

Full and reduced-order observer-based controller design for H_2 -optimization

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In this paper the most general H_2 control problem is considered. We derive necessary and sufficient conditions when the infimum is attained by state feedback. We do the same for the measurement feedback case where we derive necessary and sufficient conditions when the infimum is attained by proper dynamic compensators. We also investigate reduced-order compensators if some states are observable without noise. We discuss, for all of these cases, the freedom that the non-uniqueness of optimal compensators gives us in assigning the closed-loop eigenvalues. The second half of this paper investigates the case when the infimum cannot be attained. We give a constructive algorithm to find a minimizing sequence of stabilizing controllers and discuss the freedom in the asymptotic locations of the closed-loop eigenvalues. Again we do the above for three different cases: static-state feedback, full-order measurement feedback and reduced-order measurement feedback.

1. Introduction

During the last two decades the H_2 control problem and its stochastic interpretation, the linear quadratic Gaussian (LQG) control problem, have been thoroughly investigated (see for example, Anderson and Moore 1989, Fleming and Rishel 1975, Kailath 1974, Kwakernaak and Sivan 1972 and Willems 1978 and the references contained therein). Recently, the LQG theory has been investigated in the form of the so-called mixed LQG/ H_∞ control problems (see for example, Bernstein and Haddad 1989 a, 1989 b, Mustafa and Glover 1990 and Rotea and Khargonekar 1991). However, in most of these papers a number of standard assumptions are made.

- (i) The subsystem from noise w to the measurement y should not have invariant zeros on the imaginary axis and its direct feedthrough matrix should be surjective.
- (ii) The subsystem from control input u to the controlled output z should not have invariant zeros on the imaginary axis and its direct feedthrough matrix should be injective.

More recently, the H_2 control problem was investigated without these assumptions. Geerts (1989) and Willems *et al.* (1986), discussed the state feedback case while Schumacher (1985), investigated the filtering side. Stoorvogel (1990) noted that for the problem of attaining the infimum, the separation principle does not hold: there are systems for which we can attain the infimum (over all static-state

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feedbacks) with some static-state feedback and there also exists an optimal observer. Yet the interconnection of observer and state feedback does not attain the infimum for the measurement feedback case. This shows that we cannot, in general, use classical techniques for this problem. Therefore, the measurement feedback case deserves a separate investigation.

This paper intends to unify and extend the results from the papers mentioned above. We investigate systems without any assumptions. On the other hand we restrict ourselves to proper controllers and we do not consider the stochastic interpretation.

We give necessary and sufficient conditions when the infimum of the H_2 norm can be attained. In Stoorvogel (1990), existence conditions were given only for the case of optimal strictly proper compensators. However, here we give conditions for the existence of optimal proper compensators. A novel aspect of this paper is an analysis of the behaviour and the freedom of the closed-loop eigenvalues. In general, the optimal H_2 controller is not unique and we study how to use this non-uniqueness to obtain suitable locations for the closed-loop eigenvalues.

We discuss this problem for three classes of compensators.

- (1) Static-state feedback.
- (2) Full-order proper dynamic output feedback. (By 'full-order compensator' we mean a compensator with the same dynamical order as the given plant; and by 'reduced-order compensator' we mean a compensator with dynamical order less than the dynamical order of the plant.)
- (3) Reduced-order proper dynamical output feedback.

In the last case, we investigate the situation when some of the states can be observed without noise. It turns out that this yields the possibility of attaining the infimum of the H_2 norm by a lower-order compensator.

We also discuss the suboptimal case when we cannot attain the infimum by a proper compensator. Again we consider the classes of compensators mentioned above. A novel aspect is a derivation of a minimizing sequence of reduced-order compensators in case the infimum is not attained. For these classes we also investigate the asymptotic behaviour of the closed-loop eigenvalues and the freedom we have in changing these asymptotic properties by using different minimizing sequences. We will give explicit algorithms to achieve this freedom in the closed-loop eigenvalues. We do this both for suboptimal design and for optimal design.

This paper has the following structure: in § 2 we give the problem statement. In § 3 we give a transformation of our system into a new system. This transformation replaces the role of the separation principle which, as mentioned before, is not applicable to show all the goals stated above. Then, in § 4 we discuss optimal design for the three classes mentioned. We investigate when we can attain the infimum and the available freedom in assigning the closed-loop eigenvalues. In § 5 we investigate suboptimal design: we derive minimizing sequences of stabilizing controllers in case the infimum is not attained. Moreover, we investigate the available freedom in the asymptotic locations of the closed-loop eigenvalues. We end the paper in § 6 with some concluding remarks. In Appendix A we recall a special coordinate basis from Sannuti and

Saberi (1987) and Saberi and Sannuti (1990), which will be instrumental in the proofs and algorithms of this paper. We also delegate some of the more technical proofs and algorithms for the constructions of optimal and suboptimal control laws to Appendices B to E.

Throughout this paper, A^T denotes the transpose of A , and I denotes an identity matrix with appropriate dimension. \mathbb{C} , \mathbb{C}^- , \mathbb{C}^0 and \mathbb{C}^+ respectively denote the whole complex plane, the open left-half complex plane, the imaginary axis, and the open right-half complex plane. $\text{Ker}[V]$ and $\text{Im}[V]$ denote, respectively, the kernel and the image of V . We will denote, for a given subspace \mathcal{X} and a matrix C_1 , by $C_1^{-1}\{\mathcal{X}\}$ the set $\{x|C_1x \in \mathcal{X}\}$.

2. Problem statement

Consider the following system

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu + Ew \\ y = C_1x + D_1w \\ z = C_2x + D_2u \end{cases} \quad (2.1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^l$ is the unknown disturbance, $y \in \mathbb{R}^p$ is the measured output and $z \in \mathbb{R}^q$ is the controlled output. The closed-loop transfer function from w to z after applying a dynamic compensator Σ_F to the system will be denoted by $T_{zw}(\Sigma_F)$. Let us define

$$\gamma^* := \inf \{ \|T_{zw}(\Sigma_F)\|_{H_2} \mid \Sigma_F \text{ internally stabilizes } \Sigma \} \quad (2.2)$$

where the H_2 norm is defined as

$$\|G\|_{H_2}^2 := \int_0^\infty \text{Trace } G^T(-j\omega)G(j\omega) d\omega$$

The goal of this paper is twofold. First we would like to design stabilizing control laws to minimize the H_2 -norm of $T_{zw}(\Sigma_F)$. We study this problem for three classes of control laws, namely, static-state feedback, dynamic output feedback and reduced-order dynamic-output feedback laws. Secondly, in the case that the infimum cannot be attained exactly we design suboptimal control laws. Here, we mean by suboptimal laws, a parametrized family of controllers which achieves the infimum, γ^* , asymptotically as the parameter goes to infinity. We study suboptimal design for the same three classes of control laws. We will also study the locations of the closed-loop eigenvalues, either the exact locations or the asymptotic locations depending on whether the infimum can be attained or not.

In our problem formulation, we have assumed that the direct feedthrough matrix from w to z is equal to 0. This can be done without loss of generality: if this matrix is unequal to 0 and there exists a compensator which makes the closed-loop H_2 norm finite then there always exists a preliminary static output feedback which makes the direct feedthrough matrix from w to z equal to 0. We have also assumed that the direct feedthrough matrix from u to y is 0. The absence of such an assumption forces one to handle some extra problems with respect to the fact that the closed-loop system should be well-posed. Although not difficult, this yields results which are somewhat messier. To prevent these difficulties we assume this matrix to be 0.

3. Preliminaries

Our intention in this section is to recall some known results from Stoorvogel (1990) and to develop some new results which are pertinent to our present work. It was shown by Stoorvogel (1990) that the H_2 optimal control problem for the given plant Σ can be reformulated as a disturbance (or almost disturbance) decoupling problem via measurement feedback with internal stability for an auxiliary system Σ_{PQ} . Here, in this section, we first state the dynamic equations of Σ_{PQ} and then develop its properties as to its invertibility, finite and infinite zero structure. Next, a theorem is recalled which connects the given H_2 problem for Σ to a disturbance decoupling problem for Σ_{PQ} .

The auxiliary system Σ_{PQ} is described by

$$\Sigma_{PQ}: \begin{cases} \dot{x}_{PQ} = Ax_{PQ} + Bu_{PQ} + E_Q w_{PQ} \\ y_{PQ} = C_1 x_{PQ} + D_Q w_{PQ} \\ z_{PQ} = C_P x_{PQ} + D_P u_{PQ} \end{cases} \quad (3.1)$$

with C_P , D_P , E_Q and D_Q satisfying: (i) $[C_P \ D_P]$ and $[E_Q^T \ D_Q^T]^T$ are of maximal rank; and (ii)

$$F(P) = \begin{bmatrix} C_P^T \\ D_P^T \end{bmatrix} [C_P \ D_P] \text{ and } G(Q) = \begin{bmatrix} E_Q \\ D_Q \end{bmatrix} [E_Q^T \ D_Q^T] \quad (3.2)$$

Here

$$F(P) := \begin{bmatrix} A^T P + PA + C_2^T C_2 & PB + C_2^T D_2 \\ B^T P + D_2^T C_2 & D_2^T D_2 \end{bmatrix} \quad (3.3)$$

$$G(Q) := \begin{bmatrix} AQ + QA^T + EE^T & QC_1^T + ED_1^T \\ C_1 Q + D_1 E^T & D_1 D_1^T \end{bmatrix} \quad (3.4)$$

and furthermore, P and Q are the largest solutions of the respective matrix inequalities $F(P) \geq 0$ and $G(Q) \geq 0$.

The following lemma characterizes the properties of the auxiliary system Σ_{PQ} .

Lemma 3.1: Consider system (2.1). Assume that (A, B) is stabilizable and (C_1, A) is detectable. Then (3.1) has the following properties.

- (1) (A, B, C_P, D_P) is right-invertible with no invariant zeros in \mathbb{C}^+ and has the same infinite zero structure as $\Sigma_{ci} := (A, B, C_2, D_2)$. Moreover, (A, B, C_P, D_P) has a total number of $n_a^-(\Sigma_{ci}) + n_a^0(\Sigma_{ci}) + n_a^+(\Sigma_{ci}) + n_b(\Sigma_{ci})$ invariant zeros which are given by:
 - (a) the stable invariant zeros of Σ_{ci} ,
 - (b) the invariant zeros of Σ_{ci} which are on the imaginary axis,
 - (c) the mirror images with respect to the imaginary axis of the invariant zeros of Σ_{ci} in \mathbb{C}^+ , and
 - (d) some fixed locations in the open left-half plane which contain the stable input decoupling zeros (but not invariant zeros) of Σ_{ci} .
- (2) (A, E_Q, C_1, D_Q) is left-invertible with no invariant zeros in \mathbb{C}^+ and has the same infinite zero structure as $\Sigma_{di} := (A, E, C_1, D_1)$. Moreover, (A, E_Q, C_1, D_Q) has a total number of $n_a^-(\Sigma_{di}) + n_a^0(\Sigma_{di}) + n_a^+(\Sigma_{di}) +$

$n_c(\Sigma_{di})$ invariant zeros which are given by:

(e) the stable invariant zeros of Σ_{di} ,

(f) the invariant zeros of Σ_{di} which are on the imaginary axis,

(g) the mirror images with respect to the imaginary axis of the invariant zeros of Σ_{di} in \mathbb{C}^+ , and

(h) some fixed locations in the open left-half plane which contain the stable output decoupling zeros (but not invariant zeros) of Σ_{di} .

Here $n_a^-(\Sigma_*)$, $n_a^0(\Sigma_*)$, $n_a^+(\Sigma_*)$, $n_b(\Sigma_*)$ and $n_c(\Sigma_*)$ are the constants n_a^- , n_a^0 , n_a^+ , n_b and n_c as defined in Appendix A when we bring Σ_* to the special coordinate basis.

Proof: For the proof see Appendix B. □

We would like to give an interpretation of the constants $n_a^-(\Sigma_*)$, $n_a^0(\Sigma_*)$, $n_a^+(\Sigma_*)$, $n_b(\Sigma_*)$ and $n_c(\Sigma_*)$ which appear in the above lemma, and the constant $n_f(\Sigma_*)$ used later.

- (i) $n_a^-(\Sigma_*)$, $n_a^0(\Sigma_*)$ and $n_a^+(\Sigma_*)$ are the number (counting multiplicity) of invariant zeros in \mathbb{C}^- , \mathbb{C}^0 and \mathbb{C}^+ respectively.
- (ii) $n_f(\Sigma_*)$ is the number of infinite zeros.
- (iii) $n_c(\Sigma_*)$ is the dimension of the intersection $\mathcal{T}_g(\Sigma_*) \cap \mathcal{V}_g(\Sigma_*)$ which are both subspaces of the state-space defined in Definition 3.1. This intersection is the largest subspace of the state-space which is completely controllable by the input u while maintaining an output equal to 0.
- (iv) $n_b(\Sigma_*)$ equals the dimension of the state space n minus the numbers defined above. It is equal to n minus the dimension of $\mathcal{T}_g(\Sigma_*) + \mathcal{V}_g(\Sigma_*)$.

Next, we recall from Stoorvogel (1990) the following theorem which reformulates the given H_2 optimal control problem for Σ in terms of another problem for Σ_{PQ} . Such a reformulation indeed plays a significant role in the development of subsequent sections.

Theorem 3.1: Let an arbitrary compensator Σ_F be given as

$$\Sigma_F: \begin{cases} \dot{v} = Kv + Ly \\ -u = Mv + Ny \end{cases} \quad (3.5)$$

Then the following two statements are equivalent.

- (i) The compensator Σ_F applied to the system Σ defined by (2.1) is internally stabilizing and the resulting closed-loop transfer function from w to z is strictly proper and has H_2 norm γ .
- (ii) The compensator Σ_F applied to the new system Σ_{PQ} defined by (3.1) is internally stabilizing and the resulting closed-loop transfer function from w to z_{PQ} is strictly proper and has H_2 norm

$$\{\gamma^2 - \text{Trace } E^T P E - \text{Trace } (A^T P + P A + C_2^T C_2) Q\}^{1/2}$$

Since (A, B, C_p, D_p) and (A, E_Q, C_1, D_Q) are right- and left-invertible respectively and since neither one of these systems has invariant zeros in \mathbb{C}^+ , we

know that for Σ_{PQ} the H_2 almost disturbance decoupling problem with measurement feedback and stability is solvable (see Stoorvogel 1990), i.e. we can find a stabilizing compensator for Σ_{PQ} which makes the H_2 norm of the closed-loop system arbitrarily small. Note that this is, in general, not true if we replace the H_2 norm by the H_∞ norm (except when we exclude invariant zeros on the imaginary axis, see for example Trentelman 1986, Weiland and Willems 1989, Willems 1981 and 1982). Combining this result with Theorem 3.1 yields the following corollary.

Corollary 3.1: *Let γ^* be defined by (2.2). Moreover, let P and Q be the largest solutions of the respective matrix inequalities $F(P) \geq 0$ and $G(Q) \geq 0$. Then we have*

$$\gamma^* = \{\text{Trace } E^T P E + \text{Trace } (A^T P + P A + C_2^T C_2) Q\}^{1/2}$$

In this section we deal with the design problems for the cases when the infimum, γ^* , can be attained exactly. We will consider the following types of controllers.

- (1) Static-state feedback controllers.
- (2) Dynamic output feedback controllers based on either full or reduced-order observers.

For each type of controller considered, we first develop the necessary and sufficient conditions under which an optimal controller exists, and then proceed with the design of controller. Our conditions for the existence of an optimal controller, are expressed in subspace inclusions. The required geometric subspaces are defined as follows.

Definition 3.1: We define the detectable strongly controllable subspace $\mathcal{T}_g(A, B, C, D)$ as the *smallest subspace* \mathcal{T} of \mathbb{R}^n for which there exists a linear mapping K such that the following subspace inclusions are satisfied

$$(A - KC)\mathcal{T} \subseteq \mathcal{T} \tag{3.6}$$

$$\text{Im}(B - KD) \subseteq \mathcal{T} \tag{3.7}$$

and such that $A - KC|_{\mathbb{R}^n/\mathcal{T}}$ is asymptotically stable. We also define the stabilizable weakly unobservable subspace $\mathcal{V}_g(A, B, C, D)$ as the *largest subspace* \mathcal{V} for which there exists a mapping F such that the following subspace inclusions are satisfied

$$(A - BF)\mathcal{V} \subseteq \mathcal{V} \tag{3.8}$$

$$(C - DF)\mathcal{V} = \{0\} \tag{3.9}$$

and such that $A - BF|_{\mathcal{V}}$ is asymptotically stable. □

The subspaces $\mathcal{V}_g(A, B, C, D)$ and $\mathcal{T}_g(A, B, C, D)$ defined above can be computed by means of well-known algorithms (see for example Stoorvogel 1992, Sannuti and Saberi 1987, Saberi and Sannuti 1990 and Wonham 1985). We note also that if (A, B) is stabilizable then for $\mathcal{V}_g(A, B, C, D)$ there exists an F such that (3.8) and (3.9) are satisfied and moreover $A - BF$ is asymptotically stable. A similar comment can be made for $\mathcal{T}_g(A, B, C, D)$ in case (C, A) is detectable.

3.1. Static-state feedback design

In this subsection, we deal with the problem when all states of the given system (2.1) are available for feedback, i.e. $C_1 = I$ and $D_1 = 0$. We have the following theorem regarding the existence of an optimal static-state feedback law.

Theorem 3.2: *Consider the given system (2.1) with $C_1 = I$ and $D_1 = 0$. The infimum, γ^* , can be attained by a static-state feedback law if and only if (A, B) is stabilizable and*

$$\text{Im}(E) \subseteq \mathcal{V}_g(A, B, C_P, D_P) \quad (3.10)$$

Proof: By Theorem 3.1 and Corollary 3.1 the infimum γ^* is attained if and only if there exists a state feedback law which stabilizes Σ_{PQ} and which yields a closed loop transfer matrix equal to 0. By Stoorvogel and van der Woude (1991), this is equivalent to condition (3.10). \square

Design

The next step is to design state feedback control laws which attain the infimum. In general, the state feedback laws which attain the infimum are not unique. However, it can be shown that the closed loop transfer matrix is unique. We are going to see how this freedom yields a freedom in the location of the closed-loop poles. Because the closed-loop transfer matrix is unique we cannot affect the external behaviour but we can vary the responses of the internal signals.

Clearly, if $\text{Im}(E) = \{0\}$ any stabilizing feedback will do. On the other hand, if we have equality in (3.10) then our choice will be severely limited. We can build a design method around any subspace \mathcal{V} for which a matrix F exists such that $A - BF$ is stable, (3.8) and (3.9) are satisfied for the system (A, B, C_P, D_P) and such that $\text{Im}(E) \subseteq \mathcal{V}$. A smaller subspace \mathcal{V} yields extra freedom in our design. However no method is available to generate the smallest subspace \mathcal{V} satisfying all these conditions. Therefore we will only discuss design techniques based on the largest subspace $\mathcal{V}_g(A, B, C_P, D_P)$ satisfying all these conditions. In particular, we have the following lemma.

Lemma 3.2: *Suppose $\text{Im}(E) = \mathcal{V}_g(A, B, C_P, D_P)$. Let F_{op} denote the set of feedback gains F such that $A - BF$ is asymptotically stable and such that the conditions of Definition 3.1 are satisfied for $\mathcal{V}_g(A, B, C_P, D_P)$. Then, any member of the set F_{op} , say F , is an optimal solution, i.e. the feedback law $u = -Fx$ applied to Σ is stabilizing and the closed-loop H_2 norm is equal to the infimum (2.2). Moreover, any feedback $u = -Fx$ which is stabilizing and yields a closed-loop H_2 norm equal to γ^* is in F_{op} , i.e. $F \in F_{\text{op}}$.*

Proof: For the proof see Appendix C. \square

This set F_{op} can be constructed using the algorithm given in Appendix C. We will compare different state feedbacks by investigating the location of the closed-loop poles. We can derive the following result.

Lemma 3.3: *Let F_{op} be as defined in Lemma 3.2. Define Σ_{ci} to be the system (A, B, C_2, D_2) .*

Any $F \in F_{\text{op}}$ yields a feedback law $u = -Fx$ which is stabilizing and, when

applied to Σ , the resulting closed loop H_2 norm is equal to γ^* defined by (2.2). Moreover, by varying the feedback gain over the set F_{op} , we have the following freedom in assigning the closed-loop eigenvalues

- (1) $n_a^-(\Sigma_{ci})$ eigenvalues lie at the locations of the stable invariant zeros of Σ_{ci} ,
- (2) $n_a^+(\Sigma_{ci})$ eigenvalues lie at the locations of the mirror images with respect to the imaginary axis of the invariant zeros of Σ_{ci} in the open right-half plane.
- (3) $n_b(\Sigma_{ci})$ eigenvalues lie at fixed (independent of our specific choice for F) locations in the open left-half plane which contain the stable input decoupling zeros (but not invariant zeros) of Σ_{ci} .
- (4) $n_a^0(\Sigma_{ci}) + n_c(\Sigma_{ci}) + n_f(\Sigma_{ci})$ eigenvalues can be assigned to arbitrary locations in the open left-half plane depending on the specific choice of F .

Here $n_a^-(\Sigma_{ci})$, $n_a^+(\Sigma_{ci})$, $n_a^0(\Sigma_{ci})$, $n_b(\Sigma_{ci})$, $n_c(\Sigma_{ci})$ and $n_f(\Sigma_{ci})$ are the constants n_a^- , n_a^+ , n_a^0 , n_b , n_c and n_f as defined in Appendix A when we transform Σ_{ci} to the special coordinate basis.

If we have strict inclusion in (3.10) then, in general, there exist feedback laws $u = -Fx$ which are stabilizing and attain the infimum γ^* but which are not in F_{op} , i.e. $F \notin F_{op}$.

Proof: For the proof see Appendix C. □

This lemma completely characterizes the available freedom and the constraints on the closed-loop eigenvalues for the case that $\text{Im}(E) = \mathcal{V}_g(A, B, C_p, D_p)$. For the case when $\text{Im}(E)$ is a proper subset of $\mathcal{V}_g(A, B, C_p, D_p)$, the result of this lemma is valid only for a subclass of all state feedback laws which attain the infimum. This is to be expected. For example, as mentioned before, for the case when $E = 0$, any stabilizing feedback will attain the infimum and we clearly cannot say anything specific about the closed-loop eigenvalues.

It is obvious that, in general, an H_2 optimal controller is not unique. However, it is easy to see that if

- (1) D_2 injective
- (2) E surjective, and
- (3) (A, B, C_2, D_2) has no invariant zeros on the imaginary axis,

then F_{op} is a singleton and hence an H_2 optimal controller is unique. This observation was also noted in Rotea and Khargonekar (1991).

We illustrate our results in the following example.

Example: Consider a given plant characterized by

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 5 & 0 & 9 \\ 0 & 0 & 0 \\ 0 & -0.2929 & 0 \\ 0 & 0 & 0 \\ 3 & 3 & 5 \\ 0 & 2 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_1 = I_6$ and $D_1 = 0$. It can be verified that the system Σ_{ci} characterized by (A, B, C_2, D_2) is neither right nor left-invertible with three invariant zeros at $-1, 0$ and 2 , and one infinite zero of order one, i.e. $n_a^-(\Sigma_{ci}) = n_a^0(\Sigma_{ci}) = n_a^+(\Sigma_{ci}) = n_b(\Sigma_{ci}) = n_c(\Sigma_{ci}) = n_f(\Sigma_{ci}) = 1$. Using the software package developed by Lin *et al.* (1991), we obtain

$$C_P = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6.828427 & -0.414214 & 0 & 1 \end{bmatrix}, \quad D_P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathcal{V}_g(A, B, C_P, D_P) = \text{Im} \left[\begin{array}{cccc} -0.7662610 & 0.640502 & -0.051008 & 0 \\ -0.1915653 & -0.176888 & 0.656593 & 0 \\ -0.0957826 & -0.115702 & -0.013974 & 0 \\ -0.1915653 & -0.176888 & 0.656593 & 0 \\ 0 & 0 & 0 & 0.707107 \\ 0.5746958 & 0.716793 & 0.367389 & 0 \end{array} \right]$$

It is straightforward to verify that the quadruple (A, B, C_P, D_P) is right invertible with four invariant zeros at $0, -1, -\sqrt{2}$ and -2 , and one infinite zero of order one. Moreover, $\text{Im}(E) \subseteq \mathcal{V}_g(A, B, C_P, D_P)$. Then, following the procedure given in Appendix C, we obtain that the set F_{op} consists of matrices of the form

$$\begin{bmatrix} -0.957826 & 0.463613 & 0.605585 & -0.447214 + f_{11} & 0 & f_{12} \\ 0.478913 & 2.770753 & 7.992790 & f_{21} & 0.707107 & f_{22} \\ \star & \star & \star & \star & f_c & \star \end{bmatrix} \Gamma_s^{-1}$$

where

$$\Gamma_s^{-1} = \begin{bmatrix} -0.766261 & -1.532522 & -4.020051 & 1.387438 & 0 & 0 \\ 0.640502 & 1.281003 & -5.010269 & -1.337875 & 0 & 0 \\ -0.051008 & -0.102017 & -2.522660 & 1.567380 & 0 & 0 \\ 0 & 2.236068 & 0 & -2.236068 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.414213 & 0 \\ 0 & 0 & 6.828427 & -0.414214 & 0 & 1 \end{bmatrix}$$

f_{11}, f_{12}, f_{21} and f_{22} are free parameters such that

$$\lambda \left(\begin{bmatrix} 0 & 0 \\ -0.447214 & 7.414214 \end{bmatrix} - \begin{bmatrix} 2.236068 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \right) \subset \mathbb{C}^-$$

the parameter f_c is constrained by $f_c > 1/\sqrt{2}$ while \star denotes completely free parameters.

It is simple to verify that $\{-1, -\sqrt{2}, -2\} \subset \lambda(A - BF)$ for any $F \in F_{op}$ while the three eigenvalues of $A - BF$ can be anywhere in \mathbb{C}^- by varying F over the set F_{op} . \square

3.2. Full-order dynamic output feedback design

In this subsection and the next we consider dynamic output feedback controllers. More specifically, this subsection considers full-order observer-based controllers, while the next subsection considers reduced-order observer-based controllers. As we already noted, attaining the minimal H_2 norm for Σ is

nothing else than disturbance decoupling with measurement feedback and stability for Σ_{PQ} , i.e. construction of compensators which make the closed-loop transfer matrix equal to 0. This fact is a key in getting our results.

We consider general proper, i.e. not necessarily strictly proper controllers. Perhaps some justification for the use of such controllers is warranted here. In all books related to the linear quadratic Gaussian control problem (see for example, Kwakernaak and Sivan 1972 and Davis 1977), which is simply the stochastic interpretation of the H_2 control problem, only strictly proper controllers are considered. This was necessary since the disturbance w is white-noise. Only integrated white-noise has a thorough mathematical interpretation via stochastic integrals (see for example Davis 1977). Since we clearly want a well-defined input process this yields the extra requirement that the closed-loop transfer matrix from w to u should be strictly proper. Together with the classical assumption that D_1 is surjective, this yields the requirement that the controller should be strictly proper. Also, if D_2 is injective and D_1 is surjective, which are *both* classical assumptions, then a strictly proper compensator is necessary for a finite H_2 norm (since this requires a strictly proper transfer matrix from w to z). On the other hand, in this paper we do not make these classical assumptions nor do we use this stochastic interpretation. Therefore, it makes sense to investigate non-strictly proper compensators as well.

The following theorem gives necessary and sufficient conditions under which we can attain the infimum γ^* for general proper (not necessarily strictly proper) stabilizing compensators:

Theorem 3.3: *Consider the system (2.1). The following two statements are equivalent.*

- (1) *There exists a proper internally stabilizing compensator of the form (3.5) such that the infimum γ^* is attained.*
- (2) *(A, B) is stabilizable, (C_1, A) is detectable and*
 - (a) $\text{Im}(E_Q) \subseteq \mathcal{V}_g(A, B, C_P, D_P) + B \text{Ker}(D_P)$,
 - (b) $\text{Ker}(C_P) \supseteq \mathcal{T}_g(A, E_Q, C_1, D_Q) \cap C_1^{-1}\{\text{Im}(D_Q)\}$,
 - (c) $\mathcal{T}_g(A, E_Q, C_1, D_Q) \subseteq \mathcal{V}_g(A, B, C_P, D_P)$.

Proof: Again, this is a combination of the results in Stoorvogel and van der Woude (1991) and Theorem 3.1. □

The conditions of Theorem 3.3 are weaker than the conditions derived by Stoorvogel (1990) in the sense that they are implied by the conditions of Stoorvogel (1990) but not vice versa. This is due to the fact that in Stoorvogel (1990) only strictly proper compensators are considered. It is possible for a system to have an optimal non-strictly proper output feedback control law but not to have any optimal strictly proper output feedback control law. Also, a system can have an optimal non-strictly proper output feedback control law while not having any optimal state feedback law. This is possible since, in general, in the case of a non-strictly proper output feedback law, we have a feedback of the disturbance (which is part of the measurement) as well; often this will hurt the design, but in some cases one can use this to attain the infimum which would not have been possible by feedback of the state only.

It should be noted that the above conditions *do not* satisfy the separation

principle. The separation principle tells us that we can look separately at state feedback and observer design and afterwards we can simply interconnect them. Condition (a) in the above theorem guarantees that we can attain the infimum of the closed loop H_2 norm in the case of state feedback. Condition (b) yields a dual result for observer design. On the other hand, condition (c) is a requirement which expresses whether we can couple the state feedback and the observer in a suitable way. The above conclusion is the reason why we cannot use the classical technique of the separation principle in the most general case. It should be noted that the separation principle does not hold for the question of attaining the infimum. The separation principle does indeed hold for the infimum itself, i.e. we can find a state feedback and an observer which are each separately arbitrarily close to the infimum (for their respective objectives) such that 'the interconnection' is close to the infimum γ^* defined by (2.2).

We will now, under the assumptions of Theorem 3.3, proceed with the design of compensators which attain the infimum. As in the previous subsection we will also investigate the freedom available in assigning the closed-loop eigenvalues.

Design

We will only discuss controllers of a particular structure. Given a static full-information feedback of the form $u = -Fx + ND_1w$, where F and N are suitably chosen gains, we replace x by its estimate \hat{x} and $D_1w = y - C_1x$ by the estimate $y - C_1\hat{x}$. Here \hat{x} is an estimate of x obtained via a suitably chosen observer. In this way we obtain an observer of the form

$$\left. \begin{aligned} \dot{v} &= Av + Bu + K(y - C_1v) \\ u &= -Fv + N(y - C_1v) \end{aligned} \right\} \quad (3.11)$$

Note that whenever the infimum can be attained by a compensator of the form (3.5), it can also be attained by a compensator of the form (3.11). In Stoorvogel and van der Woude (1991) a necessary condition was derived which the direct feedthrough matrix has to satisfy whenever it achieves disturbance decoupling with measurement feedback and stability. This is the condition given in the following lemma.

Lemma 3.4: Define X and Y such that

$$\mathcal{V}_g(A, B, C_P, D_P) = \text{Ker}(X), \quad \mathcal{T}_g(A, E_Q, C_1, D_Q) = \text{Im}(Y)$$

Define the following linear equation

$$\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & E_Q \\ C_P & 0 \end{bmatrix} + \begin{bmatrix} B \\ D_P \end{bmatrix} N [C_1 \ D_Q] \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} = 0 \quad (3.12)$$

Under the conditions of Theorem 3.3 there exists at least one matrix N satisfying this equation.

Proof: For the proof see Stoorvogel and van der Woude (1991). \square

Next, we define C_{op} to be the set of proper output feedback control laws of the form (3.11) where F and K are arbitrary elements of the sets F_{op} and K_{op} , and N is any solution of (3.12). Here F_{op} is the set defined in Lemmas 3.2 while

K_{op} denotes the set of matrices K which satisfies the conditions of Definition 3.1 for $\mathcal{T}_g(A, E_Q, C_1, D_Q)$ and which is such that $A - KC_1$ is asymptotically stable.

We note that such a set K_{op} can be constructed first by constructing a set using the algorithm from Appendix C for the system $(A^T, C_1^T, E_Q^T, D_Q^T)$, and then taking the transpose of the elements in the resulting set.

Lemma 3.5: *Under the conditions given in Theorem 3.3, any member of C_{op} applied to Σ is stabilizing and attains the infimum γ^* .*

Proof: Combine Theorem 3.1 with Stoorvogel and van der Woude (1991). \square

The following lemma shows the available freedom that exists in assigning the closed-loop eigenvalues while using the above class of compensators.

Lemma 3.6: *Assume the conditions of Theorem 3.3 are satisfied. Define Σ_{ci} and Σ_{di} to be the systems (A, B, C_2, D_2) and (A, E, C_1, D_1) respectively.*

By varying the output feedback control laws over the set C_{op} , we have the following freedom and constraints in assigning the closed-loop eigenvalues

- (1) $n_a^-(\Sigma_{ci}) + n_a^-(\Sigma_{di})$ eigenvalues lie at the locations of the stable invariant zeros of Σ_{ci} and Σ_{di} .
- (2) $n_a^+(\Sigma_{ci}) + n_a^+(\Sigma_{di})$ eigenvalues lie at the locations of the mirror images with respect to the imaginary axis of the invariant zeros of Σ_{ci} and Σ_{di} in the open left-half plane.
- (3) $n_b(\Sigma_{ci}) + n_c(\Sigma_{di})$ eigenvalues lie at fixed (independent of our specific choice for the compensator) locations in the open left-half plane which contain the stable input decoupling zeros (but not invariant zeros) of Σ_{ci} and the stable output decoupling zeros (but not invariant zeros) of Σ_{di} .
- (4) $n_a^0(\Sigma_{ci}) + n_c(\Sigma_{ci}) + n_f(\Sigma_{ci}) + n_a^0(\Sigma_{di}) + n_b(\Sigma_{di}) + n_f(\Sigma_{di})$ eigenvalues can be assigned to arbitrary locations in the open left-half plane depending on the specific choice for the compensator.

Here $n_a^-(\Sigma_*)$, $n_a^0(\Sigma_*)$, $n_a^+(\Sigma_*)$, $n_b(\Sigma_*)$, $n_c(\Sigma_*)$ and $n_f(\Sigma_*)$ are the constants n_a^- , n_a^0 , n_a^+ , n_b , n_c and n_f as defined in Appendix A when we bring Σ_* to the special coordinate basis.

Proof: It is straightforward to check that the closed-loop eigenvalues of the interconnection of the controller defined by (3.11) and the system Σ_{PQ} described by (3.1) are the sum of the eigenvalues of $A - BF$ and the eigenvalues of $A - KC_1$. Lemma 3.3 gives us the freedom and constraints in the eigenvalues of $A - BF$ when we vary F over the set F_{op} . Dualizing the result of Lemma 3.3 yields similar results for the freedom and constraints on the eigenvalues of $A - KC_1$ when we vary K over the set K_{op} . Finally, we use that $n_a^-(\Sigma_*) = n_a^-(\Sigma_*^T)$, $n_a^0(\Sigma_*) = n_a^0(\Sigma_*^T)$, $n_a^+(\Sigma_*) = n_a^+(\Sigma_*^T)$, $n_b(\Sigma_*) = n_c(\Sigma_*^T)$, $n_c(\Sigma_*) = n_b(\Sigma_*^T)$ and $n_f(\Sigma_*) = n_f(\Sigma_*^T)$. The above lemma is then a straightforward combination of these results. \square

Remark 3.1: It should be noted that varying N over the set of solutions to equation (3.12) has no effect on the closed-loop eigenvalues, and the freedom given in this lemma in assigning the closed loop eigenvalues is independent of N . However, it is not true, in general, that we have the same freedom in assigning the closed-loop eigenvalues by using proper (not necessarily strictly

proper) compensators as one can have by using strictly proper compensators. In our case this is true because of the structure we imposed on the controller. Hence, in general, the direct feedthrough matrix N of the controller does have an effect on the closed-loop eigenvalues, but N will always satisfy (3.12) if the resulting compensator attains the infimum. \square

3.3. Reduced-order observer-based controller design

In this subsection we will discuss the case where our system Σ is such that some states are directly available for feedback without perturbation by noise. It will be shown that the order of the compensator in these cases can be reduced and we will discuss the effect of using reduced-order observers on the freedom available in assigning the closed-loop eigenvalues.

We assume that there exists a proper compensator which attains the infimum γ^* . In other words, conditions (a)–(c) of Theorem 3.3 are satisfied. The technique presented in this section is completely similar to the method used in Stoorvogel *et al.* (1991). It should be noted that if we can attain the infimum γ^* by some proper compensator of the form (3.5) then we can also attain the infimum by a reduced-order observer-based compensator. The reduction in McMillan degree for the compensator is equal to the number of states we can observe without noise.

Without loss of generality but for simplicity of presentation, we assume that the matrices C_1 and D_Q have already been transformed to the following form

$$C_1 = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \text{ and } D_Q = \begin{bmatrix} D_0 \\ 0 \end{bmatrix} \quad (3.13)$$

Thus, the system Σ_{PQ} as in (3.1) can be partitioned as follows

$$\left. \begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{PQ} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} w \\ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} &= \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_0 \\ 0 \end{bmatrix} w \\ z_{PQ} &= C_P x_{PQ} + D_P u_{PQ} \end{aligned} \right\} \quad (3.14)$$

where $[x_1^T, x_2^T]^T = x_{PQ}$ and $[y_0^T, y_1^T]^T = y_{PQ}$. The idea behind the construction of a reduced-order observer-based controller is that we only need to build an observer for x_2 . Our techniques are based on the method discussed in § 7.2 of Anderson and Moore (1989). The differential equation for x_1 is given by

$$\dot{x}_1 = A_{22}x_1 + [A_{21} \ B_2] \begin{bmatrix} y_1 \\ u_{PQ} \end{bmatrix} + E_2 w$$

where (y_1, u_{PQ}) are known. Observations of x_2 are made via y_1 and

$$\tilde{y} = A_{12}x_2 + E_1 w = \dot{y}_1 - A_{11}x_1 - B_1 u_{PQ} \quad (3.15)$$

If we do not worry about the differentiation for the moment we note that we have to build an observer for the following system

$$\Sigma_r: \left\{ \begin{aligned} \dot{x}_2 &= A_{22}x_2 + [A_{21} \ B_2] \begin{bmatrix} y_1 \\ u_{PQ} \end{bmatrix} + E_2 w \\ \begin{bmatrix} y_0 \\ \tilde{y} \end{bmatrix} &= \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix} x_2 + \begin{bmatrix} D_0 \\ E_1 \end{bmatrix} w \end{aligned} \right. \quad (3.16)$$

The following lemma identifies the properties of Σ_r .

Lemma 3.7: *Let the system Σ_{re} be defined by the quadruple*

$$\left(A_{22}, E_2, \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix}, \begin{bmatrix} D_0 \\ E_1 \end{bmatrix} \right) \quad (3.17)$$

Then we have

- (1) Σ_{re} is detectable if and only if (A, E_Q, C_1, D_Q) is detectable.
- (2) The invariant zeros of Σ_{re} are the same as the invariant zeros of (A, E_Q, C_1, D_Q) .
- (3) The infinite zeros of Σ_{re} are the infinite zeros of (A, E_Q, C_1, D_Q) with order larger than 1. Their order is reduced by 1 when compared with the order of zeros of (A, E_Q, C_1, D_Q) .
- (4) Σ_{re} is left invertible if and only if (A, E_Q, C_1, D_Q) is left invertible.
- (5) $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{V}_g(\Sigma_{re}) \subseteq \mathcal{V}_g(A, E_Q, C_1, D_Q)$.
- (6) $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{F}_g(\Sigma_{re}) \subseteq \mathcal{F}_g(A, E_Q, C_1, D_Q) \cap C^{-1}\{\text{Im}(D_Q)\}$.

Proof: For the proof see Appendix D. □

Next, we build a full-order observer for the system Σ_r defined by (3.16). Using (3.15) we find the following observer which utilizes the gain K_r

$$\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + A_{21}y_1 + B_2u_{PQ} + K_r \left(\begin{bmatrix} y_0 \\ \dot{y}_1 - A_{11}x_1 - B_1u_{PQ} \end{bmatrix} - \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix} \hat{x}_2 \right)$$

We factorize $K_r = [K_{r0} \ K_{r1}]$, compatible with the sizes of (y_0, \dot{y}) . Then, using the change of variables $v := \hat{x}_2 - K_{r1}y_1$ results in a reduced-order observer

$$\begin{aligned} \dot{v} &= (A_{22} - K_{r0}C_{02} - K_{r1}A_{12})v + (B_2 - K_{r1}B_1)u_{PQ} \\ &\quad + [K_{r0}, A_{21} - K_{r1}A_{11} + (A_{22} - K_{r0}C_{02} - K_{r1}A_{12})K_{r1}]y_{PQ} \\ \hat{x}_{PQ} &= \begin{bmatrix} 0 \\ I_{n-p+m_0} \end{bmatrix} v + \begin{bmatrix} I_{p-m_0} \\ K_{r1} \end{bmatrix} y_1 \end{aligned}$$

Next we use this reduced-order observer to implement the full information feedback $u_{PQ} = -Fx_{PQ} + ND_Qw$ given in the previous subsection and we also replace u_{PQ} and y_{PQ} by u and y respectively. This leads to a reduced-order proper control law of the form

$$\left. \begin{aligned} \dot{v} &= (A_{22} - K_{r0}C_{02} - K_{r1}A_{12})v + (B_2 - K_{r1}B_1)u \\ &\quad + [K_{r0}, A_{21} - K_{r1}A_{11} + (A_{22} - K_{r0}C_{02} - K_{r1}A_{12})K_{r1}]y \\ u &= - (F + NC_1) \begin{bmatrix} 0 \\ I_{n-p+m_0} \end{bmatrix} v \\ &\quad - \left(F \begin{bmatrix} 0 & I_{p-m_0} \\ 0 & K_{r1} \end{bmatrix} + N \begin{bmatrix} -I_{n-p+m_0} & C_{02}K_{r1} \\ 0 & 0 \end{bmatrix} \right) y \end{aligned} \right\} \quad (3.18)$$

Let us first define the set K'_{op} as the set of matrices $K_r = [K_{r0}, K_{r1}]$ which satisfies the conditions of Definition 3.1 for $\mathcal{F}_g(\Sigma_{re})$ and which are such that

$A_{22} - K_{r0}C_{02} - K_{r1}A_{12}$ is asymptotically stable. The element of this set can be constructed by applying the algorithm of Appendix C to a dual system of Σ_{re} , i.e. $(A_{22}^T, [C_{02}^T, A_{12}^T], E_2^T, [D_0^T, E_1^T])$. Next, we define C_{op}^r to be the set of non-strictly proper output feedback control laws of the form (3.18) where F and K_r are the arbitrary elements of F_{op} and K_{op}^r respectively, and N is any solution of (3.12). Here F_{op} is the set defined in Lemma 3.2.

Lemma 3.8: *Under the conditions given in Theorem 3.3, any member of C_{op}^r applied to Σ is stabilizing and attains the infimum γ^* .*

Proof: Combine Theorem 3.1 and Lemma 3.7 with Stoorvogel and van der Woude (1991). \square

As in the previous subsections, in what follows we give a lemma that shows the freedom and constraints the above class of compensators yields in assigning the closed-loop eigenvalues.

Lemma 3.9: *Assume that the conditions of Theorem 3.3 are satisfied. Define Σ_{ci} and Σ_{di} to be the systems (A, B, C_2, D_2) and (A, E, C_1, D_1) respectively.*

By varying the output feedback control laws over the set C_{op}^r , we have the following freedom and constraints in assigning the closed-loop eigenvalues

- (1) $n_a^-(\Sigma_{ci}) + n_a^-(\Sigma_{di})$ eigenvalues lie at the locations of the stable invariant zeros of Σ_{ci} and Σ_{di} .
- (2) $n_a^+(\Sigma_{ci}) + n_a^+(\Sigma_{di})$ eigenvalues lie at the locations of the mirror images with respect to the imaginary axis of the invariant zeros of Σ_{ci} and Σ_{di} in the open right half-plane.
- (3) $n_b(\Sigma_{ci}) + n_c(\Sigma_{di})$ eigenvalues lie at fixed (independent of our specific choice for the compensator) locations in the right half-plane which contain the stable input decoupling zeros (but not invariant zeros) of Σ_{ci} and the stable output decoupling zeros (but not invariant zeros) of Σ_{di} .
- (4) $n_a^0(\Sigma_{ci}) + n_c(\Sigma_{ci}) + n_f(\Sigma_{ci}) + n_a^0(\Sigma_{di}) + n_b(\Sigma_{di}) + n_f(\Sigma_{di}) - (p - m_0)$ eigenvalues can be assigned to arbitrary locations in the open right half-plane depending on the specific choice for the compensator.

Here $n_a^-(\Sigma_*)$, $n_a^0(\Sigma_*)$, $n_a^+(\Sigma_*)$, $n_b(\Sigma_*)$, $n_c(\Sigma_*)$ and $n_f(\Sigma_*)$ are the constants n_a^- , n_a^0 , n_a^+ , n_b , n_c and n_f as defined in Appendix A when we bring Σ_* to the special coordinate basis.

Proof: To show that these controllers attain the infimum we use the techniques outlined in Stoorvogel (1990) in combination with properties of Σ_{re} given in Lemma 3.7, to conclude that this controller achieves disturbance decoupling with measurement feedback and stability when applied to Σ_{PQ} . The available freedom and constraints for the closed-loop eigenvalues are shown by noting that the closed-loop eigenvalues are the eigenvalues of the matrices $A - BF$ and $A_{22} - K_{r0}C_{02} - K_{r1}A_{12}$. By Lemma 3.3 choosing elements K_r and F from the sets K_{op}^r and F_{op} yields the required freedom and constraints. \square

To conclude, we note that whenever we can attain the infimum γ^* , we can reduce the dynamic order of the compensator in case some states are observed without noise. We can clearly assign less closed-loop eigenvalues for the closed-loop system but we have the same constraints as we encountered using

full-order dynamic compensators. In principle, the above theorem allows us to reobtain the results of Theorem 3.2 and Lemmas 3.2 and 3.3 by setting $C_1 = I$ and $D_1 = 0$. The above result then yields static-state feedbacks, i.e. dynamic compensators with 0-dimensional state-space. Moreover, if there are no states which are observed without noise, namely when $p = m_0$, then the set C_{op}^r reduces to the set C_{op} (i.e. $C_{op}^r = C_{op}$) and the results given in this subsection allow us to re-obtain the results of the previous subsection.

4. Suboptimal H_2 design

It must be clear from the previous section that we cannot always attain the infimum γ^* . In this section we discuss techniques to approach this infimum. As in the previous section, we consider both static-state feedback, and dynamic output feedback controllers. We start this section by introducing the definition of a suboptimal solution for the H_2 optimization problem.

Definition 4.1: A sequence of state or output feedback control laws $\{\Sigma_F(\varepsilon) | \varepsilon > 0\}$ is said to be *suboptimal* if there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the closed-loop system comprising of the given system Σ and controller $\Sigma_F(\varepsilon)$ is asymptotically stable and the H_2 -norm of the corresponding closed-loop transfer function $T_{zw}[\Sigma_F(\varepsilon)]$ decreases as $\varepsilon \downarrow 0$ with limit γ^* . \square

In the previous section we showed that we cannot always attain the infimum. In contrast, we can, of course, always find a suboptimal sequence as long as the system is stabilizable and detectable. The intention here is to construct such suboptimal sequences. Also, a secondary goal here is to characterize the closed-loop eigenvalues as to the available freedom and constraints in their asymptotic locations.

We again split this section in three parts: state feedback, full-order measurement feedback, and reduced-order observer-based controllers.

4.1. Static-state feedback

In this subsection, we will generate a sequence of suboptimal state feedback controllers utilizing a variation of the cheap control approach. This technique is outlined in Appendix E.

We have the following results.

Lemma 4.1: Consider the system (2.1) with $C_1 = I$, $D_1 = 0$. Define Σ_{ci} to be the system (A, B, C_2, D_2) .

Denote the class of sequences of static-state feedbacks described in Appendix E for the system Σ_{ci} by F_{op}^{seq} . Any sequence in the set F_{op}^{seq} is a suboptimal sequence. Moreover, by choosing an appropriate sequence in the set F_{op}^{seq} we have the following freedom and constraints in the asymptotic behaviour of the closed-loop eigenvalues.

- (1) $n_a^-(\Sigma_{ci})$ eigenvalues converge to the locations of the stable invariant zeros of Σ_{ci} .
- (2) $n_a^+(\Sigma_{ci})$ eigenvalues converge to the locations of the mirror images with respect to the imaginary axis of the invariant zeros of Σ_{ci} in the open right half-plane.

- (3) $n_a^0(\Sigma_{ci})$ eigenvalues converge to the locations of the invariant zeros of Σ_{ci} on the imaginary axis,
- (4) $n_b(\Sigma_{ci})$ eigenvalues converge to certain fixed (independent of the specific sequence in F_{op}^{seq}) locations in \mathbb{C}^- , which include the input-decoupling zeros (but not invariant zeros) of Σ_{ci} .
- (5) $n_c(\Sigma_{ci})$ eigenvalues can be assigned with arbitrary asymptotic behaviour in \mathbb{C}^- .
- (6) $n_f(\Sigma_{ci})$ eigenvalues go to infinity in the open left half-plane and remain bounded away from the imaginary axis.

Here $n_a^-(\Sigma_{ci})$, $n_a^+(\Sigma_{ci})$, $n_a^0(\Sigma_{ci})$, $n_b(\Sigma_{ci})$, $n_c(\Sigma_{ci})$ and $n_f(\Sigma_{ci})$ are the constants n_a^- , n_a^+ , n_a^0 , n_b , n_c and n_f as defined in Appendix A when we transform Σ_{ci} to the special coordinate basis.

Proof: The construction of suboptimal sequences with the above freedom and constraints is given in Appendix E. \square

Remark 4.1: In the appendix we find two methods to design a suboptimal sequence. The first one, i.e. using Lemma E.1 directly, does not need the special coordinate basis and therefore is very easy to use. However, we lose the freedom to assign poles arbitrarily. A second method is described using the special coordinate basis although it is easy to see that for its implementation one needs only the space $\mathcal{X}_c = \mathcal{V}_g \cap \mathcal{T}_g$. Normally, one could use the simple first algorithm. However, if the resulting positions of the closed-loop poles is unsatisfactory, then the second algorithm can be used as it has some freedom to change the location of poles. \square

It should be noticed that if $E = 0$ then any sequence of stabilizing feedback will do and the above constraints in the asymptotic locations of the closed-loop eigenvalues need not be satisfied. On the other hand, if E is surjective then we conjecture that any suboptimal sequence will satisfy the above constraints on the asymptotic locations of the closed-loop eigenvalues. Unfortunately, we have not been able to find a proof of this conjecture and we believe that this is a very hard problem.

Remark 4.2: Another approach to generate suboptimal sequences is the asymptotic time-scale and eigenstructure assignment (ATEA) technique used in almost disturbance-decoupling (see Saberi and Sannuti 1989, Ozcetin *et al.* 1990, 1991). The ATEA approach has the advantage that, contrary to the approach presented in this paper, it can assign the time-scales of eigenvalues that go off to infinity arbitrarily. Moreover, it is free of numerical problems related to stiffness which are present in the perturbation methods. These properties yield large improvements in the design of the suboptimal sequences. However, at this moment the ATEA method cannot handle invariant zeros on the imaginary axis. For more details on this alternative method, we refer to Saberi and Sannuti 1989, Ozcetin *et al.* 1990, 1991. \square

4.2. Full-order dynamic compensators

In this subsection we investigate full-order dynamic compensators. Since we can always find a suboptimal sequence of strictly proper compensators and since

the direct feedthrough matrix did not give us extra freedom in the asymptotic pole locations, we only investigate strictly proper full-order compensators.

Lemma 4.2: Consider the system (2.1). Define Σ_{ci} to be the system (A, B, C_2, D_2) and Σ_{di} to be the system (A, E, C_1, D_1) . Denote by \mathbf{K}_{op}^{seq} the set of sequences we get by applying the construction of Appendix E to the system Σ_{di}^1 and then transposing every element of the sequences thus obtained.

Let $F(\varepsilon)$ be a sequence in the set \mathbf{F}_{op}^{seq} . Moreover, let $K(\varepsilon)$ be a sequence in the set \mathbf{K}_{op}^{seq} . We can then construct the following sequence of strictly proper dynamic compensators

$$\left. \begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K(\varepsilon)(y - C_1\hat{x}) \\ u &= -F(\varepsilon)\hat{x} \end{aligned} \right\} \quad (4.1)$$

The class of sequences of dynamic compensators we thus obtain will be denoted by \mathbf{C}_{op}^{seq} . Any sequence in this set is a suboptimal sequence for the H_2 control problem under investigation. Moreover, by choosing an appropriate sequence in this set \mathbf{C}_{op}^{seq} we have the following freedom and constraints in the asymptotic behaviour of the closed-loop eigenvalues

- (1) $n_a^-(\Sigma_{ci}) + n_a^-(\Sigma_{di})$ eigenvalues converge to the locations of the stable invariant zeros of Σ_{ci} and Σ_{di} .
- (2) $n_a^+(\Sigma_{ci}) + n_a^+(\Sigma_{di})$ eigenvalues converge to the locations of the mirror images with respect to the imaginary axis of the invariant zeros of Σ_{ci} and Σ_{di} in the open right half-plane.
- (3) $n_a^0(\Sigma_{ci}) + n_a^0(\Sigma_{di})$ eigenvalues converge to the locations of the invariant zeros of Σ_{ci} and Σ_{di} on the imaginary axis.
- (4) $n_b(\Sigma_{ci}) + n_c(\Sigma_{di})$ eigenvalues converge to fixed locations in \mathbb{C}^- that include the stable input decoupling zeros (but not invariant zeros) of Σ_{ci} and the stable output decoupling zeros (but not invariant zeros) of Σ_{di} .
- (5) $n_c(\Sigma_{ci}) + n_b(\Sigma_{di})$ eigenvalues can be assigned arbitrary asymptotic properties in \mathbb{C}^- .
- (6) $n_f(\Sigma_{ci}) + n_f(\Sigma_{di})$ eigenvalues go to infinity in the open left half-plane and remain bounded away from the imaginary axis.

Here $n_a^-(\Sigma_*)$, $n_a^+(\Sigma_*)$, $n_a^0(\Sigma_*)$, $n_b(\Sigma_*)$, $n_c(\Sigma_*)$ and $n_f(\Sigma_*)$ are the constants n_a^- , n_a^+ , n_a^0 , n_b , n_c and n_f as defined in Appendix A when we transform Σ_* to the special coordinate basis.

Proof: We first have to consider whether we can indeed obtain a suboptimal sequence via this construction. If $n_c(\Sigma_{ci}) + n_b(\Sigma_{di}) = 0$ then the compensator in (4.1) is an H_2 optimal controller for the system:

$$\Sigma_\varepsilon: \begin{cases} \dot{x} = Ax + Bu + (E \ \varepsilon I \ 0)\tilde{w} \\ y = C_1x + (D_1 \ 0 \ \varepsilon I)\tilde{w} \\ \tilde{z} = \begin{bmatrix} C_2 \\ \varepsilon I \\ 0 \end{bmatrix} x + \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I \end{bmatrix} u \end{cases}$$

It is then easy to show via some classical continuity arguments that (4.1) yields a suboptimal sequence for Σ . If $n_c(\Sigma_{ci}) + n_b(\Sigma_{di}) > 0$ then this argument needs to be refined but remains basically the same.

The closed-loop eigenvalues are the sum of the eigenvalues of $A - BF(\varepsilon)$ and $A - K(\varepsilon)C_1$. Therefore, the freedom and constraints in the asymptotic behaviour can be found from the state feedback case as described in Lemma 4.1. □

Again, we would like to know whether any suboptimal sequence which is not an element of C_{op}^{seq} , will satisfy the above constraints on the asymptotic behaviour of the closed-loop eigenvalues. We conjecture that this is true if E is surjective and C_2 injective (which, in a certain way, is the worst case), but we have not been able to find a proof of this conjecture.

If the systems Σ_{ci} and Σ_{di} do not have invariant zeros on the imaginary axis we could, and probably should, use the algorithm based on the almost disturbance decoupling mentioned earlier.

4.3. Reduced-order observer

We conclude our investigation of suboptimal design by investigating whether, in suboptimal design as well, it is possible to reduce the order of the compensators if we observe one or more states without noise. This is indeed possible and we find the following result.

Lemma 4.3: *Consider the system (2.1) with its corresponding Σ_{PQ} partitioned as in (3.14). Define Σ_{ci} to be the system (A, B, C_2, D_2) , Σ_{di} to be the system (A, E, C_1, D_1) and Σ_{re} to be the system defined by the quadruple (3.17). Denote by F_{op}^{seq} the set of sequences defined in Lemma 4.1. Let $K_{op,r}^{seq}$ denote the set of sequences which are the transposes of the elements of the set we obtain by applying the construction of Appendix E to Σ_{re}^T .*

Let $F(\varepsilon)$ and $K_r(\varepsilon) = [K_{r0}(\varepsilon), K_{r1}(\varepsilon)]$ be arbitrary sequences from the sets F_{op}^{seq} and $K_{op,r}^{seq}$ respectively. We then construct a sequence of reduced-order observer based controllers as follows

$$\left. \begin{aligned} \dot{v} &= [A_{22} - K_{r0}(\varepsilon)C_{02} - K_{r1}(\varepsilon)A_{12}]v + [B_2 - K_{r1}(\varepsilon)B_1]u \\ &\quad + [K_{r0}(\varepsilon), A_{21} - K_{r1}(\varepsilon)A_{11} + (A_{22} - K_{r0}(\varepsilon)C_{02} - K_{r1}(\varepsilon)A_{12})K_{r1}(\varepsilon)]y \\ u &= -F(\varepsilon) \begin{bmatrix} 0 \\ I_{n-p+m_0} \end{bmatrix} v - F(\varepsilon) \begin{bmatrix} 0 & I_{p-m_0} \\ 0 & K_{r1}(\varepsilon) \end{bmatrix} y \end{aligned} \right\} \quad (4.2)$$

The set of compensators we thus obtain will be denoted by $C_{op,r}^{seq}$. Any element of this set is a suboptimal sequence for the H_2 control problem for the system Σ . Moreover, by choosing an appropriate sequence in the set $C_{op,r}^{seq}$ we have the following freedom and constraints in the asymptotic behaviour of the closed-loop eigenvalues.

- (1) $n_a^-(\Sigma_{ci}) + n_a^-(\Sigma_{di})$ eigenvalues converge to the locations of the stable invariant zeros of Σ_{ci} and Σ_{di} .
- (2) $n_a^0(\Sigma_{ci}) + n_a^0(\Sigma_{di})$ eigenvalues converge to the invariant zeros of Σ_{ci} and Σ_{di} on the imaginary axis.

- (3) $n_a^+(\Sigma_{ci}) + n_a^+(\Sigma_{di})$ eigenvalues converge to the mirror images with respect to the imaginary axis of the invariant zeros of Σ_{ci} and Σ_{di} in the open right half-plane.
- (4) $n_b(\Sigma_{ci}) + n_c(\Sigma_{di})$ eigenvalues converge to some fixed locations in \mathbb{C}^- that include the stable input decoupling zeros which are not invariant zeros of Σ_{ci} and the stable output decoupling zeros which are not the invariant zeros of Σ_{di} .
- (5) $n_c(\Sigma_{ci}) + n_b(\Sigma_{re})$ eigenvalues can be assigned arbitrarily in \mathbb{C}^- .
- (6) $n_f(\Sigma_{ci}) + n_f(\Sigma_{re})$ eigenvalues go to infinity in the open left half-plane and remain bounded away from the imaginary axis.

Here $n_a^-(\Sigma_*)$, $n_a^+(\Sigma_*)$, $n_a^0(\Sigma_*)$, $n_b(\Sigma_*)$, $n_c(\Sigma_*)$ and $n_f(\Sigma_*)$ are the constants n_a^- , n_a^+ , n_a^0 , n_b , n_c and n_f as defined in Appendix A when we transform Σ_* to the special coordinate basis. Also, note that $n_b(\Sigma_{di}) + n_f(\Sigma_{di}) = n_b(\Sigma_{re}) + n_f(\Sigma_{re}) + p - m_0$.

Proof: We use the same arguments as in the proof of Lemma 4.2.

We first have to consider whether we can indeed obtain a suboptimal sequence via this construction. If $n_c(\Sigma_{ci}) + n_b(\Sigma_{re}) = 0$ then the compensator in (4.2) is an H_2 optimal reduced-order observer based controller for the system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{PQ} + \begin{bmatrix} E_1 & 0 & 0 & \varepsilon I \\ E_2 & \varepsilon I & 0 & 0 \end{bmatrix} w \\ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} &= \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_0 & 0 & \varepsilon I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} w \\ \tilde{z}_{PQ} &= \begin{bmatrix} C_P \\ \varepsilon I \\ 0 \end{bmatrix} x_{PQ} + \begin{bmatrix} D_P \\ 0 \\ \varepsilon I \end{bmatrix} u_{PQ} \end{aligned}$$

It is then easy to show via some classical continuity arguments that (4.2) yields a suboptimal sequence for Σ_{PQ} and therefore also a suboptimal sequence for Σ . If $n_c(\Sigma_{ci}) + n_b(\Sigma_{re}) > 0$ then this argument needs to be refined but remains basically the same.

The closed-loop eigenvalues are the sum of the eigenvalues of $A - BF(\varepsilon)$ and $A - K_{r0}(\varepsilon)C_{02} - K_{r1}(\varepsilon)A_{12}$. Therefore, the freedom and constraints in the asymptotic behaviour can be found from the state feedback case as described in Lemma 4.1. Note that $K_r(\varepsilon)$ is an observer gain for the system Σ_{re} . We use Lemma 3.7 and Lemma 3.1 to relate the system Σ_{re} with Σ_{di} . \square

This concludes the results for the suboptimal sequences. The main open problem remains the necessity of the constraints on the asymptotic locations of the closed-loop eigenvalues.

5. Conclusions

In this paper we have tried to give a fairly complete treatment of the H_2 control problem. We have given explicit characterizations of when we can attain the infimum of the H_2 norm. Moreover, we have shown how the freedom given to us by the non-uniqueness of the optimal controller can be used up to a certain level to manipulate the closed-loop eigenvalues. We also investigated

minimizing sequences for the case when the infimum cannot be obtained, and in that case as well, we investigated the freedom available in the asymptotic locations of the closed-loop eigenvalues. We have considered both measurement feedback and state feedback. We have also investigated reduced-order observer-based measurement feedback.

If the infimum could be attained, we investigated three specific architectures for the controller. In each case we show that 'in the worst case', namely where we have certain subspace equalities, our constraints on the closed-loop eigenvalues are necessary and our characterizations of optimal solutions are complete. However, in general—namely not just in the 'worst case'—the optimal solutions obtained here are incomplete and constitute only a subclass of all possible optimal solutions. It is an interesting problem to characterize all possible optimal solutions for these three architectures of control laws and examine the freedom and constraints on their closed-loop eigenvalues.

In suboptimal design we investigated three different architectures for the controller. We give the freedom and constraints for the asymptotic locations of the closed-loop eigenvalues. However, even in a 'worst case' we cannot prove that these constraints are necessary. This remains an interesting yet very difficult open problem.

The algorithm presented in Appendix E is numerically not very reliable. However alternatives are available, as mentioned in § 4. This alternative method cannot treat invariant zeros on the imaginary axis. However, since our method shows that invariant zeros on the imaginary axis result in suboptimal sequences with eigenvalues converging to the imaginary axis, this is already—from a practical point of view—a very undesirable situation.

Notwithstanding the above, it is our belief that this paper gives a fairly complete picture of the H_2 control problem in its full generality.

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Appendix A

The special coordinate basis

In this section we will present the special coordinate basis introduced in Sannuti and Saberi (1987) and Saberi and Sannuti (1990). Such a special coordinate basis has a distinct feature of explicitly displaying the finite and infinite zero structure of a given system. Consider the system characterized by (A, B, C, D) . We first choose a new basis in the input and output spaces such that the direct feedthrough matrix gets a nice form

$$\tilde{D} := UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = [B_0, B_1] := BV, \quad \tilde{C} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} := UC$$

where U and V are non-singular matrices. In this way, we obtain the transformed system $(A, \tilde{B}, \tilde{C}, \tilde{D})$. It follows from Sannuti and Saberi (1987)

and Saberi and Sannuti (1990) that there exists new bases in the state, input and output spaces represented by the base transformations Γ_s , Γ_i and Γ_o respectively such that the resulting matrices have a special form

$$\Gamma_s^{-1}(A - B_0C_0)\Gamma_s = \begin{bmatrix} A_{aa}^- & 0 & 0 & 0 & L_{ab}^-C_b & L_{af}^-C_f \\ B_cE_{ca}^- & A_{cc} & B_cE_{ca}^0 & B_cE_{ca}^+ & L_{ab}C_b & L_{cf}C_f \\ 0 & 0 & A_{aa}^0 & 0 & L_{ab}^0C_b & L_{af}^0C_f \\ 0 & 0 & 0 & A_{aa}^+ & L_{ab}^+C_b & L_{af}^+C_f \\ 0 & 0 & 0 & 0 & A_{bb} & L_{bf}C_f \\ B_fE_{fa}^- & B_fE_{fc} & B_fE_{fa}^0 & B_fE_{fa}^+ & B_fE_{fb} & A_f \end{bmatrix} \tag{A 1}$$

$$\Gamma_s^{-1}[B_0, B_1]\Gamma_i = \begin{bmatrix} B_{a0}^- & 0 & 0 \\ B_{c0} & 0 & B_c \\ B_{a0}^0 & 0 & 0 \\ B_{a0}^+ & 0 & 0 \\ B_{b0} & 0 & 0 \\ B_{f0} & B_f & 0 \end{bmatrix} \tag{A 2}$$

$$\Gamma_o^{-1} \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \Gamma_s = \begin{bmatrix} C_{0a}^- & C_{0c} & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0f} \\ 0 & 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & 0 & 0 & C_b & 0 \end{bmatrix} \tag{A 3}$$

and

$$\Gamma_o^{-1} \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A 4}$$

Here C_f is chosen to be surjective. Note that we have decomposed the state space into six parts: $\mathcal{X} = \mathcal{X}_a^- \oplus \mathcal{X}_c \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_b \oplus \mathcal{X}_f$. \mathcal{X}_a^- is related to the stable invariant zeros, i.e. the eigenvalues of A_{aa}^- are exactly the stable invariant zeros of Σ . Similarly \mathcal{X}_a^0 and \mathcal{X}_a^+ are related to the invariant zeros of Σ on the imaginary axis and open right half-plane respectively. \mathcal{X}_c is related to left-invertibility, i.e the system is left-invertible if and only if $\mathcal{X}_c = \{0\}$. Similarly, \mathcal{X}_b is related to right-invertibility, i.e. the system is right-invertible if and only if $\mathcal{X}_b = \{0\}$. Finally, \mathcal{X}_f is related to zeros of Σ at infinity. The space $\mathcal{V}_g(A, B, C, D)$ introduced earlier in this paper is equal to $\mathcal{X}_a^- \oplus \mathcal{X}_c$. On the other hand, the space $\mathcal{T}_g(A, B, C, D)$ which was also introduced earlier equals $\mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_c \oplus \mathcal{X}_f$.

We denote the dimensions of \mathcal{X}_a^- , \mathcal{X}_c , \mathcal{X}_a^0 , \mathcal{X}_a^+ , \mathcal{X}_b and \mathcal{X}_f by n_a^- , n_c , n_a^0 , n_a^+ , n_b and n_f respectively.

Appendix B

Proof of Lemma 3.2—Properties of Σ_{PQ} : Without loss of generality but for simplicity of presentation, we assume that the system (A, B, C_2, D_2) is in the form of the special coordinate basis as described by (A 1) to (A 4). Then, it is straightforward to verify that (since C_f is surjective) the unique positive

semidefinite matrix P satisfying conditions (i) to (iii) of Lemma 3.1 is given by

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{P}_{11} & \tilde{P}_{21}^T & 0 \\ 0 & 0 & 0 & \tilde{P}_{21} & \tilde{P}_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{B 1}$$

where

$$\tilde{P} := \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{21}^T \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix}$$

is the positive definite solution of the following ARE

$$\begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix}^T \tilde{P} + \tilde{P} \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C_b^T C_b \end{bmatrix} - \tilde{P} \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix} \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix}^T \tilde{P} = 0 \tag{B 2}$$

with

$$\lambda \left\{ \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix} - \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix} \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix}^T \tilde{P} \right\} \subset \mathbb{C}^-$$

We define

$$\tilde{F} := \begin{bmatrix} F_{a0}^+ & F_{b0} \\ F_{a1}^+ & F_{b1} \end{bmatrix} := \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix}^T \tilde{P}$$

It is simple to show that $F(P)$ can be factorized as

$$F(P) = \begin{bmatrix} C_P^T \\ D_P^T \end{bmatrix} [C_P \ D_P]$$

where

$$C_P = \begin{bmatrix} C_{0a}^- & C_{0c} & C_{0a}^0 & C_{0a}^+ + F_{a0}^+ & C_{0b} + F_{b0} & C_{0f} \\ 0 & 0 & 0 & F_{a1}^+ & F_{b1} & C_f \end{bmatrix}$$

and

$$D_P = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Next, let us define

$$\bar{A} = A - B_0 C_0 - B_0 [0 \ 0 \ 0 \ F_{a0}^+ \ F_{b0} \ 0], \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & B_c \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ B_f & 0 \end{bmatrix}$$

and

$$\bar{C} = [0 \ 0 \ 0 \ F_{a1}^+ \ F_{b1} \ C_f]$$

Then, using the same techniques as in Appendix B of Chen *et al.* (1992) and using the properties of the special coordinate basis, it is easily shown that $(\bar{A}, \bar{B}, \bar{C}, 0)$ has the following properties.

- (1) $(\bar{A}, \bar{B}, \bar{C}, 0)$ is right invertible;
- (2) $(\bar{A}, \bar{B}, \bar{C}, 0)$ has the same infinite zero structure as (A, B, C_2, D_2) ; and
- (3) $(\bar{A}, \bar{B}, \bar{C}, 0)$ has invariant zeros at

$$\lambda \left[\begin{array}{cc|cc} A_{aa}^- & 0 & & \star \\ 0 & A_{aa}^0 & & \star \\ \hline 0 & 0 & \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix} & - \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix} \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix}^T \tilde{P} \end{array} \right]$$

where \star denotes matrices of not much interest.

It is trivial to verify that (A, B, C_p, D_p) satisfies the same properties (1) to (3) which $(\bar{A}, \bar{B}, \bar{C}, 0)$ satisfies. Obviously, this implies that the stable and $j\omega$ axis invariant zeros of (A, B, C_2, D_2) , i.e. $\lambda(A_{aa}^-)$ and $\lambda(A_{aa}^0)$, are also invariant zeros of (A, B, C_p, D_p) . The remaining invariant zeros of (A, B, C_p, D_p) are given by

$$\lambda \left\{ \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix} - \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix} \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix}^T \tilde{P} \right\} \subset \mathbb{C}^-$$

Let

$$\tilde{P}^{-1} := \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{21}^T \\ \tilde{S}_{21} & \tilde{S}_{22} \end{bmatrix}$$

Then, post-multiplying equation (B 2) by \tilde{P}^{-1} , one obtains

$$\begin{aligned} \tilde{P} \left\{ \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix} - \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix} \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix}^T \tilde{P} \right\} \tilde{P}^{-1} \\ = \begin{bmatrix} -(A_{aa}^+)^T & 0 \\ \star & -(A_{bb}^T + C_b^T C_b \tilde{S}_{22}) \end{bmatrix} \end{aligned}$$

Thus, the mirror images of the unstable invariant zeros of (A, B, C_2, D_2) , $\lambda(-A_{aa}^+)$, are also contained among the invariant zeros of (A, B, C_p, D_p) . The remaining invariant zeros of (A, B, C_p, D_p) are at $\lambda(-A_{bb} - \tilde{S}_{22} C_b^T C_b) \subset \mathbb{C}^-$. It is worth noting that \tilde{S}_{22} is the unique positive definite solution of

$$\tilde{S}_{22} A_{bb}^T + A_{bb} \tilde{S}_{22} + \tilde{S}_{22} C_b^T C_b \tilde{S}_{22} - B_{b0} B_{b0}^T - L_{bf} L_{bf}^T = 0$$

In summary, the invariant zeros of (A, B, C_p, D_p) are at: $\lambda(A_{aa}^-) \cup \lambda(A_{aa}^0) \cup \lambda(-A_{aa}^+) \cup \lambda(-A_{bb} - \tilde{S}_{22} C_b^T C_b)$. This completes the proof of the properties of (A, B, C_p, D_p) . The proof for the properties of (A, E_Q, C_1, D_Q) follows from the same arguments as above since the roles of (A, B, C_p, D_p) and (A, E_Q, C_1, D_Q) are dual. \square

Appendix C

Construction of F_{op}

In Lemma 3.2 we introduced the set F_{op} . We will now give a constructive method of determining the elements of this set. Given a stabilizable system

characterized by (A, B, C, D) , the goal is to generate all the possible stabilizing state feedback gains F such that $(A - BF)$ is asymptotically stable and such that the conditions of Definition 3.1 for $\mathcal{V}_g(A, B, C_P, D_P)$ are satisfied. The following is a step-by-step description of an algorithm which accomplishes this goal.

Step 1. Transform the given system (A, B, C, D) into the form of the special coordinate basis as in (A 1) to (A 4).

Step 2.

(2.1) One of the properties of the special coordinate basis derived in Sannuti and Saberi (1987) and Saberi and Sannuti (1990) guarantees that the pair (A_{cc}, B_c) is controllable. Therefore, we can select a gain matrix F_{cc} such that the eigenvalues of A_{cc}^c , where $A_{cc}^c := A_{cc} - B_c F_{cc}$, are in arbitrary desired locations in \mathbb{C}^- .

(2.2) Form matrices A_x and B_x as follows

$$A_x := \begin{bmatrix} A_{aa}^0 & 0 & L_{ab}^0 C_b & L_{af}^0 C_f \\ 0 & A_{aa}^+ & L_{ab}^+ C_b & L_{bf}^+ C_f \\ 0 & 0 & A_{bb} & L_{bf} C_f \\ B_f E_{fa}^0 & B_f E_{fa}^+ & B_f E_{fb} & A_f \end{bmatrix}, \quad B_x := \begin{bmatrix} B_{a0}^0 & 0 \\ B_{a0}^+ & 0 \\ B_{b0} & 0 \\ B_{f0} & B_f \end{bmatrix} \tag{C1}$$

Another property of the special coordinate basis guarantees us that the uncontrollable eigenvalues of (A_x, B_x) are exactly the uncontrollable (stable) eigenvalues of (A, B) which are not invariant zeros of (A, B, C, D) . We select a gain matrix F_x such that the eigenvalues of A_x^c , where $A_x^c := A_x - B_x F_x$, are in arbitrary desired locations in \mathbb{C}^- subject to the constraint that certain stable eigenvalues are uncontrollable. Next, we partition F_x compatible with the partitions of A_x and B_x as

$$F_x = \begin{bmatrix} F_{a0}^0 & F_{a0}^+ & F_{b0} & F_{f0} \\ F_{a1}^0 & F_{a1}^+ & F_{c1} & F_{f1} \end{bmatrix} \tag{C2}$$

Step 3. Let

$$F = V \Gamma_i \begin{bmatrix} C_{0a}^- & C_{0c} & C_{0a}^0 + F_{a0}^0 & C_{0a}^+ + F_{a0}^+ & C_{0b} + F_{b0} & C_{0f} + F_{f0} \\ E_{fa}^- & E_{fc} & F_{a1}^0 & F_{a1}^+ & F_{b1} & F_{f1} \\ F_{ca}^- & F_{cc} & F_{ca}^0 & F_{ca}^+ & F_{cb} & F_{cf} \end{bmatrix} \Gamma_s^{-1} \tag{C3}$$

where F_{ca}^- , F_{ca}^0 , F_{ca}^+ , F_{cb} and F_{cf} in (C 3) are some arbitrary sub-matrices with appropriate dimensions which do not affect the closed-loop eigenvalues.

We have the following lemma which can be proven by some simple algebra.

Lemma C.1: *Let a system (A, B, C, D) be given which is written in the special coordinate basis. A state feedback gain F is an element of the set F_{op} , i.e. F satisfies the conditions of Definition 3.1 for $\mathcal{V}_g(A, B, C, D)$ and is such that $(A - BF)$ is asymptotically stable, if and only if F can be written in the form (C 3) where F_{cc} and F_x (defined by (C 2)) are such that:*

- (i) $A_{cc} - B_c F_{cc}$ is asymptotically stable, and
(ii) $A_x - B_x F_x$ is asymptotically stable (A_x and B_x are defined by (C 1)).

Remark: It is simple to see from (C 3) that F_{op} contains only a single element if and only if $\mathcal{X} = \mathcal{X}_a^-$. In case both $[B^T, D^T]$ and $[C, D]$ have full rank, this is equivalent to the fact that (A, B, C, D) is of minimum-phase, invertible and has no infinite zeros. Moreover, the only element of F_{op} is then $F = D^{-1}C$. \square

The proofs of Lemmas 3.2 and 3.3 follow in a straightforward way by using the properties of the special coordinate basis and Lemma C.1.

Appendix D

Proof of Lemma 4.8—Properties of Σ_{re} : Consider a given linear time-invariant system characterized by

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad (D 1)$$

First, we note that without loss of generality, one can assume that the matrices C and D are of the form

$$C = \begin{bmatrix} 0 & C_{02} \\ I & 0 \end{bmatrix}, \quad D = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, we can partition the given system (D 1) as

$$\Sigma: \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{01} & B_{11} \\ B_{02} & B_{12} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \\ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & C_{02} \\ I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \end{cases} \quad (D 2)$$

In the proof of Lemma 3.7 we need a modified version of the special coordinate basis: we can choose the transformations Γ_s , Γ_i and Γ_o such that we find the following decomposition

$$\Gamma_s^{-1} \begin{bmatrix} A_{11} & A_{12} - B_{01}C_{02} \\ A_{21} & A_{22} - B_{02}C_{02} \end{bmatrix} \Gamma_s = \begin{bmatrix} A_a & L_{ab} & 0 & 0 & L_{af1} & L_{af0} & 0 \\ 0 & A_{b11} & A_{b12} & 0 & L_{bf11} & L_{bf10} & 0 \\ 0 & A_{b21} & A_{b22} & 0 & L_{bf21} & L_{bf20} & 0 \\ B_c E_{ca} & L_{cb} & 0 & A_c & L_{cf1} & L_{cf0} & 0 \\ E_{fa1} & E_{fb11} & E_{fb12} & E_{fc1} & A_{f11} & A_{f10} & A_{f12} \\ 0 & 0 & 0 & 0 & A_{f01} & A_{f00} & A_{f02} \\ B_{f2} E_{fa2} & B_{f2} E_{fb21} & B_{f2} E_{fb22} & B_{f2} E_{fc2} & A_{f21} & A_{f20} & A_{f22} \end{bmatrix} \quad (D 3)$$

$$\Gamma_s^{-1} \begin{bmatrix} B_{01} & B_{11} \\ B_{02} & B_{12} \end{bmatrix} \Gamma_i = \begin{bmatrix} B_{a0} & 0 & 0 & 0 \\ B_{b01} & 0 & 0 & 0 \\ B_{b02} & 0 & 0 & 0 \\ B_{c0} & 0 & 0 & B_c \\ B_{f01} & I & 0 & 0 \\ B_{f00} & 0 & 0 & 0 \\ B_{f02} & 0 & B_{f2} & 0 \end{bmatrix} \quad (D 4)$$

$$\Gamma_o^{-1} \begin{bmatrix} 0 & C_{02} \\ I & 0 \end{bmatrix} \Gamma_s = \begin{bmatrix} C_{0a} & 0 & C_{0b2} & C_{0c} & 0 & 0 & C_{0f2} \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (D 5)$$

$$\Gamma_o^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Gamma_s = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (D 6)$$

Here we have decomposed our state-space into seven parts: $\mathcal{X} = \mathcal{X}_a \oplus \mathcal{X}_{b1} \oplus \mathcal{X}_{b2} \oplus \mathcal{X}_c \oplus \mathcal{X}_{f1} \oplus \mathcal{X}_{f0} \oplus \mathcal{X}_{f2}$. These spaces have a strong relationship to the decomposition introduced in Appendix A: we have $\mathcal{X}_a = \mathcal{X}_a^+ \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^-$, i.e. \mathcal{X}_a is related to all finite invariant zeros of the system. We have $\mathcal{X}_b = \mathcal{X}_{b1} \oplus \mathcal{X}_{b2}$ where $\mathcal{X}_{b2} = \mathcal{X}_b \cap \text{Ker}(C_b)$. \mathcal{X}_c is the same space as in Appendix A. Finally we have $\mathcal{X}_f = \mathcal{X}_{f1} \oplus \mathcal{X}_{f0} \oplus \mathcal{X}_{f2}$. Note that \mathcal{X}_f was related to the infinite zeros of the structure. This time we refine this: \mathcal{X}_{f1} is related to infinite zeros of order 1, \mathcal{X}_{f0} is related to infinite zeros of order larger than 1, and \mathcal{X}_{f2} is $\mathcal{X}_f \cap \text{Ker}(C_f)$.

Our goal in this appendix is to prove Lemma 3.7. We note that

$$\begin{aligned} & \left(A_{22}, [B_{02}, B_{12}], \Gamma \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix}, \Gamma \begin{bmatrix} I & 0 \\ B_{01} & B_{11} \end{bmatrix} \right) \\ & = \left(A_{22}, [B_{02}, B_{12}], \begin{bmatrix} C_{02} \\ A_{12} - B_{01}C_{02} \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & B_{11} \end{bmatrix} \right) \end{aligned}$$

where Γ is non-singular and is given by

$$\Gamma = \begin{bmatrix} I & 0 \\ -B_{01} & I \end{bmatrix}$$

Hence, it is sufficient to prove Lemma 3.7 for the new reduced-order system characterized by

$$\left(A_{22}, [B_{02}, B_{12}], \begin{bmatrix} C_{02} \\ A_{12} - B_{01}C_{02} \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & B_{11} \end{bmatrix} \right)$$

From (D 3) to (D 6), we obtain

$$\begin{aligned} A_{22} - B_{02}C_{02} &= \begin{bmatrix} A_a & 0 & 0 & 0 \\ 0 & A_{b22} & 0 & 0 \\ B_c E_{ca} & 0 & A_{cc} & 0 \\ B_{f2} E_{fa2} & B_{f2} E_{b22} & B_{f2} E_{c2} & A_{f22} \end{bmatrix} \\ [B_{02} \ B_{12}] &= \begin{bmatrix} B_{a0} & 0 & 0 & 0 \\ B_{b02} & 0 & 0 & 0 \\ B_{c0} & 0 & 0 & B_c \\ B_{f02} & 0 & B_{f2} & 0 \end{bmatrix} \\ \begin{bmatrix} C_{02} \\ A_{12} - B_{01}C_{02} \end{bmatrix} &= \begin{bmatrix} C_{0a} & C_{0b2} & C_{0c} & C_{0f2} \\ E_{a1} & E_{b12} & E_{c1} & A_{f12} \\ 0 & 0 & 0 & A_{f02} \\ 0 & A_{b12} & 0 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{bmatrix} I & 0 \\ 0 & B_{11} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is interesting to observe from the above that

$$\left(A_{22}, [B_{02}, B_{12}], \begin{bmatrix} C_{02} \\ A_{12} - B_{01}C_{02} \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & B_{11} \end{bmatrix} \right)$$

is already in the form of special coordinate basis. Then all the properties listed in Lemma 3.7 follows trivially from the properties of the special coordinate basis. \square

Appendix E

An algorithm for suboptimal design of static-state feedbacks

In this appendix we will describe the algorithm which is instrumental in the results derived in § 5. The algorithm is generally applicable, but because here we do not have the freedom of assigning the time-scales of eigenvalues that go off to infinity, the alternative mentioned briefly in § 5 might be preferable. However, that algorithm is *not* generally applicable.

A well-known approach to treat the H_2 control problem with state feedback in case the direct feedthrough matrix from u to z is not injective is adding in the output z a small extra weight on the input u (in case the system is strictly proper this results in the so-called cheap control problem). For the cheap control problem the following result can be derived.

Lemma E.1: *Consider the system (2.1) with $C_1 = I$, $D_1 = 0$ and $D_2 = 0$. Define Σ_{ci} to be the system $(A, B, C_2, 0)$. For each $\varepsilon > 0$ there exists $P_\varepsilon \geq 0$ such that*

$$A^T P_\varepsilon + P_\varepsilon A - \varepsilon^{-2} P_\varepsilon B B^T P_\varepsilon + C_2^T C_2 = 0$$

If we apply the feedback $u = -\varepsilon^{-2} B B^T P_\varepsilon$ then the closed-loop system is asymptotically stable and as $\varepsilon \downarrow 0$ the H_2 norm of the closed-loop transfer matrix decreases to γ^ .*

Moreover, the closed-loop eigenvalues have the following asymptotic behaviour.

- (1) $n_a^-(\Sigma_{ci})$ eigenvalues converge to the locations of the stable invariant zeros of Σ_{ci} .
- (2) $n_a^+(\Sigma_{ci})$ eigenvalues converge to the locations of the mirror images with respect to the imaginary axis of the invariant zeros of Σ_{ci} in the open right half-plane.
- (3) $n_a^0(\Sigma_{ci})$ eigenvalues converge to the invariant zeros on the imaginary axis.
- (4) $n_b(\Sigma_{ci}) + n_c(\Sigma_{ci})$ eigenvalues converge to fixed locations in \mathbb{C}^- which include the stable input decoupling zeros and output decoupling zeros which are not invariant zeros of Σ_{ci} .
- (5) $n_f(\Sigma_{ci})$ eigenvalues converge to infinity in the open left half-plane and remain bounded away from the imaginary axis.

Here $n_a^-(\Sigma_{ci})$, $n_a^+(\Sigma_{ci})$, $n_a^0(\Sigma_{ci})$, $n_b(\Sigma_{ci})$, $n_c(\Sigma_{ci})$ and $n_f(\Sigma_{ci})$ are the constants n_a^- , n_a^+ , n_a^0 , n_b , n_c and n_f as defined in Appendix A when we transform Σ_{ci} to the special coordinate basis.

Proof: This result is given in Saberi and Sannuti (1987) for the case that Σ_{ci} has no invariant zeros on the imaginary axis. The result in that paper can be extended to yield the above result. Unfortunately, a formal presentation is too extensive to include in this paper. \square

The question is whether we can adapt the above lemma, to derive a general perturbation approach which yields a sequence of stabilizing compensators for which the H_2 norm of the closed-loop transfer matrix approaches γ^* . Moreover, we would like to know the asymptotic behaviour of the closed-loop eigenvalues.

It should be noted that we can apply this kind of perturbation approach both to Σ as well as to Σ_{PQ} . Both systems have the same invariant zeros on the imaginary axis and therefore this nasty behaviour arises in both cases.

We can now construct the set F_{op}^{seq} and prove its properties as given in Lemma 4.1.

Let $\Sigma_{ci} = (A, B, C, D)$. Define the special coordinate basis of Appendix A for the system Σ_{ci} as given in (A 1)–(A 4). After a suitable preliminary feedback the subspace \mathcal{X}_c becomes unobservable. We can then remove this unobservable part and obtain the system $\tilde{\Sigma}_{ci} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ where

$$\tilde{A} := \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ab}^- C_b & L_{af}^- C_f \\ 0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & L_{af}^0 C_f \\ 0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & L_{af}^+ C_f \\ 0 & 0 & 0 & A_{bb} & L_{bf} C_f \\ B_f E_{fa}^- & B_f E_{fa}^0 & B_f E_{fa}^+ & B_f E_{fb} & A_f \end{bmatrix} \quad (E 1)$$

$$\tilde{B} := [\tilde{B}_0 \quad \tilde{B}_1] := \begin{bmatrix} B_{a0}^- & 0 \\ B_{a0}^0 & 0 \\ B_{a0}^+ & 0 \\ B_{b0} & 0 \\ B_{f0} & B_f \end{bmatrix} \quad (E 2)$$

$$\tilde{C} := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & 0 & C_b & 0 \end{bmatrix} \quad (E 3)$$

and

$$\tilde{D} := \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (E 4)$$

$\tilde{\Sigma}_{ci}$ has the same finite and infinite zero structure as Σ_{ci} and it can be shown that $n_a^-(\tilde{\Sigma}_{ci}) = n_a^-(\Sigma_{ci})$, $n_a^0(\tilde{\Sigma}_{ci}) = n_a^0(\Sigma_{ci})$, $n_a^+(\tilde{\Sigma}_{ci}) = n_a^+(\Sigma_{ci})$, $n_b(\tilde{\Sigma}_{ci}) = n_b(\Sigma_{ci})$, $n_c(\tilde{\Sigma}_{ci}) = 0$ and $n_f(\tilde{\Sigma}_{ci}) = n_f(\Sigma_{ci})$.

For all $\epsilon > 0$ let $P_\epsilon \geq 0$ be the solution of the following Riccati equation

$$\tilde{A}^T P_\epsilon + P_\epsilon \tilde{A} - P_\epsilon \tilde{B} (\tilde{D}^T \tilde{D} + \epsilon^2 I)^{-1} \tilde{B}^T P + \tilde{C}^T \tilde{C} + \epsilon^2 I = 0$$

Write P_ε compatible with the decomposition of the state space for $\tilde{\Sigma}_{cl}$

$$P_\varepsilon := \begin{bmatrix} P_{11,\varepsilon} & P_{12,\varepsilon} & P_{13,\varepsilon} & P_{14,\varepsilon} & P_{15,\varepsilon} \\ P_{21,\varepsilon} & P_{22,\varepsilon} & P_{23,\varepsilon} & P_{24,\varepsilon} & P_{25,\varepsilon} \\ P_{31,\varepsilon} & P_{32,\varepsilon} & P_{33,\varepsilon} & P_{34,\varepsilon} & P_{35,\varepsilon} \\ P_{41,\varepsilon} & P_{42,\varepsilon} & P_{43,\varepsilon} & P_{44,\varepsilon} & P_{45,\varepsilon} \\ P_{51,\varepsilon} & P_{52,\varepsilon} & P_{53,\varepsilon} & P_{54,\varepsilon} & P_{55,\varepsilon} \end{bmatrix}$$

Moreover, choose for every $\varepsilon > 0$ a matrix $F_c(\varepsilon)$ such that the eigenvalues of $A_{cc} - B_c F_c(\varepsilon)$ are at some desired locations in the open left half-plane (remember that by the properties of the special coordinate basis the pair (A_{cc}, B_c) is controllable). Let us partition

$$(\tilde{D}^T \tilde{D} + \varepsilon^2 I)^{-1} \tilde{B}^T P_\varepsilon = \begin{bmatrix} F_{a0}^-(\varepsilon) & F_{a0}^0(\varepsilon) & F_{a0}^+(\varepsilon) & F_{b0}(\varepsilon) & F_{f0}(\varepsilon) \\ F_{a1}^-(\varepsilon) & F_{a1}^0(\varepsilon) & F_{a1}^+(\varepsilon) & F_{b1}(\varepsilon) & F_{f1}(\varepsilon) \end{bmatrix}$$

We then define

$$F(\varepsilon) := \begin{bmatrix} F_{a0}^-(\varepsilon) & 0 & F_{a0}^0(\varepsilon) & F_{a0}^+(\varepsilon) & F_{b0}(\varepsilon) & F_{f0}(\varepsilon) \\ F_{a1}^-(\varepsilon) & 0 & F_{a1}^0(\varepsilon) & F_{a1}^+(\varepsilon) & F_{b1}(\varepsilon) & F_{f1}(\varepsilon) \\ \star & 0 & \star & \star & \star & \star \end{bmatrix} + \begin{bmatrix} C_{0a}^- & C_{0c} & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0f} \\ 0 & E_{fc} & 0 & 0 & 0 & 0 \\ 0 & F_c(\varepsilon) & 0 & 0 & 0 & 0 \end{bmatrix}$$

where \star are arbitrary matrices with appropriate dimensions. Then it is straightforward to show that if we apply the feedback $u = -F(\varepsilon)x$ to our system Σ then the closed-loop system is asymptotically stable and the closed-loop eigenvalues are the eigenvalues of

$$\tilde{A}_{cl} := \tilde{A} - \tilde{B}(\tilde{D}^T \tilde{D} + \varepsilon^2 I)^{-1} \tilde{B}^T P_\varepsilon$$

and $A_{cc} - B_c F_c(\varepsilon)$. Moreover, as $\varepsilon \downarrow 0$ the H_2 norm of the closed-loop transfer matrix decreases to γ^* . The latter can be shown by using that P_ε converges to

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{P}_{11} & \tilde{P}_{21}^T & 0 \\ 0 & 0 & \tilde{P}_{21} & \tilde{P}_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By our choice of $F_c(\varepsilon)$ we can indeed place $n_c(\Sigma_{cl})$ poles arbitrarily. It remains to show the asymptotic behaviour of the eigenvalues of \tilde{A}_{cl} . We will use Lemma E.1. It is well known that eigenvalues of \tilde{A}_{cl} are the stable eigenvalues of the following Hamiltonian matrix

$$H = \begin{bmatrix} \tilde{A} & -(1 + \varepsilon^2)^{-1} \tilde{B}_0 \tilde{B}_0^T - \varepsilon^{-2} \tilde{B}_1 \tilde{B}_1^T \\ -\tilde{C}^T \tilde{C} - \varepsilon^2 I & -\tilde{A}^T \end{bmatrix}$$

Let $K_\varepsilon \geq 0$ be the solution of the following Riccati equation

$$\tilde{A} K_\varepsilon + K_\varepsilon \tilde{A}^T - K_\varepsilon (\tilde{C}^T \tilde{C} + \varepsilon^2) K_\varepsilon + (1 + \varepsilon^2)^{-1} \tilde{B}_1 \tilde{B}_1^T = 0$$

Moreover define

$$\tilde{A}_\varepsilon := \tilde{A} - K_\varepsilon \tilde{C}^T \tilde{C} - \varepsilon^2 K_\varepsilon$$

We know that $K_\varepsilon \rightarrow K_0$ and $\tilde{A}_\varepsilon \rightarrow \tilde{A}_0$ as $\varepsilon \downarrow 0$. H has the same eigenvalues as

$$\begin{aligned} \tilde{H} &:= \begin{bmatrix} I & K_\varepsilon \\ 0 & I \end{bmatrix} H \begin{bmatrix} I & -K_\varepsilon \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A}_\varepsilon & -\varepsilon^{-2} \tilde{B}_1 \tilde{B}_1^T \\ -\tilde{C}^T \tilde{C} - \varepsilon^2 I & -\tilde{A}_\varepsilon^T \end{bmatrix} \end{aligned}$$

On the other hand, the asymptotic behaviour of the eigenvalues of \tilde{H} is the same as the asymptotic behaviour of the eigenvalues of

$$\begin{bmatrix} \tilde{A}_0 & -\varepsilon^{-2} \tilde{B}_1 \tilde{B}_1^T \\ -\tilde{C}^T \tilde{C} - \varepsilon^2 I & -\tilde{A}_0^T \end{bmatrix}$$

However, this is a Hamiltonian matrix corresponding to the Riccati equation given in Lemma E.1. Therefore, we know the asymptotic behaviour of the eigenvalues. The proof is completed by noting that the finite and infinite zeros of $(\tilde{A}_0, \tilde{B}_1, \tilde{C}, 0)$ are the same as the invariant zeros of (A, B, C_2, D_2) . \square

REFERENCES

- ANDERSON, B. D. O., and MOORE, J. B., 1989, *Optimal Control: Linear Quadratic Methods* (Englewood Cliffs, NJ: Prentice Hall).
- BERNSTEIN, D. S., and HADDAD, W. M., 1989 a, Steady state Kalman filtering with an H_∞ error bound. *Systems and Control Letters*, **12**, 9–16; 1989 b, LQG control with an H_∞ performance bound: a Riccati equation approach. *IEEE Transactions on Automatic Control*, **34**, 293–305.
- CHEN, B. M., SABERI, A., and SANNUTI, P., 1992, Explicit expressions for cascade factorization of general nonminimum phase systems. *IEEE Transactions on Automatic Control*, **37**, 358–363.
- DAVIS, M. H. A., 1977, *Linear Estimation and Stochastic Control* (New York: Chapman and Hall).
- FLEMING, W. H., and RISHEL, R. W., 1975, *Deterministic and Stochastic Optimal Control* (New York: Springer-Verlag).
- GEERTS, T. 1989, All optimal controls for the singular linear-quadratic problem without stability; a new interpretation of the optimal cost. *Linear Algebra and its Applications*, **122**, 65–104.
- KAILATH, T., 1974, A view of three decades of linear filtering theory. *IEEE Transactions on Information Theory*, **20**, 146–180.
- KWAKERNAAK, H., and SIVAN, R., 1972, *Linear Optimal Control Theory* (New York: Wiley).
- LIN, Z. L., SABERI, A., and CHEN, B. M., 1991, *Linear Systems Toolbox*. Washington State University Technical Report No: EE/CS 0097.
- MACFARLANE, A. G. J., and KARCANIAS, N., 1976, Poles and zeros of linear multivariable systems: a survey of the algebraic, geometric and complex variable theory. *International Journal of Control*, **24**, 33–74.
- MUSTAFA, D., and GLOVER, K., 1990, *Minimum entropy H_∞ Control*. Lecture Notes in Control and Information Sciences, Vol. 146 (New York: Springer-Verlag).
- OZCETIN, H. K., SABERI, A., and SANNUTI, P., 1990, Almost disturbance decoupling problem with internal stability via state or measurement feedback—singular perturbation approach. *Proceedings of the 29th IEEE Conference on Decision and Control*, Honolulu, Hawaii, pp. 1042–1047. Also as: 1992, *International Journal of Control*, **55**, 901–944.
- OZCETIN, H. K., SABERI, A., and SHAMASH, Y., 1991, H^∞ -almost disturbance decoupling for non-strictly proper systems—A singular perturbation approach. *Proceedings of the 30th IEEE Conference on Decision and Control*, Brighton, U.K.
- ROTEA M. A., and KHARGONEKAR, P. P. 1991, H_2 optimal control with an H_∞ constraint: the state feedback case. *Automatica*, **27**, 307–316.
- SABERI, A., and SANNUTI, P., 1987, Cheap and singular controls for linear quadratic

- regulators. *IEEE Transactions on Automatic Control*, **32**, 208–219; 1989, Time-scale structure assignment in linear multivariable systems using high-gain feedback. *International Journal of Control*, **49**, 2191–2213; 1990, Squaring down of non-strictly proper systems. *Ibid.*, **51**, 621–629.
- SANNUTI, P., and SABERI, A., 1987, A special coordinate basis of multivariable linear systems—Finite and infinite zero structure, squaring down and decoupling. *International Journal of Control*, **45**, 1655–1704.
- SCHUMACHER, J. M., 1985, A geometric approach to the singular filtering problem. *IEEE Transactions on Automatic Control*, **30**, 1075–1082.
- STOORVOGEL, A. A., 1990, The singular H_2 control problem. Memorandum COSOR 90–43, Eindhoven University of Technology; also as: 1992, *Automatica*, **28**, 627–631; 1992, *The H_∞ Control Problem: a State Space Approach* (Englewood Cliffs, NJ: Prentice Hall).
- STOORVOGEL, A. A., and VAN DER WOUDE, J. W., 1991, The disturbance decoupling problem with measurement feedback and stability for systems with direct feedthrough matrices. *Systems and Control Letters*, **17**, 217–226.
- STOORVOGEL, A. A., SABERI, A., and CHEN, B. M., 1991, A reduced order observer based controller design for H_2 -optimization. *Proceedings of AIAA Guidance, Navigation and Control Conference*, New Orleans, Louisiana, Vol. 2, pp. 716–722.
- TRENTELMAN, H. L., 1986, *Almost Invariant Subspaces and High Gain Feedback*, Vol. 29 (Amsterdam: CWI Tracts).
- WEILAND, S., and WILLEMS, J. C., 1989, Almost disturbance decoupling with internal stability. *IEEE Transactions on Automatic Control*, **34**, 277–286.
- WILLEMS, J. C., 1978, Recursive filtering. *Statistica Neerlandica*, **32**, 1–39; 1981, Almost invariant subspaces: an approach to high gain feedback design—Part I: almost controlled invariant subspaces. *IEEE Transactions on Automatic Control*, **26**, 235–252; 1982, Almost invariant subspaces: an approach to high gain feedback design—Part II: almost conditionally invariant subspaces. *Ibid.* **27**, 1071–1084.
- WILLEMS, J. C., KITAPCI, A., and SILVERMAN, L. M., 1986, Singular optimal control: a geometric approach. *SIAM Journal on Control and Optimization*, **24**, 323–337.
- WONHAM, W. M., 1985, *Linear Multivariable Control: a Geometric Approach*, third edition (New York: Springer-Verlag).