CONTROL-THEORY AND ADVANCED TECHNOLOGY
Vol. 9, No. 2, pp.565–576, June, 1993

C92084sR
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CTAT *Short Paper*

CHARACTERIZATION OF ALL CLOSED-LOOP TRANSFER FUNCTION MATRICES IN H_∞-OPTIMIZATION*

A. A. STOORVOGEL,[1] A. SABERI[2] AND B. M. CHEN[3]

Abstract. In this paper, we derive a characterization of all stable closed loop systems with H_∞-norm strictly less than 1 which we can obtain via a suitable stabilizing feedback. We give an exact characterization. However, this characterization contains relatively implicit constraints on the free parameter. We also introduce an "approximate" characterization parameterized via a stable system X with H_∞-norm less than 1 (and no other conditions on X). A element of this approximate characterization can be arbitrarily well approximated by a closed loop system we can obtain via a suitable stabilizing feedback.

Key Words—H_∞-optimization, robust control, disturbance decoupling.

1. Introduction

In H_∞ control (see e.g., Doyle et al., 1989; Stoorvogel, 1991; Tadmor, 1990) it is well-known that suitable controllers are not unique. This is in part because we in general investigate suboptimal design (make the H_∞ norm less than some *a priori* given number γ) and also because even optimal controllers are in the MIMO case non-unique.

An interesting question one might therefore ask is the following: characterize all closed loop systems with H_∞ norm less than γ that we can obtain via a suitable stabilizing feedback. In several papers (see e.g., Doyle et al., 1989; Tadmor, 1990) a characterization is given of all time-invariant controllers which stabilize a given linear time-invariant system and result in a closed loop system with H_∞ norm strictly less than γ. This can be used in a straightforward manner to characterize the closed loop systems these controllers generate. However, this is done under some assumptions on the direct feedthrough matrices of the system (the so-called regular case). Without these conditions (the singular case) necessary and sufficient conditions for the existence of a suitable controller are available (see Stoorvogel, 1991). On the other hand, for this singular case relatively little is known about closed loop systems one can obtain.

* Received by the editors May 21, 1992 and in revised form December 25, 1992.
*1 This research has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences and this author is also supported by the Netherlands Organization for Scientific Research (NWO).
*2,3 This work is supported in part by Boeing Commercial Airplane Group and in part by NASA Langley Research Center under grant contract NAG-1-1210.

1 Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.
2 School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752, U.S.A.
3 Department of Electrical Engineering, State University of New York at Stony Brook, Stony Brook, NY 11794-2350, U.S.A.

In this paper, we derive a characterization of all stable closed loop systems with H_∞ norm strictly less than 1 which we can obtain via applying a suitable stabilizing feedback to the given system Σ. We can replace 1 by γ via simple scaling. The closed-loop systems are parameterized via a stable system X with H_∞-norm less than 1. However, these systems X have to satisfy two other, relatively implicit, extra conditions. Therefore, we also give an approximate characterization. It is the same characterization except that X does not have to satisfy these extra two conditions; the system X is an arbitrary stable system with H_∞ norm strictly less than 1. The trade-off is that it is an approximate characterization. For each stable system X with H_∞ norm less than 1 we generate a system which can be arbitrarily well approximated with a closed loop system which we obtain by applying a suitable stabilizing controller to our system Σ. Conversely any closed loop system with H_∞ norm strictly less than 1, which we can obtain by applying a stabilizing controller to Σ, is identical to a system we obtain for a suitable choice of the parameter X in our characterization.

In other words, we find a simpler characterization of the "closure" (the approximate set is not actually closed but lies between the set itself and its closure) of the set of attainable closed loop systems. Finally, we would like to note that this approximate characterization and the actual characterization are equal in the regular case.

The formal problem statement will be given in the next section. In Sec. 3, we will recall some preliminary results. In Sec. 4, we will give an exact characterization of all closed loop systems. Finally, in Sec. 5, we give the much simpler approximate characterization. We conclude with some final remarks in Sec. 6.

2. Problem Statement

We consider the linear, time-invariant, finite-dimensional system

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = C_1 x + D_1 w, \\ z = C_2 x + D_2 u, \end{cases} \tag{2.1}$$

where $x \in \mathscr{R}^n$ is the state, $u \in \mathscr{R}^m$ is the control input, $w \in \mathscr{R}^l$ is the unknown disturbance, $y(t) \in \mathscr{R}^p$ is the measured output and $z \in \mathscr{R}^q$ is the unknown output to be controlled. A, B, E, C_1, C_2, D_1 and D_2 are matrices of appropriate dimensions. The following assumptions are made:

(a) (A, B) is stabilizable and (A, B, C_2, D_2) has no invariant zeros on the $j\omega$-axis, and

(b) (A, C_1) is detectable and (A, E, C_1, D_1) has no invariant zeros on the $j\omega$-axis.

Throughout this paper, we will assume that there exists an internally stabilizing controller of the form

$$\Sigma_F: \begin{cases} \dot{v} = Kv + Ly, \\ u = Mv + Ny, \end{cases} \tag{2.2}$$

such that the H_∞-norm of the closed-loop transfer function from z to w, $T_{zw}(s)$, is strictly less than 1. To be precise, let us define the sets

$$\boldsymbol{C} \trianglelefteq \{\Sigma_F \mid \Sigma \times \Sigma_F \quad \text{is stable and} \quad \| T_{zw} \|_\infty < 1\} \tag{2.3}$$

and

$$\boldsymbol{T} \trianglelefteq \{T_{zw} \mid \Sigma_F \in \boldsymbol{C}\}. \tag{2.4}$$

Elements of the sets $\boldsymbol{C}$ and $\boldsymbol{T}$ will sometimes be called suitable controllers and suitable closed loop systems, respectively. The goal of this paper is to characterize the set $\boldsymbol{T}$, i.e., all the closed-loop transfer function $T_{zw}(s)$ satisfying $\| T_{zw} \|_\infty < 1$.

3. Preliminary

In this section, we recall some results from Stoorvogel (1991). A central role in our study of the above problem will be played by the quadratic matrix inequality. For matrix $P \in \mathscr{R}^{n \times n}$ we consider the following matrix:

$$F(P) \trianglelefteq \begin{bmatrix} A^TP + PA + C_2^T C_2 + PEE^TP & PB + C_2^T D_2 \\ B^TP + D_2^T C_2 & D_2^T D_2 \end{bmatrix}.$$

If $F(P) \geq 0$, we say that P is a solution of the quadratic matrix inequality.

We also define a dual version of this quadratic matrix inequality. For $Q \in \mathscr{R}^{n \times n}$ we define the following matrix:

$$G(Q) \trianglelefteq \begin{bmatrix} AQ + QA^T + EE^T + QC_2^T C_2 Q & QC_1^T + ED_1^T \\ C_1 Q + D_1 E^T & D_1 D_1^T \end{bmatrix}.$$

If $G(Q) \geq 0$, we say that Q is a solution of the dual quadratic matrix inequality. In addition to these two matrices, we define two matrices pencils, which play dual roles

$$L(P, s) \trianglelefteq [sI - A - EE^TP \quad - B],$$

$$M(Q, s) \trianglelefteq \begin{bmatrix} sI - A - QC_2^T C_2 \\ - C_1 \end{bmatrix}.$$

Finally, we define the following two transfer matrices:

$$G_{ci}(s) \trianglelefteq C_2(sI - A)^{-1}B + D_2,$$

$$G_{di}(s) \trianglelefteq C_1(sI - A)^{-1}E + D_1.$$

Let $\varrho(M)$ denote the spectral radius of the matrix M. Then the following theorem characterizes the existence of suitable controllers.

Theorem 3.1. Consider the system (2.1). Assume that both the subsystem (A, B, C_2, D_2) as well as the subsystem (A, E, C_1, D_1) have no invariant zeros on the imaginary axis. Then, the following two statements are equivalent:

1. For the system (2.1) there exists a time-invariant, finite-dimensional dynamic compensator Σ_F of the form (2.2), such that the resulting closed-loop system, with transfer matrix $T_{zw}(s)$, is internally stable and has H_∞ norm less than 1, i.e., $\| T_{zw} \|_\infty < 1$.
2. There exist positive semi-definite solutions P, Q of the quadratic matrix inequalities $F_\gamma(P) \geq 0$ and $G(Q) \geq 0$ satisfying $\varrho(PQ) < 1$, such that the following rank conditions are satisfied:
   (a) $\operatorname{rank} F(P) = \operatorname{rank}_{R(s)} G_{ci}$,
   (b) $\operatorname{rank} G(Q) = \operatorname{rank}_{R(s)} G_{di}$,
   (c) $\operatorname{rank}\begin{bmatrix} L(P, s) \\ F(P) \end{bmatrix} = n + \operatorname{rank}_{R(s)} G_{ci}, \quad \forall s \in C^0 \cup C^+$,
   (d) $\operatorname{rank}[M(Q, s) \quad G(Q)] = n + \operatorname{rank}_{R(s)} G_{di}, \quad \forall s \in C^0 \cup C^+$.

Our goal is to characterize the set of all closed loop systems with H_∞ norm less than 1 which we can obtain by applying a suitable stabilizing controller. By the above theorem, this set is empty if the conditions in part 2 are not met. Therefore, in the remainder of this paper we will assume that there exist matrices P and Q satisfying the conditions in part 2 of the above theorem. We can now start with the derivation of the characterization of all suitable closed loop systems.

Next, we construct a new system,

$$\Sigma_{P,Q}: \begin{cases} \dot{x}_{P,Q} = A_{P,Q} x_{P,Q} + B_{P,Q} u_{P,Q} + E_{P,Q} w, \\ y_{P,Q} = C_{1,P} x_{P,Q} + D_{P,Q} w, \\ z_{P,Q} = C_{2,P} x_{P,Q} + D_P u_{P,Q}, \end{cases} \tag{3.1}$$

where

$$F(P) = \begin{bmatrix} C_{2,P}^T \\ D_P^T \end{bmatrix} [C_{2,P} \quad D_P], \quad G(Q) = \begin{bmatrix} E_Q \\ D_{P,Q} \end{bmatrix} [E_Q^T \quad D_{P,Q}^T],$$

such that $[C_{2,P} \quad D_P]$ and $[E_Q^T \quad D_{P,Q}^T]$ are both surjective. Moreover,

$$A_{P,Q} \triangleq A + EE^T P + (I - QP)^{-1} Q C_{2,P}^T C_{2,P},$$

$$B_{P,Q} \triangleq B + (I - QP)^{-1} Q C_{2,P}^T D_P,$$

$$E_{P,Q} \triangleq (I - QP)^{-1} E_Q,$$

$$C_{1,P} \triangleq C_1 + D_1 E^T P.$$

It has been shown in Stoorvogel (1991) that this new system has the following properties:
1. $(A_{P,Q}, B_{P,Q}, C_{2,P}, D_P)$ is right invertible and minimum phase.
2. $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$ is left invertible and minimum phase.

In Stoorvogel (1991), the transformation to $\Sigma_{P,Q}$ is done in two stages. In the first stage (the transformation into a system Σ_P), a system Σ_U is constructed which connects Σ and Σ_P, i.e., the following systems have the same realization except for some extra stable uncontrollable dynamics on the right hand side (Fig. 1). Here the system Σ_P is given by

$$\Sigma_P : \begin{cases} \dot{x}_P = (A + EE^TP)x_P + Bu_P + Ew_P, \\ y_P = (C_1 + D_1 E^TP)x_P + D_1 w_P, \\ z_P = C_{2,P}x_P + D_P u_P, \end{cases} \tag{3.2}$$

and Σ_U, given in Appendix, is due to its complexity. Moreover, it is shown in Stoorvogel (1991) that Σ_U is inner, i.e., the system is stable and the transfer function of Σ_U from (w_U, u_U) to (z_U, y_U), say G_U, has the following property:

$$G_U^T(-s_0)G_U(s_0) = G_U(s_0)G_U^T(-s_0) = I,$$

for any $s_0 \in C$ which is not a pole of the system $G_U(s)$. Finally, the subsystem from w_U to y_U has a stable inverse. Similarly, we can connect Σ_P and $\Sigma_{P,Q}$ via some system Σ_V, which can be defined using a dual argument of Σ_U.

In this way, we can derive that the original system Σ in (2.1), and the new system $\Sigma_{P,Q}$ have a similar connection. In other words, we can construct a system Σ_C (which is simply the interconnection of Σ_U and Σ_V), such that the following two interconnections have the same realization for every controller Σ_F except for some extra stable uncontrollable or unobservable dynamics on the right hand side (Fig. 2).

Due to the properties of Σ_U and Σ_V, it can be easily shown that Σ_C is inner. Moreover, by applying Redheffer's lemma (see Doyle et al., 1989) and its dual version we can derive the following theorem.

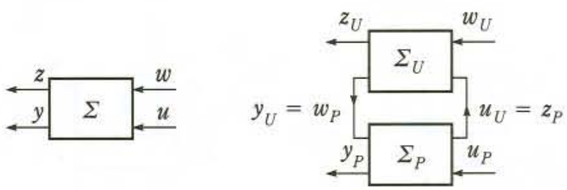

Fig. 1.

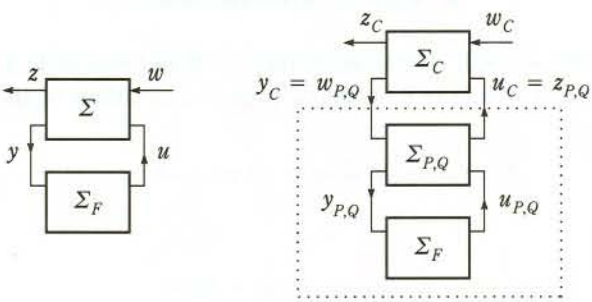

Fig. 2.

Theorem 3.2. For any given compensator Σ_F of the form (2.2) the following two statements are equivalent:

(i) Σ_F applied to the system Σ defined by (2.1) is internally stabilizing and the resulting closed loop transfer function from w to z has H_∞-norm less than 1, i.e., $\| T_{zw} \|_\infty < 1$.

(ii) Σ_F applied to the new system $\Sigma_{P,Q}$ defined by (3.1) is internally stabilizing and the resulting closed loop transfer function from $w_{P,Q}$ to $z_{P,Q}$, $T_{z_{P,Q} w_{P,Q}}(s)$, has H_∞-norm less than 1, i.e., $\| T_{z_{P,Q} w_{P,Q}} \|_\infty < 1$.

Next, we denote the system inside the dashed box of Fig. 2 by $X(s)$. We can then simplify the picture (Fig. 3).

Our goal of this paper is to characterize all suitable closed-loop systems $\Sigma \times \Sigma_F$ as in the left of Fig. 3, i.e., the set $\boldsymbol{T}$ as defined in the previous section. By the previous theorem if the closed loop system on the left in Fig. 3 is stable and has H_∞-norm strictly less than 1 then X, defined to be equal to the dashed box in Fig. 2, is stable and has H_∞-norm strictly less than 1. Our goal is to show the "converse": for any stable system X with H_∞-norm strictly less than 1 the interconnection on the right hand side of Fig. 3 is asymptotically stable and has H_∞-norm strictly less than 1. Moreover, we can find a system Σ_F, such that the two interconnections in Fig. 3 are both stable and arbitrarily close in H_∞-norm. In the next section, we will show for which systems X we can make the interconnections equal. In Sec. 5, we show that for all strictly proper X which are stable and have H_∞-norm strictly less than 1 we can always make the interconnections arbitrarily close in H_∞-norm.

We would like to conclude this section by stressing that the construction of Σ_C is an straightforward application of the results in Stoorvogel (1991). It is only because of space limitations that we do not give this explicit construction in this paper.

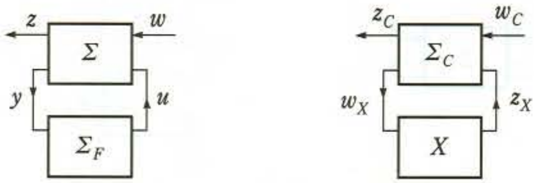

Fig. 3.

4. Exact Characterization

In this section, we will characterize the set $\boldsymbol{T}$ defined in (2.4). We first give the following result which is a straightforward application of the results in the previous section.

Lemma 4.1. Let X be a stable system described by

$$X: \begin{cases} \dot{x}_x = A_x x_x + B_x w_x, \\ z_x = C_x x_x + D_x w_x, \end{cases} \tag{4.1}$$

where A_x is stable and the transfer matrix of X has H_∞-norm strictly less than

1. Then the interconnection $\Sigma_C \times \Sigma_X$ as given on the right hand side in Fig. 3 is internally stable and the resulting closed-loop transfer function from w_C to z_C has H_∞-norm less than 1.

For a system X satisfying the conditions of the above lemma we define the following auxiliary system:

$$\Sigma_a : \begin{cases} \dot{x}_a = \begin{bmatrix} A_x & 0 \\ 0 & A_{P,Q} \end{bmatrix} x_a + \begin{bmatrix} 0 \\ B_{P,Q} \end{bmatrix} u + \begin{bmatrix} B_x \\ E_{P,Q} \end{bmatrix} w, \\ y = [0 \quad C_{1,P}] x_a + D_{P,Q} w, \\ z = [C_x \quad - C_{2,P}] x_a - D_P u + D_x w. \end{cases} \tag{4.2}$$

For economy of notation, let us define

$$\tilde{A} = \begin{bmatrix} A_x & 0 \\ 0 & A_{P,Q} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ B_{P,Q} \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} B_x \\ E_{P,Q} \end{bmatrix}$$

and

$$\tilde{C}_1 = [0 \quad C_{1,P}], \quad \tilde{D}_1 = D_{P,Q}, \quad \tilde{C}_2 = [C_x \quad - C_{2,P}], \quad \tilde{D}_2 = - D_P.$$

In order to proceed we need a number of definitions.

Definition 4.1. Let $\Sigma = (A, B, C, D)$. By $T_g(\Sigma)$, we denote the smallest subspace T of $\mathscr{R}^n$ for which there exists a linear mapping K, such that the following conditions are satisfied:

$$(A - KC)T \subseteq T, \quad \operatorname{Im}(B - KD) \subseteq T,$$

and such that $A - KC | \mathscr{R}^n/T$ is asymptotically stable. Similarly, by $V_g(\Sigma)$ we denote the largest subspace V for which there exists a mapping F, such that the following conditions are satisfied:

$$(A - BF)V \subseteq V, \quad (C - DF)V = \{0\},$$

and such that $A - BF|V$ is asymptotically stable.

Definition 4.2. Let $\boldsymbol{X}_e$ denote the set of systems X satisfying the conditions of Lemma 4.1, such that the corresponding auxiliary system Σ_a satisfies the following conditions:
1. $T_g(\tilde{\Sigma}_{di}) \subseteq V_g(\tilde{\Sigma}_{ci})$,
2. There exists a matrix $\tilde{N}$, such that

$$\left(\begin{bmatrix} \tilde{A} & \tilde{E} \\ \tilde{C}_2 & D_x \end{bmatrix} + \begin{bmatrix} \tilde{B} \\ \tilde{D}_2 \end{bmatrix} \tilde{N} [\tilde{C}_1 \quad \tilde{D}_1] \right) (T_g(\tilde{\Sigma}_{di}) \oplus \mathscr{R}^q) \subseteq (V_g(\tilde{\Sigma}_{ci}) \oplus \{0\}). \tag{4.3}$$

Here $\tilde{\Sigma}_{ci} \trianglelefteq (\tilde{A}, \tilde{B}, \tilde{C}_2, \tilde{D}_2)$ and $\tilde{\Sigma}_{di} \trianglelefteq (\tilde{A}, \tilde{E}, \tilde{C}_1, \tilde{D}_1)$.

Next, we note that since the matrix A_x is asymptotically stable and because of the properties of $\Sigma_{P,Q}$ as given in the previous section, it is simple to verify that

1. $(\tilde{A}, \tilde{B}, \tilde{C}_2, \tilde{D}_2)$ is right invertible and minimum phase, and
2. $(\tilde{A}, \tilde{E}, \tilde{C}_1, \tilde{D}_1)$ is left invertible and minimum phase.

This immediately implies that $(\tilde{A}, \tilde{B})$ is stabilizable and $(\tilde{A}, \tilde{C}_1)$ is detectable. Moreover, these conditions combined with the Conditions 1 and 2 given in Definition 4.2 guarantee (see Stoorvogel and van der Woude, 1991) that for the system Σ_a the disturbance decoupling with measurement feedback and internal stability is solvable. In other words, there exists a compensator Σ_F of the form (2.2), such that the interconnection $\Sigma_a \times \Sigma_F$ is internally stable and its transfer matrix is equal to 0.

Therefore, if we define the set

$$\boldsymbol{T}_e \triangleq \{T_{z_e w_e} \mid X \in \boldsymbol{X}_e\}, \tag{4.4}$$

then we have the following result.

Theorem 4.1. The set $\boldsymbol{T}$ defined by (2.4) and the set $\boldsymbol{T}_e$ defined by (4.4) are equal, i.e., $\boldsymbol{T} = \boldsymbol{T}_e$.

Proof. ($\Rightarrow$): Let $T_{zw}(s) \in \boldsymbol{T}$. Hence, by definition, there exists a controller $\Sigma_F \in \boldsymbol{C}$, which makes the closed-loop system on the left of Fig. 2 internally stable. It then follows from Theorem 3.2 that such a controller makes the closed-loop system inside the dashed box on the right of Fig. 2 internally stable and yields $\| T_{z_{P,Q} w_{P,Q}} \|_\infty < 1$. Next, define X to be equal to the dashed box of Fig. 2. It is trivial to see that $\| X \|_\infty = \| T_{z_{P,Q} w_{P,Q}} \|_\infty < 1$ and that this system Σ_F solves the disturbance decoupling problem with measurement feedback and internal stability for the related auxiliary system Σ_a. It then follows from Stoorvogel and van der Woude (1991) that X must be, such that the corresponding auxiliary system Σ_a satisfies the two conditions in Definition 4.2. Hence, $X \in \boldsymbol{X}_e$ and $T_{z_C w_C} = T_{zw} \in \boldsymbol{T}_e$.

($\Leftarrow$): Conversely, assume that $T_{z_C w_C} \in \boldsymbol{T}_e$. By definition, there exists a system X of the form (4.1), such that $\| X \|_\infty < 1$ and the conditions in Definition 4.2 hold. Hence, by Stoorvogel and van der Woude (1991), the disturbance decoupling problem with measurement feedback and internal stability is solvable for the corresponding auxiliary system Σ_a. Hence, there exists a stabilizing controller Σ_F, such that the resulting system inside the dashed box of Fig. 2 is equal to X. By Theorem 3.2, we have $\Sigma_F \in \boldsymbol{C}$. Moreover, the corresponding $T_{zw} = T_{z_C w_C} \in \boldsymbol{T}$.

5. 'Almost' Characterization

It turns out that it is easier to define a bigger set $\boldsymbol{T}_a$ which contains set $\boldsymbol{T}$ and transfer matrices that are not in $\boldsymbol{T}$ are arbitrarily close to the set $\boldsymbol{T}$. To be more precise, for each element of $\boldsymbol{T}_a$ and for any positive scalar, say ε, there exists an element of $\boldsymbol{T}$, such that the difference between these two transfer matrices has H_∞-norm less than ε. Next, we will give a precise definition of the set $\boldsymbol{T}_a$.

Definition 5.1. Let $\boldsymbol{X}_a$ denote the set of systems X of the form (4.1) where A_x is asymptotically stable, $\| X \|_\infty < 1$ and $D_x = D_2 N D_1$ for some constant matrix N.

Moreover, define the set $\boldsymbol{T}_a$ by,

$$\boldsymbol{T}_a \triangleq \{T_{z_C w_C} | X \in \boldsymbol{X}_a\}. \tag{5.1}$$

Now we can derive the following theorem.

Theorem 5.1. The set $\boldsymbol{T}$ of (2.4) and the set $\boldsymbol{T}_a$ of (5.1) has the following relationship:

$$\boldsymbol{T} \subseteq \boldsymbol{T}_a \subseteq \overline{\boldsymbol{T}}, \tag{5.2}$$

where the closure of the set $\boldsymbol{T}$ is with respect to the topology induced by the H_∞-norm. In other words, for any $\varepsilon > 0$ and for any $T_{z_C w_C} \in \boldsymbol{T}_a$, there exists a $T_{zw} \in \boldsymbol{T}$, such that $\| T_{zw} - T_{z_C w_C} \|_\infty < \varepsilon$.

Proof.

(Part 1) It is trivial to verify that $\boldsymbol{X}_e \subseteq \boldsymbol{X}_a$. Hence, by definition, $\boldsymbol{T} = \boldsymbol{T}_e \subseteq \boldsymbol{T}_a$.

(Part 2) For any $T_{z_C w_C} \in \boldsymbol{T}_a$, again by definition, there exists a system X of the form (4.1) with A_x stable, $\| X \|_\infty < 1$ and, moreover, there exists a matrix N, such that $D_x = D_2 N D_1$. Since the range of D_2 and D_P are equal and since the kernel of $D_{P,Q}$ and D_1 are equal there exists a matrix $\tilde{N}$, such that

$$D_x = D_2 N D_1 = D_P \tilde{N} D_{P,Q} = \tilde{D}_2(-\tilde{N})\tilde{D}_1. \tag{5.3}$$

Finally, recall the following properties:
1. $(\tilde{A}, \tilde{B}, \tilde{C}_2, \tilde{D}_2)$ is right invertible and of minimum phase, and
2. $(\tilde{A}, \tilde{E}, \tilde{C}_1, \tilde{D}_1)$ is left invertible and of minimum phase.

It follows from Özcetin et al. (1991; 1992) that the H_∞-almost disturbance decoupling problem with internal stability for the corresponding auxiliary system Σ_a is solvable. Hence, there exists Σ_F, such that the corresponding closed-loop system inside the dashed box of Fig. 2, which we denote by X_F, is internally stable and is arbitrarily close to X in H_∞-norm. Let $T_{zw}(s)$ and $T_{z_C w_C}(s)$ be the closed-loop transfer matrices of the systems on the left and right respectively in Fig. 2. It is straightforward since Σ_C is stable that by making the difference $X_F - X$ small enough that $T_{zw}(s)$ is also arbitrarily close to $T_{z_C w_C}(s)$ in H_∞-norm. More specifically, for any $\varepsilon > 0$, there exists Σ_F, such that the corresponding T_{zw} satisfies $\| T_{zw} - T_{z_C w_C} \|_\infty < \varepsilon$. Furthermore, $\| T_{zw} \|_\infty < 1$ and hence, $T_{zw} \in \boldsymbol{T}$. This completes the proof of the theorem.

6. Conclusion

In this paper, we have given a characterization of all stable closed loop systems we can obtain via a suitable feedback. The closed loop systems are parameterized via a stable system X with H_∞-norm less than 1. An exact characterization requires an extra constraint on X which is relatively difficult. However, if we are satisfied with an approximate characerization then the system X has to satisfy only one extra constraint which is very simple.

No explicit characterization of all suitable controllers is given. It is our belief that a simple characterization of all controllers as given in Doyle et al.

(1989) and Tadmor (1990) cannot be given in the singular case. This still remains an interesting open problem.

Construction of suitable controllers in the singular case can be done via solving an almost disturbance decoupling problem. Algorithms can e.g., be found in Ozcetin et al. (1991; 1992).

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Appendix: Construction of Σ_U

It is well-known that there exist orthogonal transformations U and V of appropriate dimensions (for example, using singular value decomposition technique), such that

$$UD_2V^T = \begin{bmatrix} \hat{D} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\hat{D}$ is invertible. Because these orthogonal transformations do not change the norm $\| z \|$, hereafter without loss of generality, we assume that D_2 is in the above form. Moreover, let us partition B and C_2 as

$$B = [B_1 \quad B_2] \quad \text{and} \quad C_2 = \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix}.$$

Let

$$F_0 = \begin{bmatrix} -\hat{D}^{-1}\hat{C}_1 \\ 0 \end{bmatrix}.$$

It is easy to see that

$$C_2 + D_2F_0 = \begin{bmatrix} 0 \\ \hat{C}_2 \end{bmatrix}.$$

We now choose a basis of the state space $\mathscr{R}^n$. Let $\mathscr{R}^n = X_1 \oplus X_2 \oplus X_3$ with $X_2 = T_g(\Sigma_{ci}) \cap \{v | C_2 v \in \mathrm{Im}(D_2)\}$, $X_2 \oplus X_3 = T_g(\Sigma_{ci})$ and X_3 arbitrary, where $\Sigma_{ci} \triangleq (A, B, C_2, D_2)$. It is shown in Stoorvogel (1991) that in this new coordinate,

$$A + BF_0 = \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix},$$

$$C_2 + D_2 F_0 = \begin{bmatrix} 0 & 0 & 0 \\ C_{21} & 0 & C_{23} \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then system Σ_U is given by

$$\Sigma_U : \begin{cases} \dot{x}_U = \bar{A} x_U + [\bar{B}_{11} \quad \bar{B}_{12}] u_U + E_1 w_U, \\ y_U = -E_1^T P_1 x_U + w_U, \\ z_U = \begin{pmatrix} \bar{C}_1 \\ \bar{C}_2 \end{pmatrix} x_U + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} u_U, \end{cases}$$

where

$$\bar{A} \triangleq A_{11} - A_{13}(C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{21}) - B_{11}(\hat{D}^T \hat{D})^{-1} B_{11}^T P_1,$$

$$\bar{C}_1 \triangleq -(\hat{D}^T)^{-1} B_{11}^T P_1,$$

$$\bar{C}_2 \triangleq C_{21} - C_{23}(C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{21}),$$

$$\bar{B}_{11} \triangleq B_{11} \hat{D}^{-1},$$

$$\hat{B}_{12} \triangleq A_{13}(C_{23}^T C_{23})^{-1} C_{23}^T - P_1^\dagger C_{21}^T [I - C_{23}(C_{23}^T C_{23})^{-1} C_{23}^T].$$

Here † denotes the Moore-Penrose inverse.

Anton A. Stoorvogel received the M.Sc. degree in Mathematics from the University of Leiden in 1987 and the Ph.D. degree from Eindhoven University of Technology in 1990.

He has been associated with Eindhoven University of Technology since 1987, currently as a researcher of the Royal Netherlands Academy of Sciences and Arts. In 1991 he visited the Dept. of Electrical Engineering of the University of Michigan for one year.

He is the author of a book entitled "*The H_∞ control problem: a state space approach*", published with Prentice-Hall. His research interests span a broad range of control problems and he currently concentrates on H_∞ and robust control.

He serves as Guest Editor of a special issue on H_∞ control of the International Journal on Robust and Nonlinear Control.

Ali Saberi: see p.515 in this issue.

Ben M. Chen: see p.515 in this issue.