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A non-iterative method for computing the infimum in H_∞-optimization

BEN M. CHEN†, ALI SABERI‡ and UY-LOI LY§

This paper presents a simple and non-iterative procedure for the computation of the exact value of the infimum in the singular H_∞-optimization problem, and is an extension of our earlier work. The problem formulation is general and does not place any restriction on the direct feedthrough terms between the control input and the controlled output variables, and between the disturbance input and the measurement output variables. Our method is applicable to a class of singular H_∞-optimization problems for which the transfer functions from the control input to the controlled output and from the disturbance input to the measurement output have no invariant zeros on the jω axis and also satisfy certain geometric conditions. The computation of the infimum in our method involves solving two well-defined Riccati and two Lyapunov equations.

Conventions and notation

A^{T}	transpose of A
I	identity matrix
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	whole complex plane
$\mathbb{C}^-$	open left-half complex plane
$\mathbb{C}^+$	open right-half complex plane
$\mathbb{C}^0$	imaginary axis jω
$\sigma_{\max}(A)$	maximum singular value of A
$\lambda(A)$	set of eigenvalues of A
$\lambda_{\max}(A)$	maximum eigenvalue of A where $\lambda(A) \subset \mathbb{R}$
$\rho(A)$	spectral radius of A
$\mathrm{Ker}\,(V)$	kernel of V
$\mathrm{Im}\,(V)$	image of V

1. Introduction

The past decade has witnessed a proliferation of literature on H_∞-optimal control since it was first introduced by Zames (1981). The main focus of the work has been, and continues to be, on the formulation of the problem for robust multivariable control and its solution. Since the original formulation of the H_∞-problem in Zames (1981), a great deal of the work has been on the solution to this problem. Practically all research results of early years involved a mixture of time-domain and frequency-domain techniques (Doyle 1984, Francis

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† Department of Electrical Engineering, State University of New York at Stony Brook, Stony Brook, NY 11794-2350, U.S.A.

‡ School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752, U.S.A.

§ Department of Aeronautics and Astronautics, FS-10, University of Washington, Seattle, WA 98195, U.S.A.

1987, Glover 1984). Recently, considerable attention has been focused on purely time-domain methods based on algebraic Riccati equations (ARE) (Doyle *et al.* 1989, Doyle and Glover 1988, Khargonekar *et al.* 1988, Petersen 1987, 1988; Sampei *et al.* 1990, Stoorvogel 1991, Stoorvogel and Trentelman 1990, Zhou and Khargonekar 1988). Along this line of research, connections are also made between H_∞-optimal control and differential games (Basar and Bernard 1989, Papavassilopoulos and Safonov 1989). Typically in ARE approaches to H_∞-optimal control problems, the achieved design solution is suboptimal in the sense that the H_∞-norm of the closed-loop system transfer function from the disturbances to the controlled outputs is less than a prescribed value. For the regular case, (this refers to a system where the feedthrough matrix from the disturbance to the measurement output is surjective and the feedthrough matrix from the control input to the controlled output is injective) the existence of suboptimal state (output) feedback laws is formulated in terms of the existence of a stabilizing positive semi-definite solution(s) for one (two) 'indefinite' algebraic Riccati equation(s) and the satisfaction of a coupling condition for the case of output feedback. A recent paper by Stoorvogel (1991) has shown that conditions for the existence of suboptimal output feedback laws for the general singular case (i.e. not a regular case) can be expressed in terms of the existence of solutions to two quadratic matrix inequalities. Solutions of these inequalities must also satisfy two rank conditions and a coupling condition. The latter condition requires that the spectral radius of the product of the two solutions to be smaller than a certain prior given upper bound.

In this paper, we address the problem of computing the infimum in H_∞-optimization for the output feedback case. The ARE-based approach to this problem simply provides an iterative scheme of approximating the infimum (denoted here by γ_0^*) of the H_∞-norm of the closed-loop transfer function using output feedback compensators. For example, in the regular case and utilizing the results of Doyle *et al.* (1989), an iterative procedure for approximating γ_0^* would proceed as follows: one starts with a value of γ and determines whether $\gamma > \gamma_0^*$ by solving two 'indefinite' algebraic Riccati equations and checking the positive semi-definiteness and stabilizing properties of these solutions. In the case where such positive semi-definite solutions exist and satisfy a coupling condition, then we have $\gamma > \gamma_0^*$ and one simply repeats the above steps using a smaller value of γ. In principle, one can approximate the infimum γ_0^* to within any degree of accuracy in this manner. However this search procedure is exhaustive and can be very costly. More significantly, due to the possible high-gain occurrence as γ gets close to γ_0^*, numerical solutions for these AREs can become highly sensitive and ill-conditioned. This difficulty also arises in the coupling condition. Namely, as γ decreases, evaluation of the coupling condition would generally involve finding eigenvalues of stiff matrices. These numerical difficulties are likely to be more severe for problems associated with the singular case. So, in general, the iterative procedure for the computation of γ_0^* based on AREs is not reliable and thus should not be used to determine the infimum γ_0^*.

In a recent paper of Chen *et al.* (1992), a non-iterative algorithm was proposed to calculate γ_0^* for a class of systems that satisfy the following conditions: (i) the transfer function from the control input to the controlled output is right-invertible and has no invariant zeros on the $j\omega$ axis; and (ii) the transfer function from the disturbance to the measurement output is left-invertible and has no invariant zeros on the jω axis. The goal of this paper is to generalize and extend the results of Chen *et al.* (1992) by relaxing the above assumptions; namely, to replace the right- and left-invertibility assumptions imposed on the given plant by two geometric conditions, which are much weaker than the former ones. We would like to point out that under the assumptions of Chen *et al.* (1992), the computation of γ_0^* involves solving four Lyapunov equations. However, the computation of γ_0^* under the new assumptions of this paper requires solving two well-defined Riccati and two Lyapunov equations. The new algorithm has been implemented efficiently in a MATLAB-software environment for numerical solutions.

The outline of this paper is as follows. In § 2 we introduce the problem statement. In § 3 we provide some preliminaries on the special coordinate basis (s.c.b) and its properties for non-strictly proper systems, and the main results of Stoorvogel (1991) in notations consistent with the problem statement of § 2. The s.c.b transformation and Stoorvogel's theorem are both instrumental in the derivation of the main results given in § 4 for the exact computation of γ_0^*. Section 5 gives other related results on problems of almost disturbance decoupling with internal stability, and conditions under which the infimum of the output feedback case is equal to that of the state feedback case. Finally in § 6 we draw the conclusion.

We refer to the linear dynamical system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{0.1}$$

as the system (A, B, C, D). We also refer to $T_{yu}(s) = C(sI - A)^{-1}B + D$ as the transfer function matrix of the system (A, B, C, D) between the input u and the output y. For any real rational matrix $T(s)$,

$$\|T\|_\infty := \sup\{\sigma_{\max}[T(\mathrm{j}\omega)] : \omega \in \mathbb{R}\} \tag{0.2}$$

then $\|T\|_\infty$ coincides with the L_∞-norm of $T(s)$ if $T(s)$ is proper and has no poles in $\mathbb{C}^0$, and with the H_∞-norm of $T(s)$ if it is proper and stable. We also define the following subspaces:

(i) $\mathcal{V}^g(A, B, C, D)$—the maximal subspace of $\mathbb{R}^n$ which is $(A + BF)$-invariant and contained in $\operatorname{Ker}(C + DF)$ such that the eigenvalues of $(A + BF)|\mathcal{V}^g$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some F.

(ii) $\mathcal{S}^g(A, B, C, D)$—the minimal $(A + KC)$-invariant subspace of $\mathbb{R}^n$ containing $\operatorname{Im}(B + KD)$ such that the eigenvalues of the map which is induced by $(A + KC)$ on the factor space $\mathbb{R}^n/\mathcal{S}^g$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some K.

For the cases that $\mathbb{C}_g = \mathbb{C}$, $\mathbb{C}_g = \mathbb{C}^-$ and $\mathbb{C}_g = \mathbb{C}^0 \cup \mathbb{C}^+$, we replace the index g in $\mathcal{V}^g$ and $\mathcal{S}^g$ by '$*$', '$-$' and '$+$' respectively.

2. Problem formulation

Let us consider the following linear system,

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew \\ y = C_1 x \qquad\quad + D_1 w \\ z = C_2 x + D_2 u \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $w \in \mathbb{R}^p$ is the disturbance, $y \in \mathbb{R}^r$ is the measured output available for feedback control and $z \in \mathbb{R}^q$ is the controlled output. Let $T_{zw}(s)$ denote the closed-loop transfer function matrix from the disturbance w to the controlled output z. The standard H_∞-optimal control problem is concerned with the construction of stabilizing feedback control-laws that minimize the H_∞-norm of $T_{zw}(s)$. We consider three different classes of control laws: static-state feedback, dynamic-state feedback and dynamic-output feedback laws. Furthermore, we denote the infimum of the H_∞-norm achieved under these three classes of feedback laws as γ_s^*, γ_d^* and γ_0^* respectively. Namely,

$\gamma_s^* := \inf\{\|T_{zw}\|_\infty$ where $u(s) = Fx(s)$ for any F which internally stabilizes the system of (2.1), i.e. $A + BF$ is a stability matrix$\}$

$\gamma_d^* := \inf\{\|T_{zw}\|_\infty$ where $u(s) = F_s(s)x(s)$ for any proper transfer function matrix $F_s(s)$ which internally stabilizes the system of (2.1)$\}$

$\gamma_0^* := \inf\{\|T_{zw}\|_\infty$ where $u(s) = F_0(s)y(s)$ for any proper transfer function matrix $F_0(s)$ which internally stabilizes the system of (2.1)$\}$

Zhou and Khargonekar (1988) have shown that $\gamma_d^* = \gamma_s^*$ which also implies that $\gamma_s^* \leqslant \gamma_0^*$. It is also well-known that, in general, γ_0^* is not equal to γ_s^*. In this paper we give a simple and non-iterative procedure for determining γ_0^*. The method is applicable to the general system of (2.1) satisfying the following assumptions.

(A1) The system (A, B, C_2, D_2) is stabilizable and has no invariant zeros in $\mathbb{C}^0$.

(A2) $\operatorname{Im}(E) \subseteq \mathcal{V}^-(A, B, C_2, D_2) \cup \mathcal{S}^-(A, B, C_2, D_2)$.

(B1) The system (A, E, C_1, D_1) is detectable and has no invariant zeros in $\mathbb{C}^0$.

(B2) $\operatorname{Ker}(C_2) \supseteq \mathcal{V}^-(A, E, C_1, D_1) \cap \mathcal{S}^-(A, E, C_1, D_1)$.

Here we would like to note that the above (A1) and (B1) are the standard assumptions on H_∞-optimization literature. On the other hand, (A2) and (B2) generalize the results of Chen *et al.* (1992), in which the subsystems (A, B, C_2, D_2) and (A, E, C_1, D_1) are required to be right- and left-invertible, respectively. In fact, if (A, B, C_2, D_2) and (A, E, C_1, D_1) are respectively right- and left-invertible, then (A2) and (B2) are automatically satisfied.

One of the key components of our method is to put the problem in a special coordinate basis (s.c.b) introduced in Sannuti and Saberi (1987) and Saberi and Sannuti (1990) which explicitly exhibits the finite and infinite zero structures of the system. The other component utilizes the results of Stoorvogel (1991).

3. Preliminaries

In the following section we shall recall the definition of the special coordinate basis (s.c.b) for a linear time-invariant non-strictly proper system (Saberi and Sannuti 1990), and the theorem of Stoorvogel (1991). Such a coordinate basis has a distinct feature of explicitly displaying the finite and infinite zero structures of a given system as well as other system geometric properties. The results of Stoorvogel provide conditions for the existence of an

H_∞-norm bound solution in the output feedback case. They are both instrumental in the derivation of the method described in § 4.

3.1. *Special coordinate basis*

In the following we recapitulate the main results in a theorem and some properties of the special coordinate basis while leaving detailed derivation and proofs to be found in Sannuti and Saberi (1987) and Saberi and Sannuti (1990). Consider the system described by

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ z &= Cx + Du \end{aligned}\right\} \tag{3.1}$$

It can be easily shown that using singular value decomposition one can always find an orthogonal transformation U and a non-singular matrix V that put the direct feedthrough matrix D into the following form

$$\bar{D} = UDV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{3.2}$$

where r is the rank of D. Without loss of generality one can assume that the matrix D in (3.1) has the form as shown in (3.2). Thus, the system in (3.1) can be rewritten as

$$\left.\begin{aligned} \dot{x} &= Ax + [B_0 \quad B_1]\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\ \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} &= \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{aligned}\right\} \tag{3.3}$$

where B_0, B_1, C_0 and C_1 are the matrices of appropriate dimensions. Note that the inputs u_0 and u_1, and the outputs z_0 and z_1 are those of the transformed system. Namely,

$$u = V\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = Uz$$

Note that the H_∞-norm of the system transfer function $T_{zw}(s)$ is unchanged when we apply an orthogonal transformation on the output z, and also under any non-singular transformations on the states and control inputs. We have the following main theorem.

Theorem 3.1: *There exist non-singular transformations Γ_s, Γ_0 and Γ_i such that*

$$x = \Gamma_s[(x_a^+)^{\mathrm{T}}, x_b^{\mathrm{T}}, (x_a^-)^{\mathrm{T}}, x_c^{\mathrm{T}}, x_f^{\mathrm{T}}]^{\mathrm{T}}$$

$$[z_0^{\mathrm{T}}, z_1^{\mathrm{T}}]^{\mathrm{T}} = \Gamma_0[z_0^{\mathrm{T}}, z_f^{\mathrm{T}}, z_b^{\mathrm{T}}]^{\mathrm{T}}, \; [u_0^{\mathrm{T}}, u_1^{\mathrm{T}}]^{\mathrm{T}} = \Gamma_i[u_0^{\mathrm{T}}, u_f^{\mathrm{T}}, u_c^{\mathrm{T}}]^{\mathrm{T}}$$

and

$$\bar{A} := \Gamma_s^{-1}(A - B_0C_0)\Gamma_s = \begin{bmatrix} A_{aa}^+ & L_{ab}^+C_b & 0 & 0 & L_{af}^+C_f \\ 0 & A_{bb} & 0 & 0 & L_{bf}C_f \\ 0 & L_{ab}^-C_b & A_{aa}^- & 0 & L_{af}^-C_f \\ B_cE_{ca}^+ & L_{cb}C_b & B_cE_{ca}^- & A_{cc} & L_{cf}C_f \\ B_fE_{fa}^+ & B_fE_{fb} & B_fE_{fa}^- & B_fE_{fc} & A_{ff} \end{bmatrix} \tag{3.4}$$

$$\bar{B} := \Gamma_s^{-1}[B_0 \;\; B_1]\Gamma_i = \begin{bmatrix} B_{0a}^{+} & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0a}^{-} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0f} & B_f & 0 \end{bmatrix} \tag{3.5}$$

$$\bar{C} := \Gamma_0^{-1}\begin{bmatrix} C_0 \\ \\ C_1 \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0a}^{+} & C_{0b} & C_{0a}^{-} & C_{0c} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & C_b & 0 & 0 & 0 \end{bmatrix} \tag{3.6}$$

and

$$\bar{D} := \Gamma_0^{-1}D\Gamma_i = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.7}$$

where the pair (A_{cc}, B_c) *is controllable, pair* (A_{bb}, C_b) *is observable and the subsystem* (A_{ff}, B_f, C_f) *is invertible with no invariant zeros.*

The proof of this theorem can be found in Sannuti and Saberi (1987) and Saberi and Sannuti (1990). We also note that the output transformation Γ_0 is of form

$$\Gamma_0 = \begin{bmatrix} I_r & 0 \\ 0 & \Gamma_{0r} \end{bmatrix} \tag{3.8}$$

In what follows, we state some important properties of the s.c.b which are pertinent to our present work. For further details regarding s.c.b and its properties, interested readers are referred to Saberi *et al.* (1991).

Property 3.1: *The given system* (A, B, C, D) *is right-invertible if and only if* x_b *and hence* z_b *are non-existent, left-invertible if and only if* x_c *and hence* u_c *are non-existent, invertible if and only if both* x_c *and* x_b *are non-existent.*

Property 3.2: *Invariant zeros of* (A, B, C, D) *are the eigenvalues of* A_{aa}^{-} *and* A_{aa}^{+}. *Moreover, the stable and unstable invariant zeros of* (A, B, C, D) *are the eigenvalue of* A_{aa}^{-} *and* A_{aa}^{+} *respectively.*

Property 3.3: *The pair* (A, B) *is stabilizable if and only if* (A_{con}, B_{con}) *is stabilizable where*

$$A_{con} = \begin{bmatrix} A_{aa}^{+} & L_{ab}^{+}C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{con} = \begin{bmatrix} B_{0a}^{+} & L_{af}^{+} \\ B_{0b} & L_{bf} \end{bmatrix} \tag{3.9}$$

There are interconnections between the s.c.b and various invariant and almost-invariant geometric subspaces. We list in the following the geometrical interpretations of some state vector components of s.c.b.

Property 3.4:

(1) $x_a^{-} \oplus x_a^{+} \oplus x_c$ *spans* $\mathcal{V}^*(A, B, C, D)$.

(2) $x_a^{-} \oplus x_c$ *spans* $\mathcal{V}^{-}(A, B, C, D)$.

(3) $x_a^{+} \oplus x_c$ *spans* $\mathcal{V}^{+}(A, B, C, D)$.

(4) $x_c \oplus x_f$ *spans* $\mathcal{S}^*(A, B, C, D)$.

(5) $x_a^{-} \oplus x_c \oplus x_f$ *spans* $\mathcal{S}^{+}(A, B, C, D)$.

(6) $x_a^{+} \oplus x_c \oplus x_f$ *spans* $\mathcal{S}^{-}(A, B, C, D)$.

3.2. *Stoorvogel's theorem*

We recall in this subsection a main theorem of Stoorvogel (1991) that will play an important role in our present work. Before we introduce the theorem, let us define the following quadratic matrices,

$$F_\gamma(P) := \begin{bmatrix} A^{\mathrm T}P + PA + C_2^{\mathrm T}C_2 + \gamma^{-2}PEE^{\mathrm T}P & PB + C_2^{\mathrm T}D_2 \\ B^{\mathrm T}P + D_2^{\mathrm T}C_2 & D_2^{\mathrm T}D_2 \end{bmatrix} \quad (3.10)$$

and

$$G_\gamma(Q) := \begin{bmatrix} AQ + QA^{\mathrm T} + EE^{\mathrm T} + \gamma^{-2}QC_2^{\mathrm T}C_2Q & QC_1^{\mathrm T} + ED_1^{\mathrm T} \\ C_1Q + D_1E^{\mathrm T} & D_1D_1^{\mathrm T} \end{bmatrix} \quad (3.11)$$

It should be noted that the above matrices are dual of each other. In addition to these two matrices, we define two polynomial matrices whose role is again completely dual.

$$L(P, s) := [sI - A - \gamma^{-2}EE^{\mathrm T}P \;\; -B] \quad (3.12)$$

and

$$M(Q, s) := \begin{bmatrix} sI - A - \gamma^{-2}QC_2^{\mathrm T}C_2 \\ -C_1 \end{bmatrix} \quad (3.13)$$

Now we are ready to introduce the theorem of Stoorvogel (1991). We have the following theorem.

Theorem 3.2: *Consider system (2.1). Assume that (A, B, C_2, D_2) and (A, E, C_1, D_1) have no invariant zeros in $\mathbb{C}^0$. Then the following statements are equivalent.*

(1) *There exists a linear, time-invariant and proper dynamic compensator $F_0(s)$ such that by applying $u(s) = F_0(s)y(s)$ in (2.1) the resulting closed-loop system is internally stable. Moreover, the H_∞-norm of the closed-loop transfer function from the disturbance input w to the controlled output z is less than γ.*

(2) *There exist positive semi-definite solutions P, Q of the quadratic matrix inequalities $F_\gamma(P) \geq 0$ and $G_\gamma(Q) \geq 0$ satisfying $\rho(PQ) < \gamma^2$, such that the following rank conditions are satisfied:*

(*a*) $\operatorname{rank}\{F_\gamma(P)\} = \operatorname{normrank}\{G_2(s)\}$

(*b*) $\operatorname{rank}\{G_\gamma(Q)\} = \operatorname{normrank}\{G_1(s)\}$

(*c*) $\operatorname{rank}\begin{bmatrix} L(P, s) \\ F_\gamma(P) \end{bmatrix} = n + \operatorname{normrank}\{G_2(s)\}, \ \forall s \in \mathbb{C}^0 \cup \mathbb{C}^+$

(*d*) $\operatorname{rank}[M(Q, s), G_\gamma(Q)] = n + \operatorname{normrank}\{G_1(s)\}, \ \forall s \in \mathbb{C}^0 \cup \mathbb{C}^+$

where $G_1(s) = C_1(sI - A)^{-1}E + D_1$, $G_2(s) = C_2(sI - A)^{-1}B + D_2$ and 'normrank' *denotes the rank of a matrix with entries in the field of rational functions.*

Proof: For the proof see Stoorvogel (1991). □

4. Computational algorithm for γ_0^*

The algorithm for γ_0^* involves the computation of two non-negative scalars γ_P^* and γ_Q^* which are respectively the infima in H_∞-optimization of the system Σ

and its dual, where in each case the measurement output is replaced by the system state. Computation of γ_P^* and γ_Q^* provides the necessary preliminary for the computation of γ_0^*.

The following § 4.1 and 4.2 deal with the definition and computation of γ_P^* and γ_Q^* respectively, while in § 4.3 we present our main theorem regarding the computation of γ_0^*.

4.1 *Computation of* γ_P^*

We define non-negative scalar γ_P^* as the infimum of H_∞-optimization for the system,

$$\Sigma_P: \begin{cases} \dot{x} = Ax + Bu + Ew \\ y = x \\ z = C_2x + D_2u \end{cases} \tag{4.1}$$

By definition, γ_P^* is clearly equal to γ_S^*. However we use the terms γ_P^* and γ_Q^* in the next subsection to conform with the notation of matrix inequalities in Stoorvogel's theorem. In what follows, we introduce a step-by-step procedure to compute γ_P^*.

Step 1

Transform the system (A, B, C_2, D_2) into the special coordinate basis (s.c.b) described in § 3. To all sub-matrices and transformations in the s.c.b of Σ_P, we append the subscript '$_P$' to signify their relation to the system Σ_P. Next we compute

$$\Gamma_{sP}^{-1}E = [(E_{aP}^+)^T \quad (E_{bP})^T \quad (E_{aP}^-)^T \quad (E_{cP})^T \quad (E_{fP})^T]^T \tag{4.2}$$

It is simple to verify from the properties of s.c.b that the assumption (A2) implies $E_{bP} = 0$. Also, for economy of notation, we denote n_P the dimension of $\mathbb{R}^n/\mathscr{S}^+(A, B, C_2, D_2)$. We note that $n_P = 0$ if and only if the system (A, B, C_2, D_2) is right-invertible and is of minimum phase.

Step 2

If the system (A, B, C_2, D_2) is of non-minimum phase and/or not right invertible, we define

$$A_{11P} := \begin{bmatrix} A_{aaP}^+ & L_{abP}^+C_{bP} \\ 0 & A_{bbP} \end{bmatrix}, \quad B_{11P} := \begin{bmatrix} B_{0aP}^+ \\ B_{0bP} \end{bmatrix}, \quad A_{13P} := \begin{bmatrix} L_{afP}^+ \\ L_{bfP} \end{bmatrix}$$

$$C_{21P} := \Gamma_{0rP}\begin{bmatrix} 0 & 0 \\ 0 & C_{bP} \end{bmatrix}, \quad C_{23P} := \Gamma_{0rP}\begin{bmatrix} C_{fP}C_{fP}^T \\ 0 \end{bmatrix}$$

and

$$A_P := A_{11P} - A_{13P}(C_{23P}^TC_{23P})^{-1}C_{23P}^TC_{21P}$$

$$B_PB_P^T := B_{11P}B_{11P}^T + A_{13P}(C_{23P}^TC_{23P})^{-1}A_{13P}^T$$

$$C_P^TC_P := C_{21P}^TC_{21P} + C_{21P}^TC_{23P}(C_{23P}^TC_{23P})^{-1}C_{23P}^TC_{21P}$$

Then we solve for the positive definite solution S_P of the algebraic matrix Riccati equation,

$$A_P S_P + S_P A_P^T - B_P B_P^T + S_P C_P^T C_P S_P = 0 \tag{4.3}$$

together with the matrix T_P defined by

$$T_P := \begin{bmatrix} T_{aaP} & 0 \\ 0 & 0 \end{bmatrix}$$

where T_{aaP} is the unique solution of the algebraic matrix Lyapunov equation,

$$A_{aaP}^+ T_{aaP} + T_{aaP}(A_{aaP}^+)^T = E_{aP}^+(E_{aP}^+)^T \tag{4.4}$$

Here we note that $(-A_P, C_P)$ is detectable since $-A_{aaP}$ is stable and (A_{bbP}, C_{bP}) is observable. Also, the assumption (A1) implies that (A_P, B_P) is stabilizable. Hence the existence and uniqueness of the solutions S_P and T_{aaP} follow from the results of Richardson and Kwong (1986).

Step 3

The scalar γ_P^* is given by

$$\gamma_P^* = \begin{cases} \sqrt{\lambda_{\max}(T_P S_P^{-1})} & \text{if } n_P > 0 \\ 0 & \text{if } n_P = 0 \end{cases} \tag{4.5}$$

Here we note that the eigenvalues of $(T_P S_P^{-1})$ are real and non-negative. (It is shown in Wielandt (1973) that AB has as many positive, zero and negative eigenvalues as A, if A is hermitian and B is hermitian and positive definite.)

Theorem 4.1: *Consider the system Σ_P given by* (4.1). *Then under the assumptions* (A1) *and* (A2),

(1) *γ_P^* is the infimum of H_∞-optimization for Σ_P.*

(2) *for $\gamma > \gamma_P^*$, the positive semi-definite matrix $P(\gamma)$ given by*

$$P(\gamma) = (\Gamma_{sP}^{-1})^T \begin{bmatrix} P_0(\gamma) & 0 \\ 0 & 0 \end{bmatrix} \Gamma_{sP}^{-1} \tag{4.6}$$

where

$$P_0(\gamma) = \begin{cases} (S_P - \gamma^{-2} T_P)^{-1} & \text{if } n_P > 0 \\ 0 & \text{if } n_P = 0 \end{cases} \tag{4.7}$$

is the unique solution of the matrix inequality $F_\gamma(P(\gamma)) \geq 0$ and satisfies both rank conditions (a) and (c) of Theorem 3.2. Moreover, such a solution $P(\gamma)$ does not exist when $\gamma < \gamma_P^*$.

Proof: This is a slight generalization of the result in Chen *et al.* (1990). It can be easily shown following arguments similar to those in Chen *et al.* (1990). □

Remark 4.1: Note that item (2) of the above theorem implies that $\gamma_0^* \geq \gamma_P^*$. We would also like to note that a similar result was obtained by Scherer (1990) under the assumption that the system (A, B, C_2, D_2) has no infinite zeros. □

The next lemma provides the necessary and sufficient condition for $\gamma_P^* = 0$.

Lemma 4.1: $\gamma_P^* = 0$ *if and only if* $\operatorname{Im}(E) \subseteq \mathscr{S}^+(A, B, C_2, D_2)$.

Proof: Again, this is a slight generalization of the result in Chen *et al.* (1990). In fact, the above result still holds when assumption (A2) is removed. □

4.2. *Computation of* γ_Q^*

As in the definition of γ_P^*, the non-negative scalar γ_Q^* is defined as the infimum in H_∞-optimization for the dual system,

$$\Sigma_Q : \begin{cases} \dot{x} = A^T x + C_1^T u + C_2^T w \\ y = x \\ z = E^T x + D_1^T u \end{cases} \tag{4.8}$$

Determination of γ_Q^* follows exactly the procedure described in § 4.1 for the computation of γ_P^* where it now applies to the subsystem Σ_Q of (4.8). For completeness and to define properly matrices required in the computation γ_0^* and in our main theorem of § 4.3, we re-iterate here the three steps involved in the computation of γ_Q^*.

Step 1

Transform the system (A^T, C_1^T, E^T, D_1^T) into the special coordinate basis (s.c.b) described in § 3. Again we add here the subscript '$_Q$' to all submatrices and transformations in the s.c.b of the system Σ_Q. Next we compute

$$\Gamma_{sQ}^{-1} C_2^T = [(E_{aQ}^+)^T \ (E_{bQ})^T \ (E_{aQ}^-)^T \ (E_{cQ})^T \ (E_{fQ})^T]^T \tag{4.9}$$

It is simple to show from the properties of s.c.b that the assumption (B2) implies $E_{bQ} = 0$. Again, for the economy of notation, we denote n_Q the dimension of $\mathcal{V}^+(A, E, C_1, D_1)$. Note that $n_Q = 0$ if and only if the system (A, E, C_1, D_1) is left-invertible and is of minimum phase.

Step 2

If the system (A, E, C_1, D_1) is of non-minimum phase and/or not left invertible, we define

$$A_{11Q} := \begin{bmatrix} A_{aaQ}^+ & L_{abQ}^+ C_{bQ} \\ 0 & A_{bbQ} \end{bmatrix}, \quad B_{11Q} := \begin{bmatrix} B_{0bQ}^+ \\ B_{0bQ} \end{bmatrix}, \quad A_{13Q} := \begin{bmatrix} L_{afQ}^+ \\ L_{bfQ} \end{bmatrix}$$

$$C_{21Q} := \Gamma_{0rQ} \begin{bmatrix} 0 & 0 \\ 0 & C_{bQ} \end{bmatrix}, \quad C_{23Q} := \Gamma_{0rQ} \begin{bmatrix} C_{fQ} C_{fQ}^T \\ 0 \end{bmatrix}$$

and

$$\begin{aligned} A_Q &:= A_{11Q} - A_{13Q}(C_{23Q}^T C_{23Q})^{-1} C_{23Q}^T C_{21Q} \\ B_Q B_Q^T &:= B_{11Q} B_{11Q}^T + A_{13Q}(C_{23}^T C_{23Q})^{-1} A_{13Q}^T \\ C_Q^T C_Q &:= C_{21Q}^T C_{21Q} + C_{21Q}^T C_{23Q}(C_{23Q}^T C_{23Q})^{-1} C_{23Q}^T C_{21Q} \end{aligned}$$

Then we solve for the positive definite solution S_Q of the algebraic matrix Riccati equation,

$$A_Q S_Q + S_Q A_Q^T - B_Q B_Q^T + S_Q C_Q^T C_Q S_Q = 0 \tag{4.10}$$

together with the matrix T_Q defined by

$$T_Q := \begin{bmatrix} T_{aaQ} & 0 \\ 0 & 0 \end{bmatrix}$$

where T_{aaQ} is the unique solution of the algebraic matrix Lyapunov equation,

$$A_{aaQ}^{+}T_{aaQ} + T_{aaQ}(A_{aaQ}^{+})^{T} = E_{aQ}^{+}(E_{aQ}^{+})^{T} \tag{4.11}$$

Again, existence of the solutions for S_Q and T_Q follows from the assumption (B1) and the properties of s.c.b.

Step 3

The scalar γ_Q^* is given by

$$\gamma_Q^* = \begin{cases} \sqrt{\lambda_{\max}}(T_Q S_Q^{-1}) & \text{if } n_Q > 0 \\ 0 & \text{if } n_Q = 0 \end{cases} \tag{4.12}$$

We note that the eigenvalues of $(T_Q S_Q^{-1})$ are real and non-negative.

Theorem 3.2: *Consider the system Σ_Q given by* (4.8). *Then under the assumptions* (B1) *and* (B2),

(1) *γ_Q^* is the infimum of H_∞-optimization for Σ_Q.*

(2) *for $\gamma > \gamma_Q^*$, the positive semi-definite matrix $Q(\gamma)$ given by*

$$Q(\gamma) = (\Gamma_{sQ}^{-1})^{T} \begin{bmatrix} Q_0(\gamma) & 0 \\ 0 & 0 \end{bmatrix} \Gamma_{sQ}^{-1} \tag{4.13}$$

where

$$Q_0(\gamma) = \begin{cases} (S_Q - \gamma^{-2} T_Q)^{-1} & \textit{if } n_Q > 0 \\ 0 & \textit{if } n_Q = 0 \end{cases} \tag{4.14}$$

is the unique solution of the matrix inequality $G_\gamma(Q(\gamma)) \geqslant 0$ and satisfies both rank conditions (b) and (d) of Theorem 3.2. Moreover, such a solution $Q(\gamma)$ does not exist when $\gamma < \gamma_Q^*$.

Proof: This is a dual version of Theorem 4.1. □

Remark 4.2: Note that item (2) of the above theorem also implies that $\gamma_0^* \geqslant \gamma_Q^*$. □

Again analogous to Lemma 4.1 we have

Lemma 4.2: $\gamma_Q^* = 0$ *if and only if* $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \operatorname{Ker}(C_2)$.

Proof: This is a dual version of Lemma 4.1. Again, the result is still true when assumption (B2) is removed. □

4.3. *Computation of γ_0^**

In this subsection, we provide our main results on a simple and non-iterative procedure for the computation of γ_0^*. First of all, we reformulate the computation of γ_0^* in the following lemma.

Lemma 4.3: *Let $\gamma_{PQ}^* = \max\{\gamma_P^*, \gamma_Q^*\}$. Then*

$$\gamma_0^* = \inf\{\gamma \in (\gamma_{PQ}^* \ \infty) | f(\gamma) < \gamma^2\} \tag{4.15}$$

where $f(\gamma) = \rho[P(\gamma)Q(\gamma)]$, and $P(\gamma)$ and $Q(\gamma)$ are given by (4.6) *and* (4.13) *respectively.*

Proof: It follows from Remarks 4.1 and 4.2 that $\gamma_0^* \geqslant \gamma_{PQ}^*$. Next, for any $\hat{\gamma} \in (\gamma_{PQ}^*, \infty)$ such that $f(\hat{\gamma}) < \hat{\gamma}^2$, i.e. $\rho[P(\hat{\gamma})Q(\hat{\gamma})] < \hat{\gamma}^2$, then the corresponding $P(\hat{\gamma})$ and $Q(\hat{\gamma})$ as given in (4.6) and (4.13) satisfy the conditions of Theorem 3.2. Hence, $\hat{\gamma} > \gamma_0^*$. □

One straightforward computation of γ_0^* can be done via an iterative search algorithm that involves in each step the multiplication of two matrices $P(\gamma)$ and $Q(\gamma)$ of dimensions $n \times n$ and the determination of the spectral radius of the product $P(\gamma)Q(\gamma)$. This iterative search is costly and usually involves computation of eigenvalues of stiff matrices since the product $P(\gamma)Q(\gamma)$ could become ill-conditioned as γ approaches γ_{PQ}^* from above. Hence, the overall procedure tends to be ill-conditioned. Note that as γ gets close to γ_P^*, $P(\gamma)$ contains the inverse of an amost singular submatrix and, similarly $Q(\gamma)$ contains the inverse of an almost singular submatrix as γ approaches γ_Q^* (see (4.6) and (4.13)).

In contrast to the above iterative procedure, here we present an elegant, well-conditioned and non-iterative algorithm for the exact computation of γ_0^*. First we derive an explicit expression for $f(\gamma)$ using (4.6) and (4.13). Then in the case where $\min\{n_P, n_Q\} > 0$, we partition the product of the inverses of the s.c.b state transformations as follows,

$$\Gamma_{sP}^{-1}(\Gamma_{sQ}^{-1})^T = \begin{bmatrix} \Gamma & \star \\ \star & \star \end{bmatrix} \tag{4.16}$$

where Γ is of dimension $n_P \times n_Q$.

It is then straightforward to show that the scalar function $f(\gamma)$ is given by

$$f(\gamma) = \begin{cases} \lambda_{\max}[(S_P - \gamma^{-2}T_P)^{-1}\Gamma(S_Q - \gamma^{-2}T_Q)^{-1}\Gamma^T] & \text{if } \min\{n_P, n_Q\} > 0 \\ 0 & \text{if } \min\{n_P, n_Q\} = 0 \end{cases} \tag{4.17}$$

The function $f(\gamma)$ of (4.17) is a well-defined mapping from (γ_{PQ}^*, ∞) to $[0, \infty)$. Its evaluation involves the computation of the maximum eigenvalue of a matrix of dimension $n_P \times n_P$, which is normally of a much smaller dimension than the original product $P(\gamma)Q(\gamma)$. We establish some important properties of the function $f(\gamma)$ in the following proposition.

Proposition 4.1: *$f(\gamma)$ is a continuous, non-negative and non-increasing function of γ on (γ_{PQ}^*, ∞).*

Proof: The proof follows from Observation 4.1 of Chen *et al.* (1992). □

The function $f(\gamma)$ defined above can be extended as a mapping from $[\gamma_{PQ}^*, \infty)$ to $[0, \infty)$ by setting $f(\gamma_{PQ}^*) = \lim_{\gamma \to \gamma_{PQ}^*} f(\gamma)$. It follows from Proposition 4.1 that the limit $f(\gamma_{PQ}^*)$ exists and could be finite or infinite.

Before stating our main result of this subsection regarding the computation of γ_0^*, we need to establish several important propositions.

Proposition 4.2: *$f(\gamma) = \gamma^2$ has either no solution or a unique solution in the interval (γ_{PQ}^*, ∞).*

Proof: The result follows from Proposition 4.1 and the fact that γ^2 is strictly increasing for positive γ. □

Proposition 4.3: *If $f(\gamma)=\gamma^2$ has no solution in the interval (γ^*_{PQ},∞) then γ^*_0 is equal to γ^*_{PQ}. Otherwise, γ^*_0 is equal to the unique solution of $f(\gamma)=\gamma^2$ in the interval (γ^*_{PQ},∞).*

Proof: $f(\gamma)=\gamma^2$ has no solution in the interval (γ^*_{PQ},∞) implies that $f(\gamma)<\gamma^2$ for all $\gamma\in(\gamma^*_{PQ},\infty)$ and hence according to Lemma 4.3, $\gamma^*_0=\gamma^*_{PQ}$. On the other hand, it is obvious that γ^*_0 is equal to the unique solution of $f(\gamma)=\gamma^2$ when such a solution exists. □

At a first glance, it seems that the solution of $f(\gamma)=\gamma^2$ would involve the rooting of a highly nonlinear algebraic equation in γ. Actually its solution can be achieved in one-step. Namely, the problem of solving $f(\gamma)=\gamma^2$, if such a solution exists in the interval (γ^*_{PQ},∞), can be converted to the problem of calculating the maximum eigenvalue of a constant matrix. In fact, we also show that, when $f(\gamma)=\gamma^2$ has no solution in the interval (γ^*_{PQ},∞), the maximum eigenvalue of this matrix is equal to γ^*_{PQ}, which is γ^*_0 as well. Define

$$N(\gamma)=:\begin{cases}(S_P-\gamma^{-2}T_P)^{-1}\Gamma(S_Q-\gamma^{-2}T_Q)^{-1}\Gamma^T-\gamma^2 I & \text{if } \min\{n_P,n_Q\}>0\\ -\gamma^2 I & \text{if } \min\{n_P,n_Q\}=0\end{cases}\tag{4.18}$$

and

$$M:=\begin{cases}\begin{bmatrix}T_PS_P^{-1}+\Gamma S_Q^{-1}\Gamma^TS_P^{-1} & -\Gamma S_Q^{-1}\\ -T_QS_Q^{-1}\Gamma^TS_P^{-1} & T_QS_Q^{-1}\end{bmatrix} & \text{if } n_P>0 \text{ and } n_Q>0\\ T_PS_P^{-1} & \text{if } n_P>0 \text{ and } n_Q=0\\ T_QS_Q^{-1} & \text{if } n_P=0 \text{ and } n_Q>0\\ 0 & \text{if } n_P=0 \text{ and } n_Q=0\end{cases}\tag{4.19}$$

We have the following propositions on the matrices M and $N(\gamma)$.

Proposition 4.4: *Eigenvalues of M are real and non-negative.*

Proof: It is trivial when $\min\{n_P,n_Q\}=0$. For the case where $\min\{n_P,n_Q\}>0$, we have

$$\begin{aligned}\lambda[M]&=\lambda\left\{\begin{bmatrix}I&0\\0&T_Q\end{bmatrix}\begin{bmatrix}T_P+\Gamma S_Q^{-1}\Gamma^T & -\Gamma S_Q^{-1}\\ -S_Q^{-1}\Gamma^T & S_Q^{-1}\end{bmatrix}\begin{bmatrix}S_P^{-1}&0\\0&I\end{bmatrix}\right\}\\ &=\lambda\left\{\begin{bmatrix}S_P^{-1}&0\\0&I\end{bmatrix}\begin{bmatrix}I&0\\0&T_Q\end{bmatrix}\begin{bmatrix}T_P+\Gamma S_Q^{-1}\Gamma^T & -\Gamma S_Q^{-1}\\ -S_Q^{-1}\Gamma^T & S_Q^{-1}\end{bmatrix}\right\}\\ &=\lambda\left\{\begin{bmatrix}S_P^{-1}&0\\0&T_Q\end{bmatrix}\begin{bmatrix}T_P+\Gamma S_Q^{-1}\Gamma^T & -\Gamma S_Q^{-1}\\ -S_Q^{-1}\Gamma^T & S_Q^{-1}\end{bmatrix}\right\}\end{aligned}\tag{4.20}$$

Now, it is trivial to verify that both sub-matrices in (4.20) are symmetric and positive semidefinite. Then using the result of Weilandt (1973) (i.e. Theorem 3), it is simple to show that the eigenvalues of M are real and non-negative. □

Proposition 4.5:

(i) *$N(\gamma)$ has real eigenvalues for all $\gamma\in(\gamma^*_{PQ},\infty)$.*

(ii) $\lambda_{\max}[N(\gamma)] = f(\gamma) - \gamma^2$ *is continuous and strictly decreasing on* (γ^*_{PQ}, ∞).

Proof: Again it is trivial when $\min\{n_P, n_Q\} = 0$. For the case where $\min\{n_P, n_Q\} > 0$, we have

(i) It is straightforward to show that $(S_P - \gamma^{-2}T_P)^{-1} > 0$ and $(S_Q - \gamma^{-2}T_Q)^{-1} > 0$ for all $\gamma \in (\gamma^*_{PQ}, \infty)$. Hence, all the eigenvalues of $N(\gamma)$ are real for $\gamma \in (\gamma^*_{PQ}, \infty)$.

(ii) It follows from Proposition 4.1. □

Proposition 4.6: *If* $\min\{n_P, n_Q\} > 0$, *then the roots of* $\det[N(\gamma)] = 0$ *are real. Moreover, the largest root of* $\det[N(\gamma)] = 0$ *in the interval* (γ^*_{PQ}, ∞) *is equal to* $\sqrt{\lambda_{\max}(M)}$.

Proof: Using the definition of $N(\gamma)$ in (4.18), we have

$$\begin{aligned}
\det[N(\gamma)] &= (-1)^{n_P} \det[\gamma^2 I - (S_P - \gamma^{-2}T_P)^{-1}\Gamma(S_Q - \gamma^{-2}T_Q)^{-1}\Gamma^T] \\
&= \frac{(-1)^{n_P}}{\det[S_P - \gamma^{-2}T_P]} \det[\gamma^2 S_P - T_P - \gamma^2\Gamma(\gamma^2 S_Q - T_Q)^{-1}\Gamma^T] \\
&= \frac{(-1)^{n_P}}{\det[S_P - \gamma^{-2}T_P]\det[\gamma^2 S_Q - T_Q]} \det\begin{bmatrix} \gamma^2 S_P - T_P & \Gamma \\ \gamma^2\Gamma^T & \gamma^2 S_Q - T_Q \end{bmatrix} \\
&= \frac{(-1)^{n_P}\det[S_P]\det[S_Q]}{\det[S_P - \gamma^{-2}T_P]\det[\gamma^2 S_Q - T_Q]} \det[\gamma^2 I - M] \qquad (4.21)
\end{aligned}$$

Now it is simple to see that the roots of $\det[N(\gamma)] = 0$ are real since all the roots of $\det[\gamma^2 S_P - T_P] = 0$, $\det[\gamma^2 S_Q - T_Q] = 0$ and $\det[\gamma^2 I - M] = 0$ are real. Moreover, it follows from (4.5) and (4.12) that $\det[S_P - \gamma^{-2}T_P] \neq 0$ and $\det[\gamma^2 S_Q - T_Q] \neq 0$ for all $\gamma \in (\gamma^*_{PQ}, \infty)$. Hence, the largest root of $\det[N(\gamma)] = 0$ in (γ^*_{PQ}, ∞) is equal to the largest root of $\det[\gamma^2 I - M] = 0$, which is equal to $\sqrt{\lambda_{\max}(M)}$. □

The main result of this subsection is summarized in the following theorem.

Theorem 4.3:

$$\gamma^*_0 = \sqrt{\lambda_{\max}(M)}$$

where M is defined in (4.19).

Proof: The result is obvious for the case where $\min\{n_P, n_Q\} = 0$. In what follows, we proceed to prove our claim for the case where $\min\{n_P, n_Q\} > 0$.

First, we will show that γ^*_0 is equal to the largest root of $\det[N(\gamma)] = 0$ when $f(\gamma) = \gamma^2$ has a unique solution in the interval (γ^*_{PQ}, ∞). It is simple to observe that $\det[N(\gamma^*_0)] = 0$ since $\lambda_{\max}[N(\gamma^*_0)] = f(\gamma^*_0) - (\gamma^*_0)^2 = 0$. Now suppose that there exists a γ_1 such that $\det[N(\gamma_1)] = 0$ and $\gamma_1 > \gamma^*_0$. This implies that there exists an eigenvalue of $N(\gamma_1)$, say $\lambda_i[N(\gamma_1)]$, such that $\lambda_i[N(\gamma_1)] \neq \lambda_{\max}[N(\gamma_1)]$ and $\lambda_i[N(\gamma_1)] = 0$. Thus, we have

$$\lambda_{\max}[N(\gamma_1)] > \lambda_i[N(\gamma_1)] = 0 = \lambda_{\max}[N(\gamma^*_0)],$$

contradicting the findings in Proposition 4.5 that $\lambda_{\max}[N(\gamma)]$ must be a non-increasing function. Hence, γ^*_0 is the largest root of $\det[N(\gamma)] = 0$ and it is equal to $\sqrt{\lambda_{\max}(M)}$ as shown in Proposition 4.6.

Now we consider the situation when $f(\gamma)=\gamma^2$ has no solution in the interval (γ^*_{PQ},∞). In this case, clearly we have $\gamma^*_0=\gamma^*_{PQ}$ and $0\leqslant f(\gamma^*_{PQ})\leqslant(\gamma^*_{PQ})^2$. The last inequality and the definition of $N(\gamma)$ in (4.18) imply that $-(\gamma^*_{PQ})^2\leqslant\lambda_i[N(\gamma^*_{PQ})]\leqslant 0$. Thus, the determinant of $N(\gamma^*_{PQ})$ is bounded. Evaluatng (4.21) at $\gamma=\gamma^*_{PQ}$, we have

$$\det[N(\gamma^*_{PQ})]\det[S_P-(\gamma^*_{PQ})^{-2}T_P]\det[(\gamma^*_{PQ})^2S_Q-T_Q] = (-1)^{n_P}\det[S_P]\det[S_Q]\det[(\gamma^*_{PQ})^2I-M] \quad (4.22)$$

Note that from (4.5) and (4.12) and the definition of γ^*_{PQ}, we have

$$\det[S_P-(\gamma^*_{PQ})^{-2}T_P]\det[(\gamma^*_{PQ})^2S_Q-T_Q]=0$$

and since $\det[N(\gamma^*_{PQ})]$ is bounded, it follows from (4.22) that

$$\det[(\gamma^*_{PQ})^2I-M]=0$$

or $(\gamma^*_{PQ})^2$ is an eigenvalue of M. Furthermore since $\det[N(\gamma)]=0$ and similarly $\det[\gamma^2I-M]=0$ do not have a root in (γ^*_{PQ},∞), hence $\gamma^*_{PQ}=\sqrt{\lambda_{\max}(M)}$. □

We illustrate our main result in the following example.

Example: Consider a given system characterized by

$$A=\begin{bmatrix}1&1&1&0&1\\0&1&0&0&1\\0&1&1&0&1\\1&1&1&1&1\\1&1&1&1&0\end{bmatrix},\quad B=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad E=\begin{bmatrix}5&1\\0&0\\0&0\\2&3\\1&4\end{bmatrix}$$

$$C_1=\begin{bmatrix}0&-2&-3&-2&-1\\1&2&3&2&1\end{bmatrix},\quad D_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

$$C_2=\begin{bmatrix}0&0&1&0&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\end{bmatrix},\quad D_2=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Step 1

It is simple to verify that the subsystem (A,B,C_2,D_2) is neither left- nor right-invertible with one invariant zero at $s=1$. Also, assumption (A 2) is satisfied. Moreover, it is already in the form of s.c.b with

$$A_P=\begin{bmatrix}1&1&1\\0&1&0\\0&1&0\end{bmatrix},\quad B_PB_P^T=\begin{bmatrix}1&1&1\\1&1&1\\1&1&2\end{bmatrix},\quad C_P^TC_P=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

and

$$A^+_{aaP}=1,\quad E^+_{aP}=[5\ \ 1]$$

Then, solving equations (4.3) and (4.4), we obtain

$$S_P=\begin{bmatrix}0{\cdot}556281&0{\cdot}185427&-0{\cdot}305593\\0{\cdot}185427&0{\cdot}395142&0{\cdot}231469\\-0{\cdot}305593&0{\cdot}231469&1{\cdot}217984\end{bmatrix},\quad T_P=\begin{bmatrix}13&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Step 2

The subsystem (A, E, C_1, D_1) is invertible and of non-minimum phase with invariant zeros at $\{-1{\cdot}630662, -3{\cdot}593415, 0{\cdot}521129 \pm j0{\cdot}363043\}$. Hence, assumption (B 2) is automatically satisfied. Applying s.c.b transformation to $(A^{\mathrm{T}}, C_1^{\mathrm{T}}, E^{\mathrm{T}}, D_1^{\mathrm{T}})$, we obtain

$$\Gamma_{sQ} = \begin{bmatrix} -0{\cdot}011218 & -0{\cdot}106028 & -0{\cdot}906482 & -0{\cdot}212184 & 0{\cdot}090909 \\ 0{\cdot}185213 & -0{\cdot}745725 & 0{\cdot}194520 & -0{\cdot}119195 & 0{\cdot}181818 \\ -0{\cdot}919232 & 0{\cdot}096732 & 0{\cdot}326906 & -0{\cdot}603079 & 0{\cdot}272727 \\ 0{\cdot}279141 & 0{\cdot}532936 & 0{\cdot}087364 & -0{\cdot}581308 & 0{\cdot}181818 \\ -0{\cdot}206551 & -0{\cdot}373195 & 0{\cdot}161098 & 0{\cdot}489027 & 0{\cdot}090909 \end{bmatrix}$$

$$\Gamma_{0rQ} = 1$$

$$A_Q = A^+_{aaQ} = \begin{bmatrix} 0{\cdot}433179 & -0{\cdot}253237 \\ 0{\cdot}551005 & 0{\cdot}609080 \end{bmatrix}, \quad B_Q B_Q^{\mathrm{T}} = \begin{bmatrix} 0{\cdot}033508 & -0{\cdot}018630 \\ -0{\cdot}018630 & 0{\cdot}030289 \end{bmatrix}$$

and

$$C_Q^{\mathrm{T}} C_Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^+_{aQ} = \begin{bmatrix} -0{\cdot}769496 & 0{\cdot}010023 & 0{\cdot}448951 & -0{\cdot}769496 \\ -0{\cdot}090061 & 0{\cdot}655677 & -1{\cdot}044466 & -0{\cdot}090061 \end{bmatrix}$$

Again, solving equations (4.10) and (4.11), we obtain

$$S_Q = \begin{bmatrix} 0{\cdot}026333 & -0{\cdot}021114 \\ -0{\cdot}021114 & 0{\cdot}043965 \end{bmatrix}, \quad T_Q = \begin{bmatrix} 1{\cdot}274771 & -0{\cdot}555799 \\ -0{\cdot}555799 & 1{\cdot}764580 \end{bmatrix}$$

Step 3

Evaluate

$$M = 10^2 \times \begin{bmatrix} 0{\cdot}500695 & -0{\cdot}334250 & 0{\cdot}245016 & 0{\cdot}082332 & 0{\cdot}052125 \\ -0{\cdot}442374 & 0{\cdot}992368 & -0{\cdot}260321 & 0{\cdot}032515 & 0{\cdot}253182 \\ 0{\cdot}616882 & -0{\cdot}513348 & 0{\cdot}588766 & 0{\cdot}501907 & 0{\cdot}261525 \\ 1{\cdot}074941 & -1{\cdot}295698 & 0{\cdot}921909 & 0{\cdot}622391 & 0{\cdot}172484 \\ -0{\cdot}583103 & 1{\cdot}526365 & -0{\cdot}286520 & 0{\cdot}180099 & 0{\cdot}487850 \end{bmatrix}$$

We obtain

$$\gamma_0^* = 13{\cdot}638725$$

□

5. Other related results

Results developed in § 4 can also be used to examine solvability conditions of almost disturbance decoupling problems with internal stability and to establish exact conditions where $\gamma_0^* = \gamma_s^*$.

5.1. *Almost disturbance decoupling with stability*

The problem of almost disturbance decoupling was first introduced by Willems (see Weiland and Willems (1989) for a recent result and related references). The basic problem is the design of a linear time-invariant internally stabilizing controller using output feedback such that the controlled output z is approximately decoupled from the disturbance input w. The more precise definitions of these problems are given below.

Definition 5.1: Consider the system of (2.1) with $C_1 = I$ and $D_1 = 0$, i.e., $y = x$. Then we say that the H_∞-Almost Disturbance Decoupling Problem with internal Stability $(\mathrm{ADDPS})_{H_\infty}$ is solvable if for all $\varepsilon > 0$ there exists a state feedback law $u = Fx$ for the system defined above such that the closed-loop system is internally stable and the H_∞-norm of the transfer function between the disturbance input w and the controlled output z is less than ε. □

Definition 5.2: Consider the system of (2.1), we say that the H_∞-Almost Disturbance Decoupling Problem with Measurement feedback and internal Stability $(\mathrm{ADDPMS})_{H_\infty}$ is solvable if for all $\varepsilon > 0$ there exists an output feedback law $u(s) = F_0(s)y(s)$ such that the closed-loop system is internally stable and the H_∞-norm of the transfer function between the disturbance input w and the controlled output z is less than ε. □

From the above formulation, it is obvious that solvability conditions for $(\mathrm{ADDPS})_{H_\infty}$ and $(\mathrm{ADDPMS})_{H_\infty}$ are exactly the conditions where $\gamma_s^* = 0$ and $\gamma_0^* = 0$ respectively. Solvability conditions for $(\mathrm{ADDPS})_{H_\infty}$ with $D_2 = 0$ and for $(\mathrm{ADDPMS})_{H_\infty}$ with $D_1 = 0$ and $D_2 = 0$ are well-known (see Weiland and Willems 1989). In the following theorem, we extend these results to the general case when $D_1 \neq 0$ and/or $D_2 \neq 0$.

Theorem 5.1: *Consider the system Σ as given by* (2.1). *Let $C_1 = I$ and $D_1 = 0$, i.e. $y = x$. Then $(ADDPS)_{H_\infty}$ is solvable under the assumption* (A1) *if and only if* $\mathrm{Im}(E) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$.

Proof: The proof follows from Lemma 4.1. □

Theorem 5.2: *Consider the system Σ as given by* (2.1). *Then $(ADDPMS)_{H_\infty}$ is solvable under the assumptions* (A1) *and* (B1) *if and only if*

(1) $\mathrm{Im}(E) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$,

(2) $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathrm{Ker}(C_2)$,

(3) $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$.

Proof. (⇒): It follows from Lemmas 4.1 and 4.2 that the firt two conditions imply $\gamma_\mathrm{P}^* = \gamma_\mathrm{Q}^* = 0$ and

$$T_\mathrm{P} = 0 \quad \text{and} \quad T_\mathrm{Q} = 0 \tag{5.1}$$

Also, it is simple to verify that

$$\mathcal{V}^+(A, E, C_1, D_1) = \mathrm{Im}\left\{(\Gamma_{s\mathrm{Q}}^{-1})^\mathrm{T}\begin{bmatrix} I_{n_\mathrm{Q}} \\ 0 \end{bmatrix}\right\}$$

and

$$\mathcal{S}^+(A, B, C_2, D_2) = \mathrm{Ker}\{[I_{n_\mathrm{P}} \ \ 0]\Gamma_{s\mathrm{P}}^{-1}\}$$

Then, it is easy to see that the condition

$$\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2) \tag{5.2}$$

holds if and only if

$$[I_{n_\mathrm{P}} \ \ 0]\Gamma_{s\mathrm{P}}^{-1}(\Gamma_{s\mathrm{Q}}^{-1})^\mathrm{T}\begin{bmatrix} I_{n_\mathrm{Q}} \\ 0 \end{bmatrix} = \Gamma = 0 \tag{5.3}$$

Equations (5.1) to (5.3) imply that $M = 0$ and hence $\gamma_0^* = 0$.

(⇐): Conversely, it follows from Lemma 4.3 that $\gamma_0^* = 0$ implies $\gamma_P^* = \gamma_Q^* = 0$. Then, by Lemmas 4.1 and 4.2, we have

$$\mathrm{Im}(E) \subseteq \mathcal{S}^+(A, B, C_2, D_2), \mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathrm{Ker}(C_2)$$

and

$$T_P = 0, \quad T_Q = 0$$

Thus,

$$M = \begin{bmatrix} \Gamma S_Q^{-1} \Gamma^T S_P^{-1} & -\Gamma S_Q^{-1} \\ 0 & 0 \end{bmatrix}$$

Now, it is simple to see that $\gamma_0^* = \sqrt{\lambda_{\max}(M)} = 0$ implies $\Gamma = 0$ and hence

$$\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$$

This completes the proof of Theorem 5.2. □

5.2. *When γ_0^* is equal to γ_s^**

An interesting question in H_∞-optimization problem is under what conditions the infimum in H_∞-optimization via output feedback is equal to that achieved using state feedback. In the following theorem we provide a necessary and sufficient condition under which $\gamma_0^* = \gamma_s^*$.

Theorem 5.3: *Consider the system Σ given by* (2.1) *that satisfies the assumptions* (A1), (A2), (B1) *and* (B2). *Then $\gamma_0^* = \gamma_s^*$ if and only if*

$$\lambda_{\max}(M) = \begin{cases} \lambda_{\max}(T_P S_P^{-1}) & \text{if } n_P > 0 \\ 0 & \text{if } n_P = 0 \end{cases}$$

Proof: The proof follows from Theorems 4.1 and 4.3. □

Corollary 5.1:

(1) *If (A, E, C_1, D_1) is left-invertible and is of minimum phase, i.e. $n_Q = 0$, then $\gamma_0^* = \gamma_s^*$.*

(2) *If (A, B, C_2, D_2) is right-invertible and is of minimum phase, i.e. $n_P = 0$, then $\gamma_0^* = \gamma_s^*$ if and only if $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathrm{Ker}(C_2)$.*

(3) *If both n_P and n_Q are non-zero, then $\gamma_0^* = \gamma_s^*$ if $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$ and $\lambda_{\max}(T_Q S_Q^{-1}) \leq \lambda_{\max}(T_P S_P^{-1})$.*

Proof: Items (1) and (2) are obvious in view of Theorem 4.3 and Lemma 4.2. To prove item (3), let us consider the following. It follows from (5.2) and (5.3) that

$$\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$$

implies $\Gamma = 0$ and hence

$$M = \begin{bmatrix} T_P S_P^{-1} & 0 \\ 0 & T_Q S_Q^{-1} \end{bmatrix}$$

Thus the result is trivial in view of the fact $\gamma_P^* = \gamma_s^*$. □

6. Conclusions

In this paper we have extended the results of Chen *et al.* (1992) and presented a simple and non-iterative algorithm for the computation of the infimum for a class of singular H_∞-optimization problems using output feedback. We have shown that this infimum is equal to the square root of the maximum eigenvalue of a constant matrix that can be easily obtained from the system matrices of Σ. Our results are obtained under the assumptions that the two subsystems Σ_P and Σ_Q have no invariant zeros on the $j\omega$ axis and satisfy certain geometric conditions. The proposed algorithm for computing the infimum is applicable to the general case of a singular H_∞-optimization problem where no restrictions have been placed on the direct feedthrough matrices from the control input to the controlled output, and from the disturbance to the measurement output. Our current research effort is directed toward removing some of the assumptions imposed in this paper on Σ_P and Σ_Q.

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