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Contributed Paper

LOOP TRANSFER RECOVERY FOR GENERAL NONMINIMUM PHASE NON-STRICTLY PROPER SYSTEMS, PART 2—DESIGN*

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Abstract. This part focuses on the design of full order observer based controllers for the recovery of target loop transfer function or sensitivity and complimentary sensitivity functions for general non-strictly proper, not necessarily left invertible and not necessarily minimum phase systems. For general systems, loop transfer recovery is not completely feasible although there exists considerable amount of freedom to shape the inevitable recovery error. Here the necessary design constraints and the available design freedom are reviewed. In view of the available freedom, possible specifications on the time-scale and/or eigenstructure of the observer dynamic matrix are formulated. Then three types of design schemes are developed in detail. The first one is an asymptotic time-scale and eigenstructure assignment (ATEA) scheme, and the other two are optimization based designs; one dealing with the minimization of H_2 norm of a so called 'recovery matrix' while the other dealing with the minimization of H_∞ norm of the same. Relative advantages and disadvantages of both ATEA and optimization based design schemes are discussed. Besides the conventional LTR problem which is concerned with the recovery over the entire control space, another generalized recovery problem where the concern is with the recovery over a specified subspace of the control space is also considered. All the developed design methods are implemented in a "Matlab" software package. A bank of examples illustrate the developed design schemes.

Key Words—Loop transfer recovery, robust control, asymptotic time-scale and eigenstructure assignment, H_2- or H_∞-optimization.

1. Introduction and Problem Statement

As is well known and as discussed earlier in Part 1 (Chen, Saberi and Sannuti, 1992 a), the basic loop transfer recovery problem is concerned with analyzing and possibly designing an observer based controller which can achieve the same robustness properties as those of a state feedback controller. To be specific, consider a plant Σ,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1.1}$$

where the state vector $x \in \mathscr{R}^n$, output vector $y \in \mathscr{R}^p$ and input vector $u \in \mathscr{R}^m$.

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Without loss of generality, assume that $[B' \quad D']'$ and $[C \quad D]$ are of maximal rank. Let us also assume that Σ is stabilizable and detectable. Let the state feedback control law,

$$u = -Fx \tag{1.2}$$

be such that (a) the closed-loop system is asymptotically stable, i.e., the eigenvalues of $A-BF$ lie in the left half s-plane, and (b) the open-loop transfer function when the loop is broken at the input point of the plant meets the given frequency dependent specifications. Then $L_t(s)$, $S_t(s)$ and $T_t(s)$, respectively, the target loop transfer function, sensitivity and complimentary sensitivity functions are

$$L_t(s) = F\Phi B,$$

$$S_t(s) = [I_m + L_t(s)]^{-1}$$

and

$$T_t(s) = I_m - S_t(s) = [I_m + L_t(s)]^{-1} L_t(s), \tag{1.3}$$

where $\Phi = (sI - A)^{-1}$ and I_m denotes an identity matrix of dimension $m \times m$. On the other hand, let

$$\hat{u} = -F\hat{x},$$

$$\dot{\hat{x}} = (A - KC - BF + KDF)\hat{x} + Ky \tag{1.4}$$

be a full order observer based control law where K is an observer gain. Thus $L_o(s)$, $S_o(s)$ and $T_o(s)$, respectively, the obtainable loop transfer function and sensitivity and complimentary sensitivity functions are given by

$$L_o(s) = C(s)P(s), \quad P(s) = C\Phi B + D,$$

$$S_o(s) = [I_m + L_o(s)]^{-1}$$

and

$$T_o(s) = I_m - S_o(s) = [I_m + L_o(s)]^{-1} L_o(s), \tag{1.5}$$

where $C(s)$ is the observer based controller transfer function,

$$C(s) = F[sI_n - A + KC + BF - KDF]^{-1} K. \tag{1.6}$$

Thus, the goal of loop transfer recovery problem is to design a K such that

$$E(j\omega) = L_t(j\omega) - L_o(j\omega) \tag{1.7}$$

is either exactly zero or in some sense approximately zero over the frequency range of interest. Obviously, as we did in Part 1, $E(s)$ can be termed as *recovery* *error.* We say exact LTR (ELTR) is achieved if the recovery error $E(s)$ can be rendered zero, i.e., if

$$C(s)P(s) = L_t(s) \quad \text{for all } s.$$

Achieving ELTR is in general not possible. In an attempt to achieve "approximate" LTR, one normally parameterizes $C(s)$ as a function of a tuning parameter σ. In observer based controllers, the gain K is the only free design variable and thus parameterizing it as a function of σ, a family of controllers $C(s, \sigma)$ are obtained,

$$C(s, \sigma) = F[sI_n - A + K(\sigma)C + BF - K(\sigma)DF]^{-1}K(\sigma). \tag{1.8}$$

We say asymptotic LTR (ALTR) is achieved if

$$C(s, \sigma)P(s) \to L_t(s) \text{ pointwise in } s$$

as $\sigma \to \infty$, or equivalently $E(s, \sigma) \to 0$ pointwise in s as $\sigma \to \infty$. Achievability of ALTR enables the designer to choose a member of the family of controllers that corresponds to a particular value of σ which achieves a desired level of recovery.

In Part 1 (Chen, Saberi and Sannuti, 1992 a), we considered general non-strictly proper, not necessarily invertible and not necessarily of minimum phase plants and analyzed the mechanism of loop transfer recovery. The analysis there, while showing that neither ELTR nor ALTR can in general be achieved, focuses on four fundamental issues. The first issue is concerned with what can and what cannot be achieved for a given system and for an arbitrarily specified target loop transfer function. On the other hand, the second issue is concerned with the development of necessary and/or sufficient conditions a target loop has to satisfy so that it can be either exactly or asymptotically be recovered for a given system while the third issue is concerned with the development of necessary and/or sufficient conditions on a given system such that it has at least one, either exactly or asymptotically, recoverable target loop. The fourth issue deals with a generalization of all the above three issues when recovery is required over a subspace of the control space. It concerns with generalizing the traditional LTR concept to sensitivity recovery over a subspace and deals with method(s) to test whether projections of target and achievable sensitivity and complimentary sensitivity functions onto a given subspace match each other or not. All this analysis of Part 1 shows some fundamental limitations of the given system as a consequence of its structural properties, namely finite and infinite zero structure and invertibility. It also discovers a multitude of ways in which freedom exists to shape the recovery error in a desired way. Thus, it helps to set meaningful design goals at the onset of design.

For strictly proper systems, there exists essentially three methods of designing observer based controllers for LTR. These methods are, (1) Kalman filter formalism (Doyle and Stein, 1979), (2) direct eigenstructure placement method (Sogaard-Andersen, 1989), and (3) asymptotic eigenstructure and time-scale structure assignment (ATEA) method (Saberi and Sannuti, 1990 a; Saberi et al., 1991 b). A general discussion of these methods including their relative advantages and disadvantages, is given in detail in Saberi et al. (1991 b).

In this paper, we develop design methods for general non-strictly proper systems and for different types of design tasks. The existing ATEA design method of Saberi and Sannuti (1990 a) and Saberi et al. (1991 b) is streamlined and extended to deal with general systems. Besides the ATEA design, two optimization based designs are also developed here. One optimization based method deals with minimizing the H_2 norm of a "recovery matrix" related to the loop transfer recovery error while the other minimizes the H_∞ norm of the same. In optimization methods, one normally generates a sequence of observer gains by solving parameterized algebraic Riccati equations. As the parameter tends to a certain value, the corresponding sequence of H_2 norms (or H_∞ norms depending on the method) of the resulting recovery matrices tends to a limit which is the infimum of the H_2 norm (or H_∞ norm) of the recovery matrix over the set of all possible observer gains. A sub-optimal solution is obtained when one selects an observer gain corresponding to a particular value of the parameter. In the process of generating a sequence of suboptimal solutions, the mathematical optimization procedure follows a particular path and shapes the recovery matrix accordingly. That is, there is no freedom to shape the recovery matrix directly, and one has to be content with what the mathematical optimization procedure yields. In contrast with this, since parameterization of ATEA design procedure is explicit rather than being implicit via algebraic Riccati equations of optimization based methods, ATEA design procedure allows all the available design freedom to shape the recovery matrix as desired within the structural constraints imposed by the given system. Also, in connection with ELTR, the present existing optimization based design methods can generate the required observer gain for ELTR **only** for a subclass of all allowable problems. On the other hand, ATEA design method appropriately modified and simplified, can generate the required observer gain which achieves ELTR whenever it is possible.

The conventional LTR design task seeks the recovery over the entire control space. As discussed in Part 1, one can also formulate another generalized design task which seeks the recovery only over a specified subspace of the entire control space. Such a formulation is meaningful especially when recovery over the entire control space is not feasible. All the three design methods developed here are modified to deal with such a generalized design task.

The paper is organized as follows. Section 2 reviews the necessary design constraints and the available design freedom. Section 3 develops the general ATEA method of design. Also, in Sec. 3, a simplification of ATEA is given to arrive at a design for exact loop transfer recovery whenever it is feasible. Section 4 develops optimization based designs. Here two designs are considered; one minimizes the H_2 norm of a recovery matrix while the other minimizes the H_∞ norm of the same. Section 5 considers the generalized design task of recovering the target sensitivity and complimentary sensitivity functions over a subspace of the control space. For this purpose, here an auxiliary system of the given system is constructed so that all the three designs developed earlier can readily be applied for the new design task. Section 6 discusses the relative advantages of the ATEA and optimization based designs. Section 7 draws conclusions of our work.

As in Part 1, throughout this paper, A' denotes the transpose of A, A^H denotes the complex conjugate transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. $\lambda(A)$ denotes the set of eigenvalues of A. Similarly, $\sigma_{\max}[A]$ and $\sigma_{\min}[A]$ respectively denote the maximum and minimum singular values of A. Ker$[V]$ and Im$[V]$ denote respectively the kernel and the image of V. The open left, closed right half s-planes and the $j\omega$ axis are respectively denoted by C^-, C^+ and C^o. Also, the set $\mathbf{T}^{\mathrm{ER}}(\Sigma)$ denotes the set of exactly recoverable target loop transfer functions for any given system Σ, $\mathbf{T}^{\mathrm{R}}(\Sigma)$ denotes the set of either exactly or asymptotically recoverable target loop transfer functions, while $\mathbf{T}^{\mathrm{AR}}(\Sigma)$ denotes the set of target loop transfer functions which are asymptotically recoverable but not exactly recoverable for the given system Σ. The precise definitions of $\mathbf{T}^{\mathrm{ER}}(\Sigma)$, $\mathbf{T}^{\mathrm{R}}(\Sigma)$ and $\mathbf{T}^{\mathrm{AR}}(\Sigma)$ are given in Part 1. Some geometric subspaces defined in Part 1 are recalled next.

$V^g(A, B, C, D)$ is the maximal subspace of $\mathscr{R}^n$ which is $(A-BF)$—invariant and contained in $\mathrm{Ker}(C-DF)$ such that the eigenvalues of $(A-BF) \mid V^g$ are contained in $C_g \subseteq C$ for some F.

$S^g(A, B, C, D)$ is the minimal $(A-KC)$—invariant subspace of $\mathscr{R}^n$ containing in $\mathrm{Im}(B-KD)$ such that the eigenvalues of the map which is induced by $(A-KC)$ on the factor space $\mathscr{R}^n/S^g$ are contained in $C_g \subseteq C$ for some K.

For the cases that $C_g = C$, $C_g = C^-$ and $C_g = C^+$, we replace the index g in V^g and S^g by "*", "−" and "+", respectively.

A special coordinate basis (s.c.b) of any system Σ was discussed in Sec. 3 of Part 1. Such an s.c.b has a distinct feature of explicitly displaying the finite and infinite zero structure of Σ. Some integers associated with the s.c.b of Σ and hence with Σ, play a dominant role in our discussions to follow. We would like to recall now an interpretation of these integers:

- n_a^- and n_a^+ are the numbers (counting multiplicity) of invariant zeros of Σ in C^- and C^+ respectively.
- n_f is the number of infinite zeros of Σ.
- n_c is the dimension of the intersection of the subspaces $S^+(A, B, C, D)$ and $V^+(A, B, C, D)$ which are defined above. This intersection is the largest subspace of the state space which is completely controllable by the input while maintaining an output equal to zero.
- n_b equals the dimension of the state space n minus the numbers defined above. It is equal to n minus the dimension of $S^+(A, B, C, D) + V^+(A, B, C, D)$.

2. Design Constraints and Available Freedom

In Part 1 (Chen, Saberi and Sannuti, 1992), using full order observer based controllers, we have analyzed systematically when and under what conditions loop transfer recovery (LTR) is possible. We have also characterized the recovery error whenever either exact or asymptotic recovery is not feasible. For such an analysis, we considered a family of full order observer based controllers $C(s, \sigma)$ where the observer gain K is parameterized as a function of a tuning parameter σ. We discovered that for an appropriate recovery, there are constraints in selecting the gain $K(\sigma)$, and these constraints manifest themselves in assigning both the asymptotically finite as well as the infinite eigenstructure of the observer dynamic matrix $A-K(\sigma)C$. However, we also discovered that there exists a considerable amount of freedom in assigning certain parts of either asymptotically finite or infinite eigenstructure to $A-K(\sigma)C$. Let us next briefly review the LTR mechanism as analyzed in Part 1 so as to familiarize ourselves with the necessary design constraints and the available design freedom. We recall that the recovery error $E(s, \sigma)$ between the target loop transfer function $L_t(s)$ and the achievable one $L_o(s)$, is given by

$$E(s, \sigma) = M(s, \sigma)[I_m+M(s, \sigma)]^{-1}(I_m+F\Phi B), \tag{2.1}$$

where

$$M(s, \sigma) = F[sI_n-A+K(\sigma)C]^{-1}[B-K(\sigma)D]. \tag{2.2}$$

The matrix $M(s, \sigma)$, termed as *recovery matrix*, plays a dominant role in recovery analysis. In fact, Lemma 4.1 of Part 1 states that recovery error $E(j\omega, \sigma)$ is zero if and only if the recovery matrix $M(j\omega, \sigma)$ is zero. Thus the study of LTR is tantamount to the study of $M(s, \sigma)$. The matrix $M(s, \sigma)$ has a simple physical meaning. If one considers the observer based controller as a device with two inputs, one the plant input u and the other plant output y, then $-M(s, \sigma)$ is the transfer function from plant input point to the controller output point. This is illustrated in Fig. 2.1 where

$$\hat{U}(s) = -M(s, \sigma)U(s) - N(s, \sigma)Y(s)$$

and

$$N(s, \sigma) = F[sI_n-A+K(\sigma)C]^{-1}K(\sigma).$$

Thus the condition for loop transfer recovery demands that the controller not entail any feedback from the plant input u. Now to study $M(s, \sigma)$ systematically, assuming that $A-KC$ is nondefective, we can expand $M(s, \sigma)$ dyadically as,

$$M(s, \sigma) = \sum_{i=1}^{n} \frac{R_i(\sigma)}{s-\lambda_i(\sigma)}, \tag{2.3}$$

where the residue $R_i(\sigma)$ is given by

$$R_i(\sigma) = FW_i(\sigma)V_i^H(\sigma)[B-K(\sigma)D]. \tag{2.4}$$

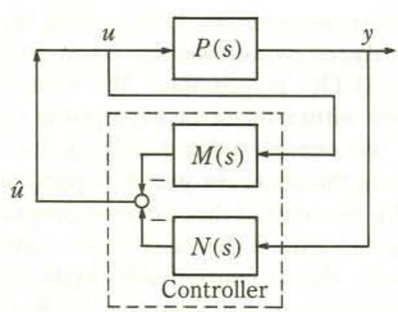

Fig. 2.1. Plant and controller configuration.

Here, $W_i(\sigma)$ and $V_i(\sigma)$ are respectively the right and left eigenvectors associated with an eigenvalue $\lambda_i(\sigma)$ of $A-KC$ and they are scaled so that $W(\sigma)V^H(\sigma) = V^H(\sigma)W(\sigma)=I_n$ where

$$\left.\begin{aligned} W(\sigma) &= [W_1(\sigma),\ W_2(\sigma),\ \cdots,\ W_n(\sigma)] \\ \text{and} \qquad & \\ V(\sigma) &= [V_1(\sigma),\ V_2(\sigma),\ \cdots,\ V_n(\sigma)] \end{aligned}\right\}. \tag{2.5}$$

As is evident from (2.3) and (2.4), one can render the ith term of $M(s, \sigma)$ zero only by two approaches:

1. by rendering the residue $R_i(\sigma)$ zero while λ_i is finite; or
2. by pushing λ_i asymptotically to infinity while keeping the residue $R_i(\sigma)$ uniformly bounded.

The first approach implies an appropriate finite eigenstructure assignment while the second one implies an appropriate asymptotically infinite eigenstructure assignment to $A-KC$. Owing to the structural properties of the given system Σ, there does not exist complete freedom to assign the needed eigenstructure to $A-KC$. The analysis of Part 1 reveals several guidelines as to when, how and to what extent such an assignment can be done. To review these guide lines, the recovery matrix $M(s, \sigma)$ is partitioned into four parts based on the analysis of Part 1:

$$M(s, \sigma) = M^-(s, \sigma) + M^b(s, \sigma) + M^\infty(s, \sigma) + M^e(s, \sigma), \tag{2.6}$$

where

$$M^-(s, \sigma) = \sum_{i=1}^{n_a^-} \frac{R_i^-(\sigma)}{s-\lambda_i^-(\sigma)}, \quad M^b(s, \sigma) = \sum_{i=1}^{n_b} \frac{R_i^b(\sigma)}{s-\lambda_i^b(\sigma)},$$

$$M^\infty(s, \sigma) = \sum_{i=1}^{n_f} \frac{R_i^\infty(\sigma)}{s-\lambda_i^\infty(\sigma)} \quad \text{and} \quad M^e(s, \sigma) = \sum_{i=1}^{n_e} \frac{R_i^e(\sigma)}{s-\lambda_i^e(\sigma)}.$$

Define the following sets:

$$\Lambda^-(\sigma) \triangleq \{\lambda_i^-(\sigma) \mid i = 1, \cdots, n_a^-\}, \quad \Lambda^b(\sigma) \triangleq \{\lambda_i^b(\sigma) \mid i = 1, \cdots, n_b\},$$

$$\Lambda^\infty(\sigma) \triangleq \{\lambda_i^\infty(\sigma) \mid i = 1, \cdots, n_f\}, \quad \Lambda^e(\sigma) \triangleq \{\lambda_i^e(\sigma) \mid i = 1, \cdots, n_e\},$$

$$V^-(\sigma) \triangleq \{V_i^-(\sigma) \mid i = 1, \cdots, n_a^-\}, \quad V^b(\sigma) \triangleq \{V_i^b(\sigma) \mid i = 1, \cdots, n_b\},$$

$$V^\infty(\sigma) \triangleq \{V_i^\infty(\sigma) \mid i = 1, \cdots, n_f\}, \quad V^e(\sigma) \triangleq \{V_i^e(\sigma) \mid i = 1, \cdots, n_e\},$$

$$W^-(\sigma) \triangleq \{W_i^-(\sigma) \mid i = 1, \cdots, n_a^-\}, \quad W^b(\sigma) \triangleq \{W_i^b(\sigma) \mid i = 1, \cdots, n_b\},$$

$$W^\infty(\sigma) \triangleq \{W_i^\infty(\sigma) \mid i = 1, \cdots, n_f\}, \quad W^e(\sigma) \triangleq \{W_i^e(\sigma) \mid i = 1, \cdots, n_e\}.$$

Hereafter, we will use an over bar on a certain variable to denote its limit as $\sigma \to \infty$ whenever it exists. For example, $\bar{M}^e(s)$ and $\bar{W}^e$ denote respectively the limits of $M^e(s, \sigma)$ and $W^e(\sigma)$ as $\sigma \to \infty$.

It turns out that irrespective of the target loop transfer function, both $M^-(s, \sigma)$ and $M^b(s, \sigma)$ can be rendered zero either exactly or asymptotically as $\sigma \to \infty$ by an appropriate finite eigenstructure assignment to $A-K(\sigma)C$. Also, $M^\infty(s, \sigma)$ can be rendered zero asymptotically as $\sigma \to \infty$ by an appropriate asymptotically infinite eigenstructure assignment to $A-K(\sigma)C$. On the other hand, *for arbitrary target loop transfer functions*, $M^e(s, \sigma)$ cannot always be rendered zero and hence $M^e(s, \sigma)$ has been termed in Part 1 as the *recovery error matrix*. Although $M^e(s, \sigma)$ cannot always be rendered zero, there exists abundant amount of freedom to shape $M^e(s, \sigma)$ within the given constraints. Nonetheless, for a particular class of target loops, namely $L_t(s) \in \mathbf{T}^{\mathrm{ER}}(\Sigma)$, $M^e(s, \sigma)$ can exactly be rendered zero, while for another class of target loops, namely $L_t(s) \in \mathbf{T}^{\mathrm{R}}(\Sigma)$, $M^e(s, \sigma)$ can asymptotically be rendered zero as $\sigma \to \infty$.

We proceed now to describe in detail the necessary design constraints and the available design freedom in assigning an appropriate eigenstructure to $A-K(\sigma)C$. We do this by considering one part of $M(s, \sigma)$ at a time.

Discussion on $M^-(s, \sigma)$: Consider an arbitrary target loop transfer function $L_t(s)$. The term $M^-(s, \sigma)$ can identically (irrespective of the value of σ) be rendered zero. To accomplish this, the set of n_a^- eigenvalues $\Lambda^-(\sigma)$ and the corresponding set of left eigenvectors $V^-(\sigma)$ of $A-K(\sigma)C$ must be selected to coincide respectively with the set of plant minimum phase invariant zeros and the corresponding left state zero directions of Σ. If one prefers, $M^-(s, \sigma)$ can be rendered zero asymptotically as $\sigma \to \infty$. This can be done if $\Lambda^-(\sigma)$ and the corresponding set of left eigenvectors $V^-(\sigma)$ of $A-K(\sigma)C$ are selected so that their asymptotic limits $\bar{\Lambda}^-$ and $\bar{V}^-$ coincide respectively with the set of plant minimum phase invariant zeros and the corresponding left state zero directions of Σ.

Discussion on $M^b(s, \sigma)$: Consider an arbitrary target loop transfer function $L_t(s)$. The term $M^b(s, \sigma)$ can identically (irrespective of the value of σ) be rendered zero. To accomplish this, the set of n_b eigenvalues $\Lambda^b(\sigma)$ can be assigned arbitrarily either at asymptotically finite or infinite locations in C^-, while the corresponding set of left eigenvectors $V^b(\sigma)$ of $A-K(\sigma)C$ is in the null space of matrix $[B-K(\sigma)D]'$. If one prefers, $M^b(s, \sigma)$ can be rendered zero asymptotically as $\sigma \to \infty$. This can be done by selecting $\Lambda^b(\sigma)$ arbitrarily either at asymptotically finite or infinite locations in C^-, while the corresponding set of left eigenvectors $V^b(\sigma)$ of $A-K(\sigma)C$ must be such that its asymptotic limit $\bar{V}^b$ is in the null space of matrix $[B-K(\sigma)D]'$. Note that assigning all the elements of $\bar{\Lambda}^b$ to finite locations, conserves the controller bandwidth.

Discussion on $M^\infty(s, \sigma)$: Consider an arbitrary target loop transfer function $L_t(s)$. $M^\infty(s, \sigma)$ can be rendered zero asymptotically as $\sigma \to \infty$. For this purpose, the set of n_f eigenvalues $\Lambda^\infty(\sigma)$ can be assigned arbitrarily at asymptotically infinite locations in C^-. However, for every $\lambda_i^\infty(\sigma) \in \Lambda^\infty(\sigma)$, the corresponding right and left eigenvectors $W_i^\infty(\sigma)$ and $V_i^\infty(\sigma)$ must be such that $W_i^\infty(\sigma)[V_i^\infty(\sigma)]^H[B-K(\sigma)D]$ is uniformly bounded as $\sigma \to \infty$. This enables each residue $R_i^\infty(\sigma)$ uniformly bounded as $\sigma \to \infty$ and thus renders $\bar{M}^\infty(s)$ zero. We note that there exists complete freedom in the way $\lambda_i^\infty(\sigma) \in \Lambda^\infty(\sigma)$ tends to infinity as $\sigma \to \infty$, i.e., the asymptotic direction and the rate at which each $\lambda_i^\infty(\sigma)$ goes to infinity can be dictated as desired by the designer.

Discussion on $M^e(s, \sigma)$: We note that if the given system is of minimum phase and left invertible, then $n_e = n_a^+ + n_c = 0$ and hence $M^e(s, \sigma)$ is nonexistent. To deal with the general case when $n_a^+ + n_c \neq 0$, let us *at first* consider an arbitrary target loop transfer function $L_t(s)$. Then in general $M^e(s, \sigma)$ cannot always be rendered zero although there exists abundant amount of freedom to assign the associated eigenvalues and eigenvectors. To be explicit, the set of $n_a^+ + n_c$ eigenvalues $\Lambda^e(\sigma)$ can be assigned arbitrarily at any (either asymptotically finite or infinite) locations in C^- subject to the condition that any unobservable but stable eigenvalues of the given system must be included among $\Lambda^e(\sigma)$. Also, there exists a complete freedom consistent with the results of Moore (1976) in assigning the right and left eigenvector sets $W^e(\sigma)$ and $V^e(\sigma)$ and hence $\bar{W}^e$ and $\bar{V}^e$. But in general $\bar{\Lambda}^e$, $\bar{W}^e$ and $\bar{V}^e$ cannot be assigned such that $\bar{M}^e(s)$ is zero. However, there exists a multitude of ways to assign $\bar{\Lambda}^e$ and $\bar{W}^e$ (and hence $\bar{V}^e$) so that $\bar{M}^e(s)$ can be shaped to have certain desired directional properties or it is as small as it could be.

3. Observer Design by 'ATEA'

The previous section summarizes the available design freedom as well as constraints in assigning the eigenstructure of observer dynamic matrix for appropriate loop transfer recovery. We develop here a design procedure which follows the asymptotic time-scale and eigenstructure assignment (ATEA) concepts proposed originally in Saberi and Sannuti (1989). Following the concepts of Saberi and Sannuti (1989), we developed earlier an observer design for left invertible and minimum phase plants in Saberi and Sannuti (1990 a) and for not necessarily left invertible and not necessarily minimum phase but strictly proper systems in Saberi et al. (1991 b). In what follows, we will present a step by step ATEA design algorithm for general non-strictly proper systems. At first in Subsection 3.1, we give a design procedure for an arbitrarily specified target loop transfer function, i.e., without taking into account any specific characteristics of F. This is the most general design procedure. Also, as discussed in Remark 5.1 of Part 1, whenever the given $L_t(s)$ is asymptotically recoverable, that is whenever $L_t(s) \in \mathbf{T}^{\mathrm{R}}(\Sigma)$, the gain F has a particular structure,

$$F = \Gamma_3 \tilde{F} \Gamma_1^{-1}, \quad \tilde{F} = \begin{bmatrix} F_{a1}^- & 0 & F_{b1} & 0 & F_{f1} \\ F_{a2}^- & 0 & F_{b2} & 0 & F_{f2} \end{bmatrix}, \tag{3.1}$$

where Γ_3 and Γ_1 are the nonsingular transformation matrices as defined in Theorem 3.1 of Part 1. Whenever F conforms to the form given by (3.1), it entails additional freedom in selecting some eigenvalues and eigenvectors. For this special case, with or without making use of such an additional freedom, the general ATEA procedure of Subsection 3.1 yields a design which asymptotically recovers $L_t(s)$. On the other hand, as discussed in Remark 5.2 of Part 1, for exactly recoverable target loop transfer functions, that is whenever $L_t(s) \in \mathbf{T}^{\mathrm{ER}}(\Sigma)$, F has another specific structure,

$$F = \Gamma_3 \tilde{F} \Gamma_1^{-1}, \quad \tilde{F} = \begin{bmatrix} F_{a1}^- & 0 & F_{b1} & 0 & 0 \\ F_{a2}^- & 0 & F_{b2} & 0 & 0 \end{bmatrix}. \tag{3.2}$$

For the case when F has the above special structure, one needs to assign only a finite eigenstructure to $A-K(\sigma)C$. For this special case, the general ATEA design procedure of Subsection 3.1 can be simplified greatly and such a simplified design is presented in Subsection 3.2.

3.1 General 'ATEA' design The ATEA design method is decentralized in nature. It uses the special coordinate basis (s.c.b) of the given system Σ (see Theorem 3.1 of Part 1 of Chen, Saberi and Sannuti, 1992 a; see also Sannuti and Saberi, 1987; Saberi and Sannuti, 1990 b). The specified finite eigenstructure of $A-K(\sigma)C$ is assigned appropriately by working with subsystems which represent the finite zero structure of the given system (see Eqs. (3.3) to (3.6) of Part 1). Similarly the specified asymptotically infinite eigenstructure of $A-K(\sigma)C$ is assigned appropriately by working with subsystems which represent the infinite zero structure of the given system (see Eq. (3.8) of Part 1 for each $i=1,\cdots,m_f$).

There are two issues in formulating the observer dynamic matrix $A-K(\sigma)C$ by an appropriate selection of $K(\sigma)$. The first issue is eigenvalue assignment and the second one is corresponding eigenvector assignment. We will focus on one issue at a time. Let us first consider the eigenvalue assignment. As discussed in Sec. 2, some eigenvalues of $A-K(\sigma)C$ are constrained while some others are free to be assigned to either asymptotically finite or infinite locations in C^-. To be specific,

1. $\Lambda^-(\sigma)$ must coincide either exactly or asymptotically with the set of plant minimum phase invariant zeros,
2. $\Lambda^b(\sigma)$ and $\Lambda^e(\sigma)$ can be assigned to either asymptotically finite or infinite locations, and
3. $\Lambda^\infty(\sigma)$ have to be assigned to asymptotically infinite locations.

In this section in order to conserve controller bandwidth, both $\Lambda^b(\sigma)$ and $\Lambda^e(\sigma)$ are assigned to asymptotically finite locations. Let us next examine carefully the freedom available in assigning $\Lambda^\infty(\sigma)$ to asymptotically infinite locations. As is clear from the discussion in Sec. 2, there exists complete freedom in the way each $\lambda_i^\infty(\sigma)\in\Lambda^\infty(\sigma)$ tends to infinity as $\sigma\to\infty$, i.e., both the asymptotic direction and the rate at which $\lambda_i^\infty(\sigma)$ goes to infinity can be dictated as desired by the designer. In other words, the freedom available in assigning every asymptotically infinite eigenvalue $\lambda_i^\infty(\sigma)$ manifests itself in two ways:

1. in choosing the asymptotic directions along which the eigenvalues tend to infinity, and
2. in choosing the rates at which the eigenvalues tend to infinity.

To reflect both these types of freedom, let $\Lambda^\infty(\sigma)$ for asymptotically large values of σ be subdivided into $r\leq n_f$ sets,

$$\frac{\Lambda_1}{\mu_1},\ \frac{\Lambda_2}{\mu_2},\ \cdots,\ \frac{\Lambda_r}{\mu_r}. \tag{3.3}$$

Here Λ_l is a set of n_l numbers all in C^- and Λ_l is closed under complex conjugation. Also $\sum_{l=1}^r n_l=n_f$. Apparently, the elements of Λ_l, $l=1,\cdots,r$, define the asymptotic directions of asymptotically infinite or fast eigenvalues while the small parameters μ_l, $l=1,\cdots,r$, which are some functions of σ, define the rates at which these eigenvalues go to infinity.

In summary, regarding the eigenvalues, a designer has the freedom to specify
i. the asymptotic limits $\bar{\Lambda}^b$ and $\bar{\Lambda}^e$ of $\Lambda^b(\sigma)$ and $\Lambda^e(\sigma)$, and
ii. Λ_l and μ_l, $l=1,\cdots,r$.
We note that $\bar{\Lambda}^b$ and $\bar{\Lambda}^e$ in addition to $\bar{\Lambda}^-$ define the asymptotically finite eigenvalues of $A-K(\sigma)C$, while Λ_l and μ_l, $l=1,\cdots,r$, define the asymptotically infinite eigenvalues.

Let us look now at the constraints and design freedom available in assigning the eigenvectors of $A-K(\sigma)C$. The set of right eigenvectors $\bar{V}^-$ is constrained to coincide with the corresponding set of state zero directions of the plant. Moreover, $\mathrm{Im}(\bar{V}^-)$ coincides with the subspace $V^*(A, B, C, D)/V^+(A, B, C, D)$. On the other hand, the set of eigenvectors $\bar{V}^b$ is constrained to be in the null space of $[B-K(\sigma)D]'$. In view of the particular structure of s.c.b, it can be seen then that every element $\bar{V}_i^b$ of $\bar{V}^b$ is constrained to be of the form $[0, 0, (V_i^b)^H, 0, 0]^H$. In other words, the set $\bar{V}^b$ can be represented in a matrix notation as $[0, 0, (V^{bb})^H, 0, 0]^H$ where V^{bb} is an $n_b \times n_b$ matrix. Thus, the selection of $\bar{V}^b$ to be in the null space of $[B-K(\sigma)D]'$ is equivalent to any arbitrary selection of V^{bb} consistent with the freedom available in assigning it (Moore, 1976). Again in view of the properties of s.c.b, we note that the columns of $\bar{V}^b$ span the subspace $\mathscr{R}^n/\{S^+(A, B, C, D) \cup S^-(A, B, C, D)\}$. There is also freedom available in specifying $\bar{W}^e$. Furthermore, it can be shown (see for example Saberi et al., 1991 a) that $\mathrm{Im}(\bar{W}^e)$ coincides with the subspace $V^+(A, B, C, D)$. Again owing to the special structure of s.c.b, $\bar{W}^e$ has the special matrix form $[(W^{e+})^H, 0, 0, (W^{ec})^H, 0]^H$ where $W^{ee} \equiv [(W^{e+})^H, (W^{ec})^H]^H$ is an $n_e \times n_e$ matrix. Thus, an appropriate selection of $\bar{W}^e$ is equivalent to any arbitrary selection of W^{ee} consistent with the freedom available in assigning it (Moore, 1976).

Now an assignment of both asymptotically finite and infinite eigenvalues and the corresponding eigenvectors to $A-K(\sigma)C$ can be viewed as an asymptotic time-scale and eigenstructure assignment (ATEA) to it. Further discussion on time-scale structure of a system can be found in Saberi and Sannuti (1989) and Saberi et al. (1991 b). In order to have a well defined separation of time-scales, we will assume throughout the paper that

$$\mu_l/\mu_{l+1} \to 0 \quad \text{as} \quad \mu_{l+1} \to 0. \tag{3.4}$$

We emphasize that the freedom that exists in specifying the asymptotically infinite eigenstructure of $A-K(\sigma)C$ reflects itself in specifying an appropriate fast time-scale structure. The asymptotic directions of asymptotically infinite eigenvalues can be specified by the sets Λ_l, $l=1,\cdots,r$, where r is an integer less than or equal to n_f. The relative fastness of time-scales is specified by specifying the small positive parameters μ_l, $l=1,\cdots,r$, which are appropriate functions of the tuning parameter σ so that (3.4) is true as $\sigma \to \infty$. We note that there is also a constraint on the infinite eigenstructure, namely, for every asymptotically infinite eigenvalue $\lambda_i^\infty(\sigma)$, the corresponding right and left eigenvectors $W_i^\infty(\sigma)$ and $V_i^\infty(\sigma)$ of $A-K(\sigma)C$ must be such that $W_i^\infty(\sigma)[V_i^\infty(\sigma)]^H[B-K(\sigma)D]$ is uniformly bounded as $\sigma \to \infty$. This constraint, however, is automatically taken into account by the ATEA design procedure given in this section.

In what follows, we give a step by step ATEA design algorithm. In view of the above discussion, the input parameters of the algorithm are $\bar{\Lambda}^b$, V^{bb}, $\bar{\Lambda}^e$, W^{ee}, Λ_l and μ_l, $l=1,\cdots,r$, as well as the integer r. In fact, the primary inputs to the algorithm are (1) $\bar{\Lambda}^e$ and W^{ee} which shape the resulting $\bar{M}^e(s)$ and (2) Λ_l and μ_l, $l=1,\cdots,r$, which control the time-scale structure of the observer and thus have a strong impact on the resulting gain of the controller. The rest of the input parameters, namely $\bar{\Lambda}^b$ and V^{bb} are secondary inputs to the algorithm. Our algorithm can be divided into three steps. Steps 1 and 2 deal respectively with subsystem designs to assign the asymptotically finite and infinite eigenstructures. In Step 3, subsystem designs of Steps 1 and 2 are put together to form a composite design for the given system.

Step 1: This step deals with the assignment of asymptotically finite eigenstructure (i.e., slow time-scale structure) and makes use of subsystems (2.3) to (2.6) of Part 1. $\lambda(A_{aa}^-)$ are the minimum phase invariant zeros of the given system Σ and these are left alone to form some of the eigenvalues of $A-K(\sigma)C$, namely the set $\bar{\Lambda}^-$, while the corresponding left eigenvectors of $A-K(\sigma)C$ coincide with the corresponding left state zero directions of Σ. To place the set of eigenvalues $\bar{\Lambda}^b$ and left eigenvectors $\bar{V}^b$, choose a gain K^b such that $\lambda(A_{bb}^c)$ coincides with $\bar{\Lambda}^b$ while V^{bb} coincides with the set of left eigenvectors of A_{bb}^c where

$$A_{bb}^c = A_{bb} - K^b C_b. \tag{3.5}$$

Note that the existence of such a K^b is guaranteed by Property 3.2 of Sec. 3 of Part 1 (Chen, Saberi and Sannuti, 1992 a) as long as the eigenvector set $\bar{V}^b$ is consistent with the freedom available in assigning it (Moore, 1976). Next, in order to place the set of eigenvalues $\bar{\Lambda}^e$ and right eigenvectors W^{ee}, let us first form matrices A^{ee} and C^e as follows:

$$A^{ee} = \begin{bmatrix} A_{aa}^+ & 0 \\ B_c E_{ca}^+ & A_{cc} \end{bmatrix}, \quad C^e = \begin{bmatrix} C^{e0} \\ C^{e1} \end{bmatrix} = \begin{bmatrix} C_{0a}^+ & C_{0c} \\ E_a^+ & E_c \end{bmatrix}, \tag{3.6}$$

where

$$E_a^+ = [(E_{1a}^+)', (E_{2a}^+)', \cdots, (E_{m_fa}^+)']',$$

$$E_{ia} = [E_{ia}^+, E_{ia}^-], \quad E_c = [E_{1c}', E_{2c}', \cdots, E_{m_fc}']'.$$

Now select a gain $K^e=[K^{e0}, K^{e1}]$ such that the set of eigenvalues and right eigenvectors of

$$A^{ee} - K^e C^e \equiv A^{ee} - K^{e0}C^{e0} - K^{e1}C^{e1}, \tag{3.7}$$

coincide with $\bar{\Lambda}^e$ and W^{ee}, respectively. Again note that the existence of such a K^e is guaranteed by Property 3.2 of Sec. 3 of Chen, Saberi and Sannuti (1992 a) as long as the eigenvector set W^{ee} is consistent with the freedom available in assigning it (Moore, 1976). For future use, let us define

$$A^{ee1} = A^{ee} - K^{e0}C^{e0}, \quad K^{e0} = \begin{bmatrix} K^{a0+} \\ K^{c0} \end{bmatrix},$$

and partition K^{e1} as

$$K^{e1} = [K^{e11}, K^{e12}, \cdots, K^{e1m_f}], \tag{3.8}$$

where K^{e1i} is an $n_e \times 1$ dimensional vector.

Step 2: This step deals with the assignment of asymptotically infinite eigenstructure (i.e., the fast time-scale structure) and makes use of subsystems, $i=1,\cdots,m_f$, represented by (3.8) of Part 1. This step exists only when $n_f>0$ since otherwise there is no need to assign any asymptotically infinite eigenstructure, and hence we assume here that $n_f>0$. As discussed earlier, there is complete freedom to specify any $r \leq n_f$ fast time-scales. In particular, one can always choose $r=1$. For generality, we will keep r as arbitrarily given. The freedom in assigning the fast time-scales is reflected in specifying the sets Λ_l, and the small positive parameters μ_l, $l=1,\cdots,r$. Our design to assign an appropriate fast time-scale structure is again decentralized. We deal with one single input single output system at a time as represented by (3.8) of Part 1 for a particular value of i, $i=1,\cdots,m_f$. Thus, to proceed with our design, we need to distribute the designer specified elements of the sets Λ_l, and the parameters μ_l, $l=1,\cdots,r$, among m_f subsystems. There exists a complete freedom in such a distribution and hence it can be done in a number of ways. Let subsystem i be assigned r_i time-scales for some $r_i \leq q_i$. Let

$$\frac{\Lambda_{ij}}{\mu_{ij}}, \quad j = 1, \cdots, r_i,$$

be the asymptotically infinite eigenvalues that need to be assigned to subsystem i. Let n_{ij} be the number of eigenvalues corresponding to the time-scale t/μ_{ij}. That is, let Λ_{ij} contain n_{ij} elements. As usual, the set Λ_{ij} is assumed to be closed under complex conjugation. Also, in order to have a well defined separation of time-scales in subsystem i, we will assume that

$$\mu_{ij}/\mu_{ij+1} \to 0 \quad \text{as} \quad \mu_{ij+1} \to 0 \quad \text{for all} \quad j = 1, \cdots, r_i - 1. \tag{3.9}$$

We note that when $r=1$, all μ_{ij} are equal to a single parameter μ and all r_i are equal to unity. That is, there is only one time-scale to be assigned to all subsystems. In this case, σ can be taken as $1/\mu$. With these preliminaries, we are now ready to design the ith subsystem. At first, we will design a gain matrix K_{ij} for each time-scale t/μ_{ij}, $j=1,\cdots,r_i$. Define an $n_{ij} \times n_{ij}$ dimensional matrix G_{ij} and a $1 \times n_{ij}$ dimensional matrix C_{ij} having the following structure:

$$G_{ij} = \begin{bmatrix} 0 & I_{n_{ij}-1} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_{ij} = [1 \quad 0].$$

Choose an $n_{ij} \times 1$ dimensional gain vector K_{ij} such that $\lambda(G_{ij}^c)$ coincides with Λ_{ij} where $G_{ij}^c = G_{ij} - K_{ij}C_{ij}$. Owing to the special structure of G_{ij} and C_{ij}, such a K_{ij} always exists. Let K_{ij} be partitioned as

$$K_{ij} = \begin{bmatrix} K_{ijc} \\ K_{ijd} \end{bmatrix},$$

where K_{ijd} is a scalar. Moreover, the nonsingularity of G_{ij}^c implies that K_{ijd} is nonzero. Next, the gains K_{ij}, $j=1,\cdots,r_i$, obtained above, are put together to form a composite gain vector which will induce the required fast time-scales in the ith subsystem. Define the scalar numbers J_{ij} as

$$J_{i1} = 1, \quad J_{ij} = \prod_{l=1}^{j-1} K_{ild} \quad \text{for} \quad j = 2, \cdots, r_i.$$

Let

$$\alpha_{i0} = 0$$

and

$$\alpha_{ij} = \sum_{k=1}^{j} n_{ik}, \quad j = 1, \cdots, r_i.$$

Note that $\alpha_{ir_i} = q_i$. Also, let for each $j=1,\cdots,r_i$,

$$\varepsilon_{i\alpha_{ij-1}+1} = \varepsilon_{i\alpha_{ij-1}+2} = \cdots = \varepsilon_{i\alpha_{ij}} = \mu_{ij}$$

and

$$\eta_i = \prod_{k=1}^{q_i} \varepsilon_{ik}. \tag{3.10}$$

Also, define a scaling matrix S_{ij} as

$$S_{ij} = \text{Diag}\left[\prod_{l=\alpha_{ij-1}+2}^{q_i} \varepsilon_{il}, \prod_{l=\alpha_{ij-1}+3}^{q_i} \varepsilon_{il}, \cdots, \prod_{l=\alpha_{ij}+1}^{q_i} \varepsilon_{il} \right]. \tag{3.11}$$

In (3.11), for $j=r_i$, the product $\prod_{l=q_i+1}^{q_i} \varepsilon_{il}$ is taken as unity. Now let,

$$\tilde{K}_{ij}(\sigma) = \frac{1}{\eta_i} J_{ij} S_{ij} K_{ij}$$

and

$$\tilde{K}_i(\sigma) = [\tilde{K}_{i1}'(\sigma), \tilde{K}_{i2}'(\sigma), \cdots, \tilde{K}_{ir_i}'(\sigma)]'. \tag{3.12}$$

The above design is rather simple when $r_i=1$. For this case, let $\bar{\mu}_i$ denote the small parameter. Then,

$$\tilde{K}_i(\sigma) = \frac{1}{(\bar{\mu}_i)^{q_i}} [(\bar{\mu}_i)^{q_i-1}\hat{K}_{i1}, (\bar{\mu}_i)^{q_i-2}\hat{K}_{i2}, \cdots, \hat{K}_{iq_i}]', \tag{3.13}$$

where $\hat{K}_{ij}$, $j=1,\cdots,q_i$, are selected such that $\lambda(G_i^c)$ are as desired, where

$$G_i^c = -\begin{bmatrix} \hat{K}_{i1}, \hat{K}_{i2}, \cdots, \hat{K}_{iq_i-1} & \hat{K}_{iq_i} \\ -I_{q_i-1} & 0 \end{bmatrix}'.$$

Here we did not discuss any eigenvector assignment. However, it turns out that our eventual design is such that the eigenvectors corresponding to the asymptotically infinite eigenvalues are naturally assigned to appropriate locations so that $M^{\infty}(j\omega,\ \sigma)\to 0$ as $\sigma\to\infty$.

Step 3: In this step, various gains calculated in Steps 1 and 2 are put together to form a composite observer gain for the given system Σ. Define $\tilde{K}^{e1}$ as

$$\left.\begin{aligned} \tilde{K}^{e1}(\sigma) &= \begin{bmatrix} \tilde{K}^{a1+}(\sigma) \\ \tilde{K}^{c1}(\sigma) \end{bmatrix} = [\tilde{K}^{e11},\ \tilde{K}^{e12},\ \cdots,\ \tilde{K}^{e1m_f}] \\ \tilde{K}^{e1i} &= \frac{1}{\eta_i} J_{ir_i} K_{ir_id} K^{e1i} \end{aligned}\right\}. \tag{3.14}$$

For the case when $r_i=1$, K_{i1d} is same as $\hat{K}_{iq_i}$ and η_i is same as $(\bar{\mu}_i)^{q_i}$. Assume $n_f>0$ and define the observer gain $K(\sigma)$ as

$$K(\sigma) = \Gamma_1 \tilde{K}(\sigma) \Gamma_2^{-1}, \tag{3.15}$$

where

$$\tilde{K}(\sigma) = \begin{bmatrix} B_{0a}^- & L_{af}^- + \tilde{H}_{af}^- & L_{ab}^- + \tilde{H}_{ab}^- \\ B_{0a}^+ + K^{a0+} & L_{af}^+ + \tilde{H}_{af}^+ + \tilde{K}^{a1+}(\sigma) & L_{ab}^+ + \tilde{H}_{ab}^+ \\ B_{0b} & L_{bf} + \tilde{H}_{bf} & K^b \\ B_{0c} + K^{c0} & L_{cf} + \tilde{H}_{cf} + \tilde{K}^{c1}(\sigma) & L_{cb} + \tilde{H}_{cb} \\ B_{0f} & L_f + \tilde{K}_f(\sigma) & 0 \end{bmatrix} \tag{3.16}$$

and where

$$\tilde{K}_f(\sigma) = \mathrm{Diag}[\tilde{K}_1(\sigma),\ \tilde{K}_2(\sigma),\ \cdots,\ \tilde{K}_{m_f}(\sigma)],$$

$$L_f = [L_1',\ L_2',\ \cdots,\ L_{m_f}']',$$

while the gains $\tilde{H}_{af}^+$, $\tilde{H}_{ab}^+$, $\tilde{H}_{af}^-$, $\tilde{H}_{ab}^-$, $\tilde{H}_{bf}$, $\tilde{H}_{cf}$ and $\tilde{H}_{cb}$ are arbitrary but finite. We have the following theorem.

Theorem 3.1. Consider a full order observer based controller with its gain given by (3.15) where n_f is assumed to be greater than zero. Then we have the following properties:

1. There exists a σ^* such that for all $\sigma>\sigma^*$, the designed observer is asymptotically stable. Furthermore, it has the time-scale structure t, t/μ_{ij}, $j=1,\cdots,r_i$ and $i=1,\cdots,m_f$. That is, the eigenvalues of the observer as $\mu_r\to 0$ are given by

$$\bar{\Lambda}^- + 0(\mu_r),\quad \bar{\Lambda}^b + 0(\mu_r),\quad \bar{\Lambda}^e + 0(\mu_r),$$

$$\frac{\Lambda_{ij}}{\mu_{ij}} + 0(1) \quad \text{for} \quad j = 1,\ \cdots,\ r_i \quad \text{and} \quad i = 1,\ \cdots,\ m_f.$$

Moreover, if $\tilde{H}_{af}^-=0$ and $\tilde{H}_{bf}=0$, some finite eigenvalues of $A-K(\sigma)C$ are exactly equal to $\bar{\Lambda}^-$ and $\bar{\Lambda}^b$ for all σ rather than asymptotically tending to $\bar{\Lambda}^-$ and $\bar{\Lambda}^b$.

2. LTR is achieved as intended in the sense that as $\sigma \to \infty$,

$$M(s, \sigma) \to \bar{M}^e(s) \text{ pointwise in } s.$$

Proof. See Appendix A.

Remark 3.1: For the case when $n_f = 0$, the observer gain obtained in the above ATEA procedure is independent of σ and is simply given by

$$K(\sigma) = \Gamma_1 \begin{bmatrix} B_{0a}^- & L_{ab}^- \\ B_{0a}^+ + K^{a0+} & L_{ab}^+ \\ B_{0b} & K^b \\ B_{0c} + K^{c0} & L_{cb} \end{bmatrix} \Gamma_2^{-1}. \tag{3.17}$$

Moreover, such an observer gain places the eigenvalues of $A - K(\sigma)C$ precisely at $\bar{\Lambda}^- \cup \bar{\Lambda}^b \cup \bar{\Lambda}^e$ and $\bar{M}^e(s)$ is exactly rather than asymptotically attained, i.e., $M(s, \sigma) = \bar{M}^e(s)$.

Remark 3.2: We emphasize that whenever $L_t(s)$ is an element of $\mathbf{T}^{\mathrm{R}}(\Sigma)$, owing to the special structure of F as in (3.1), $\bar{M}^e(s)$ is zero irrespective of the way the set of $n_a^+ + n_c$ eigenvalues belonging to $\Lambda^e(\sigma)$ and the associated right and left eigenvector sets $W^e(\sigma)$ and $V^e(\sigma)$ are selected.

As can be easily seen, ATEA design is decentralized. Required time-scale structure and eigenstructure is assigned to the subsystems of the given system Σ. The calculations involved in subsystem designs do not explicitly require the value of tuning parameter σ. σ enters only in (3.12) or (3.13) where subsystem designs are put together to form a composite gain which assigns the required time-scale structure. Thus, σ truly and directly acts as a tuning parameter and controls the degree of fastness of fast time-scales. We present next three examples to illustrate the ATEA design algorithm.

Example 3.1. Let Σ be characterized by

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let the target loop transfer function $F\Phi B$ be specified by

$$F = \begin{bmatrix} 0 & 12 & 0 & 0 & 1 \\ 1 & 0 & 30 & 1 & 10 \\ 0 & 2 & 0 & 3 & 0 \end{bmatrix}.$$

It is simple to verify that this system is neither left nor right invertible and is of nonminimum phase with two invariant zeros at $s = -1$ and $s = 1$. Also, Σ is already in the form of s.c.b with $n_a^- = n_a^+ = n_b = n_c = n_f = 1$. It can also be verified that the target loop specified by the given F is not a member of $\mathbf{T}^R(\Sigma)$. Hence, we cannot completely recover the given target loop. However, there exists a gain matrix $K(\sigma)$ such that the corresponding recovery matrix $M(s, \sigma)$ tends to some $\bar{M}^e(s)$, which is of dynamical order $n_e = n_a^+ + n_c = 2$.

In what follows, we proceed with the ATEA design. First, we need to specify the input data. We note that in this example,

$$A_{bb} = 0 \quad \text{and} \quad C_b = 1.$$

Let us choose,

$$\bar{\Lambda}^b = \{-2\} \quad \text{and} \quad V^{bb} = [1].$$

Also, in this example,

$$A^{ee} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad C^e = I_2.$$

Let us specify,

$$\bar{\Lambda}^e = \{-1.5, -2.5\} \quad \text{and} \quad W^{ee} = [W^{ee1}, W^{ee2}] = I_2,$$

so that $\bar{M}^e(s)$ is prescribed as

$$\bar{M}^e(s) = -\frac{1}{(s+1.5)(s+2.5)} \begin{bmatrix} 30(s+2.5) & 0 & 0 \\ 2(s+1.5) & 3.5(s+1.5) & -(s+1.5) \\ 11s+21.5 & 10.5(s+1.5) & -3(s+1.5) \end{bmatrix}.$$

Since $m_f = 1$ and $n_f = 1$, we can specify only one fast time-scale. Let the required Λ_1 be

$$\Lambda_1 = \{-1\}.$$

Then, following the ATEA algorithm, we obtain

$$K^b = 2, \quad K^e = \begin{bmatrix} 2.5 & 0 \\ 2 & 3.5 \end{bmatrix},$$

and the observer gain $K(\sigma)$,

$$K(\sigma) = \begin{bmatrix} 0 & 1 & 1 \\ 3.5 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 3.5\sigma & 0 \\ 0 & \sigma & 0 \end{bmatrix}.$$

This $K(\sigma)$ places three observer eigenvalues exactly at $\{-1, -1.5, -2\}$ and the remaining eigenvalues asymptotically at -2.5 and $-\sigma$. The plots of maximum and minimum singular values of the target and the achieved loop transfer functions for $\sigma=50$ are shown in Fig. 3.1. We note that for this example, the minimum singular value of $L_o(j\omega,\sigma)=C(j\omega,\sigma)P(j\omega)$ is identically zero since the given system is degenerate. Hence only the maximum singular value of $L_o(j\omega, 50)$ is shown. Also, the plots of maximum singular values of $\bar{M}^e(j\omega)$ and $M(j\omega,\sigma)$ for several values of σ as shown in Fig. 3.2, clearly demonstrate that $M(j\omega,\sigma)$ tends to $\bar{M}^e(j\omega)$ as $\sigma\to\infty$.

Example 3.2. Let Σ be characterized by

$$A=\begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & -2 \end{bmatrix},\quad B=\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

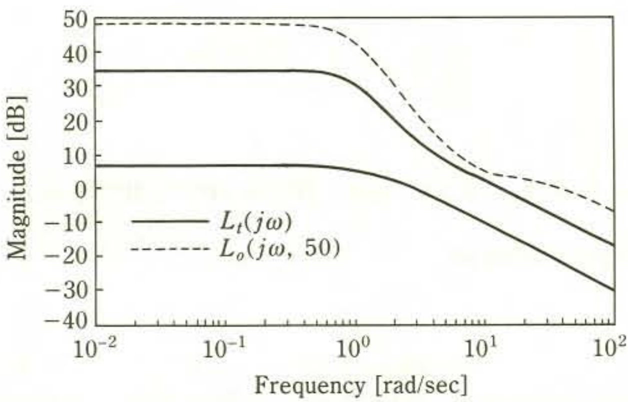

Fig. 3.1. Maximum and minimum singular values of $L_t(j\omega)$ and $L_o(j\omega, 50)$.

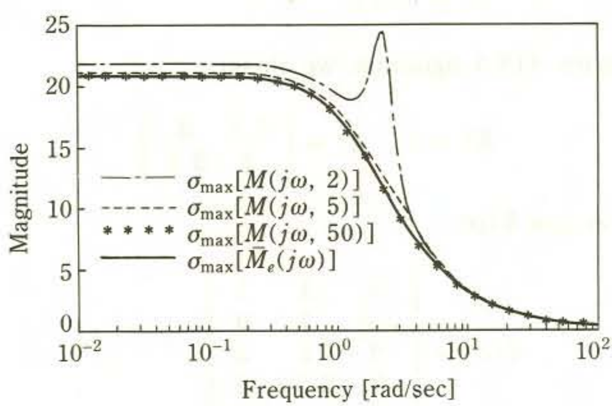

Fig. 3.2. Maximum singular values of $\bar{M}^e(j\omega)$ and $M(j\omega,\sigma)$.

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let the target loop transfer function $F\Phi B$ be specified by

$$F = [-10 \quad 10 \quad 25].$$

This system is left invertible and is of nonminimum phase with two invariant zeros at $s=-1$ and $s=1$. Also, Σ is already in the form of s.c.b with $n_a^- = n_a^+ = n_b = 1$ and $n_c = n_f = 0$. It is simple to verify that the target loop specified by the given F is not a member of $\mathbf{T}^{\mathrm{R}}(\Sigma)$. Hence, we cannot completely recover the given target loop. However, there exists a gain K, which is independent of σ owing to $n_f=0$, such that the dynamical order of the corresponding recovery matrix is equal to $n_e = n_a^+ = 1$.

We now proceed to design the observer gain using ATEA algorithm. As in the previous example, let us first specify the required input data. We note that in this example,

$$A_{bb} = -2 \quad \text{and} \quad C_b = 1.$$

Let us choose,

$$\bar{\Lambda}^b = \{-2\} \quad \text{and} \quad V^{bb} = [1].$$

Also, in this example,

$$A^{ee} = 1 \quad \text{and} \quad C^e = 1.$$

Let us specify,

$$\bar{\Lambda}^e = \{-0.1\} \quad \text{and} \quad W^{ee} = [1],$$

so that $\bar{M}^e(s)$ is prescribed as

$$\bar{M}^e(s) = -\frac{11}{s+0.1}.$$

Then following ATEA algorithm, we obtain

$$K^b = 0, \quad K^e = 1.1,$$

and the observer gain K,

$$K = \begin{bmatrix} 0 & 0 \\ 2.1 & 0 \\ 1 & 0 \end{bmatrix}.$$

This K places the observer eigenvalues at $\{-0.1, \ -1, \ -2\}$. The plots of singular values of the target and the achieved loop transfer functions is shown in

Fig. 3.3. Also, the plots of singular values of $\bar{M}^e(j\omega)$, $M(j\omega)$ and $E(j\omega)$ as shown in Fig. 3.4, clearly demonstrate that $M(j\omega)=\bar{M}^e(j\omega)$.

Example 3.3. Let Σ be characterized by

$$A=\begin{bmatrix}-1&0&0&1\\0&-2&0&1\\0&2&-3&1\\0&0&0&-4\end{bmatrix},\quad B=\begin{bmatrix}0&0\\1&0\\-1&0\\0&1\end{bmatrix},$$
$$C=\begin{bmatrix}0&-2&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\quad D=\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}.$$

The target loop transfer function $L_t(s)$ is specified by

$$F=\begin{bmatrix}0&0&-50&0\\450&0&0&46\end{bmatrix}.$$

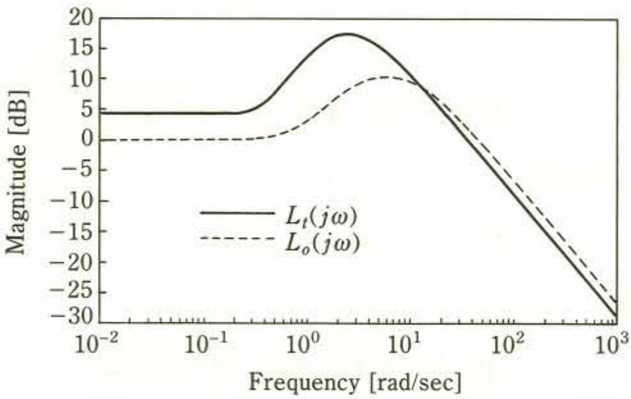

Fig. 3.3. Singular values of $L_t(j\omega)$ and $L_o(j\omega)$.

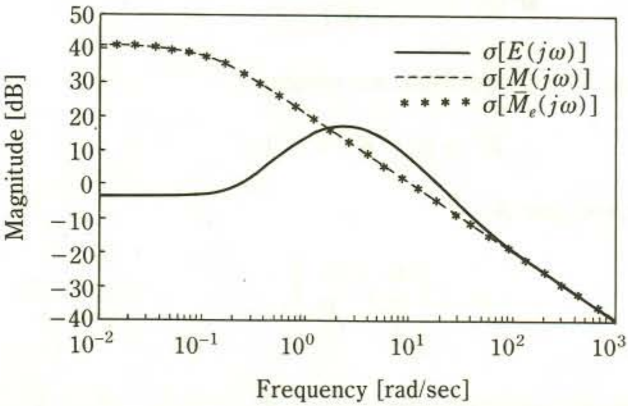

Fig. 3.4. Singular values of $\bar{M}^e(j\omega)$, $M(j\omega)$ and $E(j\omega)$.

The given system is left invertible and has invariant zeros at $s=-1$ and $s=0$. Also, it is already in the form of s.c.b with $n_a^- = n_a^+ = n_b = n_f = 1$ and $n_c = 0$. Moreover, it is straightforward to verify that the target loop specified by F is asymptotically recoverable, i.e., the given target loop is in $\mathbf{T}^{\mathrm{R}}(\Sigma)$ but not in $\mathbf{T}^{\mathrm{ER}}(\Sigma)$.

We design the observer gain for this example using ATEA to achieve ALTR. As in the previous examples, let us specify the required input data. We note that in this example,

$$A_{bb} = -3 \quad \text{and} \quad C_b = 1.$$

Let us choose,

$$\bar{\Lambda}^b = \{-3\} \quad \text{and} \quad V^{bb} = [1].$$

Also, in this example,

$$A^{ee} = 0 \quad \text{and} \quad C^e = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Let us specify,

$$\bar{\Lambda}^e = \{-2\} \quad \text{and} \quad W^{ee} = [1].$$

Note that for this example $\bar{M}^e(s) \equiv 0$. Then following the ATEA algorithm, we obtain

$$K^b = 0, \quad K^e = [-1,\ 0],$$

and an observer gain $K(\sigma)$,

$$K(\sigma) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & \sigma-4 & 0 \end{bmatrix}.$$

This gain places the eigenvalues of observer (precisely) at $\{-1, -2, -3, -\sigma\}$. The plots of maximum and minimum singular values of the target loop transfer function and the achieved loop transfer function for $\sigma=250$ in Fig. 3.5, as well as the maximum singular values of the recovery error and recovery matrix for $\sigma=250$ in Fig. 3.6 demonstrate that ALTR is achieved.

3.2. Design for exactly recoverable target loops As discussed in the previous subsection, in general in ATEA design, some eigenvalues are assigned to finite locations and some others are assigned to asymptotically infinite locations. Obviously, ATEA design discussed there yields a family of parameterized controllers $C(s, \sigma)$. Depending upon the design requirements, one then chooses a particular member of this family that corresponds to a particular value of tuning parameter σ. However, for the case when the given target loop is exactly recoverable (i.e., $L_t(s) \in \mathbf{T}^{\mathrm{ER}}(\Sigma)$), there is no necessity of generating a sequence of controllers. As discussed in Sec. 3, whenever $L_t(s) \in \mathbf{T}^{\mathrm{ER}}(\Sigma)$, F has

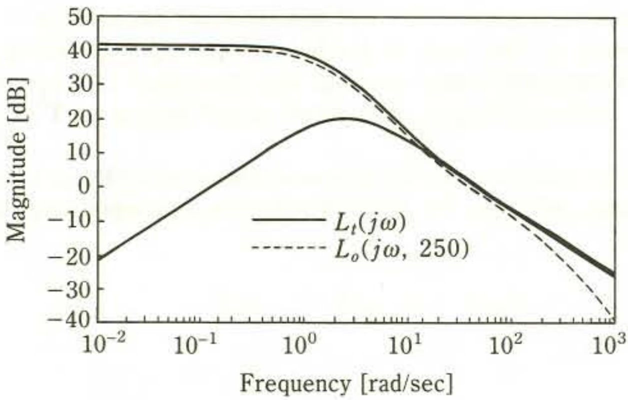

Fig. 3.5. Maximum and minimum singular values of $L_t(j\omega)$ and $L_o(j\omega,\ 250)$.

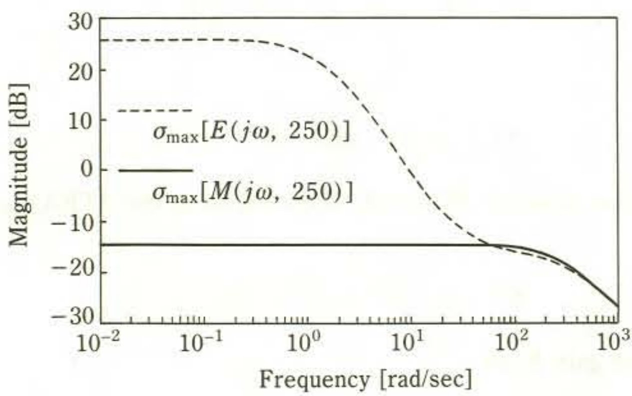

Fig. 3.6. Maximum singular values of $M(j\omega,\ 250)$ and $E(j\omega,\ 250)$.

a particular structure as given in (3.2). Owing to this particular structure of F, all eigenvalues of $A-KC$ can be assigned to finite locations and hence ATEA design procedure can drastically be simplified. In fact, in this case, design requires only finite eigenstructure assignment, and no fast time-scale structure assignment is required. The intent of this section is to describe in detail the available design freedom and step by step design for an appropriate finite eigenstructure assignment to $A-KC$ for exact loop transfer function recovery (ELTR) whenever it is feasible.

Note that for exactly recoverable case, the observer gain K is not parameterized as a function of σ and thus the presence of σ is dropped in all our notations. Following the interpretations of different partitions of $M(s)$ as in Sec. 2, in view of Lemmas 4.2 and 4.3 of Part 1, and the form of F as in (3.2), the available design freedom whenever $L_t(s)\in\mathbf{T}^{\mathrm{ER}}(\Sigma)$ can be described as follows:

1. A set of n_a^- eigenvalues of $A-KC$, namely Λ^-, must be chosen to coincide

exactly with the set of plant stable invariant zeros while the corresponding left eigenvectors of $A-KC$ must coincide exactly with the corresponding left state zero directions of Σ so that $M^{-}(s)$ is rendered zero.

2. A set of n_b eigenvalues of $A-KC$, namely Λ^b, can be assigned arbitrarily at finite locations in C^-. Moreover, the eigenvector set V^b corresponding to these eigenvalues can be selected freely within the constraints defined in Moore (1976). However, V^b must be selected to be in the null space of $(B-KD)'$ so that $M^b(s)$ is rendered zero.
3. A set of $n_e=n_a^+ + n_c$ eigenvalues of $A-KC$, namely Λ^e, can be assigned arbitrarily at finite locations in C^- subject to the condition that any unobservable but stable eigenvalues of the given system must be included among Λ^e. Moreover, the eigenvector set W^{ee} corresponding to these eigenvalues can be selected freely within the constraints defined in Moore (1976). We note that owing to the structure of F as in (3.2), $M^e(s)$ is zero irrespective of how Λ^e and W^{ee} are selected. Also, we note that $n_a^+ + n_c = 0$ if the given system is of minimum phase and left invertible.
4. A set of n_f eigenvalues of $A-KC$, namely Λ^f, can be assigned arbitrarily at any **finite** locations in C^-. (The sets Λ^∞ and V^∞ are renamed here as Λ^f and V^f because of the finiteness of the eigenvalues.) Moreover, the eigenvector set V^f corresponding to these eigenvalues can be selected freely within the constraints defined in Moore (1976). Again, owing to the structure of F as in (3.2), $M^\infty(s)$ is zero irrespective of how Λ^f and V^f are selected.

We now move on to give the design steps to obtain K which assigns an appropriate finite eigenstructure of $A-KC$ so that the observer based controller achieves ELTR.

Step 1*a*: This step deals with the assignment of finite eigenstructure to the subsystem (3.5) of Part 1. Choose a gain K^b such that $\lambda(A_{bb}^c)$ coincides with Λ^b, a set of n_b designer specified eigenvalues all in C^-, where

$$A_{bb}^c = A_{bb} - K^b C_b. \tag{3.18}$$

Note that the existence of such a K^b is guaranteed by Property 3.2 of Sec. 3 of Part 1 (Chen, Saberi and Sannuti, 1992 a). Also, in our design, the eigenvectors of A_{bb}^c can be assigned in any chosen way consistent with the freedom available in assigning them (Moore, 1976). Owing to the properties of s.c.b, our design always results in an eigenvector set V^b corresponding to the eigenvalues Λ^b of $A-KC$, in the null space of $(B-KD)'$ so that $M^b(s)=0$.

Step 1*b*: This step deals with the assignment of finite eigenstructure to the subsystems (3.4), (3.6) and (3.8) of Part 1. Let A^g and C^g be defined as

$$A^g = \begin{bmatrix} A_{aa}^+ & 0 & L_{af}^+ C_f \\ B_c E_{ca}^+ & A_{cc} & L_{cf} C_f \\ B_f E_a^+ & B_f E_c & A_f \end{bmatrix}, \quad C^g = \begin{bmatrix} C_{0a}^+ & C_{0c} & C_{0f} \\ 0 & 0 & C_f \end{bmatrix}. \tag{3.19}$$

Also, let $\Lambda^g \equiv \Lambda^e \cup \Lambda^f$ be a set of $n_a^+ + n_c + n_f$ designer specified eigenvalues all in C^- subject to the condition that any unobservable but stable eigenvalues of the given system must be included among Λ^g. Now select a gain K^g such that $\lambda(A^g - K^g C^g)$ coincides with Λ^g. Again note that the existence of such a K^g is guaranteed by Property 3.2 of Sec. 3 of Part 1. Also, the eigenvectors of $A^g - K^g C^g$ can be assigned in any chosen way consistent with the freedom available in assigning them (Moore, 1976). Let us partition K^g as

$$K^g = \begin{bmatrix} K^{a0+} & K^{a1+} \\ K^{c0} & K^{c1} \\ K^{f0} & K^{f1} \end{bmatrix}.$$

Step 2: In this step, K^b and K^g calculated in Step 1 are put together into a composite matrix. Let

$$\tilde{K} = \begin{bmatrix} B_{0a}^- & L_{af}^- & L_{ab}^- \\ B_{0a}^+ + K^{a0+} & K^{a1+} & L_{ab}^+ \\ B_{0b} & L_{bf} & K^b \\ B_{0c} + K^{c0} & K^{c1} & L_{cb} \\ B_{0f} + K^{f0} & K^{f1} & 0 \end{bmatrix}. \tag{3.20}$$

Finally define the observer gain K as

$$K = \Gamma_1 \tilde{K} \Gamma_2^{-1}. \tag{3.21}$$

We have the following theorem.

Theorem 3.2. Consider a full order observer based controller with its gain as given by (3.21). Then the eigenvalues of the observer are given by Λ^-, Λ^b and Λ^g. Moreover, the observer based controller using the gain given in (3.21) achieves ELTR.

Proof. See Appendix B.

Remark 3.3: We note that in general the observer gain which leads to ELTR is not unique.

Example 3.4. Consider the system given in Example 3.2 except that now the target loop transfer function $L_t(s)$ is specified by

$$F = \begin{bmatrix} -20 & 0 & -50 & 0 \\ 50 & 0 & 100 & 0 \end{bmatrix}.$$

It is straightforward to verify that the target loop specified by F is exactly recoverable. In fact, as illustrated in Fig. 3.7, the following observer gain,

$$K = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

does achieve ELTR while placing the observer eigenvalues precisely at $\{-1, -2, -3, -4\}$. Figure 3.7 shows the plots of maximum and minimum singular values of the target and the achieved sensitivity functions, $S_t(j\omega)$ and $S_o(j\omega)$.

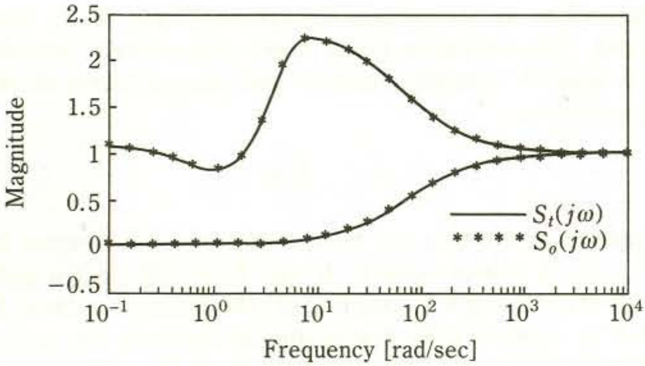

Fig. 3.7. Maximum and minimum singular values of $S_t(j\omega)$ and $S_o(j\omega)$.

4. Optimization Based Design Methods

As is clear from Sec. 2, the whole notion of LTR is to render the recovery matrix $M(s)=F(sI_n-A+KC)^{-1}(B-KD)$ small in some sense or other. The ATEA design method views this task from the perspective of asymptotic time-scale and eigenstructure assignment to the observer dynamic matrix. An alternative method is to view it as finding a gain K which minimizes some (say, either H_2 or H_∞) norm of $M(s)$. That is, one can cast the LTR design as a straightforward mathematical optimization problem. A suboptimal or optimal solution to such an optimization problem provides the needed observer gain. There is some historical basis to casting the LTR problem as such. In their seminal work, considering only left invertible and minimum phase systems, Doyle and Stein (1979) propose a design method based on Kalman filter formalism in which the intensity of a fictitious input process noise is used as a tuning parameter σ. As $\sigma\to\infty$, their method yields an observer gain which renders $M(s, \sigma)$ asymptotically zero and thus achieves ALTR. It looks, however, mysterious why and how such a gain achieves ALTR for the class of problems considered by Doyle and Stein (1979). It turns out, as proved later on by Goodman (1984), that the procedure of Doyle and Stein (1979) minimizes the H_2 norm of $M(s)$ as $\sigma\to\infty$. That is, the procedure of Doyle and Stein (1979) yields a sequence of suboptimal solutions to H_2 norm minimization of $M(s)$. These suboptimal solutions are parameterized in terms of σ; and the limit as $\sigma\to\infty$ of the sequence of corresponding $\|M(s, \sigma)\|_{H_2}$ is the infimum of $\|M(s)\|_{H_2}$ over the set of all possible gains. The infimum of $\|M(s)\|_{H_2}$ happens to be zero for left invertible and minimum phase systems. In view of this historic perspective, in this section, we cast the loop transfer recovery problem for general not necessarily left invertible and not necessarily minimum phase systems, as a standard H_2 or H_∞ optimization problem. To facilitate this, we consider the following auxiliary system,

$$\Sigma_a: \begin{cases} \dot{x} = A'x + C'u + F'w, \\ y = x, \\ z = B'x + D'u. \end{cases} \tag{4.1}$$

Here w is treated as an exogenous disturbance input to Σ_a while u is the controlling input. The variables y and z are respectively considered as the measured and desired outputs. Suppose one uses a state feedback law to generate the control u,

$$u = -K'x. \tag{4.2}$$

It is then simple to verify that the closed-loop transfer function from w to z, denoted by $T_{zw}(s)$, is indeed equal to $M'(s)$. Now LTR design problem can be cast as the task of obtaining a K such that (1) the auxiliary system Σ_a under the control law (4.2) is asymptotically stable, and (2) the norm (H_2 or H_∞) of $M(s)$ is minimized. There exists a vast literature on H_2 or H_∞ minimization methods. Borrowing from such a literature, subsection 4.1 discusses algorithms for H_2 minimization of $M(s)$ while subsection 4.2 does the same for H_∞ minimization. We want to emphasize that the optimization problem is cast here in terms of minimizing an appropriate norm of recovery matrix $M(s)$ rather than the recovery error $E(s)$.

It is well known that an optimal solution for either H_2 or H_∞ minimization of $M(s)$ does not necessarily exist, and the infimum of $\|M(s)\|_{H_2}$ or $\|M(s)\|_{H_\infty}$ is in general nonzero. For a class of target loops, however, the infimum of $\|M(s)\|_{H_2}$ or $\|M(s)\|_{H_\infty}$ is in fact zero, and it can be attained by a finite gain K. This is the class of exactly recoverable target loops $\mathbf{T}^{\mathrm{ER}}(\Sigma)$. Also, for another class of target loops, namely the class of asymptotically recoverable target loops $\mathbf{T}^{\mathrm{R}}(\Sigma)$, the infimum of $\|M(s)\|_{H_2}$ or $\|M(s)\|_{H_\infty}$ is zero, and it can be attained only asymptotically by using larger and larger gain K. Whether the infimum of $\|M(s)\|_{H_2}$ or $\|M(s)\|_{H_\infty}$ is zero or not, for general target loops, one needs to generate a sequence of gains having the property that the limit of H_2 or H_∞ norms of the correspondingly generated recovery matrices is the infimum of $\|M(s)\|_{H_2}$ or $\|M(s)\|_{H_\infty}$ over the set of all possible gains. A suboptimal solution results when one uses a gain corresponding to a particular member of the sequence. In H_2 optimization, an observer gain is generated via the solution of an algebraic Riccati equation (called here after H_2-ARE) parameterized in terms of a tuning parameter σ. A sequence of suboptimal gains is generated by tending σ to ∞. In H_∞ optimization, let γ^* be the infimum of $\|M(s)\|_{H_\infty}$ over the set of all possible gains. Then given a parameter γ greater than γ^*, in H_∞ optimization, one generates a gain by solving an algebraic Riccati equation (called hereafter H_∞-ARE) parameterized in terms of γ so that the resulting $\|M(s, \gamma)\|_{H_\infty}$ is strictly less than γ. Starting with a $\gamma>\gamma^*$, and gradually reducing γ step by step but always keeping $\gamma>\gamma^*$, one obtains a sequence of suboptimal gains.

For simplicity but without loss of generality, we assume throughout this section that the matrix D is of the form,

$$D = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}.$$

Also, we partition the matrices B and C as

$$B = [B_0, B_1] \quad \text{and} \quad C = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix},$$

and let $A_1 = A - B_0 C_0$.

4.1 H_2-optimization based design algorithms In this subsection, we consider H_2 norm minimization of $M(s)$ or equivalently $T_{zw}(s)$. At first, let us look at calculating the infimum value of $\|M(s)\|_{H_2}$ as is done very elegantly in a recent work by Stoorvogel (1990). We first recall the following lemma from Stoorvogel (1990).

Lemma 4.1. Assume that (A, C) is detectable. Then the infimum of $\|M(s)\|_{H_2}$ over all the stabilizing observer gains is given by $\mathrm{tr}\{F\bar{P}F'\}$, where $\bar{P}\in\mathscr{R}^{n\times n}$ is the unique positive semi-definite matrix satisfying:

i. $\tilde{F}(\bar{P}) = \begin{bmatrix} A\bar{P}+\bar{P}A'+BB' & \bar{P}C'+BD' \\ C\bar{P}+DB' & DD' \end{bmatrix} \geq 0,$

ii. $\mathrm{rank}\tilde{F}(\bar{P}) = \mathrm{normrank}\{C(sI_n-A)^{-1}B+D\}, \quad \forall s \in C^+/C^o,$

iii. $\mathrm{rank}\begin{bmatrix} [sI-A', \ -C'] \\ \tilde{F}(\bar{P}) \end{bmatrix} = n + \mathrm{normrank}\{C(sI_n-A)^{-1}B+D\}, \quad \forall s \in C^+/C^o.$

Here normrank$\{\cdot\}$ denotes the rank of matrix $\{\cdot\}$ over the field of rational functions.

Proof. See Stoorvogel (1990).

In general, as discussed earlier, the infimum of $\|M(s)\|_{H_2}$ can only be obtained asymptotically. In what follows, we proceed to introduce a basic algorithm of obtaining a sequence of parameterized observer gains $K(\sigma)$ for the general system Σ such that the H_2 norm of the corresponding recovery matrix, which is also parameterized by σ and is denoted by $M(s, \sigma)$, tends to the infimum of $\|M(s)\|_{H_2}$ as $\sigma\to\infty$. The algorithm consists of the following two steps:

Step 1: Solve the following parameterized algebraic Riccati equation (H_2-ARE) for a chosen fixed value of the parameter σ,

$$A_1P + PA_1' - PC_0'C_0P - \sigma PC_1'C_1P + B_1B_1' + \frac{1}{\sigma}I_n = 0, \qquad (4.3)$$

for its positive definite solution P. We note that a unique positive definite solution P of (4.3) always exists for all $\sigma>0$. Obviously, P is a function of σ and is denoted by $P(\sigma)$.

Step 2: Let

$$K(\sigma) = [B_0+P(\sigma)C_0', \ \sigma P(\sigma)C_1']. \qquad (4.4)$$

We have the following theorem.

Theorem 4.1. Consider a full order observer based controller with its gain taken as in (4.4). Let $M(s, \sigma)$ be the resulting recovery matrix. Then, we have

$$\lim_{\sigma\to\infty} P(\sigma) = \bar{P}.$$

Moreover, $\|M(s, \sigma)\|_{H_2}$ tends to the infimum of $\|M(s)\|_{H_2}$ as $\sigma \to \infty$, i.e.,

$$\lim_{\sigma \to \infty} \|M(s, \sigma)\|_{H_2} = \mathrm{tr}\{F\bar{P}F'\}.$$

Proof. See Appendix C.

In view of Theorem 4.1, it is apparent that as σ takes values larger and larger, the design algorithm given above generates a sequence of observer gains having the property that the limit of the correspondingly generated $\|M(s, \sigma)\|_{H_2}$ is the infimum of $\|M(s)\|_{H_2}$ over the set of all possible gains. A suboptimal solution results when one uses a particular value of the parameter σ. However, for some particular class of systems, e.g., the well-known *regular problems* (i.e., D is surjective implying that Σ is right invertible and has no infinite zeros, and Σ has no invariant zeros on the $j\omega$ axis), the infimum value of $\|M(s)\|_{H_2}$ can be achieved with the following observer gain (Doyle et al., 1989),

$$K = B_0 + PC_0', \tag{4.5}$$

where P is the positive semi-definite solution of

$$A_1P + PA_1' - PC_0'C_0P + B_1B_1' = 0.$$

The resulting infimum value of $\|M(s)\|_{H_2}$ is given by

$$\|M(s)\|_{H_2} = \mathrm{tr}\{FPF'\}.$$

Note that in this discussion, the observer gain K and thus the resulting recovery matrix is not parameterized as a function of σ. We note that for a regular problem when $\|M(s)\|_{H_2}=0$, the observer gain K as given in (4.5) achieves exact loop transfer recovery (ELTR). There is a larger class of systems than the class of regular systems, for which $\|M(s)\|_{H_2}=0$. However, no optimization based method exists yet in the literature to generate the needed gain to achieve $\|M(s)\|_{H_2}=0$, whenever it is possible, for systems other than the class of regular systems. On the other hand, a direct design procedure based on ATEA, which achieves ELTR *whenever it can be done*, was presented earlier in Subsection 3.2.

Another special case of design that is of interest is as follows. Consider a left invertible minimum phase system Σ which is non-strictly proper. Let the observer gain $K(\sigma)$ be given by

$$K(\sigma) = [B_0,\ \sigma P(\sigma)C_1'],$$

where $P(\sigma)\triangleq P$ is the positive definite solution of

$$A_1P + PA_1' - \sigma PC_1'C_1P + B_1B_1' = 0.$$

It is simple to show then that the observer gain $K(\sigma)$ chosen as above achieves asymptotic loop transfer recovery (ALTR), i.e., the resulting $\|M(s, \sigma)\|_{H_2}$ tends to zero asymptotically as $\sigma \to \infty$. This is a generalization, for non-strictly proper left invertible minimum phase systems, of the result given by Doyle and

Stein (1979) who treat only strictly proper left invertible minimum phase systems. The above result has been given earlier by Chen, Saberi, Bingulac and Sannuti (1990).

It is of interest to investigate what type of time-scale structure and eigenstructure is assigned to the observer dynamic matrix $A-K(\sigma)C$ by the gain $K(\sigma)$ obtained via the basic algorithm of Eqs. (4.3) and (4.4). Obviously, the basic algorithm renders $M^-(s, \sigma)$, $M^b(s, \sigma)$ and $M^\infty(s, \sigma)$ zero as $\sigma \to \infty$, while shaping $M^e(s, \sigma)$ in a particular way so that the infimum of $\|M(s)\|_{H_2}$ is attained as $\sigma \to \infty$. In so doing, among all the possible choices for the time-scale structure and eigenstructure of $A-K(\sigma)C$, it selects a particular choice which can easily be deduced from the results of cheap and singular control problems as analyzed in Saberi and Sannuti (1987) (see also, Zhang and Freudenberg, 1990; Saberi et al., 1991 b). We have the following results:

1. As $\sigma \to \infty$, the asymptotic limits of the set of n_a^- eigenvalues $\Lambda^-(\sigma)$ and the associated set of left eigenvectors $V^-(\sigma)$ of $A-K(\sigma)C$ coincide respectively with the set of plant stable invariant zeros and the corresponding left state zero directions of Σ. This renders $M^-(s, \sigma)$ zero as $\sigma \to \infty$.
2. As $\sigma \to \infty$, some of the n_b eigenvalues in $\Lambda^b(\sigma)$ coincide with the stable but uncontrollable eigenvalues of Σ while the rest of them coincide with what are called "compromise" zeros of Σ (Saberi and Sannuti, 1987). Also, the asymptotic limits of the associated left eigenvectors, namely $V^b(\sigma)$, fall in the null space of matrix $[B-K(\sigma)D]'$ so that $M^b(s, \sigma) \to 0$ as $\sigma \to \infty$.
3. As $\sigma \to \infty$, the set of n_f eigenvalues $\Lambda^\infty(\sigma)$ of $A-K(\sigma)C$ tend to asymptotically infinite locations in such a way that $M^\infty(s, \sigma) \to 0$. The time-scale structure assigned to these eigenvalues depends on the infinite zero structure of Σ (see for details in Saberi and Sannuti, 1987). Also, the eigenvalues assigned to each fast time-scale follow asymptotically a Butterworth pattern.
4. As $\sigma \to \infty$, the asymptotic limits of n_a^+ eigenvalues in $\Lambda^e(\sigma)$ coincide with the mirror images of unstable invariant zeros of Σ, while the associated set of left eigenvectors of $A-K(\sigma)C$ coincide with the corresponding right input zero directions of Σ. The rest of n_c eigenvalues of $\Lambda^e(\sigma)$, as $\sigma \to \infty$, tend to some unnamed finite locations, while the associated left eigenvectors follow some unnamed directions. This shapes the limit of recovery matrix, $M^e(s)$ in a particular way so that the infimum of $\|M(s)\|_{H_2}$ is attained as $\sigma \to \infty$.

To conclude, as ATEA design procedure does in general, the basic algorithm of Eqs. (4.3) and (4.4) renders $M^-(s, \sigma)$, $M^b(s, \sigma)$ and $M^\infty(s, \sigma)$ zero asymptotically as $\sigma \to \infty$. Moreover, it shapes $M^e(s)$ in a particular way so that the infimum of $\|M(s)\|_{H_2}$ is attained as $\sigma \to \infty$. In contrast to this, ATEA design procedure of Sec. 3 allows complete available freedom to shape the limit of recovery matrix $\bar{M}^e(s)$ in a chosen manner within the design constraints imposed by the structural properties of the given system.

Example 4.1. Let Σ be characterized by

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This system is invertible and has $n_f=2$. It is of nonminimum phase as it has two invariant zeros at $s=1$ and $s=2$. Let the target loop transfer function $F\Phi B$ be specified by

$$F = \begin{bmatrix} -1 & 5 & 1 & 1 \\ 9 & -4 & 3 & 5 \end{bmatrix}.$$

Then it can be verified that the infimum of $\|M(s)\|_{H_2}$ for this problem is equal to 639.6. Hence, we cannot completely recover the given target loop. Let the observer gain $K(\sigma)$ be calculated using H_2-optimization algorithm. The plots of maximum and minimum singular values of the target and the achieved loop transfer functions for $\sigma=1000$ are shown in Fig. 4.1.

We note that Σ is already in the form of s.c.b with

$$A^{ee} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad C^{e} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then it can be shown that H_2-optimization algorithm renders $M^{\infty}(s, \sigma)$ zero asymptotically as $\sigma \to \infty$ and places

$$\bar{\Lambda}^{e} = \{-1, -2\} \quad \text{and} \quad W^{ee} = [W^{ee1}, W^{ee2}] = \begin{bmatrix} 0.9487 & 0.8 \\ 0.3162 & 0.6 \end{bmatrix}.$$

This results in a particular $\bar{M}^{e}(s)$,

$$\bar{M}^{e}(s) = \frac{1}{5(s+1)(s+2)} \begin{bmatrix} 69(s+2) & -3(41s+18) \\ -129(s+2) & 123s-6 \end{bmatrix}.$$

The plots of maximum singular values of $\bar{M}^{e}(j\omega)$ and $M(j\omega, \sigma)$ for several values of σ as shown in Fig. 4.2, clearly demonstrate that $M(j\omega, \sigma) \to \bar{M}^{e}(j\omega)$ as $\sigma \to \infty$.

4.2 H_∞-optimization design algorithms In this subsection, we consider H_∞ norm minimization of $M(s)$ or equivalently $T_{zw}(s)$. Unlike in H_2 norm minimization case of previous subsection, for general systems, there are no direct methods available of exactly computing the infimum of $\|M(s)\|_{H_\infty}$ which is denoted here by γ^*. However, there are iterative algorithms that can approximate γ^*, at least in principle, to an arbitrary degree of accuracy (see for example, Pandey et al., 1990). Recently though, for a particular class of problems, i.e., when Σ is left invertible and has no invariant zeros on the $j\omega$ axis, such an infimum γ^* has explicitly been calculated in Chen, Saberi and Ly (1991; 1992).

We now proceed to present a basic algorithm of computing the observer gain matrix K such that the resulting H_∞-norm of the recovery matrix $M(s, \gamma)$, is less than *a priori* given desired scalar $\gamma > \gamma^*$. The algorithm is as follows:

Step 0: Choose a value $\varepsilon=1$.

Step 1: Solve the following algebraic Riccati equation (H_∞-ARE),

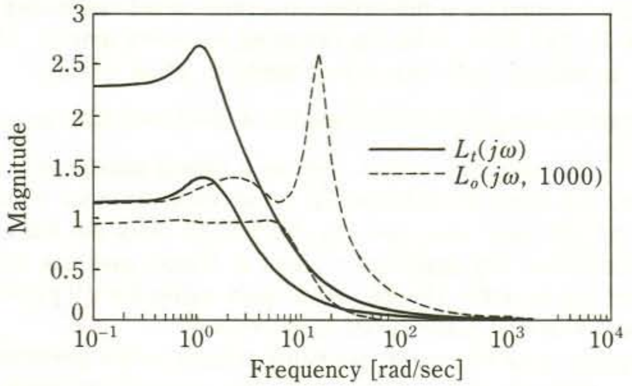

Fig. 4.1. Maximum and minimum singular values of $L_t(j\omega)$ and $L_o(j\omega,\ 1000)$.

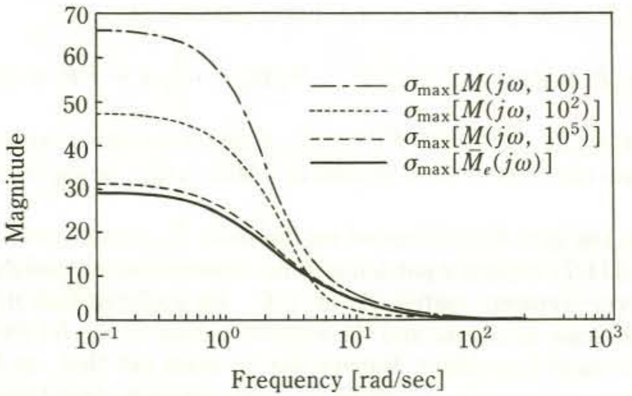

Fig. 4.2. Maximum singular values of $\bar{M}^e(j\omega)$ and $M(j\omega,\ \sigma)$.

$$A_1P + PA_1' - PC_0'C_0P - \frac{1}{\varepsilon}PC_1'C_1P + B_1B_1' + \frac{1}{\gamma^2}PF'FP + \varepsilon I_n = 0 \tag{4.6}$$

for P. Evidently, since (4.6) is parameterized in terms of γ, the solution P of it is a function of γ and is denoted by $P(\gamma)$.

Step 2: If $P(\gamma)>0$ go to Step 3. Otherwise, decrease ε and go to Step 1. Note that for $\gamma>\gamma^*$, it is shown in Zhou and Khargonekar (1988) that there always exists a sufficiently small scalar $\varepsilon^*>0$ such that the H_∞-ARE (4.6) has a unique positive definite solution $P(\gamma)$ for each $\varepsilon\in(0,\ \varepsilon^*)$.

Step 3: Let

$$K(\gamma) = \left[B_0+P(\gamma)C_0',\ \frac{1}{2\varepsilon}P(\gamma)C_1'\right]. \tag{4.7}$$

We have the following theorem.

Theorem 4.2. Consider a full order observer based controller with its gain taken as in (4.7). Let $M(s, \gamma)$ be the resulting recovery matrix. Then, we have $\|M(s, \gamma)\|_{H_\infty}$ is strictly less than γ, and tends to γ^* as $\gamma \to \gamma^*$.

Proof. It follows simply from the results of Zhou and Khargonekar (1988).

Remark 4.1: We note that γ acts here as a tuning parameter. Since to start with, one does not know γ^*, a particular prescribed value for γ may turn out to be less than γ^*. In that case, the H_∞-ARE (4.6) does not have any positive definite solution even for sufficiently small ε. Then, one has to increase the value of γ and try to solve the H_∞-ARE once again for $P(\gamma)>0$. One has to repeat this procedure as many times as necessary.

For the special case of *regular problems* defined in the previous subsection, there exists a method of generating the gain without the need to introduce another parameter ε, and is given by Doyle et al. (1989),

$$K(\gamma) = B_0 + P(\gamma)C_0', \tag{4.8}$$

where $P(\gamma) \triangleq P$ is the positive semi-definite solution of

$$A_1P + PA_1' - PC_0'C_0P + B_1B_1' + \frac{1}{\gamma^2}PF'FP = 0,$$

such that $\lambda(A_1' - C_0'C_0P + \gamma^{-2}F'FP) \subseteq C^-$. A full order observer based controller with its gain taken as in (4.8) results in $\|M(s, \gamma)\|_{H_\infty}$ being strictly less than γ.

Obviously, the gain $K(\gamma)$ obtained via the basic H_∞-optimization algorithm of Eqs. (4.6) and (4.7) assigns a particular time-scale structure and eigenstructure to the observer dynamic matrix $A - K(\gamma)C$. An investigation into the exact nature of time-scale structure and the eigenstructure of $A - K(\gamma)C$ as $\gamma \to \gamma^*$ is still an open research problem. But we like to point out that, as ATEA design procedure does in general, the basic H_∞-optimization algorithm renders the corresponding $M^-(s, \gamma)$, $M^b(s, \gamma)$ and $M^\infty(s, \gamma)$ zero asymptotically as $\gamma \to \gamma^*$. Also, the corresponding $M^e(s)$ is shaped in a particular way so that the infimum of $\|M(s)\|_{H_\infty}$ is attained as $\gamma \to \gamma^*$. In so doing, in addition to $\Lambda^\infty(\gamma)$, some elements of $\Lambda^e(\gamma)$ may be pushed to infinite locations in C^- as $\gamma \to \gamma^*$. The investigation of these and other properties of H_∞-optimization algorithm of Eqs. (4.6) and (4.7) is outside the scope of this paper.

Example 4.2. Let Σ be characterized by

$$A = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This system is neither left invertible nor right invertible with one invariant zero at $s=-1$. Let the target loop transfer function $F\Phi B$ be specified by

$$F = \begin{bmatrix} 0.2 & 2.3 & 0.3 & 0.3 \\ 0.5 & 0.3 & 0.1 & 1.7 \\ 0.1 & 0.3 & 1.0 & 0.1 \end{bmatrix}.$$

For this example, it can be verified that γ^*, the infimum of $\|M(s)\|_{H_\infty}$, is equal to $\sqrt{0.55}=0.74161985$. Hence, we cannot completely recover the given target loop. Let the observer gain $K(\gamma)$ be calculated using H_∞-optimization algorithm. The plots of maximum and minimum singular values of the target and the achieved loop transfer functions for $\gamma=1$ are shown in Fig. 4.3. We note that for this example, the minimum singular value of $L_o(j\omega, \gamma)=C(j\omega, \gamma)P(j\omega)$ is identically zero since the given system is degenerate. Hence, only the maximum singular value of $L_o(j\omega, 1)$ is shown.

We note that Σ is already in the form of s.c.b with $n_a^-=n_b=n_c=n_f=1$ and

$$A^{ee} = -1 \quad \text{and} \quad C^e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then it can be shown that H_∞-optimization procedure renders $M^-(s, \gamma)$, $M^b(s, \gamma)$ and $M^\infty(s, \gamma)$ zero asymptotically as $\gamma \to \gamma^*$ and places

$$\bar{\Lambda}^e = \{-2\} \quad \text{and} \quad W^{ee} = 1.$$

This results in a particular $\bar{M}^e(s)$,

$$\bar{M}^e(s) = \frac{1}{(s+2)} \begin{bmatrix} -0.3 & 0 & 0.3 \\ -0.1 & 0 & 0.1 \\ -1.0 & 0 & 1.0 \end{bmatrix}.$$

The plots of maximum singular values of $\bar{M}^e(j\omega)$ and $M(j\omega, \gamma)$ for several values of γ as given in Fig. 4.4, clearly demonstrate that $M(j\omega, \gamma) \to \bar{M}^e(j\omega)$ as $\gamma \to \gamma^*$.

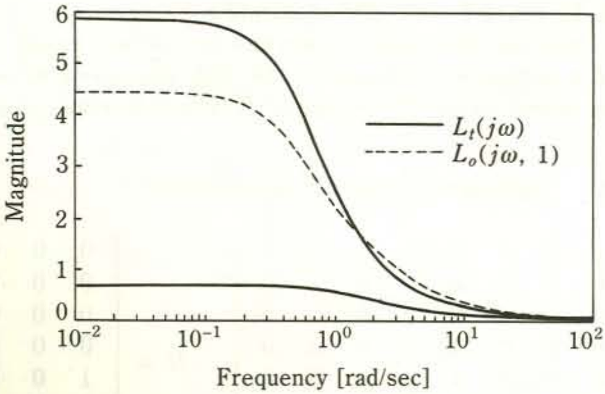

Fig. 4.3. Maximum and minimum singular values of $L_t(j\omega)$ and $L_o(j\omega, 1)$.

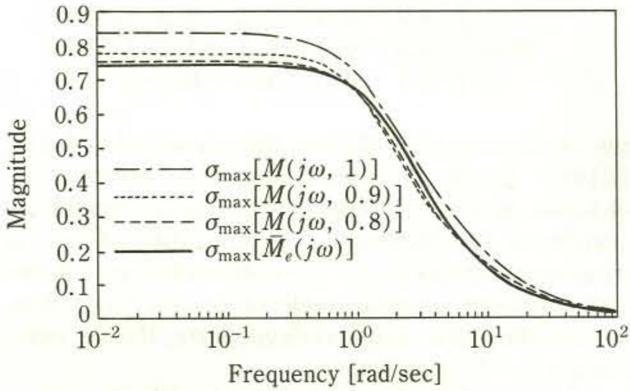

Fig. 4.4. Maximum singular values of $\bar{M}^e(j\omega)$ and $M(j\omega, \gamma)$.

5. Design for Recovery over a Specified Subspace

Sections 3 and 4 consider the conventional LTR design problem which seeks the recovery over the entire control space. Here, given a subspace S of $\mathscr{R}^m$, the interest is in designing an observer so that the achieved and target sensitivity and complimentary sensitivity functions projected onto the subspace S match each other either exactly or asymptotically. The conditions under which such a design is possible are given in Part 1. To recapitulate these conditions, let V^s be a matrix whose columns form an orthogonal basis of the given subspace S of $\mathscr{R}^m$. Also, given the system Σ characterized by (A, B, C, D), let us define an auxiliary system Σ^s characterized by the matrix triple (A, BV^s, C, DV^s). Also, let $L_t(s)=F\Phi B$ be the specified target loop transfer function. Then, the analysis given in Part 1 implies the following:

1. The projections of achievable and target sensitivity and complimentary sensitivity functions onto the subspace S match each other exactly, if and only if $S^-(A, BV^s, C, DV^s) \subseteq \mathrm{Ker}(F)$.
2. The projections of achievable and target sensitivity and complimentary sensitivity functions onto the subspace S match each other asymptotically, if and only if $V^+(A, BV^s, C, DV^s) \subseteq \mathrm{Ker}(F)$.

Thus, the task of designing observers for either exact or asymptotic recovery over a subspace collapses to the task discussed in earlier sections except that one needs to use Σ^s instead of Σ. The following example illustrates this.

Example 5.1. Consider a system Σ characterized by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This system is left invertible and of nonminimum phase with invariant zeros at $s=1$, $s=2$, $s=3$ and at $s=4$. Now consider a specified subspace S which is a span of

$$V^s = \begin{bmatrix} 0.4433 & -0.4553 & -0.0027 \\ 0.3802 & 0.5771 & -0.7128 \\ 0.6006 & -0.4719 & -0.1664 \\ 0.5462 & 0.4867 & 0.6813 \end{bmatrix}.$$

It is simple to verify that the auxiliary system Σ^s characterized by (A, BV^s, C, DV^s) is left invertible and of minimum phase. Hence the projections of target and achievable sensitivity and complimentary sensitivity functions onto V^s can match each other asymptotically. To exemplify this, let the target loop be specified by

$$F = \begin{bmatrix} 0 & 0 & 0 & 200 & 100 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 50 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 & 0 & 30 & 0 \\ 50 & 0 & 0 & 0 & 0 & 0 & 0 & 25 \end{bmatrix}.$$

Let us choose $K(\sigma)$ with $\sigma=1000$ as

$$K(\sigma) = \begin{bmatrix} 0 & 10707293 & 240.6328 & -717.12942 & -18.09924 \\ 0 & -1324480 & -35.68825 & 106.64317 & 2.70284 \\ 0 & -6883343 & -215.567 & 643.91894 & 16.36376 \\ 0 & 258551 & 9.26071 & -27.598585 & -0.701356 \\ 1 & 1.76203 & 2.185\times10^{-7} & -6.51\times10^{-7} & -1.655\times10^{-8} \\ -0.4570667 & 3978.336 & 144.08447 & 68.78912 & 1.74812 \\ 3.57747 & -9798.08346 & 40.0091 & 23.90837 & -3.03008 \\ 0.185196 & -354.2448 & 1.115087 & -3.32316 & 125.04055 \end{bmatrix}$$

so that the observer eigenvalues are placed at -1000, -1000, -1, -2, -3, -4, -5 and -6. Let the orthogonal projection matrix onto the subspace S be $P^s = V^s(V^s)'$. Then the resulting $M(j\omega, \sigma)P^s$, $E(j\omega, \sigma)P^s$, $S_o(j\omega, \sigma)P^s$ and

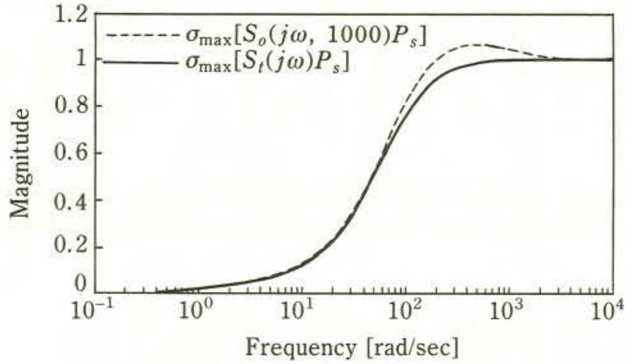

Fig. 5.1. Maximum singular values of $S_t(j\omega)P^s$ and $S_o(j\omega,\ 1000)P^s$.

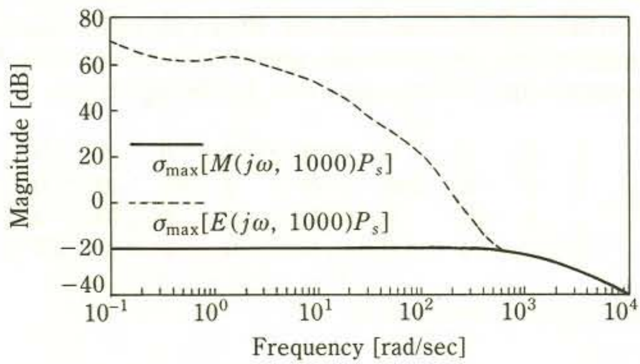

Fig. 5.2. Maximum singular values of $M(j\omega,\ 1000)P^s$ and $E(j\omega,\ 1000)P^s$.

$S_t(j\omega)P^s$ are ploted with respect to ω over a given range of ω in Figs. 5.1 and 5.2. It is easy to note that $M(j\omega,\ \sigma)P^s$ is approximately zero while $S_o(j\omega,\ \sigma)P^s$ is close to $S_t(j\omega)P^s$. Also, note that the minimum singular values of $S_o(j\omega,\ \sigma)P^s$ and $S_t(j\omega)P^s$ are identically zero due to the singularity of P^s.

6. Comparison of 'ATEA' and 'ARE' Based Design Algorithms

A comparison of optimal or suboptimal design schemes based on solving Algebraic Riccati equations (ARE's) as described in Sec. 4 and the asymptotic time-scale and eigenstructure assignment (ATEA) design schemes of Sec. 3, is in order. In this regard, our earlier paper (Saberi et al., 1991 b) discusses several relative advantages and disadvantages of ATEA and ARE based designs. Here we look at ATEA design and optimization based designs from two different perspectives, (1) numerical simplicity and (2) flexibility to use all the available freedom.

Let us first consider numerical aspects of both designs. It is clear that the central part of either optimization based design of Sec. 4 lies in obtaining the positive definite solutions of parameter dependent ARE's repeatedly for different values of the parameter. As is well known, these ARE's become numerically "stiff" when the concerned parameter takes values close to a critical value. To be specific, the H_2-ARE becomes stiff as the parameter σ takes larger and larger values, where as the H_∞-ARE becomes stiff when γ approaches γ^*. This is due to the interaction of fast and slow dynamics inherent in such equations. Thus the numerical difficulties accrue not only due to the required repeated solutions of ARE's but also due to the "stiffness" of such equations. On the other hand, as is clear from Sec. 3, ATEA adopts a decentralized design procedure and in so doing removes both the obstacles of repeated solution of algebraic equations and their stiffness. That is, in ATEA, in order not to allow the interaction between the slow and various fast time-scales, the design that assigns the appropriate asymptotically finite and infinite eigenstructure to the observer dynamic matrix is done separately. The tuning parameter which merely adjusts the relative fastness of fast time-scales is introduced only in composing the two separately designed gains together into a composite gain, and this presents no numerical difficulties whatsoever as the parameter takes larger and larger values.

Another factor that is of great importance in selecting a design procedure is the flexibility it offers to utilize all the available design freedom. As summarized in Sec. 2, there exists considerable amount of freedom to shape the recovery matrix by an appropriate eigenstructure assignment to the observer dynamic matrix $A-KC$. Such a freedom can be utilized to shape $\bar{M}^e(j\omega)$, the limit of the recovery matrix, with respect to ω. Any optimization based method adopts a particular way of shaping $\bar{M}^e(j\omega)$ as dictated by the mathematical minimization procedure. For example, as discussed earlier, in H_2 optimization $\bar{M}^e(j\omega)$ is shaped by assigning some of the eigenvalues of $A-KC$ at the mirror images of the unstable invariant zeros of Σ, while the associated set of left eigenvectors of $A-KC$ coincide with the corresponding right input zero directions of Σ. Such a shaping obviously limits the available design freedom, and may or may not be desirable from an engineering point of view. Next, available design freedom can also be utilized to characterize appropriately the behavior of asymptotically infinite or otherwise called fast eigenvalues of $A-KC$. What we mean by the behavior of fast eigenvalues is (a) their asymptotic directions and (b) the rate at which they go to infinity, i.e., the fast time-scale structure of $A-KC$. As demonstrated in Saberi et al. (1991 b), the behavior of fast eigenvalues has a dramatic effect on the resulting controller bandwidth. Again, optimization based design methods fix the behavior of fast eigenvalues in a particular way that may or may not be favorable to the designer's goals. We believe that the ability to utilize all the available design freedom is a valuable asset; in particular, exploring such a freedom in the space in which complete recovery is not feasible is of dire importance. ATEA design methods of Sec. 3 put all the available design freedom in the hands of designer and hence are preferable to optimization based designs of Sec. 4. However, a clear advantage of the optimization based schemes is that at the onset of design, they do not require much systematic planning and hence are straightforward to apply. In fact, one simply (!) solves the concerned ARE's repeatedly for several values of tuning parameter until a gain obtained for one particular value of the parameter is appropriate for a suboptimal design. Admittedly, ATEA design does not have such a simplicity. One needs in ATEA design to come up with an appropriate utilization of the available design freedom and thus the selection of available design parameters in order to meet the practical design specifications. But this perhaps can be done by a simple iterative adjustment. Such a procedure is still computationally inexpensive as the required calculations for ATEA design are straightforward and do not involve any "stiff" equations.

7. Conclusions

Full order observer design for loop transfer recovery for general not necessarily left invertible, not necessarily of minimum phase, and non-strictly proper systems is considered. After reviewing the necessary design constraints and the available design freedom, three different methods of design are developed. The first method is an asymptotic time-scale structure and eigen-structure assignment (ATEA) scheme. The other two methods are optimization based; one minimizes the H_2 norm of a recovery matrix related to the loop transfer recovery error and the other minimizes the H_∞ norm of the same. All three methods of design give explicit methods of obtaining observer gain parameterized in terms of a tuning parameter. In optimization based designs, the gain is implicitly parameterized via the solution of parameterized nonlinear algebraic Riccati equations (ARE's). On the other hand, ATEA design does not require any solution of nonlinear algebraic equations; here the tuning parameter enters the design only in forming a composite gain from several subsystem designs, and thus it truly acts as tuning parameter. All three methods of design yield a sequence of controllers as the tuning parameter takes different values. In optimization based methods, as the tuning parameter tends to a certain critical value, the corresponding sequence of H_2 norms (or H_∞ norms depending on the method) of the resulting recovery matrices tends to a limit which is the infimum of the H_2 norm (or H_∞ norm) of the recovery matrix over all possible observer gains. In so doing, the optimization based methods shape the recovery error in a particular way which may or may not be meaningful from an engineering point of view. On the other hand, the ATEA method has the flexibility to utilize all the available design freedom to shape the recovery error appropriately to meet the designer's needs within the constraints imposed by the structural properties of the given system. Also, ATEA method can easily be modified and simplified to yield an observer design that achieves ELTR whenever it is feasible. In contrast with ATEA design, optimal or suboptimal design schemes do not require much priori planning but require solving repeatedly the parameterized ARE's for different values of the parameter. However, these ARE's invariably become "stiff" as the parameter takes values closer to certain critical value.

Besides the conventional LTR design task which seeks the recovery over the entire control space, another generalized task which seeks recovery only over a specified subspace of the control space is also considered.

All the design methods developed here are implemented in a MATLAB-software package. A number of design examples illustrate several aspects of ATEA design as well as optimization based designs.

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Appendix A: Proof of Theorem 3.1

Without loss of generality, we will assume that the given system is in the form of s.c.b. Then by renaming the variables $x_0=[(x_a^-)', x_b']'$ and $x_e=[(x_a^+)', x_c']'$, we can rewrite the observer dynamic matrix $A-K(\sigma)C$ as,

$$A-K(\sigma)C = \begin{bmatrix} A_{00} & 0 & -\tilde{H}_{0f}C_f \\ A^{e0} & A^{ee1} & -[\tilde{H}_{ef}+\tilde{K}^{e1}(\sigma)]C_f \\ B_fE_0 & B_fC^{e1} & A_f-\tilde{K}_f(\sigma)C_f-L_fC_f \end{bmatrix}, \tag{A.1}$$

where

$$A_{00} = \begin{bmatrix} A_{aa}^- & -\tilde{H}_{ab}^-C_b \\ 0 & A_{bb}^c \end{bmatrix}, \quad \tilde{H}_{0f} = \begin{bmatrix} \tilde{H}_{af}^- \\ \tilde{H}_{bf} \end{bmatrix}, \quad E_0 = [E_a^- \;\; E_b],$$

$$A^{e0} = \begin{bmatrix} -K^{a0+}C_{0a}^- & -\tilde{H}_{ab}^+C_b-K^{a0+}C_{0b} \\ B_cE_{ca}^- - K^{c0}C_{0a}^- & -\tilde{H}_{cb}C_b-K^{c0}C_{0b} \end{bmatrix},$$

$$A^{ee1} = A^{ee} - K^{e0}C^{e0} = \begin{bmatrix} A_{aa}^+ - K^{a0+}C_{0a}^+ & -K^{a0+}C_{0c} \\ B_cE_{ca}^+ - K^{c0}C_{0a}^+ & A_{cc}-K^{c0}C_{0c} \end{bmatrix},$$

$$\tilde{H}_{ef} = \begin{bmatrix} K^{a0+}C_{0f}+\tilde{H}_{af}^+C_f \\ K^{c0}C_{0f}+\tilde{H}_{cf}C_f \end{bmatrix}.$$

The fact that ATEA algorithm yields an admissible observer gain $K(\sigma)$ in the sense that $A-K(\sigma)C$ is a stable matrix for sufficiently large σ and that it has the required time-scale structure follows along the lines of Appendix B of Saberi et al. (1991 b). In what follows, we will show that $K(\sigma)$ achieves LTR in the sense that

$$M(s, \sigma) = F[sI_n-A+K(\sigma)C]^{-1}[B-K(\sigma)D] \rightarrow \bar{M}^e(s) \text{ pointwise in } s \tag{A.2}$$

as $\sigma \rightarrow \infty$. In view of (3.16), let us partition $K(\sigma)$ as,

$$K(\sigma) = \tilde{K}_0 + [0 \;\; \bar{K}(\sigma)]$$

$$= \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^+ + K^{a0+} & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c}+K^{c0} & 0 & 0 \\ B_{0f} & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & L_{af}^- + \tilde{H}_{af}^- & L_{ab}^- + \tilde{H}_{ab}^- \\ 0 & L_{af}^+ + \tilde{H}_{af}^+ + \tilde{K}^{a1+}(\sigma) & L_{ab}^+ + \tilde{H}_{ab}^+ \\ 0 & L_{bf}+\tilde{H}_{bf} & K^b \\ 0 & L_{cf}+\tilde{H}_{cf}+\tilde{K}^{c1}(\sigma) & L_{cb}+\tilde{H}_{cb} \\ 0 & L_f+\tilde{K}_f(\sigma) & 0 \end{bmatrix}. \tag{A.3}$$

Then we have

$$\bar{B} = B - K(\sigma)D = B - \tilde{K}_0 D = \begin{bmatrix} 0 & 0 & 0 \\ -K^{a0+} & 0 & 0 \\ 0 & 0 & 0 \\ -K^{c0} & 0 & B_c \\ 0 & B_f & 0 \end{bmatrix}$$

and

$$\bar{A} = A - \tilde{K}_0 C$$

$$= \begin{bmatrix} A_{aa}^{-} & 0 & L_{ab}^{-}C_b & 0 & L_{af}^{-}C_f \\ -K^{a0+}C_{0a}^{-} & A_{aa}^{+}-K^{a0+}C_{0a}^{+} & L_{ab}^{+}C_b-K^{a0+}C_{0b} & -K^{a0+}C_{0c} & L_{af}^{+}C_{0f} \\ 0 & 0 & A_{bb} & 0 & L_{bf}C_f \\ B_cE_{ca}^{-}-K^{c0}C_{0a}^{-} & B_cE_{ca}^{+}-K^{c0}C_{0a}^{+} & L_{cb}C_b-K^{c0}C_{0b} & A_{cc}-K^{c0}C_{0c} & L_{cf}C_f-K^{c0}C_{0f} \\ B_fE_a^{-} & B_fE_a^{+} & B_fE_b & B_fE_c & A_f \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix}.$$

With these definitions, we can write $M(s, \sigma)$ as

$$M(s, \sigma) = F[sI_n - \bar{A} + \bar{K}(\sigma)\bar{C}]^{-1}\bar{B}.$$

Then in view of (A.3), it can be seen easily that $\bar{K}(\sigma)$ has the form,

$$\bar{K}(\sigma) = T(\sigma)\Gamma(\sigma)N + Q,$$

where

$$\Gamma(\sigma) = \text{Diag}\left[\frac{1}{\eta_1}, \frac{1}{\eta_2}, \cdots, \frac{1}{\eta_{m_f}}\right], \quad N = [I_{m_f}, 0]$$

and

$$Q = \begin{bmatrix} L_{af}^{-}+\tilde{H}_{af}^{-} & L_{ab}^{-}+\tilde{H}_{ab}^{-} \\ L_{af}^{+}+\tilde{H}_{af}^{+} & L_{ab}^{+}+\tilde{H}_{ab}^{+} \\ L_{bf}+\tilde{H}_{bf} & K^b \\ L_{cf}+\tilde{H}_{cf} & L_{cb}+\tilde{H}_{cb} \\ L_f & 0 \end{bmatrix},$$

while $T(\sigma)$ satisfies

$$T(\sigma) \to B_m T$$

as $\sigma \to \infty$, where

$$B_m = \begin{bmatrix} 0 \\ K^{a1+} \\ 0 \\ K^{c1} \\ B_f \end{bmatrix}, \quad T = \mathrm{Diag}[J_{1r_1}K_{1r_1d}, J_{2r_2}K_{2r_2d}, \cdots, J_{m_fr_{m_f}}K_{m_fr_{m_f}d}].$$

It is shown in Chen, Saberi and Sannuti (1992 b) that the triple $(\bar{C}, \bar{A}, B_m)$ form a left invertible and a minimum phase system. Thus, it follows from the results of Saberi and Sannuti (1990 a) that

$$[sI_n - A + K(\sigma)C]^{-1}B_m \to 0 \quad \text{pointwise in } s$$

as $\sigma \to \infty$. Next let

$$\bar{B} = [0, B_m, 0] + B^e,$$

where

$$B^e = \begin{bmatrix} 0 & 0 & 0 \\ -K^{a0+} & -K^{a1+} & 0 \\ 0 & 0 & 0 \\ -K^{c0} & -K^{c1} & B_c \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we have

$$M(s, \sigma) \to F[sI_n - \bar{A} + \bar{K}(\sigma)\bar{C}]^{-1}B^e$$

as $\sigma \to \infty$. Let us partition now F as

$$F = [F_0 \quad F_e \quad F_\infty]$$

and define

$$B^{ee} = \begin{bmatrix} -K^{a0+} & -K^{a1+} & 0 \\ -K^{c0} & -K^{c1} & B_c \end{bmatrix}.$$

It follows then from the results of Appendix B of Saberi et al. (1991 b) that

$$M(s, \sigma) \to \bar{M}^e(s) = \sum_{i=1}^{n_e} \frac{F_e \bar{W}^{eei}(\bar{V}^{eei})^H B^{ee}}{s - \bar{\lambda}^{ei}} = F_e(sI_{n_e} - A^{ee} + K^eC^e)^{-1}B^{ee},$$

where

$$[\bar{V}^{ee1}, \bar{V}^{ee2}, \cdots, \bar{V}^{een_e}] = [\bar{W}^{ee1}, \bar{W}^{ee2}, \cdots, \bar{W}^{een_e}]^{-H}.$$

This completes the proof of Theorem 3.1.

Appendix B: Proof of Theorem 3.1

We assume that the given system Σ is in the form of s.c.b (see Theorem 3.1 of Part 1). Then in view of (3.20), we note that

$$A - KC$$

$$= \begin{bmatrix} A_{aa}^- & 0 & 0 & 0 & 0 \\ -K^{a0+}C_{0a}^- & A_{aa}^+ - K^{a0+}C_{0a}^+ & -K^{a0+}C_{0b} & -K^{a0+}C_{0c} & (L_{af}^+ - K^{a1+})C_f - K^{a0+}C_{0f} \\ 0 & 0 & A_{bb} - K^bC_b & 0 & 0 \\ B_cE_{ca}^- - K^{c0}C_{0a}^- & B_cE_{ca}^+ - K^{c0}C_{0a}^+ & -K^{c0}C_{0b} & A_{cc} - K^{c0}C_{0c} & (L_{cf} - K^{c1})C_f - K^{c0}C_{0f} \\ B_fE_a^- - K^{f0}C_{0a}^- & B_fE_a^+ - K^{f0}C_{0a}^+ & B_fE_b - K^{f0}C_{0b} & B_fE_c - K^{f0}C_{0c} & A_f - K^{f0}C_{0f} - K^{f1}C_f \end{bmatrix}.$$

Then, it is simple to verify that the eigenvalues of $A - K(\sigma)C$ are given by $\Lambda^- \cup \Lambda^b \cup \Lambda^g$. Moreover,

$$(sI_n - A - KC)^{-1} = \begin{bmatrix} \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star \\ 0 & 0 & \star & 0 & 0 \\ \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \end{bmatrix} \quad \text{and} \quad B - KD = \begin{bmatrix} 0 & 0 & 0 \\ \star & 0 & 0 \\ 0 & 0 & 0 \\ \star & 0 & B_c \\ \star & B_f & 0 \end{bmatrix},$$

where $\star$'s represent appropriate dimensional submatrices which are not necessarily zero. Hence, we have

$$\mathrm{Im}[(sI_n - A - KC)^{-1}(B - KD)] = \mathrm{Im}\left\{\begin{bmatrix} 0 & 0 & 0 \\ \star & \star & \star \\ 0 & 0 & 0 \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}\right\} \subseteq S^-(A, B, C, D),$$

which implies $M(s) = F[sI_n - A + KC]^{-1}(B - KD) \equiv 0$ provided $S^-(A, B, C, D) \subseteq \mathrm{Ker}(F)$. Therefore, ELTR is achieved.

Appendix C: Proof of Theorem 4.1

Let $\varepsilon = 1/\sqrt{\sigma}$, and define the following perturbed system,

$$\Sigma_{a\varepsilon}: \begin{cases} \dot{x} = A'x + C'u + F'w, \\ z = B'_\varepsilon x + D'_\varepsilon u, \end{cases} \tag{C.1}$$

where

$$B_\varepsilon = [B_0,\ B_1,\ \varepsilon I_n,\ 0] \quad \text{and} \quad D_\varepsilon = \begin{bmatrix} I_{m_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon I_{m-m_0} \end{bmatrix}.$$

Consider the state feedback law,

$$u = -K'(\sigma)x \tag{C.2}$$

with gain $K'(\sigma)$ defined by,

$$K'(\sigma) = (D_\varepsilon D_\varepsilon')^{-1}(PC + D_\varepsilon B_\varepsilon'), \tag{C.3}$$

where P is the positive definite solution of

$$AP + PA' + B_\varepsilon B_\varepsilon' - (PC' + B_\varepsilon D_\varepsilon')(D_\varepsilon D_\varepsilon')^{-1}(CP + D_\varepsilon B_\varepsilon') = 0. \tag{C.4}$$

We note that D_ε' is injective. Then, it is shown in Stoorvogel (1990) that the state feedback law (C.2), minimizes the H_2 norm of the transfer function from w to z, namely $T_{zw}(s, \sigma)$, as $\sigma \to \infty$ (or $\varepsilon \to 0$). The proof of the first part of Theorem 4.1 follows now by recognizing that (C.3) and (C.4) are respectively equivalent to (4.4) and (4.3). The rest of Theorem 4.1 follows trivially from Stoorvogel (1990).

Ben M. Chen see p.35 in this issue.

Ali Saberi see p.35 in this issue.

Peddapullaiah Sannuti see p.100 in this issue.