]\tilde{G}^{-1}[\tilde{D}_1\ \tilde{D}_2]'\tilde{C}\}' \end{bmatrix}$$

Find an orthogonal matrix $\mathbf{T} \in \mathbb{R}^{2n \times 2n}$ that puts $\mathbf{H}$ in the real Schur form

$$\mathbf{T}'\mathbf{H}\mathbf{T} = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}$$

where $S_{11} \in \mathbb{R}^{n \times n}$ is a stable matrix and $S_{22} \in \mathbb{R}^{n \times n}$ is an anti-stable matrix. Partition $\mathbf{T}$ into four $n \times n$ blocks:

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

Then $\tilde{P}$ is given by

$$\tilde{P} = T_{21} T_{11}^{-1}$$

3. The stabilizing solution to the H_∞-DARE for the full information problem is given by

$$P = 2(A' + I)^{-1}\tilde{P}(A + I)^{-1} \tag{19}$$

Corollary 3.1 below shows some properties of the H_∞-DARE for the full information problem. This corollary follows from Theorem 3.1 and the well-known results in the continuous-time H_∞ optimal control theory (see, for example, Reference 18).

Corollary 3.1

Consider the H_∞-DARE for the full information problem. Assume that (A, B) is stabilizable and (A, B, C, D_1) is left-invertible with no invariant zeros on the unit circle. Then we have the following:

1. The stabilizing solution, if existent, is unique.
2. If the stabilizing solution to the H_∞-DARE for the full information problem exists for some γ_1, then the stabilizing solution exists for any $\gamma > \gamma_1$.

It is well-known that the H_∞-DARE is the generalization of the H_2-DARE. Namely, by letting $\gamma = \infty$, or equivalently $E = 0$ and $D_2 = 0$, we obtain the general H_2-DARE. For the purpose of completeness we give the following corollary that provides a noniterative method of solving the general H_2-DARE.

Corollary 3.2

Assume that A has no eigenvalues at -1. Then the following two statements are equivalent:

1. (A, B) is stabilizable and (A, B, C, D_1) is left-invertible with no invariant zeros on the unit circle. Moreover, there exists a positive semidefinite matrix P such that
$$B'PB + D_1'D_1 > 0 \tag{20}$$
$$P = A'PA + C'C - (A'PB + C'D_1)(D_1'D_1 + B'PB)^{-1}(A'PB + C'D_1)' \tag{21}$$
and such that the eigenvalues of the matrix A_{cl} are inside the unit circle, where
$$A_{cl} = A - B(D_1'D_1 + B'PB)^{-1}(A'PB + C'D_1)' \tag{22}$$
2. $(\tilde{A}, \tilde{B})$ is stabilizable, $\tilde{D}_1$ is injective and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}_1)$ has no invariant zeros on the jω axis. Moreover, there exists a positive semidefinite solution $\tilde{P}$ of the following CARE
$$0 = \tilde{P}\tilde{A} + \tilde{A}'\tilde{P} + \tilde{C}'\tilde{C} - (\tilde{P}\tilde{B} + \tilde{C}'\tilde{D}_1)(\tilde{D}_1'\tilde{D}_1)^{-1}(\tilde{P}\tilde{B} + \tilde{C}'\tilde{D}_1)' \tag{23}$$
such that the eigenvalues of $\tilde{A}_{cl}$ are in the open left half complex plane, where
$$\tilde{A}_{cl} = \tilde{A} - \tilde{B}(\tilde{D}_1'\tilde{D}_1)^{-1}(\tilde{P}\tilde{B} + \tilde{C}'\tilde{D}_1)' \tag{24}$$

Furthermore, P and $\tilde{P}$ are related by $P = 2(A' + I)^{-1}\tilde{P}(A + I)^{-1}$.

Lemmas 2.1 and 3.1, and Theorem 3.1 show the interconnection between the H_∞ γ-suboptimal control problem for the discrete-time system Σ_{FI} and the continuous-time system $\tilde{\Sigma}_{FI}$. This connection is formalized in the following lemma.

Lemma 3.2

Assume that (A, B) is stabilizable and (A, B, C, D_1) is left-invertible with no invariant zeros on the unit circle. Then the following statements are equivalent:

1. The H_∞ γ-suboptimal full information feedback control problem for the discrete-time system Σ_{FI} has a solution. Namely, for a given γ, there exists a static full information feedback $u = K_1x + K_2w$ such that the closed-loop transfer function from w to z has an H_∞-norm less than γ.
2. The H_∞ γ-suboptimal full information feedback control problem for the continuous-time system $\tilde{\Sigma}_{FI}$ has a solution. Namely, for a given γ, there exists a static full information feedback $u = \tilde{K}_1x + \tilde{K}_2w$ such that the closed-loop transfer function from w to z has an H_∞-norm less than γ.

Remark 3.3

The results of Lemma 3.2 can easily be obtained from a different route. It is well known that the Hankel norm and the H_∞ norm of a transfer function are invariant under bilinear transformation (see, for example, Reference 4). Hence one can recast the H_∞ γ-suboptimal control problem for the discrete-time system Σ_{FI} into an equivalent H_∞ γ-suboptimal control problem for an auxiliary continuous-time system obtained by performing bilinear transformation on Σ_{FI}. It can be shown that one of the state-space realizations of this auxiliary continuous-time system, Σ_{BL}, is given by

$$\Sigma_{BL}: \begin{cases} \dot{x} = \tilde{A}\ x + \tilde{B}\ u + \tilde{E}\ w \\ y = \begin{pmatrix} I \\ 0 \end{pmatrix} x + \begin{pmatrix} \tilde{D}_3 \\ 0 \end{pmatrix} u + \begin{pmatrix} \tilde{D}_2 \\ I \end{pmatrix} w \\ z = \tilde{C}\ x + \tilde{D}_1\ u + \tilde{D}_2\ w \end{cases} \tag{25}$$

where $\tilde{D}_3 = -(A+I)^{-1}B$, $\tilde{D}_4 = -(A+I)^{-1}E$, and $\tilde{A}$, $\tilde{B}$, $\tilde{E}$, $\tilde{C}$, $\tilde{D}_1$ and $\tilde{D}_2$ are as defined in (15). Consequently the H_∞ γ-suboptimal control problem for the discrete-time Σ_{FI} has a solution if and only if the H_∞ γ-suboptimal control problem for the continuous-time system Σ_{BL} has a solution. However, we note that Σ_{BL} is not completely in the full information form. This difficulty can easily be removed by redefining the measurement output in Σ_{BL} as

$$\tilde{y} := \begin{bmatrix} I & -\tilde{D}_4 \\ 0 & I \end{bmatrix} \left(y - \begin{bmatrix} \tilde{D}_3 \\ 0 \end{bmatrix} u \right) = \begin{pmatrix} I \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} w \tag{26}$$

It is now obvious that Σ_{BL} with the new measurement output $\tilde{y}$ is in fact the same as $\tilde{\Sigma}_{FI}$. Also, it is easy to show that the H_∞ γ-suboptimal problem for Σ_{BL} has a solution if and only if the H_∞ γ-suboptimal problem for Σ_{FI} has a solution and hence the result of Lemma 3.2 follows. It is important to note that the bilinear transformation approach does not establish a relationship between the stabilizing solution of the H_∞-CARE associated with the continuous-time system $\tilde{\Sigma}_{FI}$, obtained by performing bilinear transformation on discrete-time system Σ_{FI} and defining the new measurement as in (26), and the H_∞-DARE associated with the given discrete-time system Σ_{FI}. In fact, the main contribution of Theorem 3.1 is to establish such a relationship.

4. PROOF OF RESULTS

Before we proceed to prove Theorem 3.1, we first introduce a nonrecursive method for solving the following discrete-time algebraic Riccati equation, which is even more general than the H_∞-DARE and which plays a critical role in this paper,

$$P = A'PA - (A'PM + N)(R + M'PM)^{-1}(M'PA + N') + Q \tag{27}$$

where A, M, N, R and Q are real matrices of dimensions $n \times n$, $n \times m$, $n \times m$, $m \times m$ and $n \times n$, respectively, and with Q and R being symmetric matrices. We will show that the DARE of (27) can be converted to a continuous-time Riccati equation. Our approach is motivated by that of Reference 17 in which only the result for the case $R = I$, $Q \geqslant 0$ and $N = 0$, was given without proof.*

Assume that matrix A has no eigenvalues at -1. We define

$$\left.\begin{aligned} F &:= (A+I)^{-1}(A-I) \\ G &:= 2(A+I)^{-2}M \\ H &:= -Q(A+I)^{-1}M + N \\ W &:= R + M'(A'+I)^{-1}Q(A+I)^{-1}M - N'(A+I)^{-1}M - M'(A'+I)^{-1}N \end{aligned}\right\} \tag{28}$$

We have the following lemma.

* A reviewer has brought to our attention that in a recent paper by Kondo and Hara,[6] an attempt has also been made to connect CARE and DARE, again without proofs, for a special class of regular H-∞ problems.

Lemma 4.1

Assume that matrix A has no eigenvalues at -1. Then the following two statements are equivalent.

1. P is a symmetric solution to the DARE (27) and matrix W is nonsingular.
2. $\tilde{P}$ is a symmetric solution to the continuous algebraic Riccati equation (CARE),
$$\tilde{P}F + F'\tilde{P} - (\tilde{P}G + H)W^{-1}(\tilde{P}G + H)' + Q = 0 \tag{29}$$
and $R + 2G'(I - F')^{-1}\tilde{P}(I - F)^{-1}G$ is nonsingular.

Moreover, P and $\tilde{P}$ are related by $P = 2(A' + I)^{-1}\tilde{P}(A + I)^{-1}$.

Proof. First, let us consider the following reductions:

$$\begin{aligned} A'PA - P + Q &= 2A'(A' + I)^{-1}\tilde{P}(A + I)^{-1}A - 2(A' + I)^{-1}\tilde{P}(A + I)^{-1} + Q \\ &= 2(A' + I)^{-1}A'\tilde{P}A(A + I)^{-1} - 2(A' + I)^{-1}\tilde{P}(A + I)^{-1} + Q \\ &= (A' + I)^{-1}(2A'\tilde{P}A - 2\tilde{P})(A + I)^{-1} + Q \\ &= (A' + I)^{-1}[(A' + I)\tilde{P}(A - I) + (A' - I)\tilde{P}(A + I)](A + I)^{-1} + Q \\ &= \tilde{P}(A - I)(A + I)^{-1} + (A' + I)^{-1}(A' - I)\tilde{P} + Q \\ &= \tilde{P}F + F'\tilde{P} + Q \end{aligned} \tag{30}$$

(1. ⇒ 2.) Let us start with the following trivial equality,

$$A'PA - P + (A' + I)P(A + I) - (A' + I)PA - A'P(A + I) = 0$$

which implies that

$$P - PA(A + I)^{-1} + (A' + I)^{-1}A'P + (A' + I)^{-1}A'PA(A + I)^{-1} - (A' + I)^{-1}P(A + I)^{-1} = 0$$

Then we have

$$\begin{aligned} W &= R + M'(A' + I)^{-1}Q(A + I)^{-1}M - N'(A + I)^{-1}M - M'(A' + I)^{-1}N \\ &= R + M'(A' + I)^{-1}Q(A + I)^{-1}M - N'(A + I)^{-1}M - M'(A' + I)^{-1}N \\ &\quad + M'PM - M'PA(A + I)^{-1}M - M'(A' + I)^{-1}A'PM \\ &\quad + M'(A' + I)^{-1}A'PA(A + I)^{-1}M - M'(A' + I)^{-1}P(A + I)^{-1}M \\ &= R + M'PM - (M'PA + N')(A + I)^{-1}M - M'(A' + I)^{-1}(A'PM + N) \\ &\quad + M'(A' + I)^{-1}(A'PA + Q - P)(A + I)^{-1}M && (31) \\ &= R + M'PM - (M'PA + N')(A + I)^{-1}M - M'(A' + I)^{-1}(A'PM + N) \\ &\quad + M'(A' + I)^{-1}(A'PM + N)(R + M'PM)^{-1}(M'PA + N')(A + I)^{-1}M && (32) \\ &= [I - M'(A' + I)^{-1}(A'PM + N)(R + M'PM)^{-1}] \\ &\quad \times (R + M'PM)[I - (R + M'PM)^{-1}(M'PA + N')(A + I)^{-1}M] && (33) \end{aligned}$$

Here we note that we have used (27) to get (32) from (31). By the assumption that W is nonsingular, we have

$$\begin{aligned} R + M'PM = {} & [I - M'(A' + I)^{-1}(A'PM + N)(R + M'PM)^{-1}]^{-1}W \\ & \times [I - (R + M'PM)^{-1}(M'PA + N')(A + I)^{-1}M]^{-1} \end{aligned}$$

Hence,

$$\begin{aligned}
&(A'PM+N)(R+M'PM)^{-1}(M'PA+N') \\
&\quad = (A'PM+N)[I-(R+M'PM)^{-1}(M'PA+N')(A+I)^{-1}M]\,W^{-1} \\
&\qquad \times [I-(R+M'PM)^{-1}(M'PA+N')(A+I)^{-1}M]'(M'PA+N') \\
&\quad = [A'PM-(A'PM+N)(R+M'PM)^{-1}(M'PA+N')(A+I)^{-1}M+N]\,W^{-1} \\
&\qquad \times [A'PM-(A'PM+N)(R+M'PM)^{-1}(M'PA+N')(A+I)^{-1}M+N]' \qquad (34) \\
&\quad = [A'PM+(P-A'PA-Q)(A+I)^{-1}M+N]\,W^{-1} \\
&\qquad \times [A'PM+(P-A'PA-Q)(A+I)^{-1}M+N]' \qquad (35) \\
&\quad = [(A'P+P-Q)(A+I)^{-1}M+N]\,W^{-1}[(A'P+P-Q)(A+I)^{-1}M+N]' \\
&\quad = [(A'+I)P(A+I)(A+I)^{-2}M-Q(A+I)^{-1}M+N]\,W^{-1} \\
&\qquad \times [(A'+I)P(A+I)(A+I)^{-2}M-Q(A+I)^{-1}M+N]' \\
&\quad = (\tilde{P}G+H)\,W^{-1}(\tilde{P}G+H)' \qquad (36)
\end{aligned}$$

Again, we have used (27) to get (35) from (34). Finally, (27), (30) and (36) imply that

$$\tilde{P}F+F'\tilde{P}-(\tilde{P}G+H)W^{-1}(\tilde{P}G+H)'+Q=0$$

(2. ⇒ 1.) It follows from (28) that

$$\left.\begin{aligned}
A&=(I+F)(I-F)^{-1} \\
M&=2(I-F)^{-2}G \\
H&=-Q(I-F)^{-1}G+N \\
P&=(I-F')\tilde{P}(I-F)/2 \\
W&=R+G'(I-F')^{-1}Q(I-F)^{-1}G-N'(I-F)^{-1}G-G'(I-F')^{-1}N \\
R+M'PM&=R+2G'(I-F')^{-1}\tilde{P}(I-F)^{-1}G
\end{aligned}\right\}$$

Then we have

$$\begin{aligned}
&R+M'PM \\
&\quad = R+G'(I-F')^{-1}[Q+(\tilde{P}-\tilde{P}F-Q)+(\tilde{P}-F'\tilde{P}-Q)+(\tilde{P}F+F'\tilde{P}+Q)](I-F)^{-1}G \\
&\quad = R+G'(I-F')^{-1}Q(I-F)^{-1}G-N'(I-F)^{-1}G-G'(I-F')^{-1}N \\
&\qquad +G'(I-F')^{-1}[\tilde{P}G-Q(I-F)^{-1}G+N]+[\tilde{P}G-Q(I-F)^{-1}G+N]'(I-F)^{-1}G \\
&\qquad +G'(I-F')^{-1}(\tilde{P}F+F'\tilde{P}+Q)(I-F)^{-1}G \qquad (38) \\
&\quad = W+G'(I-F')^{-1}(\tilde{P}G+H)+(\tilde{P}G+H)'(I-F)^{-1}G \\
&\qquad +G'(I-F')^{-1}(\tilde{P}G+H)W^{-1}(\tilde{P}G+H)'(I-F)^{-1}G \qquad (39) \\
&\quad = [I+W^{-1}(\tilde{P}G+H)'(I-F)^{-1}G]'\,W[I+W^{-1}(\tilde{P}G+H)'(I-F)^{-1}G] \qquad (40)
\end{aligned}$$

Here we note that we have used (29) to get (39) from (38). By assumption, we have $R+M'PM$ nonsingular. Thus, we can rewrite (40) as,

$$W=[I+G'(I-F')^{-1}(\tilde{P}G+H)W^{-1}]^{-1}(R+M'PM)[I+W^{-1}(\tilde{P}G+H)'(I-F)^{-1}G]^{-1}$$

We have the following reductions,

$$\begin{aligned}
&(\tilde{P}G+H)W^{-1}(\tilde{P}G+H)' \\
&\quad = (\tilde{P}G+H)[I+W^{-1}(\tilde{P}G+H)'(I-F)^{-1}G] \\
&\qquad \times (R+M'PM)^{-1}[I+W^{-1}(\tilde{P}G+H)'(I-F)^{-1}G]'(\tilde{P}G+H)' \\
&\quad = [\tilde{P}G+H+(\tilde{P}G+H)W^{-1}(\tilde{P}G+H)'(I-F)^{-1}G](R+M'PM)^{-1} \\
&\qquad \times [\tilde{P}G+H+(\tilde{P}G+H)W^{-1}(\tilde{P}G+H)'(I-F)^{-1}G]' \qquad (41)
\end{aligned}$$

$$
\begin{aligned}
&= [\tilde{P}G - Q(I-F)^{-1}G + (\tilde{P}F + F'\tilde{P} + Q)(I-F)^{-1}G + N]\,(R + M'PM)^{-1} \\
&\quad \times [\tilde{P}G - Q(I-F)^{-1}G + (\tilde{P}F + F'\tilde{P} + Q)(I-F)^{-1}G + N]\,' \qquad (42) \\
&= [(I+F')\tilde{P}(I-F)^{-1}G + N]\,(R + M'PM)^{-1}[G'(I-F')^{-1}\tilde{P}(I+F) + N'] \\
&= (A'PM + N)(R + M'PM)^{-1}(M'PA + N') \qquad (43)
\end{aligned}
$$

Again, we have used (29) to get (42) from (41). Finally, it follows from (29), (30) and (43) that

$$A'PA - (A'PM + N)(R + M'PM)^{-1}(M'PA + N') + Q - P = 0$$

This completes the proof of Lemma 4.1. □

Now, we are ready to prove the main results in Section 3.

Proof of Theorem 3.1. Without loss of generality but for simplicity of presentation, we prove Theorem 3.1 for the case that $\gamma = 1$.

(1. ⇒ 2.) We first proceed to prove that $(\tilde{A}, \tilde{B})$ is stabilizable, $\tilde{D}_1$ is injective and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}_1)$ has no invariant zeros on the unit circle. For any complex scalar s, we consider

$$
\begin{aligned}
\operatorname{rank}[sI - \tilde{A},\ \tilde{B}] &= \operatorname{rank}[sI - (A+I)^{-1}(A-I),\ 2(A+I)^{-2}B] \\
&= \operatorname{rank}[s(A+I) - (A-I),\ 2(A+I)^{-1}B] \\
&= \operatorname{rank}[s(A+I) - (A-I),\ 2(A+I)^{-1}B - sB + (A-I)(A+I)^{-1}B] \\
&= \operatorname{rank}[(1+s)I - (1-s)A,\ (1-s)B] \qquad (44)
\end{aligned}
$$

which implies that $(\tilde{A}, \tilde{B})$ is stabilizable, i.e., all the uncontrollable modes, if any, are in the open left half plane, provided that (A, B) is stabilizable, i.e., all the uncontrollable modes, if any, are inside the unit circle. Also, noting that

$$\tilde{D}_1 = D_1 - C(A+I)^{-1}B = D_1 + C(-I-A)^{-1}B$$

together with the facts that (A, B, C, D_1) is left-invertible and has no invariant zeros on the unit circle, it follows that $\tilde{D}_1$ is of maximal column rank, i.e., $\tilde{D}_1$ is injective, and hence $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}_1)$ is left-invertible. Now, for any $s \neq 1$, we consider

$$
\begin{aligned}
\operatorname{rank}\begin{bmatrix} sI - \tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D}_1 \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} sI - (A+I)^{-1}(A-I) & -2(A+I)^{-2}B \\ C & D_1 - C(A+I)^{-1}B \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} s(A+I) - (A-I) & -2(A+I)^{-1}B \\ C & D_1 - C(A+I)^{-1}B \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} (1+s)I - (1-s)A & -(1-s)B \\ C & D_1 \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} (1+s)/(1-s)I - A & -B \\ C & D_1 \end{bmatrix}
\end{aligned}
$$

which implies that $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}_1)$ has no invariant zeros on the jω axis provided that the system (A, B, C, D_1) has no invariant zeros on the unit circle.

Next, we will show that (14) holds. Let

$$\left.\begin{aligned} M &:= [B\ \ E] \\ N &:= C'[D_1\ \ D_2] \\ R &:= \begin{bmatrix} D_1'D_1 & D_1'D_2 \\ D_2'D_1 & D_2'D_2 - I \end{bmatrix} \\ Q &:= C'C \\ F &:= \tilde{A} \\ G &:= 2(A+I)^{-2}M \\ H &:= -Q(A+I)^{-1}M + N \\ W &:= R + M'(A'+I)^{-1}Q(A+I)^{-1}M - N'(A+I)^{-1}M - M'(A'+I)^{-1}N \\ X &:= I - (R+M'PM)^{-1}(M'PA+N')(A+I)^{-1}M \end{aligned}\right\} \qquad (45)$$

It is simple to verify that

$$W = \begin{bmatrix} \tilde{D}_1'\tilde{D}_1 & \tilde{D}_1'\tilde{D}_2 \\ \tilde{D}_2'\tilde{D}_1 & \tilde{D}_2'\tilde{D}_2 - I \end{bmatrix}$$

Then, (3) and (13) are, respectively, reduced to (27) and (29), and (5) and (17) can be written, respectively, as

$$A_{cl} = A - M(R + M'PM)^{-1}(M'PA + N') \qquad (46)$$

and

$$\tilde{A}_{cl} = F - GW^{-1}(\tilde{P}G + H)' \qquad (47)$$

Noting that

$$\begin{aligned} \det[X] &= \det[I - (R+M'PM)^{-1}(M'PA+N')(A+I)^{-1}M] \\ &= \det[I - M(R+M'PM)^{-1}(M'PA+N')(A+I)^{-1}] \\ &= \det[I + A_{cl}]\det[(A+I)^{-1}] \end{aligned}$$

it follows that X is nonsingular provided that the eigenvalues of A_{cl} are inside the unit circle. Recalling (33) in the proof of Lemma 4.1, we have W nonsingular and

$$W^{-1} = X^{-1}(R + M'PM)^{-1}(X^{-1})' \qquad (48)$$

which implies that the inertia of W^{-1} is equal to the inertia of $(R + M'PM)^{-1}$ (see, for example, Theorem 4.9 of Reference 1). Again, noting that

$$W^{-1} = \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} (\tilde{D}_1'\tilde{D}_1)^{-1} & 0 \\ 0 & [\tilde{D}_2'(I - \tilde{D}_1(\tilde{D}_1'\tilde{D}_1)^{-1}\tilde{D}_1')\tilde{D}_2 - I]^{-1} \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}'$$

and

$$(R + M'PM)^{-1} = \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} \begin{bmatrix} V(P)^{-1} & 0 \\ 0 & -R(P)^{-1} \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix}'$$

where $Y = -(\tilde{D}_1'\tilde{D}_1)^{-1}\tilde{D}_1'\tilde{D}_2$ and $Z = -V(P)^{-1}B'PE$, together with (48) and the facts that $V(P) > 0$ and $R(P) > 0$, it follows that

$$\tilde{D}_2'(I - \tilde{D}_1(\tilde{D}_1'\tilde{D}_1)^{-1}\tilde{D}_1')\tilde{D}_2 < I$$

Using the fact that W is nonsingular, it follows from Lemma 4.1 that $\tilde{P}$ is a positive semidefinite solution of (13).

Finally, we are ready to prove that $\tilde{A}_{cl}$ has all its eigenvalues in the open left half complex plane. It follows from (36) in the proof of Lemma 4.1 that

$$\begin{aligned}
\tilde{A}_{cl} &= F - GW^{-1}(\tilde{P}G + H)' \\
&= F - GX^{-1}(R + M'PM)^{-1}(M'PA + N') \\
&= (A+I)^{-1}(A-I) - 2(A+I)^{-2}M[I - (R+M'PM)^{-1}(M'PA+N')(A+I)^{-1}M]^{-1} \\
&\quad \times (R+M'PM)^{-1}(M'PA+N') \\
&= (A+I)^{-1}\{A - I - 2[I - (A+I)^{-1}M(R+M'PM)^{-1}(M'PA+N')]^{-1} \\
&\quad \times (A+I)^{-1}M(R+M'PM)^{-1}(M'PA+N')\} \\
&= (A+I)^{-1}\{A - I - 2[I + A - M(R+M'PM)^{-1}(M'PA+N')]^{-1} \\
&\quad \times M(R+M'PM)^{-1}(M'PA+N')\} \\
&= (A+I)^{-1}(A_{cl}+I)^{-1}\{[I + A - M(R+M'PM)^{-1}(M'PA+N')](A-I) \\
&\quad - 2M(R+M'PM)^{-1}(M'PA+N')\} \\
&= (A+I)^{-1}(A_{cl}+I)^{-1}(A_{cl}-I)(A+I) \qquad (49)
\end{aligned}$$

which implies that the eigenvalues of $\tilde{A}_{cl}$ are in the open left half plane provided that the eigenvalues of A_{cl} are inside the unit circle.

(2. ⇒ 1.) Noting that

$$\begin{aligned}
\det[I + W^{-1}(\tilde{P}G+H)'(I-F)^{-1}G] &= \det[I + GW^{-1}(\tilde{P}G+H)'(I-F)^{-1}] \\
&= \det[I - F + GW^{-1}(\tilde{P}G+H)']\det[(I-F)^{-1}] \\
&= \det[I - \tilde{A}_{cl}]\det[(I-F)^{-1}]
\end{aligned}$$

and $\tilde{A}_{cl}$ has all its eigenvalue in the open left half plane, it follows from (40) that $R + M'PM$ is nonsingular. Thus, the condition in part 2 of Lemma 4.1 holds. The rest of the proof in reverse direction of Theorem 3.1 follows from an almost identical procedure as (1. ⇒ 2.). This completes our proof. □

5. AN EXAMPLE

In this section, a numerical example is presented to illustrate our results with a comparison of our solution to that generated by recursive algorithm of Stoorvogel and Weeren.[16] Let us consider a discrete-time H_∞-DARE for the full information problem with

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 0 \\ 0{\cdot}5 \end{bmatrix}$$

and $\gamma = 1$. It is simple to verify that (A, B, C, D_1) is left-invertible with an invariant zero at

0. Following (15), we obtain the auxiliary H_∞-CARE with

$$\tilde{A}=\begin{bmatrix}1&-2&6&-4&2\\-1&3&-8&6&-3\\2&-4&11&-8&4\\-1&2&-4&3&-1\\0&0&-2&2&-1\end{bmatrix},\quad \tilde{B}=\begin{bmatrix}68&-50\\-92&68\\128&-94\\-52&38\\-18&14\end{bmatrix},\quad \tilde{E}=\begin{bmatrix}-20\\28\\-40\\16\\6\end{bmatrix}$$

$$C=\begin{bmatrix}0&0&0&0&0\\1&0&1&0&1\\0&1&0&1&0\end{bmatrix},\quad D_1=\begin{bmatrix}1&0\\10&-8\\-9&6\end{bmatrix},\quad D_2=\begin{bmatrix}0\cdot0\\-4\cdot0\\3\cdot5\end{bmatrix}$$

Solving (29) in MATLAB, we obtain the stabilizing solution to the auxiliary H_∞-CARE as

$$\tilde{P}=10^3\times\begin{bmatrix}0\cdot76776739&1\cdot11008084&0\cdot18072033&-0\cdot30729570&-0\cdot61782810\\1\cdot11008084&1\cdot60729693&0\cdot26077549&-0\cdot44862283&-0\cdot89732214\\0\cdot18072033&0\cdot26077549&0\cdot04634303&-0\cdot06470382&-0\cdot13931814\\-0\cdot30729570&-0\cdot44862283&-0\cdot06470382&0\cdot14315024&0\cdot26428462\\-0\cdot61782810&-0\cdot89732214&-0\cdot13931814&0\cdot26428462&0\cdot51164431\end{bmatrix}$$

and the stabilizing solution to the H_∞-DARE for the full information problem is given by,

$$P=\begin{bmatrix}127\cdot14349353&187\cdot05748066&1\cdot00000000&-84\cdot67188015&-134\cdot86468041\\187\cdot05748066&278\cdot73088652&0\cdot00000000&-124\cdot06141851&-201\cdot39615252\\1\cdot00000000&0\cdot00000000&1\cdot00000000&0\cdot00000000&1\cdot00000000\\-84\cdot67188015&-124\cdot06141851&0\cdot00000000&61\cdot07801465&92\cdot56971658\\-134\cdot86468041&-201\cdot39615252&1\cdot00000000&92\cdot56971658&147\cdot98293455\end{bmatrix}$$

It is straightforward to verify that the above P satisfies (1), (2) and (3). Moreover, the eigenvalues of A_{cl} are given by $\{0, 0, 0, 0\cdot4125 \pm j0\cdot0733\}$, which are inside the unit circle.

Next, we apply the recursive algorithm of Stoorvogel and Weeren[16] to our example. It turns out that their algorithm has some difficulties in numerical convergence as it is shown below. These difficulties were also observed in some other examples. Following the algorithm of,[16] we first find a matrix K_0,

$$K_0=\begin{bmatrix}0\cdot69049759&0\cdot94309047&0\cdot11901628&-1\cdot27396910&-1\cdot01216128\\-1\cdot45340888&-1\cdot84754325&-1\cdot00816434&0\cdot05019067&0\cdot44225622\end{bmatrix}$$

such that the eigenvalues of $A+BK_0$ are placed at $\{0, 0\cdot2, 0\cdot4, 0\cdot6, 0\cdot8\}$. Then solving a Lyapunov equation,

$$L_0=(A+BK_0)'L_0(A+BK_0)+(C+D_1K_0)'(C+D_1K_0)$$

we obtain

$$L_0=\begin{bmatrix}99\cdot60958006&151\cdot72740313&2\cdot04298478&-38\cdot82884610&-93\cdot43361087\\151\cdot72740313&235\cdot84092991&1\cdot14037647&-59\cdot16676651&-146\cdot58047558\\2\cdot04298478&1\cdot14037647&1\cdot18045587&-0\cdot30514413&0\cdot42335637\\-38\cdot82884610&-59\cdot16676651&-0\cdot30514413&18\cdot04403237&38\cdot35832162\\-93\cdot43361087&-146\cdot58047558&0\cdot42335637&38\cdot35832162&93\cdot00248529\end{bmatrix}$$

Using L_0 as an initial condition, we solve the following H_2-DARE by a recursive procedure,

$$L = A'LA + C'C - (A'LB + C'D_1)(B'LB + D_1'D_1)^{-1}(B'LA + D_1'C)$$

for L,

$$L = \begin{bmatrix} 20\cdot10111499 & 29\cdot20116123 & 1\cdot00000000 & -9\cdot97132874 & -18\cdot58624499 \\ 29\cdot20116123 & 45\cdot70610888 & 0\cdot00000000 & -14\cdot47762395 & -30\cdot09186642 \\ 1\cdot00000000 & 0\cdot00000000 & 1\cdot00000000 & 0\cdot00000000 & 1\cdot00000000 \\ -9\cdot97132874 & -14\cdot47762395 & 0\cdot00000000 & 7\cdot51218973 & 10\cdot99490684 \\ -18\cdot58624499 & -30\cdot09186642 & 1\cdot00000000 & 10\cdot99490684 & 21\cdot54338663 \end{bmatrix}$$

Finally, by another recursive procedure, we obtain the following P for the H_∞-DARE for the full information, which is the best solution we are able to get using MATLAB,

$$P = \begin{bmatrix} 127\cdot14363741 & 187\cdot05767024 & 1\cdot00006027 & -84\cdot67182312 & -134\cdot86471655 \\ 187\cdot05769951 & 278\cdot73117587 & 0\cdot00009269 & -124\cdot06133653 & -201\cdot39620985 \\ 0\cdot99999886 & -0\cdot00000581 & 1\cdot00000000 & 0\cdot00002231 & 1\cdot00001406 \\ -84\cdot67200819 & -124\cdot06159512 & -0\cdot00005695 & 61\cdot07800463 & 92\cdot56977140 \\ -134\cdot86485442 & -201\cdot39638840 & 0\cdot99992518 & 92\cdot56968173 & 147\cdot98299766 \end{bmatrix}$$

Obviously, the above P is not desirable because it is even not truly symmetric. Moreover, by increasing the number of iterations, we observe divergence for this example. Let us define the solution error to the H_∞-DARE for the full information as

$$\left\| A'PA + C'C - \begin{bmatrix} B'PA + D_1'C \\ E'PA + D_2'C \end{bmatrix}' G(P)^{-1} \begin{bmatrix} B'PA + D_1'C \\ E'PA + D_2'C \end{bmatrix} - P \right\|_2$$

The detailed comparison of recursive and nonrecursive methods for the above example given in Table I clearly shows that our approach is much better than the recursive algorithm.

Table I. Comparison of nonrecursive and recursive algorithms

Method	Solution error	Computing Flops
Nonrecursive	$4\cdot7445 \times 10^{-9}$	37033
Recursive	$3\cdot9779 \times 10^{-3}$	220581

6. CONCLUDING REMARKS

In this paper we have proposed a nonrecursive method for obtaining the discrete-time Riccati equation related to the H_∞ control problem (H_∞-DARE). This was done by defining an auxiliary H_∞-CARE for the given system data and connecting the stabilizing solution to the given H_∞-DARE to the stabilizing solution of this auxiliary H_∞-CARE. The advantages of our method were also discussed.

ACKNOWLEDGEMENT

The work of A. Saberi is supported in part by NASA Langley Research Center under grant contract NAG-1-1210.

REFERENCES

1. Barnett, S., *Matrices in Control Theory*, Krieger, Malabar, Florida, 1984.
2. Başar, T., and P. Bernhard, *H_∞-optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, Birkhäuser, Boston, 1991.
3. Chan, S. W., G. C. Goodwin and K. S. Sin, 'Convergence properties of the Riccati difference equation in optimal filtering of nonstabilizable systems', *IEEE Transactions on Automatic Control*, **AC-29**, 110–118 (1984).
4. Glover, K., 'All optimal Hankel-norm approximations of linear multivariable systems and their $\mathscr{L}_\infty$ error bounds', *International Journal of Control*, **39**, 1115–1193 (1984).
5. Iglesias, P. A., and K. Glover, 'State space approach to discrete time H_∞ control', *International Journal of Control*, **54**, 1031–1073 (1991).
6. Kondo, R., and S. Hara, 'A unified approach to continuous/discrete time H_∞ problems via J-spectral factorizations', *Proceedings of the 1991 MTNS*, Vol. 1, pp. 247–252, Mita Press, Japan, 1992.
7. Kučera, V. 'The discrete Riccati equation of optimal control', *Kybernetika*, **8**, 430–447 (1972).
8. Lancaster, P., A. C. M. Ran and L. Rodman, 'State space approach to discrete time H_∞ control', *International Journal of Control*, **44**, 777–802 (1986).
9. Laub, A. J., 'A Schur method for solving algebraic Riccati equations', *IEEE Transactions on Automatic Control*, **AC-24**, 913–921 (1979).
10. Limebeer, D. J. N., M. Green and D. Walker, 'Discrete time H_∞ control', *Proceedings of IEEE Conference on Decision and Control*, Tampa, FL, 1989, pp. 392–396.
11. Pappas, T., A. J. Laub and N. R. Sandell, Jr., 'On the numerical solution of the discrete-time algebraic Riccati equation', *IEEE Transactions on Automatic Control*, **AC-25**, 631–641 (1980).
12. Ran, A. C. M., and R. Vreugdenhill, 'Existence and comparison theorems for algebraic Riccati equations for continuous- and discrete-time systems', *Linear Algebra and its Applications*, **99**, 63–83 (1988).
13. Silverman, L., 'Discrete Riccati equations: alternative algorithms, asymptotic properties, and system theory interpretations', *Control and Dynamical Systems*, **12**, 313–386 (1976).
14. Stoorvogel, A. A., *The H_∞ Control Problem: A State Space Approach*, Prentice Hall, Englewood Cliffs, NJ, 1992.
15. Stoorvogel, A. A., 'The discrete time H_∞ control problem with measurement feedback', *SIAM Journal on Control and Optimization*, **30**, 182–202 (1992).
16. Stoorvogel, A. A., and A. J. T. M. Weeren, 'The discrete time Riccati equation related to the H_∞ control problem', *Proceedings of 1992 American Control Conference*, Chicago, IL, 1128–1132, June 1992. Also to appear in *IEEE Transactions on Automatic Control*.
17. Suda, N., S. Kodama and M. Ikeda, *Matrix Theory in Automatic Control*. (edited and translated by C.-X. Cao), Science Press, Beijing, 1979 (in Chinese).
18. Zhou, K., and P. Khargonekar, 'A algebraic Riccati equation approach to H_∞-optimization', *Systems and Control Letters*, **11**, 85–91 (1988).