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Necessary and sufficient conditions under which an H_2 optimal control problem has a unique solution

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A set of necessary and sufficient conditions under which a general H_2-optimal control problem has a unique solution is derived. It is shown that the solution for an H_2-optimal control problem, if it exists, is unique if and only if (i) the transfer function from the control input to the controlled output is left invertible, and (ii) the transfer function from the disturbance to the measurement output is right invertible.

1. Introduction

This paper deals with the issue of uniqueness of the solution, whenever existent, to a general H_2-optimization problem. During the last two decades the H_2 optimal control problem and its stochastic interpretation, the linear quadratic Gaussian (LQG) control problem, have been thoroughly investigated (see e.g. Anderson and Moore 1989, Fleming and Rishel 1975, Geerts 1989, Kailath 1974, Kwakernaak and Sivan 1972, Saberi and Sannuti 1987, Schumacher 1985, Stoorvogel 1990, Stoorvogel *et al.* 1992, Willems 1978 and Willems *et al.* 1986 and the references contained therein). The H_2-optimization problem can be divided into two groups. The first group consists of those problems that satisfy the following essential assumptions.

(i) The subsystem from the disturbance to the measurement output should not have invariant zeros on the $j\omega$ axis, and its direct feedthrough matrix should be surjective.

(ii) The subsystem from the control input to the controlled output should not have invariant zeros on the $j\omega$ axis, and its direct feedthrough matrix should be injective.

This group of problems is referred to as the *regular H_2-optimization problem*. The second group is called the *singular H_2-optimization problem* (contrary to the regular one) and refers to those problems which do not satisy at least one of the two above assumptions. Most of the research in H_2-optimization problems is restricted to regular problems (see for example Doyle *et al.* 1988 and the references therein). However, more recently, a number of papers have dealt with the singular H_2-optimization problem (see for example Chen *et al.* 1992, Geerts 1989, Stoorvogel 1990, Stoorvogel *et al.* 1992 and Willems *et al.* 1986). It is well known that the solution to a regular H_2-optimization problem is unique (Doyle *et al.* 1988). However, this is not true for the singular case. In this paper we develop a set of necessary and sufficient conditions for the uniqueness of the solution to a general H_2-optimization problem. Our results indicate that a large

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class of singular H_2-optimization problems, contrary to common belief, have a unique solution. These results would shed more light on some mixed H_2/H_∞ problems such as the simultaneous H_2/H_∞ problem that was proposed in Rotea and Khargonekar (1991).

The paper is organized as follows. In § 2, we introduce the problem formulation of the H_2-optimal control problem, while in § 3, we briefly review the conditions of the existence of H_2-optimal controllers. We state our results in § 4 and give the proofs in § 5. Finally, in § 6 we draw the conclusion.

Throughout this paper, A' denotes the transpose of A and I denotes an identity matrix with appropriate dimensions. $\mathrm{Ker}[V]$ and $\mathrm{Im}[V]$ denote, respectively, the kernal and the image of V. We say a matrix A is of maximal rank if it is either injective or surjective. We will denote, for a given subspace χ and a matrix C_1, by $C_1^{-1}\chi$ the set $\{x \mid C_1 x \in \chi\}$. Given a strictly proper and stable transfer function $G(s)$, as usual, its H_2-norm is defined by $\|G\|_2$. Also, RH^2 denotes the set of real-rational transfer functions which are stable and strictly proper. RH^∞ denotes the set of real-rational transfer functions which are stable and proper.

2. Problem statement

Consider the following standard plant,

$$\Sigma: \quad \begin{cases} \dot{x} = Ax + Bu + Ew \\ y = C_1 x + D_3 u + D_1 w \\ z = C_2 x + D_2 u + D_4 w \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^l$ is the unknown disturbance, $y \in \mathbb{R}^p$ is the measured output and $z \in \mathbb{R}^q$ is the controlled output. Without loss of generality, we assume that the matrices $[C_2, D_2, D_4]$, $[C_1, D_3, D_1]$, $[B', D_3', D_2']'$ and $[E', D_1', D_4']'$ are of maximal rank. Also, consider an arbitrary proper controller Σ_F given by,

$$\Sigma_F: \quad \begin{cases} \dot{v} = Jv + Ly \\ u = Mv + Ny \end{cases} \tag{2.2}$$

The controller Σ_F is said to be admissible if it provides internal stability for the closed loop system comprising Σ and Σ_F. Let $T_{zw}(\Sigma_F)$ denote the closed-loop transfer function from w to z after applying a dynamic controller Σ_F to the system Σ. The H_2-optimization problem for Σ is to find an admissible control law which minimizes $\|T_{zw}\|_2$. The following definitions will be convenient in the following.

Definition 2.1 The regular H_2-optimization problem: A *regular H_2-optimization problem* refers to a problem for which the given plant Σ satisfies:

(i) D_1 is surjective and D_2 is injective;

(ii) the systems (A, B, C_2, D_2) and (A, E, C_1, D_1) have no invariant zeros on the $j\omega$ axis. □

Definition 2.2. The singular H_2-optimization problem. A *singular H_2-optimization problem* refers to a problem for which the given plant Σ does not satisfy either one or both of the conditions (i) and (ii) in Definition 2.1. □

Definition 2.3. The infimum of H_2-optimization. For a given plant Σ, the *infimum of the H_2-norm* of the closed-loop transfer function $T_{zw}(\Sigma_F)$ over all the stabilizing proper controllers Σ_F is denoted by γ^*, namely

$$\gamma^* := \inf \{\|T_{zw}(\Sigma_F)\|_2 | \ \Sigma_F \text{ internally stabilizes } \Sigma\} \tag{2.3}$$

□

Definition 2.4 H_2-optimal controller. A stabilizing proper controller Σ_F is said to be an *H_2-optimal controller* for Σ if $\|T_{zw}(\Sigma_F)\|_2 = \gamma^*$.

Definition 2.5 Geometric subspaces. Given a system Σ_* characterized by a matrix quadruple (A, B, C, D) we define the *detectable strongly controllable subspace* $\mathcal{T}_g(\Sigma_*)$ as the smallest subspace $\mathcal{T}$ of $\mathbb{R}^n$ for which there exists a linear mapping K such that the following subspace inclusions are satisfied:

$$(A + KC)\mathcal{T} \subseteq \mathcal{T}, \quad \operatorname{Im}(B + KD) \subseteq \mathcal{T} \tag{2.4}$$

and such that $A + KC|\mathbb{R}^n/\mathcal{T}$ is asymptotically stable. We also define the *stabilizable weakly unobservable subspace* $\mathcal{V}_g(\Sigma_*)$ as the largest subspace $\mathcal{V}$ for which there exists a mapping F such that the following subspace inclusions are satisfied:

$$(A + BF)\mathcal{V} \subseteq \mathcal{V}, \quad (C + DF)\mathcal{V} = \{0\} \tag{2.5}$$

and such that $A + BF|\mathcal{V}$ is asymptotically stable. □

To facilitate exposition, throughout this paper we assume that:

$$D_3 = 0 \quad \text{and} \quad D_4 = 0 \tag{2.6}$$

These assumptions are without loss of generality but for simplicity of presentation. We justify these in the following. If D_3 is non-zero, one can define a new output

$$y_n := y - D_3 u = C_1 x + D_1 w \tag{2.7}$$

Let the controller $u = \mathbf{C}_n(s) y_n$ be such that the resulting closed-loop is internally stable and the infimum γ^* is attained. Then, it is simple to verify that the controller

$$u = \mathbf{C}_n(s)[I + D_3 \mathbf{C}_n(s)]^{-1} y \tag{2.8}$$

also attains the infimum and stabilizes the closed-loop system provided that the closed-loop system is well-posed. On the other hand, if D_4 is non-zero, it is simple to see that there must exist a static output pre-feedback law $u = Sy + u_n$ to our system such that

$$D_4 + D_2(I - SD_3)^{-1} S D_4 = 0 \tag{2.9}$$

and thus the resulting new system has no direct feedthrough term from w to z. Otherwise, $\gamma^* = \infty$ and this is not the case we are interested in for this paper.

The goal of this paper is to derive a set of necessary and sufficient conditions under which Σ has a unique H_2-optimal controller.

3. Existence of optimal controllers

Our intention in this section is to recall from Stoorvogel (1990) and Stoorvogel *et al.* (1992) the necessary and sufficient conditions under which an

H_2-optimization problem has a solution. These conditions can be expressed as some geometric subspace inclusions of two auxiliary systems Σ_P and Σ_Q, which are respectively characterized by the matrix quadruples (A, B, C_P, D_P) and (A, E_Q, C_1, D_Q) with C_P, D_P, E_Q and D_Q satisfying: (i) $[C_P, D_P]$ and $[E'_Q, D'_Q]'$ are of maximal rank; and (ii)

$$F(P) = \begin{bmatrix} C'_P \\ D'_P \end{bmatrix} [C_P \ D_P] \quad \text{and} \quad G(Q) = \begin{bmatrix} E_Q \\ D_Q \end{bmatrix} [E'_Q \ D'_Q] \tag{3.1}$$

where

$$F(P) := \begin{bmatrix} A'P + PA + C'_2C_2 & PB + C'_2D_2 \\ B'P + D'_2C_2 & D'_2D_2 \end{bmatrix} \tag{3.2}$$

and

$$G(Q) := \begin{bmatrix} AQ + QA' + EE' & QC'_1 + ED'_1 \\ C_1Q + D_1E' & D_1D'_1 \end{bmatrix} \tag{3.3}$$

Furthermore, here P and Q are the largest solutions of the respective matrix inequalities $F(P) \geqslant 0$ and $G(Q) \geqslant 0$. It is shown in Stoorvogel (1990) and Stoorvogel *et al.* (1992) that Σ_P and Σ_Q have no invariant zeros in the open right-half plane, and are respectively right and left invertible.

The following theorem taken from Stoorvogel *et al.* (1992) gives the necessary and sufficient conditions under which the infimum, γ^*, can be attained.

Theorem 3.1: *Consider the given system Σ as in (2.1). Then the infimum, γ^*, can be attained by a proper controller of the form (2.2) if and only if*

(1) (A, B) *is stabilizable*,

(2) (A, C_1) *is detectable,*

(3) $\mathrm{Im}\,(E_Q) \subseteq \mathcal{V}_g(\Sigma_P) + B\,\mathrm{Ker}\,(D_P)$

(4) $\mathrm{Ker}\,(C_P) \supseteq \mathcal{T}_g(\Sigma_Q) \cap C_1^{-1}\,\mathrm{Im}\,(D_Q)$,

(5) $\mathcal{T}_g(\Sigma_Q) \subseteq \mathcal{V}_g(\Sigma_P)$.

Proof: For the proof see Stoorvogel *et al.* (1992). (□)

4. Statement of Results

We state in the following theorem the set of necessary and sufficient conditions under which a given plant Σ has a unique H_2-optimal controller.

Theorem 4.1: *Consider a plant Σ given by (2.1). Then the H_2-optimal controller for Σ is unique if and only if the following conditions hold:*

(1) (A, B) *is stabilizable,*

(2) (A, C_1) *is detectable,*

(3) $\mathrm{Im}\,(E_Q) \subseteq \mathcal{V}_g(\Sigma_P) + B\,\mathrm{Ker}\,(D_P)$,

(4) $\mathrm{Ker}\,(C_P) \supseteq \mathcal{T}_g(\Sigma_Q) \cap C_1^{-1}\,\mathrm{Im}\,(D_Q)$,

(5) $\mathcal{T}_g(\Sigma_Q) \subseteq \mathcal{V}_g(\Sigma_P$,

(6) (A, B, C_2, D_2) *is left invertible,*

(7) (A, E, C_1, D_2) *is right invertible.*

Moreover, the unique optimal controller is given by

$$\left.\begin{aligned}\dot{\xi} &= (A + BF + KC_1 - BNC_1)\xi + (BN - K)y \\ u &= (F - NC_1)\xi + Ny\end{aligned}\right\} \tag{4.1}$$

where F and K are any constant matrices that satisfy the conditions

$$\lambda(A + BF) \subseteq \mathbb{C}^-, \ \mathrm{Ker}\,[(C_P + D_P F)(sI - A - BF)^{-1}] = \mathcal{V}_g(\Sigma_P) \tag{4.2}$$

and

$$\lambda(A + KC_1) \subseteq \mathbb{C}^-, \ \mathrm{Im}\,[(sI - A - KC_1)^{-1}(E_Q + KD_Q)] = \mathcal{T}_g(\Sigma_Q) \tag{4.3}$$

respectively, and N is given by

$$\begin{aligned} N = &-(B'X'XB + D_P'D_P)^{-1}[B'X' \ D_P'] \\ &\times \begin{bmatrix} XAY & XE_Q \\ C_P Y & 0 \end{bmatrix} \begin{bmatrix} Y'C_1' \\ D_Q' \end{bmatrix} (C_1 YY'C_1' + D_Q D_Q')^{-1} \end{aligned} \tag{4.4}$$

Here X and Y are any constant matrices such that $\mathcal{V}_g(\Sigma_P) = \mathrm{Ker}\,(X)$ and $\mathcal{T}_g(\Sigma_Q) = \mathrm{Im}\,(Y)$. Also, note that there always exist F and K such that (4.2) and (4.3) hold provided that (A, B) is stabilizable and (A, C_1) is detectable (see for example the construction algorithm in Chen et al. *1992).*

Remark 4.1: We would like to note that conditions (1)–(5) are necessary and sufficient for the existence of the H_2-optimal controller for Σ. The additional conditions, namely (6) and (7), are required for the uniqueness. □

The following are two interesting corollaries.

Corollary 4.1. Regular case: *If Σ satisfies the conditions of regular case, then Σ has a unique H_2-optimal controller if and only if*

(1) *(A, B) is stabilizable,*

(2) *(A, C_1) is detectable.*

Moreover, in this case, the unique H_2-optimal controller for Σ is given by

$$\left.\begin{aligned}\dot{\xi} &= (A + BF + KC_1)\xi - Ky \\ u &= F\xi\end{aligned}\right\} \tag{4.5}$$

where $F = -(D_2'D_2)^{-1}(D_2'C_2 + B'P)$ and $K = -(QC_1' + ED_1')(D_1D_1')^{-1}$ with $P \geqslant 0$ and $Q \geqslant 0$ being respectively the stabilizing solutions of the algebraic Riccati equations:

$$PA + A'P + C_2'C_2 - (PB + C_2'D_2)(D_2'D_2)^{-1}(D_2'C_2 + B'P) = 0 \tag{4.6}$$

and

$$QA' + AQ + EE' - (QC_1' + ED_1')(D_1D_1')^{-1}(D_1E' + C_1Q) = 0 \tag{4.7}$$

This coincides with the result of Doyle et al. *(1988) when the orthogonality assumptions are made,* i.e. *$C_2'D_2 = 0$ and $D_1E' = 0$.*

Remark 4.2. Consider the standard LQG problem (see e.g. Doyle 1983)

$$\dot{x} = Ax + Bu + Gd$$

$$y = Cx + Nn, \quad N > 0$$

$$z = \begin{bmatrix} Hx \\ Ru \end{bmatrix}, \quad R > 0, \quad w = \begin{bmatrix} d \\ n \end{bmatrix}$$

where x is the state, u is the control, d and n white noise with identity covariance, and y the measured output. It is assumed that (A, B) is stabilizable and (A, C) is detectable. The control objective is to design a linear controller Σ_F that minimizes $\mathscr{E}[|z|^2]$. It was shown by Doyle (1983) that the above LQG problem can be solved via the H_2 optimal control problem for the following auxiliary system

$$\Sigma_{LQG}: \quad \begin{cases} \dot{x} = Ax + Bu + [G\ 0]w \\ y = Cx \qquad\quad + [0\ N]w \\ z = \begin{bmatrix} H \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R \end{bmatrix} u \end{cases}$$

Utilizing the above auxiliary H_2 optimal control problem, it follows from Corollary 4.1 that the standard LQG problem has a unique solution provided that all the input decoupling zeros of (A, G) and output decoupling zeros of (A, H) are not on the jω axis. □

Corollary 4.2. State feedback case: *If $C_1 = I$ and $D_1 = 0$, i.e. the state feedback case, then Σ has a unique H_2-optimal controller if and only if the following conditions hold:*

(1) *(A, B) is stabilizable,*
(2) *D_2 is injective,*
(3) *(A, B, C_2, D_2) has no invariant zeros on the $j\omega$ axis,*
(4) $\operatorname{Im}(E) = \mathbb{R}^n$

Moreover, in this case, the unique H_2-optimal controller for Σ is given by

$$u = -D_P^{-1} C_P x = -(D_2' D_2)^{-1}(D_2' C_2 + B'P)x \tag{4.8}$$

where $P \geqslant 0$ is the stabilizing solution of (4.6). This result coincides with the one obtained by Chen et al. *(1992).*

Remark 4.3. For the standard LQR case, the system to be controlled is given by the state-space time-domain equations (see e.g. Wilson 1989)

$$\dot{x} = Ax + Bu + Gw$$

$$y = x$$

where w is an external white noise with unity intensity. The performance index, which contains the errors, z, to be regulated, is given by

$$J = \lim_{t \to \infty} \mathscr{E}[z'(t) z(t)]$$

where $\mathscr{E}$ denotes the expectation operator and

$$z = \begin{bmatrix} H^{1/2}x \\ R^{1/2}u \end{bmatrix}, \quad H \geqslant 0, \quad R > 0$$

It was shown by Wilson (1989) that such a problem can be solved via the H_2 optimal control problem for the following auxiliary system,

$$\Sigma_{\text{LQR}}: \quad \begin{cases} \dot{x} = Ax + Bu + Gw \\ y = x \\ z = \begin{bmatrix} H^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \end{cases}$$

Utilizing the above auxiliary H_2 optimal control problem, it follows from corollary 4.2 that the LQR problem has a unique solution if and only if the following conditions hold:

(1) (A, B) is stabilizable,

(2) $(A, H^{1/2})$ has no output decoupling zeros on the jω axis,

(3) $\text{Im}\,(G) = \mathbb{R}^n$. □

5. Proofs of results

5.1. Proof of Theorem 4.1: Our proof of Theorem 4.1 involves two stages. In the first stage we obtain a special parameterization of all H_2-optimal controllers, whenever existent, for the given plant Σ. The second stage involves the examination of the set of all optimal solutions, which are identified and parametrized in the first stage, to derive the necessary and sufficient conditions for the uniqueness of the solution of the H_2-optimal control problem. Our development utilizes an interesting reformulation of the H_2-optimal control problem which was proposed by Stoorvogel (1990) and Stoorvogel *et al.* (1992). In Stoorvogel (1990) and Stoorvogel *et al.*. (1992) it was shown that the H_2-optimal control problem for a given plant Σ can be cast as a disturbance decoupling problem via measurement feedback with internal stability for an auxiliary system Σ_{PQ}, where

$$\Sigma_{\text{PQ}}: \quad \begin{cases} \dot{x}_{\text{PQ}} = Ax_{\text{PQ}} + Bu_{\text{PQ}} + E_{\text{Q}}w_{\text{PQ}} \\ y_{\text{PQ}} = C_1 x_{\text{PQ}} + \qquad + D_{\text{Q}}w_{\text{PQ}} \\ z_{\text{PQ}} = C_{\text{P}}x_{\text{PQ}} + D_{\text{P}}u_{\text{PQ}} \end{cases} \tag{5.1}$$

where C_{P}, D_{P}, C_{Q} and D_{Q} are as defined in (3.1). The following lemma which is recalled from Stoorvogel *et al.* (1992) states precisely such a reformulation of the H_2-optimal control problem.

Lemma 5.1: *The following two statements are equivalent.*

(i) *The controller Σ_{F} as in (2.2) when applied to the system Σ defined by (2.1) is internally stabilizing and the resulting closed-loop transfer function from w to z is strictly proper and has the H_2-norm γ^*.*

(ii) *The controller Σ_{F} as in (2.2) when applied to the new system Σ_{PQ} defined* *by (5.1) is internally stabilizing and the resulting closed-loop transfer function from w_{PQ} to z_{PQ} is strictly proper and has the H_2-norm 0.*

Proof: For the proof see Stoorvogel (1990). □

The above lemma shows that obtaining all the H_2-optimal controllers for Σ is equivalent to obtaining all the controllers that achieve disturbance decoupling with internal stability (DDS) for the auxiliary system Σ_{PQ}. It turned out that the characterization of the controllers that achieve DDS for Σ_{PQ} is easier than the characterization of the H_2-optimal controllers for Σ. It is well known (see for example Maciejowski 1989) that the general class of stabilizing controllers for Σ_{PQ} can be parametrized as

$$\left.\begin{aligned} \dot{\xi} &= (A + BF + KC_1)\xi - Ky + By_1 \\ u &= F\xi + y_1 \end{aligned}\right\} \tag{5.2}$$

and

$$y_1 = Q(s)(y - C_1\xi) \tag{5.3}$$

where F and K are any fixed gain matrices that satisfy

$$\lambda(A + BF) \subset \mathbb{C}^- \quad \text{and} \quad \lambda(A + KC_1) \subset \mathbb{C}^- \tag{5.4}$$

respectively, and $Q(s) \in \mathrm{RH}^\infty$ with the appropriate dimension is a free parameter. In order for controller (5.2) and (5.3) to achieve DDS for Σ_{PQ}, the free parameter $Q(s)$ must satisfy some additional conditions. Before attempting to state such additional properties of $Q(s)$, we need to recall the following lemma which reinterprets the conditions of Theorem 3.1.

Lemma 5.2: *Let X and Y be full rank matrices such that $\mathcal{V}_g(\Sigma_P) = \mathrm{Ker}\,(X)$ and $\mathcal{T}_g(\Sigma_Q) = \mathrm{Im}\,(Y)$. Then conditions (3)–(5) of Theorem 3.1 are equivalent to the following conditions: $\mathcal{T}_g(\Sigma_Q) \subseteq \mathcal{V}_g(\Sigma_P)$ and there is a matrix N such that*

$$\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & E_Q \\ C_P & 0 \end{bmatrix} + \begin{bmatrix} B \\ D_P \end{bmatrix}[N\ (C_1 \quad D_Q)]\begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} = 0 \tag{5.5}$$

Moreover, matrix N in any controller of form (2.2) that achieves DDS for Σ_{PQ} satisfies (5.5).

Proof: For the proof see Stoorvogel and van der Woude (1991) and Stoorvogel *et al.* (1992). □

It turned out that with the choice of F and K that satisfy (4.2) and (4.3), respectively, the controller (5.2) and (5.3) achieves DDS for Σ_{PQ} if and only if $Q(s) \in \mathbf{Q}$, where

$$\mathbf{Q} := \{Q(s) = Q_s(s) + N \mid Q_s(s) \in \mathbf{Q}_s \text{ and } N \in \mathbf{N}\} \tag{5.6}$$

and where

$$\begin{aligned} \mathbf{Q}_s := \{Q_s(s) \in \mathrm{RH}^2 \mid &[(C_P + D_PF)(sI - A - BF)^{-1}B + D_P] \\ &Q_s(s)\,[C_1(sI - A - KC_1)^{-1}(E_Q + KD_Q) + D_Q] = 0\} \end{aligned} \tag{5.7}$$

and

$$\mathbf{N} := \{N \in \mathbb{R}^{m \times p} \mid N \text{ satisfies Equation (5.5)}\} \tag{5.8}$$

This claim is proved in the following lemma.

Lemma 5.3: *Consider the auxiliary system Σ_{PQ} given by (5.1). Assume that the conditions in Theorem 3.1 are satisfied. Then, any controller Σ_F that achieves DDS for Σ_{PQ} if and only if it can be written in the form of (5.2) and (5.3) with F and K satisfying (4.2) and (4.3), respectively, and some $Q(s) \in \mathbf{Q}$.*

Proof: Let (A_q, B_q, C_q, N) be a state-space realization of $Q(s)$. It can be shown by some simple algebraic manipulations that controller (5.2) and (5.3) when applied to Σ_{PQ} yield the closed-loop transfer function from w_{PQ} to z_{PQ} as,

$$T_{z_{PQ}w_{PQ}}(\Sigma_F) = C_e(sI - A_e)^{-1}B_e + D_e \tag{5.9}$$

where

$$A_e = \begin{bmatrix} A + BF & BC_q & BNC_1 - BF \\ 0 & A_q & B_qC_1 \\ 0 & 0 & A + KC_1 \end{bmatrix}, \quad B_e = \begin{bmatrix} E_Q + BND_Q \\ B_qD_Q \\ E_Q + KD_Q \end{bmatrix} \tag{5.10}$$

and

$$C_e = [C_P + D_PF \quad DC_q \quad D_PNC_1 - D_PF], \quad D_e = D_PND_Q \tag{5.11}$$

Thus, it is trivial to see that the closed-loop system is internally stable if and only if (5.4) holds and $Q(s) \in \mathrm{RH}^\infty$. It is also simple to verify that

$$T_{z_{PQ}w_{PQ}}(\Sigma_F) = T_0 - T_q$$

where

$$\begin{aligned} T_0 = {} & (C_P + D_PF)(sI - A - BF)^{-1}(E_Q + BND_Q) \\ & -(C_P + D_PF)(sI - A - BF)^{-1}(sI - A - BNC_1) \\ & \times (sI - A - KC_1)^{-1}(E_Q + KD_Q) \\ & + (C_P + D_PNC_1)(sI - A - KC_1)^{-1}(E_Q + KD_Q) + D_PND_Q \end{aligned}$$

and

$$\begin{aligned} T_q = {} & [(C_P + D_PF)(sI - A - BF)^{-1}B + D_P]Q_s(s)[C_1(sI - A - KC_1)^{-1}(E_Q \\ & + KD_Q) + D_Q] \end{aligned}$$

It follows from Lemma 5.2 that whenever the controller achieves DDS for Σ_{PQ}, then N must belong to the set $\boldsymbol{N}$. Moreover, it was shown in Stoorvogel and van der Woude (1991) that $T_0 \equiv 0$ provided the conditions in Theorem 3.1 are satisfied and F and K are such that (4.2) and (4.3) hold. Hence, $T_{z_{PQ}w_{PQ}}(\Sigma_F) = 0$ (i.e. Σ_F achieves DDS for Σ_{PQ}) is equivalent to $T_q = 0$ or $Q_s(s) \in \mathbf{Q}_s$. The result follows. □

Now we are ready to move to the second stage of the proof of Theorem 4.1. The following lemma, which characterizes the conditions under which (5.5) has a unique solution, plays a crucial role in our development.

Lemma 5.4: *If equation (5.5) has at least one solution, then it is unique if and only if the matrix quadruples (A, B, C_2, D_2) and (A, E, C_1, D_1) are respectively left and right invertible. Moreover, in this case, the unique solution N is given by (4.4).*

Proof: Rewrite (5.5) as

$$\begin{bmatrix} XB \\ D_{\mathrm{P}} \end{bmatrix} N[C_1Y \quad D_{\mathrm{Q}}] = -\begin{bmatrix} XAY & XE_{\mathrm{Q}} \\ C_{\mathrm{P}}Y & 0 \end{bmatrix} \tag{5.12}$$

It is simple to verify that (5.12) has a unique solution, whenever existent, if and only if both

$$\begin{bmatrix} XB \\ D_{\mathrm{P}} \end{bmatrix} \quad \text{and} \quad [C_1Y \quad D_{\mathrm{Q}}]$$

are of maximal rank. Now, it follows from the results of Saberi and Sannuti (1990) that there exist non-singular transformations Γ_1 and Γ_3 such that

$$\Gamma_1^{-1}B\Gamma_3 = \begin{bmatrix} B_{01} & 0 & B_1 \\ B_{02} & B_2 & 0 \end{bmatrix}$$

$$D_{\mathrm{P}}\Gamma_3 = [D_0, 0, 0]$$

and

$$\mathcal{V}_{\mathrm{g}} = \mathrm{Im}\left\{\Gamma_1 \begin{bmatrix} I \\ 0 \end{bmatrix}\right\}$$

where D_0, B_1 and B_2 are of maximal rank. Moreover, Σ_{P} is invertible if and only if B_1 is non-existent. Thus,

$$X = [0, I]\Gamma_1^{-1}$$

Then, it is straightforward to verify that $[B'X', D_{\mathrm{P}}']'$ is of maximal rank if and only if B_1 is non-existent, i.e. $(A, B, C_{\mathrm{P}}, D_{\mathrm{P}})$ is invertible, which is equivalent to (A, B, C_2, D_2) being left invertible (see Chen *et al.* 1992). From this line of reasoning, one can show that $[C_1Y \ D_{\mathrm{Q}}]$ is of maximal rank if and only if (A, E, C_1, D_1) is right invertible. Hence, the first part of Lemma 5.4 follows. The unique solution N of (4.4) follows from (5.5) by some simple calculations. □

The final step of the proof of Theorem 4.1 proceeds as follows.

($\Rightarrow$): If the H_2-optimal controller for Σ is unique, i.e. there exists a unique controller that achieves DDS for Σ_{PQ}, then by Theorem 3.1 conditions (1)–(5) hold. It also implies that the set $\mathbf{N}$ is a singleton. By Lemma 5.4, conditions (6) and (7) hold.

($\Leftarrow$): Conversely, if conditions (1)–(5) hold, then Theorem 3.1 implies that there exists at least one H_2-optimal controller for Σ, which is equivalent to the existence of controllers that achieve DDS for Σ_{PQ}. Also, following the result of Chen *et al.* (1992) it can be shown that conditions (6) and (7) imply that both $(A, B, C_{\mathrm{P}}, D_{\mathrm{P}})$ and $(A, E_{\mathrm{Q}}, C_1, D_{\mathrm{Q}})$ are invertible. Hence, it follows from (5.7) that the set $\mathbf{Q}_s = \{0\}$ and from Lemma 5.4 that the set $\mathbf{N}$ is a singleton and is given by (4.4). Then, by Lemmas 5.1 and 5.3, the H_2-optimal controller for Σ is unique.

Finally, it is now trivial to verify from the above proof that the unique H_2-optimal controller for Σ is given by (4.1). This concludes the proof of Theorem 4.1. □

5.2. Proof of Corollary 4.1: It follows from Chen *et al.* (1992) that Σ_P is of minimum phase and invertible with no infinite zeros if D_2 is injective and (A, B, C_2, D_2) has no invariant zeros on the jω axis. Hence, $\mathcal{V}_g(\Sigma_P) = \mathbb{R}^n$. Also, D_2 being injective implies that (A, B, C_2, D_2) is left invertible. From this line of reasoning, one can show that $\mathcal{T}_g(\Sigma_Q) = \{0\}$ and (A, E, C_1, D_1) is right invertible provided D_1 is surjective and (A, E, C_1, D_1) has no invariant zeros on the jω axis. Hence, the first part of Corollary 4.1 follows trivially from Theorem 4.1.

Next, it is simple to verify that $\mathcal{V}_g(\Sigma_P) = \mathbb{R}^n$ and $\mathcal{T}_g(\Sigma_Q) = \{0\}$ imply

(1) $X = 0$ and $Y = 0$ and hence by (4.4), $N = 0$;

(2) $C_P + D_P F = 0$ and hence $F = -D_P^{-1} C_P$; and

(3) $E_Q + K D_Q = 0$ and hence $K = -E_Q D_Q^{-1}$.

Following (3.1) and (3.2), we obtain

$$F = -D_P^{-1} C_P = -(D_P' D_P)^{-1} D_P' C_P = -(D_2' D_2)^{-1} (B' P + D_2' C_2)$$

where $P \geqslant 0$ is the largest solution of $F(P) \geqslant 0$, which is equivalent to the stabilizing solution of (4.6) when the conditions of the regular H_2-optimization problem hold. Using the dual arguments, we have

$$K = -E_Q D_Q^{-1} = -(Q C_1' + E D_1')(D_1 D_1')^{-1}$$

where $Q \geqslant 0$ is the stabilizing solution of (4.7). Thus, (4.1) reduces to (4.5). □

5.3. Proof of Corollary 4.2: From the proof of Corollary 4.1, we know that conditions (2) and (3) of Corollary 4.2 imply that (A, B, C_2, D_2) is left invertible and $\mathcal{V}_g(\Sigma_P) = \mathbb{R}^n$. Also, the facts that $C_1 = I$ and $\mathrm{Im}\,(E) = \mathbb{R}^n$ imply that (A, E, C_1, D_1) is invertible, and $D_1 = 0$ implies that $D_Q = 0$ and thus $C_1^{-1}\,\mathrm{Im}\,(D_Q) = \mathrm{Ker}\,(C_1) = \{0\}$. Then by straightforward verifications and Theorem 4.1, it can easily be verified that Σ has a unique H_2 optimal controller.

Conversely, if Σ has a unique H_2 optimal controller, by Theorem 4.1 we have:

(*a*) Condition (1) of Theorem 4.1 holds, i.e. (A, B) stabilizable;

(*b*) Condition (7) of Theorem 4.1, $C_1 = I$ and $D_1 = 0$ imply that $\mathcal{T}_g(\Sigma_Q) = \mathrm{Im}\,(E) = \mathbb{R}^n$;

(*c*) Condition (5) of Theorem 4.1 and (*b*) imply that $\mathcal{V}_g(\Sigma_P) = \mathbb{R}^n$, which is equivalent to (A, B, C_2, D_2) having no infinite zeros and no invarient zeros on the jω axis;

(*d*) Condition (6) of Theorem 4.1 and (*c*) imply that D_2 is injective.

Hence, Conditions (1)–(4) of Corollary 4.2 hold.

It is easy to show that $\mathcal{V}_g(\Sigma_P) = \mathbb{R}^n$ and $\mathcal{T}_g(\Sigma_Q) = \mathrm{Im}\,(E)$ imply that $F = N = -D_P^{-1} C_P$. Thus, (4.1) reduces to (4.8). This completes the proof of Corollary 4.2. □

6. Conclusion

In this paper we have derived a set of necessary and sufficient conditions for the uniqueness of the solution to a general H_2-optimization problem. We have shown that the solution for an H_2-optimal control problem, if it exists, is unique if and only if the systems (A, B, C_2, D_2) and (A, E, C_1, D_1) are respectively left and right invertible. Moreover, such a unique H_2-optimal control law has been obtained.

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