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Short Paper

EXPLICIT EXPRESSIONS FOR CASCADE FACTORIZATIONS OF GENERAL NON-STRICTLY PROPER SYSTEMS*

Z. LIN,[1] B.M. CHEN[2] AND A. SABERI[1]

Abstract. This paper presents explicit expressions for two different cascade factorizations of any detectable system which is not necessarily left invertible and which is not necessarily strictly proper. The first one is a well known minimum phase/all-pass factorization by which $G(s)$ is written as $G_m(s)V(s)$, where $G_m(s)$ is left invertible and of minimum phase while $V(s)$ is a stable right invertible all-pass transfer function matrix which has all unstable invariant zeros of $G(s)$ as its invariant zeros. The second one is a generalized cascade factorization by which $G(s)$ is written as $G_M(s)U(s)$, where $G_M(s)$ is left invertible and of minimum-phase with its invariant zeros at desired locations in the open left-half s-plane while $U(s)$ is a stable right invertible system which has all "awkward" invariant zeros, including the unstable invariant zeros of $G(s)$, as its invariant zeros, and is "asymptotically" all-pass. These factorizations are useful in several applications including loop transfer recovery, H_2 and H_∞ optimal control. This paper is an extension of the results of Chen, Saberi and Sannuti (1992) who consider only strictly proper left invertible systems.

Key Words—Minimum phase/all-pass factorization, inner-outer factorization, generalized cascade factorization.

1. Introduction

Cascade factorizations of nonminimum phase systems have been used extensively in the literature. The so called minimum phase/all-pass factorization plays a significant role in several applications. The role played by it in the control literature as well as various methods available for such a factorization are well documented by Shaked (1989). Since then, minimum phase/all-pass factorization played also a substantial role in loop transfer recovery (Zhang and Freudenberg, 1990), H_2-optimization (Chen, Saberi, Sannuti and Shamash, 1992) and H_∞-optimization (Saberi et al., 1991). Traditionally, the minimum phase/all-pass factorization has been found by spectral factorization techniques, e.g., Strintzis (1972) and Tuel (1968). Recently, Chen, Saberi and Sannuti (1992) have developed explicit expressions for such a factorization. They have also introduced a generalized cascade factorization, which is a natural extension of the former one and which plays an important role in loop transfer recovery. All the above mentioned techniques, however, are confined to

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[1] School of Electrical Engineering and Computer Science, Washington State University, Pullman, Washington 99164-2752, U.S.A.
[2] Department of Electrical Engineering, State University of New York at Stony Brook, Stony Brook, N.Y. 11794-2350, U.S.A.

strictly proper left invertible systems.

Following the work of Chen, Saberi and Sannuti (1992), this paper presents explicit expressions for both cascade factorizations, the traditional minimum phase/all-pass factorization, and the generalized cascade factorization. General detectable systems which are not necessarily left invertible and which are not necessarily strictly proper, are considered. To be specific, let us consider a detectable system $\Sigma(A, B, C, D)$ characterized by the matrix quadruple (A, B, C, D), i.e.,

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\}, \tag{1.1}$$

where the state vector $x \in \mathscr{R}^n$, output vector $y \in \mathscr{R}^p$ and input vector $u \in \mathscr{R}^m$. Without loss of generality, assume that $[B', D']'$ and $[C, D]$ are of maximal rank. Let the transfer function matrix of $\Sigma(A, B, C, D)$ be $G(s)$. Then the minimum phase/all-pass factorization of $G(s)$ is expressed as

$$G(s) = G_m(s)V(s), \tag{1.2}$$

where $G_m(s)$ is left invertible and of minimum-phase, and satisfies

$$G(s)G^H(s) = G_m(s)G_m^H(s),$$

while $V(s)$ is a stable right invertible transfer function matrix satisfying

$$V(s)V^H(s) = I. \tag{1.3}$$

Here $(\cdot)^H$ denotes the Hermitian paraconjugate of $(\cdot)$. The invariant zeros of $G_m(s)$ include those stable invariant zeros of $G(s)$ and the mirror image of unstable invariant zeros of $G(s)$. The transfer function matrix $G_m(s)$ is sometimes referred to as the minimum-phase image of $G(s)$. In some applications, such as the loop transfer recovery theory, one does not necessarily require a true minimum phase image of $\Sigma(A, B, C, D)$. What is required is a model which retains the infinite zero structure of $\Sigma(A, B, C, D)$ and whose invariant zeros are appropriately assigned to some desired locations in the open left-half s-plane. With this point in view and as in Chen, Saberi and Sannuti (1992), we develop here a cascade factorization of the form,

$$G(s) = G_M(s)U(s). \tag{1.4}$$

Here $G_M(s)$ is left invertible and of minimum-phase with all its invariant zeros at the desired locations in the open left-half s-plane, and $U(s)$ is stable right invertible and "asymptotically" all-pass in the sense that

$$U(s)U^H(s) \to I \quad \text{as} \quad |s| \to \infty. \tag{1.5}$$

Furthermore, both $G_m(s)$ and $G_M(s)$ have the same infinite zero structure as that of $G(s)$.

The method of factorization that is to be presented, as in Chen, Saberi and Sannuti (1992), has some important attributes.

1. The method assumes only detectability of $\Sigma(A, B, C, D)$, i.e., $\Sigma(A, B, C, D)$ need not be controllable, observable and left invertible.
2. Guided by one's application, one can seek either one of the two cascade factorizations, a traditional minimum phase/all-pass factorization (1.2) and the generalized cascade factorization (1.4). In (1.2), $G_m(s)$ has a particular invariant zero structure dictated by the given system $\Sigma(A, B, C, D)$ while $V(s)$ is a stable right invertible all-pass transfer function matrix. On the other hand, (1.4) provides flexibility to have a chosen invariant zero structure for $G_M(s)$ but $U(s)$, although stable and right invertible, is only asymptotically all-pass.
3. Both factorizations given here retain the infinite zero structure of $\Sigma(A, B, C, D)$. This is crucial in several applications.

We emphasize that our methods can easily be implemented on a computer. In fact, we have already successfully implemented both of our factorization methods in the Matlab environment (Lin et al., 1991).

The paper is organized as follows. Sections 2 and 3 respectively give explicit methods of constructing the traditional minimum phase/all-pass factorization and the generalized cascade factorization. Section 4 draws the conclusions of our work. Throughout this paper, A' denotes the transpose of A, I denotes an identity matrix with appropriate dimension. Similarly, $\lambda(A)$ denotes the set of eigenvalues of A. The open left and closed right s-plane are respectively denoted by $\mathscr{C}^-$ and $\mathscr{C}^+$. Also, $\mathscr{RH}^\infty$ denotes the set of real-rational transfer functions which are stable and proper.

2. Minimum Phase/All-pass Factorization

In this section, we give simple and explicit expressions for the minimum phase image $G_m(s)$, and the all-pass factor $V(s)$ of $\Sigma(A, B, C, D)$. We first transform the given system $\Sigma(A, B, C, D)$ into the form of a special coordinate basis (SCB) as in Theorem A.1 of Appendix A. Let

$$A_x = \begin{bmatrix} A_{cc} & B_c E_{ca}^+ \\ 0 & A_{aa}^+ \end{bmatrix}, \quad B_x = \begin{bmatrix} 0 & 0 & B_c \\ 0 & 0 & 0 \end{bmatrix} \Gamma_3^{-1}, \tag{2.1}$$

$$C_x = \begin{bmatrix} C_{0c} & C_{0a}^+ \\ E_{fc} & E_{fa}^+ \end{bmatrix}, \quad D_x = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & B_f' B_f & 0 \end{bmatrix} \Gamma_3^{-1} \tag{2.2}$$

and

$$\Gamma_3^{-1}(\Gamma_3^{-1})' = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12}' & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{13}' & \Gamma_{23}' & \Gamma_{33} \end{bmatrix}, \quad \Gamma_m = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}' & \Gamma_{22} \end{bmatrix}^{-\frac{1}{2}}. \tag{2.3}$$

Here it should be noted that in view of the property of SCB (e.g., Property A.3 in Appendix A), the pair (A_x, C_x) is detectable whenever $\Sigma(A, B, C, D)$ is detectable. We have the following theorem.

Theorem 2.1. Consider a detectable system $\Sigma(A, B, C, D)$. Assume that $\Sigma(A, B, C, D)$ does not have any invariant zeros on the $j\omega$-axis. Then, the minimum-phase/all-pass factorization of $\Sigma(A, B, C, D)$ is obtained as follows:

1. The minimum phase factor of $\Sigma(A, B, C, D)$ is $\Sigma_m(A, B_m, C, D_m)$ which has the transfer function $G_m(s) = C(sI - A)^{-1}B_m + D_m$. Here,

$$B_m = \Gamma_1 \begin{bmatrix} B_{c0} + K_{c0} & K_{cf} \\ B_{a0}^+ + K_{a0}^+ & K_{af}^+ \\ B_{a0}^- & 0 \\ B_{b0} & 0 \\ B_{f0} & B_f \end{bmatrix} \Gamma_m^{-1}, \quad D_m = \Gamma_2 \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Gamma_m^{-1}, \tag{2.4}$$

where

$$K_x \triangleq \begin{bmatrix} K_{c0} & K_{cf} \\ K_{a0}^+ & K_{af}^+ \end{bmatrix} = (C_x P_x + D_x B_x')'(D_x D_x')^{-1}, \tag{2.5}$$

and P_x is the solution of the algebraic Riccati equation,

$$A_x P_x + P_x A_x' + B_x B_x' - (C_x P_x + D_x B_x')'(D_x D_x')^{-1}(C_x P_x + D_x B_x') = 0. \tag{2.6}$$

Also, $\Sigma_m(A, B_m, C, D_m)$ is left invertible and has the same infinite zero structure as $\Sigma(A, B, C, D)$, and satisfies

$$G(s)G^H(s) = G_m(s)G_m^H(s). \tag{2.7}$$

2. The stable right invertible all-pass factor of $\Sigma(A, B, C, D)$ is given as

$$V(s) = \Gamma_m[C_x(sI - A_x + K_x C_x)^{-1}(B_x - K_x D_x) + D_x], \tag{2.8}$$

and $V(s)$ satisfies

$$V(s)V^H(s) = I. \tag{2.9}$$

Proof. See Appendix B.

The following remark is in order.

Remark 2.1: We should emphasize that the difference between Theorem 2.1 and the result of Chen, Saberi and Sannuti (1992) is that Theorem 2.1 deals with general not necessarily strictly proper and not necessarily left invertible systems while Chen, Saberi and Sannuti (1992) deals only with strictly proper and left-invertible systems. Moreover, the procedure in Theorem 2.1 involves solving the algebraic Riccati equation instead of Lyapunov equation as in Chen, Saberi and Sannuti (1992). It is worth noting that under the condition that the system is strictly proper and left invertible, the result of Theorem 2.1 reduces to that of Chen, Saberi and Sannuti (1992).

We demonstrate the above results by the following example.

Example 2.1. Consider a system $\Sigma(A, B, C, D)$ with

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The given $\Sigma(A, B, C, D)$ has a transfer function $G(s)$,

$$G(s) = \frac{1}{s^5 - 3s^4 - 2s^3 + 3s^2 - s} \times \begin{bmatrix} s^5 - 3s^4 - 2s^3 + 3s^2 - s & s^4 + 2s - 1 & s^4 - s^3 - 3s^2 + 1 \\ 0 & s^4 - 2s^3 + 2s - 1 & s^3 - s^2 - s + 1 \\ 0 & s^3 - s^2 - s + 1 & s^2 - 1 \end{bmatrix}.$$

This system is neither left nor right invertible and has two invariant zeros at $\{-1, 1\}$. Hence, it is of nonminimum phase. Moreover, it is easy to verify that $\Sigma(A, B, C, D)$ is in the form of SCB with

$$A_x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

$$D_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Gamma_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, following the procedure given in Theorem 2.1 which involes solving a Riccati equation, we obtain,

$$K_x = \begin{bmatrix} 1.412771 & 1.063856 \\ -0.348915 & 2.255424 \end{bmatrix},$$

$$B_m = \begin{bmatrix} 1.412771 & 1.063856 \\ -0.348915 & 2.255424 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D_m = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$G_m(s) = \frac{1}{s^5 - 3s^4 - 2s^3 + 3s^2 - s} \times \begin{bmatrix} s^5 - 1.587229s^4 - 4.110602s^3 - 1.587229s^2 - 0.302169s + 1.412771 & 2.063856s^4 + 3.446991s^3 - 0.936144s^2 - 2.510847s + 0.063856 \\ 1.063856s^3 - 1.412771s^2 - 1.063856s + 1.412771 & s^4 + 1.319280s^3 - 1.063856s^2 - 1.319280s + 0.063856 \\ 1.063856s^2 - 0.348915s - 1.412771 & s^3 + 2.319280s^2 + 1.255424s - 0.063856 \end{bmatrix}$$

and

$$V(s) = \frac{1}{s^2 + 2.732051s + 1.732051} \times \begin{bmatrix} s^2 + 1.319280s - 0.063856 & -1.063856s - 1.191568 & s + 1.255424 \\ -1.063856s + 1.412771 & s^2 - 0.587229s - 0.761686 & s - 0.651085 \end{bmatrix}.$$

Our minimum-phase/all-pass factorization can be slightly modified to obtain an inner-outer factorization. We first recall the following definitions.

Definition 2.1. A matrix function $G(s) \in \mathscr{RH}^\infty$ is said to be inner if $G^H(s)G(s) = I$ and outer if it has full row rank for every s in $\mathrm{Re}(s) > 0$, equivalently, it has a right-inverse which is analytic in $\mathrm{Re}(s) > 0$.

Definition 2.2. An inner-outer factorization of a matrix $G(s) \in \mathscr{RH}^\infty$ is a factorization

$$G(s) = G_i(s)G_o(s)$$

with $G_i(s)$ an inner matrix and $G_o(s)$ an outer matrix.

Theorem 2.2. Consider a transfer function matrix $G(s) \in \mathscr{RH}^\infty$. Let $\Sigma(A, B, C, D)$ be a state space realization of $G^T(s)$. Let the SCB presentation of $\Sigma(A, B, C, D)$ be given as in Appendix A with $\lambda(A_{aa}^-)$ containing all the invariant zeros of $\Sigma(A, B, C, D)$ located on the closed left-half s-plane. Then the inner-outer factorization of $G(s)$ is given as

$$G(s) = G_i(s)G_o(s),$$

where

$$G_i(s) = [(B'_x - D'_x K'_x)(sI - A'_x + C'_x K'_x)^{-1} C'_x + D'_x]\Gamma_m$$

and

$$G_o(s) = B'_m(sI - A')^{-1}C' + D'_m,$$

with the matrices K_x, B_m and D_m as defined by (2.4)–(2.6).

Proof. The proof is a slight modification of that of Theorem 2.1 and, for the sake of brevity, is omitted.

3. A Generalized Cascade Factorization

Whenever some invariant zeros of $\Sigma(A, B, C, D)$ lie on the $j\omega$-axis, no minimum phase image of $\Sigma(A, B, C, D)$ can be obtained by any means. In what follows, we introduce a generalized cascade factorization of a given system $\Sigma(A, B, C, D)$ which is a natural extension of the minimum phase/all-pass factorization discussed above. The given system $\Sigma(A, B, C, D)$ is decomposed as

$$G(s) = G_M(s)U(s). \tag{3.1}$$

Here $G_M(s)$ is of minimum phase, left invertible and has the same infinite zero structure as that of $\Sigma(A, B, C, D)$ while $U(s)$ is a stable transfer function matrix which is asymptotically all-pass. All the invariant zeros of $G_M(s)$ are in a desired set $\mathscr{C}_d \subset \mathscr{C}^-$. If the given system $\Sigma(A, B, C, D)$ is only detectable but not observable, the set $\mathscr{C}_d$ includes all the stable output decoupling zeros of $\Sigma(A, B, C, D)$. In this way, all the awkward or unwanted invariant zeros of $\Sigma(A, B, C, D)$ (say, those in the right-half s-plane or close to $j\omega$-axis) need not be included in $G_M(s)$. Such a generalized cascade factorization has a major application in loop transfer recovery design. For instance, by applying the loop transfer recovery procedure to $G_M(s)$, one has the capability to shape the over all loop transfer recovery error over some frequency band or in some subspace of interest while placing the eigenvalues of the observer corresponding to some awkward invariant zeros of $\Sigma(A, B, C, D)$ at any desired locations (Saberi et al., 1993).

Let us assume that the given system $\Sigma(A, B, C, D)$ has been transformed into the form of SCB as in Theorem A.1 of Appendix A. Let us also assume that in the SCB formulation, x_a is decomposed into x_a^+ and x_a^-, such that the eigenvalues of A_{aa}^+ contain all the awkward invariant zeros of $\Sigma(A, B, C, D)$. We have the following theorem.

Theorem 3.1. Consider a detectable system $\Sigma(A, B, C, D)$ that is not necessarily of minimum phase and left invertible. Let the SCB representation of $\Sigma(A, B, C, D)$ be given as in Appendix A with $\lambda(A_{aa}^+)$ containing all the "awkward" but observable invariant zeros of $\Sigma(A, B, C, D)$. Then the generalized cascade factorization (3.1) is obtained as follows:

1. The minimum phase counterpart of $\Sigma(A, B, C, D)$ is $\Sigma_M(A, B_M, C, D_M)$ having the transfer function $G_M(s) = C(sI - A)^{-1}B_M + D_M$, where

$$B_M = \Gamma_1 \begin{bmatrix} B_{c0} + K_{c0} & K_{cf} \\ B_{a0}^+ + K_{a0}^+ & K_{af}^+ \\ B_{a0}^- & 0 \\ B_{b0} & 0 \\ B_{f0} & B_f \end{bmatrix} \Gamma_m^{-1}, \quad D_M = \Gamma_2 \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Gamma_m^{-1}, \tag{3.2}$$

and where

$$K_x = \begin{bmatrix} K_{c0} & K_{cf} \\ K_{a0}^{+} & K_{af}^{+} \end{bmatrix} \tag{3.3}$$

is specified such that $\lambda(A_x - K_x C_x)$ are in the desired locations in $\mathscr{C}^-$ with desired admissible eigenvectors (Moore, 1976). Moreover, $\Sigma_M(A, B_M, C, D_M)$ is also left invertible and has the same infinite zero structure as $\Sigma(A, B, C, D)$.

2. The factor $U(s)$ is given as

$$U(s) = \Gamma_m[C_x(sI - A_x + K_x C_x)^{-1}(B_x - K_x D_x) + D_x], \tag{3.4}$$

where $U(s)$ is stable right invertible and asymptotically all-pass, i.e.,

$$U(s)U^H(s) \to I \quad \text{as} \quad |s| \to \infty.$$

Proof. It follows from the same line of reasoning as in Theorem 2.1. (see also Chen, Saberi and Sannuti (1992)).

We illustrate next this generalized factorization on an example.

Example 3.1. Consider the system $\Sigma(A, B, C, D)$ given in Example 2.1. Let's choose K_x such that $\lambda(A_x - K_x C_x) = \{-2, -3\}$. We then obtain

$$K_x = \begin{bmatrix} 2 & 1 \\ -4 & 4 \end{bmatrix}, \quad B_M = \begin{bmatrix} 2 & 1 \\ -4 & 4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$G_M(s) = \frac{1}{s^5 - 3s^4 - 2s^3 + 3s^2 - s} \times \begin{bmatrix} s^5 - s^4 - 12s^3 - 7s^2 + 7s + 2 & 2s^4 + 7s^3 + 1s^2 - 6s \\ -2s^3 - 2s^2 + 2s + 2 & s^4 + 3s^3 - s^2 - 3s \\ -2s^2 - 4s - 2 & s^3 + 4s^2 + 3s \end{bmatrix}$$

and

$$U(s) = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s^2 + 3s & -s - 3 \\ 2s + 2 & s^2 - 5 \end{bmatrix}.$$

4. Conclusion

Explicit and simple expressions for two different cascade factorizations of a detectable system having a transfer function matrix $G(s)$ are given. In a traditional minimum phase/all-pass factorization, $G(s) = G_m(s)V(s)$. On the other hand, in a new cascade factorization, $G(s) = G_M(s)U(s)$. Both $G_m(s)$ and $G_M(s)$ are of minimum phase and left invertible, and retain the infinite zero structure of $G(s)$. The invariant zeros of $G_m(s)$ contain those stable invariant zeros of $G(s)$ and the mirror images of unstable invariant zeros of $G(s)$, whereas the invariant zeros of $G_M(s)$ can be assigned as desired in $\mathscr{C}^-$. $V(s)$ is an all-pass transfer function matrix, whereas $U(s)$, although stable, is only asymptotically all-pass.

Most of the existing solutions to the factorization problem deal with only left invertible and strictly proper systems and have some kind of difficulties when the invariant zeros of the given systems are not distinct. Our solution, however, does not have such a problem. Moreover, the computations involved in our method are rather simple. The implementation of both of our factorization methods in Matlab is straightforward and successful (Lin et al., 1991).

References

Chen, B.M., A. Saberi and P. Sannuti (1992). Explicit expressions for cascade factorization of general nonminimum phase systems. *IEEE Trans. Automatic Control*, **AC-37**, 3, 358–363.

Chen, B.M., A. Saberi, P. Sannuti and Y. Shamash (1992). Construction and parameterization of all static and dynamic H_2-optimal state feedback solutions, optimal fixed modes and fixed decoupling zeros. *Proceedings of the 26th Annual Conference on Information Sciences and Systems*, Princeton, N.J.; *IEEE Trans. Automatic Control*, **AC-38**, 2, 248–261 (1993).

Lin, Z.-L., A. Saberi and B.M. Chen (1991). Linear systems toolbox. Washington State University Technical Report No. EE/CS 0097.

Moore, B.C. (1976). On the flexibility offered by state feedback in multivariable systems beyond closed loop eigenvalue assignment. *IEEE Trans. Automatic Control*, **AC-21**, 5, 689–692.

Richardson, T.J. and R.H. Kwong (1986). On positive definite solutions to the algebraic Riccati equation. *Systems and Control Letters*, **7**, 99–104.

Saberi, A., B.M. Chen and P. Sannuti (1993). *Loop Transfer Recovery: Analysis and Design*. Springer-Verlag, London.

Saberi, A., B.M. Chen and Z.L. Lin (1991). Closed-form solutions to a class of H_∞-optimization problem. *Proceedings of the 29th Annual Allerton Conference on Communication, Control and Computing*, Monticello, IL, 74–83.

Saberi, A. and P. Sannuti (1990). Squaring down of non-strictly proper systems. *Int. J. Control*, **51**, 3, 621–629.

Sannuti, P. and A. Saberi (1987). A special coordinate basis of multivariable linear systems—finite and infinite zero structure, squaring down and decoupling. *Int. J. Control*, **45**, 5, 1655–1704.

Shaked, U. (1989). An explicit expression for the minimum-phase image of transfer function matrices. *IEEE Trans. Automatic Control*, **AC-34**, 12, 1290–1293.

Strintzis, M.G. (1972). A solution to the matrix factorization problem. *IEEE Trans. Information Theory*, **18**, 225–232.

Tuel, W.G., Jr. (1968). Computer algorithm for spectral factorization of rational matrices. *IBM Journal*, 163–170.

Zhang, Z. and J.S. Freudenberg (1990). Loop transfer recovery for nonminimum phase plants. *IEEE Trans. Automatic Control*, **AC-35**, 5, 547–553.

Appendix A: A Special Coordinate Basis

We recall in this appendix a special coordinate basis (SCB) of a linear time invariant system (Sannuti and Saberi, 1987; Saberi and Sannuti, 1990). Such an SCB has a distinct feature of explicitly and precisely displaying the infinite as well as the finite zero structure (i.e., the invariant zeros and zero directions), of a given system. We summarize below the SCB theorem and some properties of SCB while the detailed derivation and proofs can be found in Sannuti and Saberi (1987) and Saberi and Sannuti (1990). Consider the system $\Sigma(A, B, C, D)$,

$$\Sigma(A, B, C, D): \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du. \end{cases} \tag{A.1}$$

It can be easily shown that using singular value decomposition one can always find an orthogonal transformation U and a nonsingular matrix V that render the direct feedthrough matrix D into the following form,

$$\overline{D} = UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{A.2}$$

where m_0 is the rank of D. Thus, the system in (A.1) can be rewritten as

$$\left.\begin{aligned} \dot{x} &= Ax + [B_0 \quad B_1]\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} &= \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{aligned}\right\}, \tag{A.3}$$

where B_0, B_1, C_0 and C_1 are the matrices of appropriate dimensions. Note that the inputs u_0 and u_1, and the outputs y_0 and y_1 are those of the transformed system. Namely,

$$u = V\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = Uz.$$

We have the following main theorem.

Theorem A.1. (SCB) There exist nonsingular transformations Γ_1, Γ_2 and Γ_3, such that

$$x = \Gamma_1[x_c', (x_a)', x_b', x_f']', \quad x_a = [(x_a^+)', (x_a^-)']',$$

$$[y_0', y_1']' = \Gamma_2[y_0', y_f', y_b']', \quad [u_0', u_1']' = \Gamma_3[u_0', u_f', u_c']'$$

and

$$\Gamma_1^{-1}(A - B_0 C_0)\Gamma_1 = \begin{bmatrix} A_{cc} & B_c E_{ca}^+ & B_c E_{ca}^- & L_{cb} C_b & L_{cf} C_f \\ 0 & A_{aa}^+ & 0 & L_{ab}^+ C_b & L_{af}^+ C_f \\ 0 & 0 & A_{aa}^- & L_{ab}^- C_b & L_{af}^- C_f \\ 0 & 0 & 0 & A_{bb} & L_{bf} C_f \\ B_f E_{fc} & B_f E_{fa}^+ & B_f E_{fa}^- & B_f E_{fb} & A_f \end{bmatrix}, \tag{A.4}$$

$$\Gamma_1^{-1}[B_0,\ B_1]\Gamma_3 = \begin{bmatrix} B_{c0} & 0 & B_c \\ B_{a0}^+ & 0 & 0 \\ B_{a0}^- & 0 & 0 \\ B_{b0} & 0 & 0 \\ B_{f0} & B_f & 0 \end{bmatrix}, \tag{A.5}$$

$$\Gamma_2^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_1 = \begin{bmatrix} C_{0c} & C_{0a}^+ & C_{0a}^- & C_{0b} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & 0 & C_b & 0 \end{bmatrix} \tag{A.6}$$

and

$$\Gamma_2^{-1}\begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\Gamma_3 = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{A.7}$$

where $\lambda(A_{aa}^-) \in \mathscr{C}^-$, $\lambda(A_{aa}^+) \in \mathscr{C}^+$, $(A_{cc},\ B_c)$ is controllable, $(A_{bb},\ C_b)$ is observable and the subsystem characterized by $(A_f,\ B_f,\ C_f)$ is invertible with no invariant zeros.

The proof of this theorem can be found in Sannuti and Saberi (1987) and Saberi and Sannuti (1990). We state some important properties of the SCB which are pertinent to our present work.

Property A.1. The given system $\Sigma(A,\ B,\ C,\ D)$ is right invertible, if and only if x_b and hence, y_b are nonexistent, left invertible, if and only if x_c and hence, u_c are nonexistent, invertible, if and only if both x_c and x_b are nonexistent.

Property A.2. $\lambda(A_{aa}^-) \cup \lambda(A_{aa}^+)$ are the invariant zeros of $\Sigma(A,\ B,\ C,\ D)$. We note that $\lambda(A_{aa}^-)$ are the stable (open left-half s-plane) and $\lambda(A_{aa}^+)$ are the unstable (closed right-half s-plane) invariant zeros of $\Sigma(A,\ B,\ C,\ D)$.

Property A.3. The pair $(A,\ C)$ is detectable, if and only if $(A_x,\ C_x)$ is detectable where

$$A_x = \begin{bmatrix} A_{cc} & B_c E_{ca}^+ \\ 0 & A_{aa}^+ \end{bmatrix}, \quad C_x = \begin{bmatrix} C_{0c} & C_{0a}^+ \\ E_{fc} & E_{fa}^+ \end{bmatrix}. \tag{A.8}$$

Remark A.1. The decomposition of x_a into x_a^+ and x_a^- can be done in other ways so that the corresponding matrices A_{aa}^+ and A_{aa}^- have desired disjoint subsets of the invariant zeros of $\Sigma(A,\ B,\ C,\ D)$.

Appendix B: Proof of Theorem 2.1

We first note that since the pair $(A_x,\ C_x)$ is detectable and the pair $(-A_x,\ B_x)$ is stabilizable, it follows from Richardson and Kwong (1986) that (2.6) has a unique, symmetric and positive definite solution, i.e., $P_x = P_x' > 0$. Let us now show that $A_x - K_x C_x$ is a stable matrix. Let

$$P_x^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix}.$$

Then pre-multiplying equation (2.6) by P_x^{-1}, we obtain

$$P_x^{-1}(A_x - K_x C_x)P_x = \begin{bmatrix} -(A_{cc}^* + B_c \tilde{\Gamma} B_c' P_{11})' & 0 \\ \star & -(A_{aa}^+)' \end{bmatrix},$$

where

$$A_{cc}^* = A_{cc} - B_c[T_{13}' \quad T_{23}']\Gamma_m^2 \begin{bmatrix} C_{0c} \\ E_{fc} \end{bmatrix}$$

and

$$\tilde{\Gamma} = \Gamma_{33} - [\Gamma_{13}' \quad \Gamma_{23}']\Gamma_m^2 \begin{bmatrix} \Gamma_{13} \\ \Gamma_{23} \end{bmatrix}.$$

It is worth noting that $\tilde{\Gamma}$ is a positive definite matrix and P_{11} is the unique positive definite solution of the algebraic Riccati equation

$$P_{11}A_{cc}^* + (A_{cc}^*)'P_{11} + P_{11}B_c\tilde{\Gamma}B_c'P_{11} - [C_{0c}' E_{fc}']\Gamma_m^2 \begin{bmatrix} C_{0c} \\ E_{fc} \end{bmatrix} = 0.$$

Hence, $\lambda(A_x - K_x C_x) = \lambda(-A_{aa}^+) \cup \lambda(-A_{cc}^* - B_c\tilde{\Gamma}B_c'P_{11})$ are all in $\mathscr{C}^-$, and thus, $A_x - K_x C_x$ is indeed a stable matrix. We are now ready to prove that $\Sigma_m(A, B_m, C, D_m)$ is of minimum phase, left invertible and has the same infinite zero structure as $\Sigma(A, B, C, D)$. Without loss of generality, we assume that $\Sigma(A, B, C, D)$ is in the form of SCB of Appendix A. Let us define

$$\tilde{A} = A - B_0 C_0 - \begin{bmatrix} K_{c0} \\ K_{a0}^+ \\ 0 \\ 0 \\ 0 \end{bmatrix} C_0, \quad \tilde{B} = \begin{bmatrix} K_{cf} \\ K_{af}^+ \\ 0 \\ 0 \\ B_f \end{bmatrix}$$

and

$$\tilde{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & 0 & C_b & 0 \end{bmatrix}.$$

Then, by the construction and the properties of SCB, the system $\Sigma_m(A, B_m, C, D_m)$ and the system $\tilde{\Sigma}(\tilde{A}, \tilde{B}, \tilde{C})$ have the same finite and infinite zero structures and the same invertibility properties. Then using the same techniques as in Appendix B of Chen, Saberi and Sannuti (1992) and, using the properties of SCB, it is easily shown that the system $\tilde{\Sigma}$ has the following properties:

(1) $\tilde{\Sigma}(\tilde{A}, \tilde{B}, \tilde{C})$ is left invertible,

(2) $\tilde{\Sigma}(\tilde{A}, \tilde{B}, \tilde{C})$ has the same infinite zero structure as that of $\Sigma(A, B, C, D)$; and

(3) $\tilde{\Sigma}(\tilde{A}, \tilde{B}, \tilde{C})$ has invariant zeros at

$$\lambda\begin{bmatrix} A_x - K_x C_x & \star \\ 0 & A_{\bar{a}\bar{a}}^- \end{bmatrix} \in \mathscr{C}^-,$$

where $\star$'s denote matrices of not much interest.
Next, we proceed to show that $V(s)V^H(s) = I$. It follows from (2.6) and (2.5) that

$$A_x P_x + P_x A_x' + B_x B_x' - K_x(C_x P_x + D_x B_x') = 0$$

and

$$D_x(B_x' - D_x' K_x') = -C_x P_x.$$

We then have

$$\begin{aligned} &V(s)V^H(s) \\ &= I + \Gamma_m C_x(sI - A_x + K_x C_x)^{-1}(B_x - K_x D_x)(B_x' - D_x' K_x') \\ &\quad \times (-sI - A_x' + C_x' K_x')^{-1} C_x' \Gamma_m \\ &\quad - \Gamma_m C_x(sI - A_x + K_x C_x)^{-1} P_x C_x' \Gamma_m \\ &\quad - \Gamma_m C_x P_x(-sI - A_x' + C_x' K_x')^{-1} C_x' \Gamma_m \\ &= I + \Gamma_m C_x(sI - A_x + K_x C_x)^{-1}[(B_x - K_x D_x)(B_x' - D_x' K_x') \\ &\quad - P_x(-sI - A_x' + C_x' K_x') - (sI - A_x + K_x C_x)P_x] \\ &\quad \times (-sI - A_x' + C_x' K_x')^{-1} C_x' \Gamma_m = I. \end{aligned}$$

We are ready to show that $G(s) = G_m(s)V(s)$. Let us define

$$\overline{B}_0 = \begin{bmatrix} B_{c0} \\ B_{a0}^+ \\ B_{\bar{a}0}^- \\ B_{b0} \\ B_{f0} \end{bmatrix}, \quad \overline{B}_K = \begin{bmatrix} K_{c0} & K_{cf} \\ K_{a0}^+ & K_{af}^+ \\ 0 & 0 \\ 0 & 0 \\ 0 & B_f \end{bmatrix}, \quad \overline{B}_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ B_f \end{bmatrix}, \quad \overline{B}_c = \begin{bmatrix} B_c \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\Phi_x(s) = (sI - A_x + K_x C_x)^{-1},$$

$$\overline{\Phi}_x(s) = [\Phi_x'(s) \quad 0 \quad 0 \quad 0]'$$

It then follows that

$$B = \Gamma_1[\overline{B}_0 \quad \overline{B}_f \quad \overline{B}_c]\Gamma_3^{-1},$$

$$B_m \Gamma_m = \Gamma_1[\overline{B}_0 \quad 0] + \Gamma_1 \overline{B}_K,$$

$$B_m \Gamma_m D_x = \Gamma_1[\overline{B}_0 \quad 0 \quad 0]\Gamma_3^{-1} + \Gamma_1[\overline{B}_K \quad 0]\Gamma_3^{-1}$$

and

$$D_m \Gamma_m C_x \Phi_x(s) = C\Gamma_1 \overline{\Phi}_x(s).$$

We now have

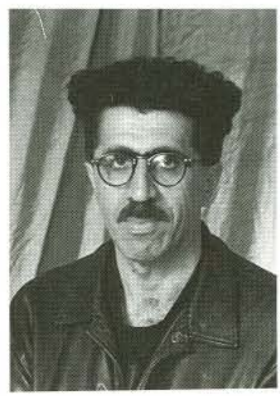
$$\begin{aligned}
&G_m(s)V(s)\\
&= [C(sI - A)^{-1}B_m + D_m]\Gamma_m[C_x(sI - A_x + K_x C_x)^{-1}\\
&\quad \times (B_x - K_x D_x) + D_x]\\
&= C(sI - A)^{-1}B_m \Gamma_m[C_x(sI - A_x + K_x C_x)^{-1}(B_x - K_x D_x) + D_x]\\
&\quad + D_m \Gamma_m C_x(sI - A_x + K_x C_x)^{-1}(B_x - K_x D_x) + D\\
&= C(sI - A)^{-1}[B_m \Gamma_m C_x \Phi_x(s)(B_x - K_x C_x) + B_m \Gamma_m D_x\\
&\quad + (sI - A)\Gamma_1 \overline{\Phi}_x(s)(B_x - K_x D_x)] + D\\
&= C(sI - A)^{-1}\Gamma_1[([\overline{B}_0 \;\; 0] + \overline{B}_K)C_x \Phi_x(s)(B_x - K_x D_x)\\
&\quad + [\overline{B}_0 \;\; 0 \;\; 0]\Gamma_3^{-1} + [\overline{B}_K \;\; 0]\Gamma_3^{-1}\\
&\quad + \Gamma_1^{-1}(sI - A)\Gamma_1 \overline{\Phi}_x(s)(B_x - K_x D_x)] + D\\
&= C(sI - A)^{-1}\Gamma_1([I \;\; 0]'(B_x - K_x D_x) + [\overline{B}_0 \;\; 0 \;\; 0]\Gamma_3^{-1}\\
&\quad + [\overline{B}_K \;\; 0]\Gamma_3^{-1}) + D\\
&= C(sI - A)^{-1}\Gamma_1([\overline{B}_0 \;\; 0 \;\; 0] + [0 \;\; \overline{B}_f \;\; 0]\\
&\quad + [0 \;\; 0 \;\; \overline{B}_c])\Gamma_3^{-1} + D\\
&= C(sI - A)^{-1}B + D\\
&= G(s).
\end{aligned}$$

Finally, the fact that $G(s)G^H(s) = G_m(s)G_m^H(s)$ follows immediately from the fact that $V(s)V^H(s) = I$, and this completes the Proof of Theorem 2.1.

Zongli Lin was born in Fuqing, Fujian, China on February 24, 1964. He received his B.S. degree in mathematics and computer science from Amoy University, Xiamen, China in 1983, and his Master of Engineering degree in automatic control from Chinese Academy of Space Technology in 1989, Beijing, China. He is currently working toward his Ph.D. degree in the School of Electrical Engineering and Computer Science at Washington State University, Pullman, Washington.

Ben M. Chen was born in Fuqing, Fujian, China on November 25, 1963. He received the B.S. degree in mathematics and computer science from Amoy University, Xiamen, China in 1983, the M.S. degree in electrical engineering from Gonzaga University, Spokane, Washington in 1988, and the Ph.D. degree in electrical and computer engineering from Washington State University, Pullman, Washington in 1991.

From 1983 to 1986 he worked as a software engineer in the South-China Computer Corporation, China. From 1991 to 1992, he was a postdoctoral associate at Washington State University. Since 1992, he has been an assistant professor in the Department of Electrical Engineering, the State University of New York at Stony Brook. His current research interests are in robust control and computer-aided design for control systems.



Ali Saberi is working in the School of Electrical Engineering and Computer Science, Washington State University, Pullman, Washington, U.S.A.