CONTROL-THEORY AND ADVANCED TECHNOLOGY
Vol.8, No.1, pp.59-100, March, 1992

C91009R

Contributed Paper

LOOP TRANSFER RECOVERY FOR GENERAL NONMINIMUM PHASE NON-STRICTLY PROPER SYSTEMS, PART 1—ANALYSIS*

B. M. CHEN,[1] A. SABERI[1] AND P. SANNUTI[2]

Abstract. A complete analysis of loop transfer recovery (LTR) using full order observer based controllers for general nonstrictly proper systems is considered. The given system need not be left invertible and of minimum phase. Our analysis of LTR focuses on four fundamental issues. The first issue is concerned with what can and what cannot be achieved for a given system and for an arbitrarily specified target loop transfer function. On the other hand, the second issue is concerned with the development of necessary and/or sufficient conditions a target loop has to satisfy so that it can either exactly or asymptotically be recovered for a given system while the third issue is concerned with the development of necessary and/or sufficient conditions on a given system such that it has at least one, either exactly or asymptotically, recoverable target loop. The fourth issue deals with a generalization of all the above three issues when recovery is required over a subspace of the control space. It concerns with generalizing the traditional LTR concept to sensitivity recovery over a subspace and deals with method(s) to test whether projections of target and achievable sensitivity and complimentary sensitivity functions onto a given subspace match each other or not. Such an analysis pinpoints the limitations of the given system for the recovery of arbitrarily specified target loops via full order observer based controllers. These limitations are the consequences of the structural properties (i.e., finite and infinite zero structure, and invertibility) of the given system. Also, the conditions developed here on a target loop transfer function for its recoverability, turn out to be constraints on its finite and infinite zero structure as related to the corresponding structure of the given system. Furthermore, the analysis given here discovers a multitude of ways in which freedom exists to shape the loops in a desired way as close as possible to the target shapes. Also, possible pole zero cancellations between the eigenvalues of the controller and the input and/or output decoupling zeros of the given system are characterized.

Key Words—Loop transfer recovery, robust control.

1. Introduction

In classical as well as modern feedback control system design, many performance and robust stability objectives can be cast in terms of maximum magnitude or maximum singular values of some particular *closed-loop* transfer functions, e.g., sensitivity and complimentary sensitivity functions at certain

* Received by the editors January 21, 1991 and in revised form November 9, 1991.
*[1] The work of B.M. Chen and A. Saberi is supported in part by Boeing Commercial Airplane Group.
[1] School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752, U.S.A.
[2] Department of Electrical and Computer Engineering, P.O. Box 909, Rutgers University, Piscataway, NJ 08855-0909, U.S.A.

points in a closed-loop. A principle idea of "loop shaping" is that such magnitude or singular value requirements on some *closed-loop* transfer functions can be directly determined by corresponding singular values of certain related *open-loop* transfer functions. A prominent design methodology for multivariable systems which is based on such loop shaping concepts is LQG/LTR. Historically, LQG/LTR design philosophy involves two steps. The first step is to design a state feedback law that yields an open-loop transfer function which accommodates satisfactorily the given design specifications on the required sensitivity functions. Such an open-loop transfer function is called a target open-loop transfer function. The second step, called loop transfer recovery (LTR), involves the design of an output feedback control law such that the resulting open-loop transfer function would be either exactly or approximately the same as the target open-loop transfer function. In other words, the idea of LTR is to come up with a measurement feedback compensator, typically observer based, to recover a specific open-loop transfer function prescribed in terms of a state feedback gain.

The above mentioned loop transfer recovery (LTR) procedure as a multivariable robust control design tool has gained significance since the seminal work of Doyle and Stein (1979). It has been studied by a number of authors including Athans (1986), Chen et al. (1990; 1991 a; b), Doyle and Stein (1981), Goodman (1984), Matson and Maybeck (1991), Niemann and Jannerup (1990), Niemann et al. (1991), Ridgely and Banda (1986), Saberi et al. (1991 a; b), Sogaard-Andersen (1989), Sogaard-Andersen and Niemann (1989), Saberi and Sannuti (1990 a), Stein and Athans (1987) and Zhang and Freudenberg (1990). Earlier literature on LTR concentrates on left invertible and minimum phase systems since this is the only class of systems for which asymptotic LTR is possible for an arbitrarily specified target loop transfer function. A variety of issues arise in analyzing the LTR mechanism in nonminimum phase systems. Recent works, Chen et al. (1991 b), Niemann and Jannerup (1990), Saberi et al. (1991 a; b) and Zhang and Freudenberg (1990), focus on some of these issues. For example, the issues discussed in Saberi et al. (1991 a) and Chen et al. (1991 b) include, (a) Characterizing the available freedom in designing controllers for a given system and for an arbitrarily specified target loop transfer function, (b) Development of necessary and/or sufficient conditions a target loop has to satisfy so that it can either exactly or asymptotically be recovered for a given system, (c) Development of necessary and/or sufficient conditions on a given system such that it has at least one recoverable (either exactly or asymptotically) target loop and (d) Development of method(s) to test whether recovery is possible in a given subspace of the control space or not, i.e., to test whether projections of target and achievable sensitivity and complimentary sensitivity functions onto a given subspace match each other or not, and in so doing generalizing the traditional notion of LTR. The theory developed in Saberi et al. (1991 a) to analyze the issues (a), (b) and (d) is fairly complete when full order observer based controllers are used and when strictly proper systems which are not necessarily left invertible and of minimum phase are considered. On the other hand, when general controllers which are not necessarily observer based are used, Chen et al. (1991 b) develops the necessary and sufficient conditions on a strictly proper system so that it has at least one, either exactly or asymptotically, recoverable target loop. As far as design is concerned, there exists essentially three methods of designing observer based controllers for LTR. These methods are,

(1) Kalman filter formalism (Doyle and Stein, 1979), (2) direct eigenstructure placement method (Sogaard-Andersen, 1989) and (3) asymptotic eigenstructure and time-scale structure assignment (ATEA) method (Saberi and Sannuti, 1990 a; Saberi et al., 1991 b).

All the above discussion pertains only to strictly proper systems. Regarding nonstrictly proper systems, the only work so far has been by Chen et al. (1990) who consider only left invertible and minimum phase systems. When a given system is nonstrictly proper and non necessarily left invertible and of minimum phase, although some aspects of Saberi et al. (1991 a) and Chen et al. (1991 b) carry over in a straight forward manner, other aspects of Saberi et al. (1991 a) present complexities which need to be examined carefully. As such, the intention of this paper is to analyze systematically the LTR mechanism using full order observer based controllers in its generality for nonstrictly proper systems which are not necessarily left invertible and of minimum phase. The basic methodology and the tools used here are akin to those in Saberi et al. (1991 a) and Chen et al. (1991 b). Also, this paper concerns itself only with the analysis of LTR mechanism. A sequel to this paper focuses on the design issues.

In order not to lengthen the paper, we concentrate throughout this paper only on full order observer based controllers. Structure of a controller can impact the recovery process in more than one way. For example, as seen in Chen et al. (1991 a), the size of a gain required for the same degree of asymptotic recovery can be vastly different in different controllers; also, the set of exactly recoverable target loop transfer functions can be enlarged by using a reduced order observer based controller instead of a full order observer based controller. Most of the results developed here for the case of full order observer based controllers can easily be extended to other controller structures; however, some results need a careful reexamination.

Throughout the paper, A' denotes the transpose of A, A^H denotes the complex conjugate transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. $\lambda(A)$ and $\mathrm{Re}[\lambda(A)]$ respectively denote the set of eigenvalues and real parts of eigenvalues of A. Similarly, $\sigma_{\max}[A]$ and $\sigma_{\min}[A]$ respectively denote the maximum and minimum singular values of A. $\mathrm{Ker}[V]$ and $\mathrm{Im}[V]$ denote respectively the kernel and the image of V. The open left and closed right half s-planes are respectively denoted by C^- and C^+. Also, R_p denotes the sub-ring of all proper rational functions of s while the set of matrices of dimension $l \times q$ whose elements belong to R_p is denoted by $M^{l \times q}(R_p)$.

2. Problem Formulation

In this section, we formulate the LTR problem in precise mathematical terms. Let us consider a nonstrictly proper system Σ,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{2.1}$$

where the state vector $x \in \mathscr{R}^n$, output vector $y \in \mathscr{R}^p$ and input vector $u \in \mathscr{R}^m$. Without loss of generality, assume that $[B', D']'$ and $[C, D]$ are of maximal rank. Let us also assume that Σ is stabilizable and detectable. In this paper, for simplicity, we concentrate on a case when plant uncertainties are modelled at the input point of a nominal plant model and hence the required loop transfer function is specified at the plant input point. However, our results can be generalized easily for the case when the required loop transfer function is specified at any arbitrary point. In fact, for the case when the required loop transfer function is specified at the plant output point (Kwakernaak, 1969), our results can easily be dualized. Let F be a full state feedback gain matrix such that (a) the closed-loop system is asymptotically stable, i.e., eigenvalues of $A-BF$ lie in the left half s-plane, and (b) the open-loop transfer function when the loop is broken at the input point of the given system meets some given frequency dependent specifications. The state feedback control is

$$u = -Fx, \tag{2.2}$$

and the loop transfer function evaluated when the loop is broken at the input point of the given system, the so called target loop transfer function, is

$$L_t(s) = F\Phi B, \tag{2.3}$$

where $\Phi=(sI-A)^{-1}$. Arriving at an appropriate value for F is concerned with the issue of loop shaping which is an engineering art and often includes the use of linear quadratic regulator (LQR) design in which the cost matrices are used as free design parameters to generate the target loop transfer function $L_t(s)$ and thus the desired sensitivity and complementary sensitivity functions. The next step of design is to recover the target loop using only a measurement feedback controller. This is the problem of loop transfer recovery (LTR) and is the focus of this paper. To explain it clearly, consider the configuration of Fig. 2.1 where $C(s)$ and $P(s)$,

$$P(s) = C\Phi B + D,$$

are respectively the transfer functions of a controller and of the given system. Given $P(s)$ and a target loop transfer function $L_t(s)$, one seeks then to design a $C(s)$ such that $E(s)$,

$$E(s) \equiv L_t(s) - C(s)P(s),$$

is either exactly or approximately equal to zero in the frequency region of interest while guaranteeing the stability of the resulting closed-loop system. Hereafter, we will call $E(s)$ as *recovery error*. Achieving exact LTR (ELTR), i.e., rendering the recovery error exactly zero, is in general not possible even for left invertible and minimum phase systems. One seeks then approximate LTR. The notion of "approximate" LTR has to be defined a little carefully. Here we seek

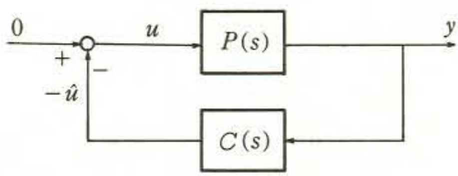

Fig. 2.1. Plant—Controller closed-loop configuration.

achieving LTR to any arbitrarily desired accuracy. In an attempt to make this feasible, one normally parameterizes $C(s)$ as a function of a scalar parameter σ and thus obtains a family of controllers $C(s, \sigma)$. We say asymptotic LTR (ALTR) is achieved if $C(s, \sigma)P(s) \to L_t(s)$ pointwise in s as $\sigma \to \infty$, i.e., $E(s, \sigma) \to 0$ pointwise in s as $\sigma \to \infty$. Achievability of ALTR enables the designer to choose a member of the family of controllers that corresponds to a particular value of σ which achieves a desired level of recovery. Traditionally, in observer based controllers, such a parameterization is done by adding a fictitious process noise of intensity proportional to σ which is injected into the system through the input into the plant. Then, the observer gain is calculated by solving the resulting filter algebraic Riccati equations (AREs). In an asymptotic and time-scale structure assignment (ATEA) procedure of Saberi and Sannuti (1990 a) and Saberi et al. (1991 b), appropriate parameterization of a controller assigns a chosen time-scale structure to the resulting closed-loop system. The relative fastness of fast time-scales is then adjusted as desired by tuning the parameter σ. We now consider the following definitions in order to impart precise meanings to ELTR and ALTR.

Definition 2.1. The set of admissible target loops $\mathbf{T}(\Sigma)$ for the given system Σ is defined by

$$\mathbf{T}(\Sigma) = \{L_t(s) \in M^{m \times m}(R_p) \,|\, L_t(s) = F\Phi B \quad \text{and} \quad \lambda(A-BF) \in C^-\}.$$

Definition 2.2. $L_t(s) \in \mathbf{T}(\Sigma)$ is said to be exactly recoverable (ELTR) if there exists a $C(s) \in M^{m \times p}(R_p)$ such that (i) the closed-loop system comprising of $C(s)$ and $P(s)$ as in the configuration of Fig. 2.1 is asymptotically stable, and (ii) $C(s)P(s) = L_t(s)$.

Definition 2.3. $L_t(s) \in \mathbf{T}(\Sigma)$ is said to be asymptotically recoverable (ALTR) if there exists a parameterized family of controllers $C(s, \sigma) \in M^{m \times p}(R_p)$, where σ is a scalar parameter taking positive values, such that (i) the closed-loop system comprising of $C(s, \sigma)$ and $P(s)$ as in the configuration of Fig. 2.1 is asymptotically stable for all $\sigma > \sigma^*$, where $0 \le \sigma^* < \infty$, and (ii) $C(s, \sigma)P(s) \to L_t(s)$ pointwise in s as $\sigma \to \infty$. Moreover, the limits, as $\sigma \to \infty$, of the finite eigenvalues of the closed-loop system should remain in C^{-}†.

Definition 2.4. $L_t(s)$ belonging to $\mathbf{T}(\Sigma)$ is said to be recoverable if $L_t(s)$ is either exactly or asymptotically recoverable.

Definition 2.5.
1. The set of exactly recoverable target loops for the given system Σ is denoted by $\mathbf{T}^{\mathrm{ER}}(\Sigma)$.
2. The set of recoverable target loops for the given system Σ is denoted by $\mathbf{T}^{\mathrm{R}}(\Sigma)$.
3. The set of target loops which are asymptotically recoverable but not exactly recoverable for the given system Σ is denoted by $\mathbf{T}^{\mathrm{AR}}(\Sigma)$.

Obviously, $\mathbf{T}^{\mathrm{R}}(\Sigma) = \mathbf{T}^{\mathrm{ER}}(\Sigma) \cup \mathbf{T}^{\mathrm{AR}}(\Sigma)$.

It is well known that for left invertible and minimum phase systems, any

† Here we have strengthened the notion of the closed-loop stability in order to exclude those cases having the limits, as $\sigma \to \infty$, of some finite eigenvalues of the closed-loop system being on the $j\omega$ axis. This avoids having an almost unstable behavior of the closed-loop system for large σ.

arbitrary admissible target loop is asymptotically recoverable and hence $\mathbf{T}^{R}(\Sigma)$ is equal to $\mathbf{T}(\Sigma)$. On the other hand, if the given system Σ is not left invertible and/or of nonminimum phase, not all target loops are recoverable, i.e., $\mathbf{T}^{R}(\Sigma)$ is not equal to $\mathbf{T}(\Sigma)$. In fact, $\mathbf{T}^{R}(\Sigma)$ might be an empty set. As mentioned in the introduction, the purpose of this paper is to analyze the LTR mechanism systematically for the general system Σ given by (2.1) when the controller $C(s)$ or $C(s, \sigma)$ uses a full order observer based structure. All the results given in this two part paper pertain only to the case when full order observer based controllers are used. In general, all the issues involved in attempting to recover a target loop transfer function depend on the architecture of the controller. For example, as is evident from Theorems 1 and 2 of Chen et al. (1991 a), the set of exactly recoverable target loops when reduced order observer based controllers are used is different from that when full order observer based controllers are used. As such the results of this paper require several modifications whenever the architecture of the controller is other than the full order observer based controller. Analysis of LTR mechanism using reduced order observer based architecture or any other compensator architecture such as the one developed in Chen et al. (1991 a), is a topic of future research.

The analysis of LTR mechanism carried out here focuses on four fundamental issues. The first issue is concerned with what can and what cannot be achieved for a given system and for an arbitrarily specified target loop transfer function. On the other hand, the second issue is concerned with the development of necessary and/or sufficient conditions a target loop has to satisfy so that it can be either exactly or asymptotically be recovered for a given system while the third issue is concerned with the development of necessary and/or sufficient conditions on a given system such that it has at least one recoverable (either exactly or asymptotically) target loop. The fourth issue deals with a generalization of all the above three issues when recovery is required over a subspace of the control space. It concerns with generalizing the traditional LTR concept to sensitivity recovery over a subspace and deals with method(s) to test whether projections of target and achievable sensitivity and complimentary sensitivity functions onto a given subspace match each other or not. All this analysis shows some fundamental limitations of the given system as a consequence of its structural properties, namely finite and infinite zero structure and invertibility. It also discovers a multitude of ways in which freedom exists to shape the recovery error in a desired way. Thus, it helps to set meaningful design goals at the onset of design.

The paper is organized as follows. As is evident from Saberi and Sannuti (1990 a) and Saberi et al. (1991 a), the finite and infinite zero structure of a given system plays a dominant role in LTR. Recognizing this, in Sec. 3, we recall a special coordinate basis (s.c.b) of Sannuti and Saberi (1987) and Saberi and Sannuti (1990 b) which displays clearly the required zero structure. Section 4 deals with analysis for an arbitrarily given target loop. This analysis includes not only the recovery of a target loop transfer function but also target sensitivity and complimentary sensitivity functions. We show that either ELTR or ALTR in general is not possible. Whenever LTR is not possible, we give explicit expressions for the asymptotic limits of loop transfer function and sensitivity and complimentary sensitivity functions. Moreover, we give explicit bounds on the attainable sensitivity and complimentary sensitivity functions in terms of the singular values of what is called a recovery error matrix to be defined later on.

These bounds can be used to analyze the inevitable trade-off between the good recovery as indicated by the maximum singular value of the recovery error matrix, and robustness and performance as reflected in the sensitivity and complimentary sensitivity functions. We next move on to the characterization of a subspace in which the target sensitivity and complimentary sensitivity functions can be recovered. All the analysis given here treats the target loop transfer function $L_t(s)$ as an arbitrarily given matrix, i.e., no particular properties of $L_t(s)$ are exploited in the analysis. However, in Sec. 5, we take into account the specific characteristics $L_t(s)$ might have. Here the necessary and sufficient conditions under which $L_t(s)$ can either exactly or approximately be recovered are given. Interestingly enough, these constraints turn out to be constraints on the finite and infinite zero structure of it. Such an interpretation of the constraints reveals that either ELTR or ALTR is possible under a variety of conditions. Also, in this section we establish the necessary and/or sufficient conditions on the given system so that it has at least one recoverable target loop. In fact, following Chen et al. (1991 b), given a general nonstrictly proper system which is not necessarily left invertible and of minimum phase, we construct here an auxiliary system from it and show that the set of recoverable target loops for the given system is nonempty, if and only if the auxiliary system is stabilizable by a static output feedback controller. This then leads to a simple and surprising necessary condition on the given system, namely, *strong stabilizability*† *of the given nonminimum phase system is a necessary condition for it to have at least one recoverable target loop*. However, the fact that the given system is strongly stabilizable itself does not guarantee that there exists at least one recoverable target loop. The analysis given in Secs. 4 and 5 stresses recoverability in the entire control space $\mathscr{R}^m$. On the other hand, Sec. 6 generalizes all the results developed in Secs. 4 and 5 in order to cover recoverability of the target sensitivity and complimentary sensitivity functions in a specified subspace and thus adds a considerable amount of flexibility to the process of design. It also shows that for left invertible systems irrespective of the number of nonminimum phase zeros and irrespective of the nature of the target loop transfer function, there exists at least one $m-1$ dimensional subspace of $\mathscr{R}^m$ in which the target sensitivity and complimentary sensitivity functions can always be recovered by an appropriate design of the controller. Also, in Secs. 4 and 5, under all the analysis conditions given above, the resulting controller eigenvalues and possible pole zero cancellations are clearly discussed. Finally in Sec. 7, we draw conclusions of our work.

3. Preliminaries

As is evident from Saberi et al. (1991 a; b), finite and infinite zero structures of both the given system and the target loop transfer function play a dominant role in the recovery analysis as well as design. Keeping this in mind, we recall in this section a special coordinate basis (s.c.b) of a linear time invariant system (Sannuti and Saberi, 1987; Saberi and Sannuti, 1990 b). Such an s.c.b has a distinct feature of explicitly displaying the finite and infinite zero structure of a given system. Consider the system Σ characterized by (A, B, C, D). It is simple to verify that there exist non-singular transformations U and V such that

† A system is said to be strongly stabilizable if there exists a stable and proper compensator which stabilizes the system (Vidyasagar, 1985).

$$UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.1}$$

where m_0 is the rank of matrix D. Hence hereafter, without loss of generality, it is assumed that the matrix D has the form given on the right hand side of (3.1). One can now rewrite the system of (2.1) as,

$$\dot{x} = Ax + [B_0 \quad B_1][u_0' \quad u_1']', \tag{3.2a}$$

$$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \tag{3.2b}$$

where the matrices B_0, B_1, C_0 and C_1 have appropriate dimensions. We have the following theorem.

Theorem 3.1. (s.c.b). Consider the system Σ characterized by (A, B, C, D). There exist nonsingular transformations Γ_1, Γ_2 and Γ_3, an integer $m_f \leq m - m_0$, and integer indexes q_i, $i = 1, \cdots, m_f$, such that

$$x = \Gamma_1 \tilde{x}, \quad y = \Gamma_2 \tilde{y}, \quad u = \Gamma_3 \tilde{u},$$

$$\tilde{x} = [x_a', x_b', x_c', x_f']', \quad x_a = [(x_a^-)', (x_a^+)']',$$

$$\tilde{x}_f = [x_1', x_2', \cdots, x_{m_f}']',$$

$$\tilde{y} = [y_0', y_f', y_b']', \quad y_f = [y_1, y_2, \cdots, y_{m_f}]',$$

$$\tilde{u} = [u_0', u_f', u_c']', \quad u_f = [u_1, u_2, \cdots, u_{m_f}]'$$

and

$$\dot{x}_a^- = A_{aa}^- x_a^- + B_{0a}^- y_0 + L_{af}^- y_f + L_{ab}^- y_b, \tag{3.3}$$

$$\dot{x}_a^+ = A_{aa}^+ x_a^+ + B_{0a}^+ y_0 + L_{af}^+ y_f + L_{ab}^+ y_b, \tag{3.4}$$

$$\dot{x}_b = A_{bb} x_b + B_{0b} y_0 + L_{bf} y_f, \quad y_b = C_b x_b, \tag{3.5}$$

$$\dot{x}_c = A_{cc} x_c + B_{0c} y_0 + L_{cb} y_b + L_{cf} y_f + B_c[E_{ca}^- x_a^- + E_{ca}^+ x_a^+] + B_c u_c, \tag{3.6}$$

$$y_0 = C_{0a}^- x_a^- + C_{0a}^+ x_a^+ + C_{0b} x_b + C_{0c} x_c + C_{0f} x_f + u_0 \tag{3.7}$$

and for each $i = 1, \cdots, m_f$,

$$\begin{aligned} \dot{x}_i = {} & A_{q_i} x_i + L_{i0} y_0 + L_{if} y_f \\ & + B_{q_i}[u_i + E_{ia} x_a + E_{ib} x_b + E_{ic} x_c + \sum_{j=1}^{m_f} E_{ij} x_j], \end{aligned} \tag{3.8}$$

$$y_i = C_{q_i} x_i, \quad y_f = C_f x_f. \tag{3.9}$$

Here, the states x_a^-, x_a^+, x_b, x_c and x_f are respectively of dimension n_a^-, n_a^+, n_b,

n_c and $n_f=\sum_{i=1}^{m_f} q_i$ while x_i is of dimension q_i for each $i=1,\cdots,m_f$. The control vectors u_0, u_f and u_c are respectively of dimension m_0, m_f and $m_c=m-m_0-m_f$ while the output vectors y_0, y_f and y_b are respectively of dimension $p_0=m_0$, $p_f=m_f$ and $p_b=p-p_0-p_f$. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the following form:

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1,\ 0,\ \cdots,\ 0]. \quad (3.10)$$

(Obviously for the case when $q_i=1$, we have $A_{q_i}=0$, $B_{q_i}=1$ and $C_{q_i}=1$.) Furthermore, we have $\lambda(A_{aa}^-)\in C^-$, $\lambda(A_{aa}^+)\in C^+$, the pair (A_{cc}, B_c) is controllable and the pair (A_{bb}, C_b) is observable. Also, assuming that x_i are arranged such that $q_i \le q_{i+1}$, the matrix L_{if} has the particular form,

$$L_{if} = [L_{i1},\ L_{i2},\ \cdots,\ L_{ii-1},\ 0,\ 0,\ \cdots,\ 0].$$

Also, the last row of each L_{if} is identically zero.

Proof. This follows from Theorem 2.1 of Sannuti and Saberi (1987) and Saberi and Sannuti (1990 b).

We can rewrite the s.c.b given by Theorem 3.1 in a more compact form,

$$\tilde{A} \triangleq \Gamma_1^{-1}(A-B_0C_0)\Gamma_1 = \begin{bmatrix} A_{aa}^- & 0 & L_{ab}^-C_b & 0 & L_{af}^-C_f \\ 0 & A_{aa}^+ & L_{ab}^+C_b & 0 & L_{af}^+C_f \\ 0 & 0 & A_{bb} & 0 & L_{bf}C_f \\ B_cE_{ca}^- & B_cE_{ca}^+ & L_{cb}C_b & A_{cc} & L_{cf}C_f \\ B_fE_a^- & B_fE_a^+ & B_fE_b & B_fE_c & A_f \end{bmatrix},$$

$$\tilde{B} \triangleq \Gamma_1^{-1}[B_0 \quad B_1]\Gamma_3 = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0f} & B_f & 0 \end{bmatrix},$$

$$\tilde{C} \triangleq \Gamma_2^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_1 = \begin{bmatrix} C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0c} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix}$$

and

$$\tilde{D} \triangleq \Gamma_2^{-1}D\Gamma_3 = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In what follows, we state some important properties of the s.c.b which are pertinent to our present work.

Property 3.1. The given system Σ is right invertible, if and only if x_b and hence y_b are nonexistent ($n_b=0$, $p_b=0$), left invertible, if and only if x_c and hence u_c are nonexistent ($n_c=0$, $m_c=0$), invertible, if and only if both x_b and x_c are nonexistent. Moreover, Σ is degenerate, if and only if it is neither left nor right invertible.

Property 3.2. We note that (A_{bb}, C_b) and (A_{q_i}, C_{q_i}) form observable pairs. Unobservability could arise only in the variables x_a and x_c. In fact, the system Σ is observable (detectable), if and only if (A_{obs}, C_{obs}) is an observable (detectable) pair, where

$$A_{obs} = \begin{bmatrix} A_{aa} & 0 \\ B_c E_{ca} & A_{cc} \end{bmatrix}, \quad A_{aa} = \begin{bmatrix} A_{aa}^- & 0 \\ 0 & A_{aa}^+ \end{bmatrix}, \quad C_{obs} = \begin{bmatrix} C_{0a} & C_{0c} \\ E_a & E_c \end{bmatrix},$$

$$C_{0a} = [C_{0a}^-, C_{0a}^+], \quad E_a = [E_a^-, E_a^+], \quad E_{ca} = [E_{ca}^-, E_{ca}^+].$$

Similarly, (A_{cc}, B_c) and (A_{q_i}, B_{q_i}) form controllable pairs. Uncontrollability could arise only in the variables x_a and x_b. In fact, Σ is controllable (stabilizable), if and only if (A_{con}, B_{con}) is a controllable (stabilizable) pair, where

$$A_{con} = \begin{bmatrix} A_{aa} & L_{ab} C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{con} = \begin{bmatrix} B_{0a} & L_{af} \\ B_{0b} & L_{bf} \end{bmatrix},$$

$$B_{0a} = \begin{bmatrix} B_{0a}^- \\ B_{0a}^+ \end{bmatrix}, \quad L_{ab} = \begin{bmatrix} L_{ab}^- \\ L_{ab}^+ \end{bmatrix}, \quad L_{af} = \begin{bmatrix} L_{af}^- \\ L_{af}^+ \end{bmatrix}.$$

Property 3.3. Invariant zeros of Σ are the eigenvalues of A_{aa}. Moreover, the stable and the unstable invariant zeros of Σ are the eigenvalues of A_{aa}^- and A_{aa}^+, respectively.

There are interconnections between the s.c.b and various invariant and almost invariant geometric subspaces. To show these interconnections, we define

$V^g(A, B, C, D)$—the maximal subspace of $\mathscr{R}^n$ which is $(A-BF)$—invariant and contained in $\text{Ker}(C-DF)$ such that the eigenvalues of $(A-BF) \mid V^g$ are contained in $C_g \subseteq C$ for some F.

$S^g(A, B, C, D)$—the minimal $(A-KC)$—invariant subspace of $\mathscr{R}^n$ containing in $\text{Im}(B-KD)$ such that the eigenvalues of the map which is induced by $(A-KC)$ on the factor space $\mathscr{R}^n/S^g$ are contained in $C_g \subseteq C$ for some K.

For the cases that $C_g = C$, $C_g = C^-$ and $C_g = C^+$, we replace the index g in V^g and S^g by "*", "−" and "+", respectively. Various components of the state vector of s.c.b have the following geometrical interpretations.

Property 3.4.
1. $x_a^- \oplus x_a^+ \oplus x_c$ spans $V^*(A, B, C, D)$.
2. $x_a^- \oplus x_c$ spans $V^-(A, B, C, D)$.
3. $x_a^+ \oplus x_c$ spans $V^+(A, B, C, D)$.
4. $x_c \oplus x_f$ spans $S^*(A, B, C, D)$.
5. $x_a^- \oplus x_c \oplus x_f$ spans $S^+(A, B, C, D)$.
6. $x_a^+ \oplus x_c \oplus x_f$ spans $S^-(A, B, C, D)$.

4. General Analysis

As mentioned in the introduction, throughout this paper, we will consider only full order observer based controllers as depicted in Fig. 4.1. The dynamic equations of the controller are

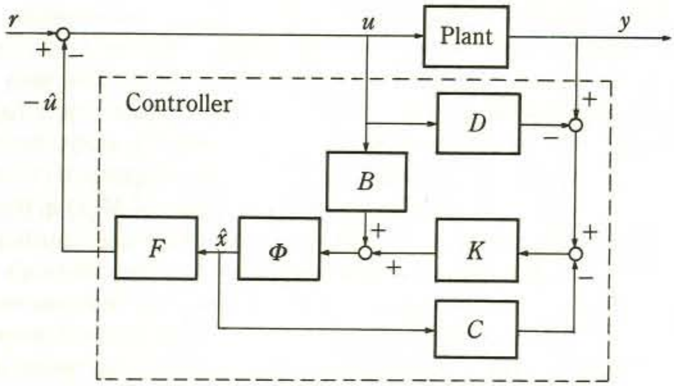

Fig. 4.1. Plant with full order observer based controller.

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x} - Du)$$

and

$$u = \hat{u} = -F\hat{x},$$

where K is the observer gain and F is the state feedback gain which prescribes the target loop transfer function $L_t(s) = F\Phi B$. The transfer function of the controller is

$$C(s) = F[sI_n - A + BF + KC - KDF]^{-1}K,$$

while the loop transfer function realized by the controller is

$$L_o(s) = C(s)P(s).$$

The recovery error $E(s)$ is

$$E(s) = L_t(s) - L_o(s). \tag{4.1}$$

A brief outline of this section is as follows. The expression for the recovery error $E(s)$ as given by (4.1) is not well suited for loop transfer recovery analysis. Realizing this, we first relate $E(s)$ to a matrix, hereafter called as *recovery matrix*, $M(s) = F(sI_n - A + KC)^{-1}(B - KD)$, and then show that $E(j\omega) = 0$, if and only if $M(j\omega) = 0$. This implies that a general loop transfer recovery (LTR) analysis is synonymous with a general study of the recovery matrix $M(s)$ which obviously is dependent explicitly both on K and F. A physical interpretation of $M(s)$ can be given. Considering the observer based controller as a device with its inputs as the plant input and the plant output, $-M(s)$ is the transfer function from the plant input point to the controller output point. Thus, whenever LTR is achieved, the controller output does not entail any feedback from the plant input point. The needed study of $M(s)$ to ascertain how and when it can be rendered zero, can be undertaken in two ways, with or without the prior knowledge of F that prescribes the target loop transfer function $L_t(s)$. Note that the study of $M(s)$ without the prior knowledge of F imitates the traditional LQG design philosophy in which the two tasks of obtaining F and K are separated. Keeping this in mind, our goal in this section is to study $M(s)$ without taking into account any specific characteristics of F. The next section, devoted to LTR analysis while taking into account appropriate characteristics of F, compliments the analysis of this section. Here, in order to study the recovery matrix $M(s)$ without having the knowledge of F, we decompose $M(s)$ as $F\tilde{M}(s)$ where $\tilde{M}(s)=(sI_n-A+KC)^{-1}(B-KD)$. A detailed study of $\tilde{M}(s)$ in this section leads to two fundamental lemmas, one dealing with finite and another dealing with asymptotically infinite eigenstructure assignment to the observer dynamic matrix $A-KC$ by an appropriate design of K. These two lemmas reveal the limitations of the given system as a consequence of its structural properties in recovering an arbitrary target loop transfer function via a full order observer based controller. Furthermore, they enable us to decompose $\tilde{M}(s)$ into three essential parts, $\tilde{M}^0(s)$, $\tilde{M}^\infty(s)$ and $\tilde{M}^e(s)$. The first part $\tilde{M}^0(s)$ can be rendered either exactly or asymptotically zero by an appropriate finite eigenstructure assignment to $A-KC$, while the second part $\tilde{M}^\infty(s)$ can be rendered asymptotically zero by an appropriate infinite eigenstructure assignment to $A-KC$. The third part $\tilde{M}^e(s)$ in general cannot be rendered zero, either exactly or asymptotically, by any means, although our analysis of $\tilde{M}^e(s)$ reveals a multitude of ways by which it can be shaped. All in all, the decomposition of $\tilde{M}(s)$ into various parts and the subsequent analysis of each part forms the core of entire analysis given throughout this paper. In particular, it leads to several important results given in this section. For example, Theorem 4.1 characterizes the asymptotic behavior of loop transfer function as well as sensitivity and complimentary sensitivity functions achievable by full order observer based controllers. On the other hand, Theorem 4.2 shows the subspace $S^e\in\mathscr{R}^m$ in which $\tilde{M}^e(s)$ can be rendered zero asymptotically, i.e., the projections of the target and achievable sensitivity and complimentary sensitivity functions onto S^e can match each other asymptotically. Furthermore, our analysis in this section reveals the mechanism of pole zero cancellation between the controller eigenvalues and the input or output decoupling zeros of Σ for the case when F is unknown.

We will now proceed with the analysis. We have the following lemma, a generalization of the result due to Goodman (1984).

Lemma 4.1. Consider any arbitrary F such that $A-BF$ is asymptotically stable. Then $E(s)$, the error between the target loop transfer function $L_t(s)$ and that realized by the controller of Fig. 4.1, is given by

$$E(s) = M(s)[I_m+M(s)]^{-1}(I_m+F\Phi B), \tag{4.2}$$

where the recovery matrix,

$$M(s) = F(sI_n-A+KC)^{-1}(B-KD). \tag{4.3}$$

Furthermore for all $\omega\in\Omega$,

$$E(j\omega) = 0, \quad \text{if and only if} \quad M(j\omega) = 0, \tag{4.4}$$

where Ω is the set of all $0\leq|\omega|<\infty$ for which $L_t(j\omega)$ and $L_o(j\omega) = C(j\omega)P(j\omega)$ are well defined (i.e., all required inverses exist).

Proof. See Appendix A.

In order to give a physical meaning to $M(s)$, one can redraw Fig. 4.1 as either Fig. 4.2 or Fig. 4.3 where the controller is viewed as a device having two inputs, (1) the plant input u and (2) the plant output y. When the controller is viewed having two inputs u and y, $-M(s)$ is the transfer function from the plant input point to the controller output point while $\tilde{M}(s)$ is the transfer function from the plant input point to the estimated state $\hat{x}$. In particular, for full order observer based controller of Fig. 4.1, we have

$$\hat{x}(s) = \tilde{M}(s)u(s) + \tilde{N}(s)y(s)$$

and

$$\hat{u}(s) = -F\hat{x}(s) = -M(s)u(s) - N(s)y(s), \tag{4.5}$$

where

$$\tilde{M}(s) = (sI_n - A + KC)^{-1}(B - KD), \quad M(s) = F\tilde{M}(s) \tag{4.6}$$

and

$$\tilde{N}(s) = (sI_n - A + KC)^{-1}K, \quad N(s) = F\tilde{N}(s).$$

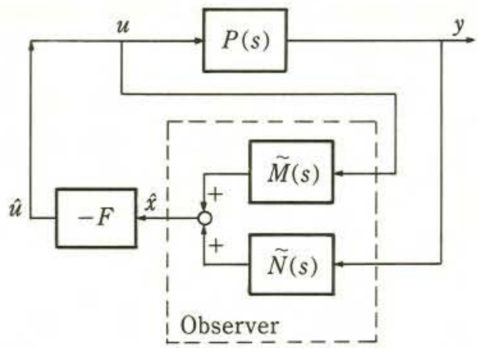

Fig. 4.2. Plant and observer configuration.

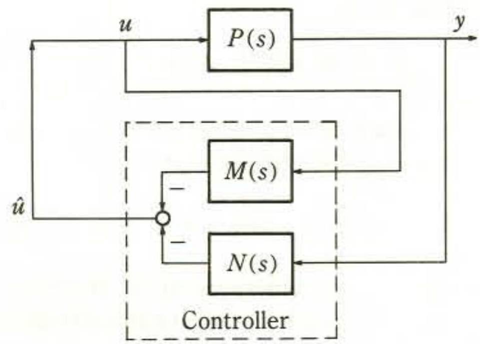

Fig. 4.3. Plant and controller configuration.

The above equations show that $-M(s)$ is the transfer function from the plant input point to the controller output point when controller is treated having two inputs, u and y. Lemma 4.1 implies then that whenever LTR is achieved by the controller, the controller output does not entail any feedback from the plant input point.

Equations (4.3) and (4.4) present a clear perspective to study the basic mechanism of LTR. In fact, they facilitate the study of $E(s)$ in terms of the study of $M(s)$. Thus, Lemma 4.1 and the expression for $M(s)$ as given by (4.3) form a basis for our study. Since in this section, F is considered as arbitrary or unknown, the only freedom we have to achieve the needed recovery is in the selection of observer gain K. First of all, in view of the well known separation principle, in order to guarantee the closed-loop stability, K must be such that the observer dynamic matrix, $A-KC$ is an asymptotically stable matrix. The remaining freedom in choosing K can then be used for the purpose of achieving LTR. Now in view of (4.3) and (4.4), exact loop transfer recovery (ELTR) is possible for an arbitrary F, if and only if

$$\tilde{M}(j\omega) = (j\omega I_n - A + KC)^{-1}(B-KD) \equiv 0.$$

However, due to the nonsingularity of $(j\omega I_n - A + KC)^{-1}$, the fact that $\tilde{M}(j\omega)\equiv 0$ implies that $B-KD\equiv 0$. The class of systems in which $B-KD$ can be rendered exactly zero is restrictive, and hence one normally attempts to achieve asymptotic loop transfer recovery (ALTR), i.e., to render $\tilde{M}(j\omega)$ approximately zero in some sense. In order to analyze whether ALTR is possible, as mentioned in the introduction, we parameterize the gain K with a tuning parameter σ and thus consider a family of controllers,

$$C(s,\sigma) = F[sI_n - A + BF + K(\sigma)C - K(\sigma)DF]^{-1}K(\sigma). \tag{4.7}$$

Now $M(s)$ and $\tilde{M}(s)$ are also functions of σ and are denoted respectively by $M(s,\sigma)$ and $\tilde{M}(s,\sigma)$. To proceed with our analysis, for clarity of presentation we will temporarily assume that $A-KC$ is nondefective. This allows us to expand $\tilde{M}(s,\sigma)$ and hence $M(s,\sigma)$ in a dyadic form,

$$\tilde{M}(s,\sigma) = \sum_{i=1}^{n} \frac{\tilde{R}_i}{s-\lambda_i}, \tag{4.8}$$

where the residue $\tilde{R}_i$ is given by

$$\tilde{R}_i = W_i V_i^H [B - K(\sigma)D]. \tag{4.9}$$

Here W_i and V_i are respectively the right and left eigenvectors associated with an eigenvalue λ_i of $A-KC$ and they are scaled so that $WV^H = V^H W = I_n$ where

$$W = [W_1, W_2, \cdots, W_n] \quad \text{and} \quad V = [V_1, V_2, \cdots, V_n]. \tag{4.10}$$

In general, all λ_i, V_i and W_i are functions of σ. However, for economy of notation we will not show the dependence on σ explicitly unless it is needed for clarity.

Remark 4.1: The assumption that $K(\sigma)$ is selected so that $A-K(\sigma)C$ is nondefective is not essential. However, it simplifies our presentation. A removal of this condition necessitates the use of generalized right and left eigenvectors of $A-K(\sigma)C$ instead of the right and left eigenvectors W_i and V_i and consequently the expansion of $\widetilde{M}(s, \sigma)$ requires a double summation in place of the single summation used in (4.8).

We are looking for conditions under which for each $i=1,\cdots,n$, the ith term of $\widetilde{M}(s, \sigma)$ in (4.8) can be made zero. There are only two possibilities to do so.

1. The first possibility is by assigning λ_i to any finite value in C^- while simultaneously rendering the corresponding residue $\widetilde{R}_i$ zero either exactly or asymptotically, i.e., $\widetilde{R}_i=W_i(\sigma)V_i^H(\sigma)[B-K(\sigma)D]=0$ or $\widetilde{R}_i \to 0$ as $\sigma \to \infty$. Thus, this possibility deals with finite eigenstructure assignment of $A-K(\sigma)C$.
2. The second possibility is to make $\widetilde{R}_i/(s-\lambda_i) \to 0$ pointwise in s as $\sigma \to \infty$. This can be done by placing the eigenvalue $\lambda_i(\sigma)$ asymptotically at infinity while making sure that the corresponding residue $\widetilde{R}_i$ is uniformly bounded as $\sigma \to \infty$. It is important to recognize that placing λ_i asymptotically at infinity alone is not beneficial unless the corresponding residue $\widetilde{R}_i$ is bounded. This amounts to assigning $W_i(\sigma)$ and $V_i(\sigma)$ such that $\widetilde{R}_i=W_i(\sigma)V_i^H(\sigma)[B-K(\sigma)D]$ remains bounded while $\lambda_i \to \infty$ as $\sigma \to \infty$. Thus, this possibility deals with infinite eigenstructure assignment of $A-K(\sigma)C$.

The above two possibilities of making a particular term of $\widetilde{M}(s, \sigma)$ zero leads to two fundamental questions that need to be answered: (1) How many left eigenvectors of $A-K(\sigma)C$ can be assigned to the null space of $[B-K(\sigma)D]'$? and (2) How many eigenvalues of $A-K(\sigma)C$ can be placed at asymptotically infinite locations in C^- so that the corresponding residues are finite? The following two lemmas respectively answer these two questions. In these lemmas and elsewhere, the geometric spaces $S^-(A, B, C, D)$, $V^*(A, B, C, D)$ and $V^+(A, B, C, D)$ are as defined in Sec. 3.

Lemma 4.2. Let λ_i and V_i be an eigenvalue and the corresponding left eigenvector of $A-K(\sigma)C$ for any gain $K(\sigma)$ such that it is asymptotically stable. Then, the maximum possible number of $\lambda_i \in C^-$ which satisfy the condition $V_i^H[B-K(\sigma)D]=0$ is $n_a^-+n_b$. A total of n_a^- of these λ_i coincide with the invariant zeros of Σ which are in C^- (the so called stable invariant zeros) and the remaining n_b eigenvalues can be assigned arbitrarily to any locations in C^-. All the eigenvectors V_i that correspond to these $n_a^-+n_b$ eigenvalues span the subspace $\mathscr{R}^n/S^-(A, B, C, D)$. Moreover, the n_a^- eigenvectors V_i which correspond to the eigenvalues which coincide with the system invariant zeros in C^- coincide with the corresponding left state zero directions and span the subspace $V^*(A, B, C, D)/V^+(A, B, C, D)$.

Proof. See Appendix B.

Remark 4.2: Instead of rendering the $n_a^-+n_b$ residues $\widetilde{R}_i$ mentioned in Lemma 4.2 exactly zero, if one prefers, they can be rendered asymptotically zero as $\sigma \to \infty$. In that case n_a^- eigenvalues coincide asymptotically with the n_a^- stable invariant zeros while the corresponding eigenvectors in the limit as $\sigma \to \infty$ coincide with the corresponding left state zero directions and span the subspace $V^*(A, B, C, D)/V^+(A, B, C, D)$.

Lemma 4.3. Let λ_i, W_i and V_i be an eigenvalue and the corresponding right and left eigenvectors of $A-K(\sigma)C$ for any gain $K(\sigma)$ such that it is asymptotically stable. The maximum number of eigenvalues of $A-K(\sigma)C$ that can be assigned arbitrarily to asymptotically infinite locations in C^- so that the corresponding $\tilde{R}_i = W_i V_i^H [B-K(\sigma)D]$ are bounded as $|\lambda_i| \to \infty$ is $n_b + n_f$. Furthermore, all the corresponding left eigenvectors V_i of such eigenvalues asymptotically span the subspace $\mathscr{R}^n / V^*(A, B, C, D)$.

Proof. It follows along the same lines as Lemma 3.3 of Saberi et al. (1991 a).

As implied by Lemma 4.2, in addition to n_a^- eigenvalues which coincide with the stable invariant zeros of the given system, there are n_b other eigenvalues which can be assigned arbitrarily to any locations in C^- such that $\tilde{R}_i \equiv 0$. This implies that $\tilde{R}_i$ corresponding to these n_b eigenvalues are identically zero and hence are bounded. Thus, these n_b eigenvalues are included among the $n_b + n_f$ eigenvalues indicated in Lemma 4.3. That is, there is a set of n_b eigenvalues which can be placed arbitrarily at either asymptotically finite locations in C^- as indicated by Lemma 4.2 or at asymptotically infinite locations in C^- as indicated by Lemma 4.3. Hereafter, in order to conserve the controller bandwidth, we will assume that these n_b eigenvalues are always assigned to asymptotically finite locations.

Remark 4.3: Consider the case when Σ is right invertible and has no infinite zeros. Note that this case includes the special case when Σ is a non-strictly proper single-input and single-output system (SISO). For this case, $n_b + n_f = 0$ and hence there is no eigenvalue, λ_i of $A-K(\sigma)C$ that can be assigned to an infinite location such that the corresponding $\tilde{R}_i$ is bounded.

Lemmas 4.2 and 4.3 together tell us all the possibilities of rendering various terms of $\tilde{M}(s, \sigma)$ zero either exactly or asymptotically. There are altogether $n_a^- + n_b + n_f$ eigenvalues which can be assigned either at finite or at asymptotically infinite locations so that the corresponding terms of $\tilde{M}(s, \sigma)$ in its dyadic expansion (4.8) are either exactly or asymptotically zero. Thus, a question arises as to under what conditions $n_a^- + n_b + n_f$ equals the dimension n of the given system. It is indeed easy to see that $n_a^- + n_b + n_f = n$, if and only if Σ is left invertible and of minimum phase. Thus, for left invertible and minimum phase systems, asymptotic LTR is always achievable irrespective of the properties of the given target loop transfer function $L_t(s)$. For strictly proper systems, this result is well known (Doyle and Stein, 1979; Matson and Maybeck, 1991).

If Σ is not left invertible and/or of nonminimum phase, there are $n_e \equiv n - n_a^- - n_b - n_f \equiv n_a^+ + n_c$ terms of $\tilde{M}(s, \sigma)$ which cannot in general be rendered zero. To emphasize explicitly the behavior of each term of $\tilde{M}(s, \sigma)$, we partition it into four parts,

$$\tilde{M}(s, \sigma) = \tilde{M}^-(s, \sigma) + \tilde{M}^b(s, \sigma) + \tilde{M}^\infty(s, \sigma) + \tilde{M}^e(s, \sigma), \tag{4.11}$$

where

$$\tilde{M}^-(s, \sigma) = \sum_{i=1}^{n_a^-} \frac{\tilde{R}_i^-}{s-\lambda_i^-}, \quad \tilde{M}^b(s, \sigma) = \sum_{i=1}^{n_b} \frac{\tilde{R}_i^b}{s-\lambda_i^b}$$

and

$$\widetilde{M}^{\infty}(s,\sigma)=\sum_{i=1}^{n_f}\frac{\widetilde{R}_i^{\infty}}{s-\lambda_i^{\infty}},\quad \widetilde{M}^{e}(s,\sigma)=\sum_{i=1}^{n_a^++n_c}\frac{\widetilde{R}_i^{e}}{s-\lambda_i^{e}}.$$

In the above partition, appropriate superscripts $-$, b, ∞ and e are added to $\widetilde{R}_i$ and λ_i in order to associate them respectively with $\widetilde{M}^-(s,\sigma)$, $\widetilde{M}^b(s,\sigma)$, $\widetilde{M}^{\infty}(s,\sigma)$ and $\widetilde{M}^e(s,\sigma)$. Next, define the following sets where $n_e=n_a^++n_c$:

$$\Lambda^-(\sigma)\triangleq\{\lambda_i^-(\sigma)\,|\,i=1,\cdots,n_a^-\},\quad \Lambda^b(\sigma)\triangleq\{\lambda_i^b(\sigma)\,|\,i=1,\cdots,n_b\},$$

$$\Lambda^{\infty}(\sigma)\triangleq\{\lambda_i^{\infty}(\sigma)\,|\,i=1,\cdots,n_f\},\quad \Lambda^e(\sigma)\triangleq\{\lambda_i^e(\sigma)\,|\,i=1,\cdots,n_e\},$$

$$V^-(\sigma)\triangleq\{V_i^-(\sigma)\,|\,i=1,\cdots,n_a^-\},\quad V^b(\sigma)\triangleq\{V_i^b(\sigma)\,|\,i=1,\cdots,n_b\},$$

$$V^{\infty}(\sigma)\triangleq\{V_i^{\infty}(\sigma)\,|\,i=1,\cdots,n_f\},\quad V^e(\sigma)\triangleq\{V_i^e(\sigma)\,|\,i=1,\cdots,n_e\},$$

$$W^-(\sigma)\triangleq\{W_i^-(\sigma)\,|\,i=1,\cdots,n_a^-\},\quad W^b(\sigma)\triangleq\{W_i^b(\sigma)\,|\,i=1,\cdots,n_b\},$$

$$W^{\infty}(\sigma)\triangleq\{W_i^{\infty}(\sigma)\,|\,i=1,\cdots,n_f\},\quad W^e(\sigma)\triangleq\{W_i^e(\sigma)\,|\,i=1,\cdots,n_e\}.$$

Hereafter, we will be using an over bar on a certain variable to denote its limit whenever it exists as $\sigma\to\infty$. For example, $\bar{\widetilde{M}}^e(s)$ and $\bar{W}^e$ denote respectively the limits of $\widetilde{M}^e(s,\sigma)$ and $W^e(\sigma)$ as $\sigma\to\infty$.

We now note that various parts of $\widetilde{M}(s,\sigma)$ have the following interpretation:

1. $\widetilde{M}^-(s,\sigma)$ contains n_a^- terms. The n_a^- eigenvalues of $A-K(\sigma)C$ represented in it form a set $\Lambda^-(\sigma)$. In accordance with the Lemma 4.2, there exists a gain $K(\sigma)$ such that $\widetilde{M}^-(s,\sigma)$ can be rendered identically zero by assigning the elements of $\Lambda^-(\sigma)$ to coincide with the stable invariant zeros of Σ while the corresponding set of left eigenvectors $V^-(\sigma)$ coincides with the corresponding set of left state zero directions. In fact, $K(\sigma)$ can also be designed such that $\Lambda^-(\sigma)$ and $V^-(\sigma)$ approach asymptotically the set of system minimum phase invariant zeros and the corresponding set of left state zero directions as $\sigma\to\infty$. In this case, $\widetilde{M}^-(s,\sigma)\to0$ as $\sigma\to\infty$.
2. $\widetilde{M}^b(s,\sigma)$ contains n_b terms. The n_b eigenvalues of $A-K(\sigma)C$ represented in it form a set $\Lambda^b(\sigma)$. In accordance with the Lemmas 4.2 and 4.3, there exists a gain $K(\sigma)$ such that $\widetilde{M}^b(s,\sigma)$ can be rendered identically zero by assigning the elements of $\Lambda^b(\sigma)$ arbitrarily to either asymptotically finite or infinite locations in C^- as $\sigma\to\infty$. As discussed earlier, in order to conserve the controller bandwidth, we will assume hereafter that these eigenvalues are assigned to asymptotically finite locations. Also, $K(\sigma)$ can be designed so that $\widetilde{M}^b(s,\sigma)\to0$ as $\sigma\to\infty$.
3. $\widetilde{M}^{\infty}(s,\sigma)$ contains n_f terms. The n_f eigenvalues of $A-K(\sigma)C$ represented in it form a set $\Lambda^{\infty}(\sigma)$. In accordance with the Lemma 4.3, there exists a gain $K(\sigma)$ such that $\widetilde{M}^{\infty}(s,\sigma)\to0$ as $\sigma\to\infty$ by assigning the elements of $\Lambda^{\infty}(\sigma)$ arbitrarily to asymptotically infinite locations in C^-.
4. $\widetilde{M}^e(s,\sigma)$ contains the remaining $n_e\equiv n_a^++n_c$ terms. It is nonexistent, i.e., $n_e=0$, if and only if Σ is left invertible and of minimum phase. The n_e eigenvalues of $A-K(\sigma)C$ represented in $\widetilde{M}^e(s,\sigma)$ form a set $\Lambda^e(\sigma)$. In view of Lemmas 4.2 and 4.3, $\widetilde{M}^e(s,\sigma)$ cannot in general be rendered zero either asymptotically or otherwise by any assignment of $\Lambda^e(\sigma)$ and the associated

sets of right and left eigenvectors, $W^e(\sigma)$ and $V^e(\sigma)$. However, as will be discussed later on, $\tilde{M}^e(s,\ \sigma)$ can be shaped to have some desirable properties. Since $(A,\ C)$ is assumed to be a detectable pair, except for the stable but unobservable eigenvalues of A, others among the remaining eigenvalues of $A-K(\sigma)C$ which are in Λ^e can be assigned to arbitrary locations in C^-. These arbitrary locations can either be asymptotically finite or infinite. Moreover, assigning elements of $\Lambda^e(\sigma)$ to asymptotically infinite locations increases unnecessarily controller bandwidth. Because of this, we assume Λ^e is confined to finite locations in C^-.

Since both $\tilde{M}^-(s,\ \sigma)$ and $\tilde{M}^b(s,\ \sigma)$ can be rendered identically zero, for future use we can combine them into one term,

$$\tilde{M}^0(s,\ \sigma) = \tilde{M}^-(s,\ \sigma) + \tilde{M}^b(s,\ \sigma),$$

and rewrite $\tilde{M}(s,\ \sigma)$ as

$$\tilde{M}(s,\ \sigma) = \tilde{M}^0(s,\ \sigma) + \tilde{M}^\infty(s,\ \sigma) + \tilde{M}^e(s,\ \sigma). \tag{4.12}$$

We define likewise, $\Lambda^0(\sigma)=\Lambda^-(\sigma)\cup\Lambda^b(\sigma)$, $W^0(\sigma)=W^-(\sigma)\cup W^b(\sigma)$ and $V^0(\sigma)=V^-(\sigma)\cup V^b(\sigma)$.

As the above discussion indicates, Lemmas 4.2 and 4.3 form the heart of the underlying mechanism of LTR as they enable us to decompose $\tilde{M}(s,\ \sigma)$ and hence $M(s,\ \sigma)$ into several parts. They show clearly what is and what is not feasible under what conditions. Although they do not directly provide methods of obtaining the gain $K(\sigma)$, they do provide structural guide lines as to how certain eigenvalues and eigenvectors are to be assigned while indicating a multitude of ways in which freedom exists in assigning the other eigenvalues and eigenvectors of $A-K(\sigma)C$. These guidelines, in turn, can appropriately be channeled to come up with a design method to obtain an appropriate gain $K(\sigma)$. As will be discussed systematically in a paper sequel to this (Chen et al., 1992), there exist essentially three methods of design to obtain appropriate $K(\sigma)$. These are (1) Kalman filter formalism which minimizes the H_2-norm of $M(s)$, (2) Methods of minimizing H_∞-norm of $M(s)$ and (3) Asymptotic time-scale and eigenstructure assignment (ATEA) method of Saberi and Sannuti (1990 a) and Saberi et al. (1991 b) by which $\tilde{M}^e(s)$ can be shaped as desired in a number of ways. Leaving aside now the methods of design, let us at this stage simply define a set of gains $K^*(\Sigma,\ \sigma)$ as follows:

Definition 4.1. $K^*(\Sigma,\ \sigma)$ is a set of gains $K(\sigma)\in\mathscr{R}^{n\times p}$ such that

(1) $A-K(\sigma)C$ is stable for all $\sigma>\sigma^*$, where $0\le\sigma^*<\infty$,
(2) the limits, as $\sigma\to\infty$, of the finite eigenvalues of $A-K(\sigma)C$ remain in C^-,
(3a) if $n_f=0$, $\tilde{M}^0(s,\ \sigma)$ is identically zero for all σ,
(3b) if $n_f\neq0$, as $\sigma\to\infty$, $\tilde{M}^0(s,\ \sigma)$ is either identically zero or asymptotically zero while the eigenvalues represented in $\tilde{M}^0(s,\ \sigma)$ tend to finite locations in C^-, and
(4) $\tilde{M}^\infty(s,\ \sigma)\to0$ as $\sigma\to\infty$.

Remark 4.4: For the case when Σ does not have any infinite zeros, i.e., $n_f=0$, any element $K(\sigma)$ of $K^*(\Sigma,\ \sigma)$ is independent of σ and hence is bounded. On the other hand, if Σ has at least one infinite zero, i.e., $n_f\neq0$, any element

$K(\sigma)$ of $K^*(\Sigma, \sigma)$ is dependent on σ. Moreover, $\|K(\sigma)\| \to \infty$ as $\sigma \to \infty$.

It is obvious that $K^*(\Sigma, \sigma)$ as defined above is a nonempty set. We note also that whenever $K(\sigma)$ is chosen as an element of $K^*(\Sigma, \sigma)$, the asymptotic limit, namely $\bar{M}^e(s) \equiv F\bar{\bar{M}}^e(s)$, as $\sigma \to \infty$ of $M^e(s, \sigma) \equiv F\tilde{M}^e(s, \sigma)$ is the ultimate error in the recovery matrix $M(s, \sigma)$. As such, hereafter $\bar{M}^e(s)$ is called as the *recovery error matrix*. Theorem 4.1 given below characterizes the asymptotic behavior of the achieved loop transfer function as well as the sensitivity and complementary sensitivity functions in terms of $\bar{M}^e(s)$. Let $S_o(s, \sigma)$ and $T_o(s, \sigma)$ be the achieved sensitivity and complementary sensitivity functions using the output feedback controller $C(s, \sigma)$ as in the configuration of Fig. 4.1 when the loop is broken at the input point of the system,

$$S_o(s, \sigma) = [I_m + C(s, \sigma)P(s)]^{-1}$$

and

$$T_o(s, \sigma) = I_m - S_o(s, \sigma) = [I_m + C(s, \sigma)P(s)]^{-1} C(s, \sigma)P(s).$$

Also, let $S_t(s)$ and $T_t(s)$ be the target sensitivity and complementary sensitivity functions corresponding to the target loop transfer function. We have the following theorem.

Theorem 4.1. Consider the closed-loop system Σ^c comprising of the given system Σ and the controller as given in Fig. 4.1. Let Σ be stabilizable and detectable. Then for any F such that $A - BF$ is asymptotically stable, and for any gain $K(\sigma) \in K^*(\Sigma, \sigma)$, the closed-loop system Σ^c is asymptotically stable. Moreover, as $\sigma \to \infty$, we have pointwise in s,

$$E(s, \sigma) \to F\bar{\bar{M}}^e(s)[I_m + F\bar{\bar{M}}^e(s)]^{-1}(I_m + F\Phi B), \tag{4.13}$$

$$S_o(s, \sigma) \to S_t(s)[I_m + F\bar{\bar{M}}^e(s)], \tag{4.14}$$

$$T_o(s, \sigma) \to T_t(s) - S_t(s)F\bar{\bar{M}}^e(s), \tag{4.15}$$

and

$$\frac{|\sigma_i[S_o(j\omega, \sigma)] - \sigma_i[S_t(j\omega)]|}{\sigma_{\max}[S_t(j\omega)]} \le \sigma_{\max}[F\bar{\bar{M}}^e(j\omega)], \tag{4.16}$$

$$\frac{|\sigma_i[T_o(j\omega, \sigma)] - \sigma_i[T_t(j\omega)]|}{\sigma_{\max}[S_t(j\omega)]} \le \sigma_{\max}[F\bar{\bar{M}}^e(j\omega)]. \tag{4.17}$$

Proof. See Appendix C.

We have the following corollaries of Theorem 4.1.

Corollary 4.1. Let Σ be stabilizable, detectable, left invertible and of minimum phase. Then $\mathbf{T}^R(\Sigma) = \mathbf{T}(\Sigma)$. Moreover, for any gain $K(\sigma) \in K^*(\Sigma, \sigma)$, the corresponding full order observer based controller achieves loop transfer recovery for any given $L_t(s) \in \mathbf{T}(\Sigma)$.

Proof. Σ is left invertible and of minimum phase implies that $n_a^+=0$, $n_c=0$. Since $n_a^+ + n_c = 0$, $\tilde{M}^e(s, \sigma)$ is nonexistent. Hence the results of Corollary 4.1 are obvious.

Remark 4.5: For strictly proper systems, the results of corollary are well known as given in the seminal work of Doyle and Stein (1979). The results of Corollary 4.1 for non-strictly proper systems are given in Chen et al. (1990).

As implied by Theorem 4.1, the recovery error matrix $\bar{\bar{M}}^e(s)$ plays a dominant role in the recovery process and hence it should be shaped to yield as best as possible the desired results. Shaping $\bar{\bar{M}}^e(s)$ involves selecting the set of eigenvalues $\bar{\Lambda}^e$ represented in $\bar{\bar{M}}^e(s)$ and the associated set of right and left eigenvectors $\bar{W}^e$ and $\bar{V}^e$. Such a selection can be done in a number of ways subject to the constraints imposed in selecting the eigenvectors (Moore, 1976). However, note that though, no shaping may be necessary if $\bar{\bar{M}}^e(s)$ turns out to be small. For certain class of systems $\bar{\bar{M}}^e(s)$ is, in fact, small in some sense or other. Following a similar result of Saberi et al. (1991 a), one can prove easily that for a left invertible nonminimum phase system which is not necessarily strictly proper but which has all its unstable invariant zeros far away from the bandwidth of the target loop transfer function, the norm of the recovery error matrix $\bar{\bar{M}}^e(s)$ is indeed always small.

In multivariable systems, one interesting aspect of Theorem 4.1 is that there could exist a subspace of the control space in which $\bar{\bar{M}}^e(s)$ can be rendered zero. To pinpoint this, let

$$e_i = [B - K(\sigma)D]'\bar{V}_i, \quad \bar{V}_i \in \bar{V}^e, \tag{4.18}$$

and let E^e be the subspace of $\mathscr{R}^m$,

$$E^e = \mathrm{Span}\{e_i \,|\, \bar{V}_i \in \bar{V}^e\}. \tag{4.19}$$

Let the dimension of E^e be m^e. Now let

$$S^e = \text{orthogonal complement of } E^e \text{ in } \mathscr{R}^m. \tag{4.20}$$

Let P^s be the orthogonal projection matrix onto S^e. Then the following theorem pinpoints the directional behavior of $\tilde{M}(s, \sigma)$ and consequently the behavior of $S_o(s, \sigma)$ and $T_o(s, \sigma)$ as $\sigma \to \infty$.

Theorem 4.2. Consider the closed-loop system Σ^c comprising of the given system Σ and the controller as given in Fig. 4.1. Let Σ be stabilizable and detectable. Then for any F such that $A-BF$ is asymptotically stable, and for any gain $K(\sigma) \in K^*(\Sigma, \sigma)$, the closed-loop system Σ^c is asymptotically stable. Moreover, considering the subspace $S^e \in \mathscr{R}^m$ as given in (4.20) and denoting the orthogonal projection matrix onto S^e as P^s, we have as $\sigma \to \infty$, pointwise in s,

$$\tilde{M}(s, \sigma)P^s \to 0,$$

$$S_o(s, \sigma)P^s \to S_t(s)P^s,$$

$$T_o(s, \sigma)P^s \to T_t(s)P^s.$$

Proof. In view of the definitions of the matrix P^s and the subspaces E^e and S^e, Theorem 4.1 implies the results of Theorem 4.2.

In view of the directional behavior of $\tilde{\bar{M}}^e(s)$ as given by Theorem 4.2, one could try to shape it in a particular way so as to obtain the recovery of sensitivity and complimentary sensitivity functions in certain desired directions or one could try to shape $\tilde{\bar{M}}^e(s)$ so that the subspace S^e has as large a dimension as possible, i.e., the subspace E^e has as small a dimension as possible. In this regard, we note that we have already selected Λ^0 and Λ^∞ and the corresponding sets of eigenvectors $\bar{V}^0$ and $\bar{V}^\infty$ so that $\tilde{M}^0(s, \sigma)$ and $\tilde{M}^\infty(s, \sigma)$ tend to zero as $\sigma \to \infty$. We also note that although all the $n_a^+ + n_c$ vectors $\bar{V}_i \in \bar{V}^e$ can be selected to be linearly independent, the corresponding $e_i \equiv [B-K(\sigma)D]'\bar{V}_i$ need not be linearly independent. In fact for a given $e \neq 0$, the equation

$$e = [B-K(\sigma)D]'V$$

has $n-m+1$ linearly independent solutions for V. Of course, not all such $n-m+1$ vectors could be admissible eigenvectors of $A-KC$ for different eigenvalues of $A-KC$ in C^-, and moreover some or all of these $n-m+1$ vectors could also be linearly dependent on already selected eigenvectors in the sets $\bar{V}^0$ and $\bar{V}^\infty$. Thus, the problem of shaping E^e is to find an admissible set of eigenvalues λ_i and vectors e_i, $i=1,\cdots,n_a^+ + n_c$, which are not necessarily linearly independent, but the associated eigenvectors V_i of $A-KC$ satisfying $e_i = [B-K(\sigma)D]'\bar{V}_i$, $i=1,\cdots,n_a^+ + n_c$, together with the vectors in the sets $\bar{V}^0$ and $\bar{V}^\infty$ form n linearly independent vectors. This problem of selecting an admissible set (λ_i, e_i) is very much related to the traditional problem of distributing the modes of a closed-loop system to various output components by an appropriate selection of the closed-loop eigenstructure. This traditional problem of "shaping the output response characteristics" of a closed-loop system has been studied first by Moore (1976) and Shaked (1977) and more recently by Sogaard-Andersen (1987) although to this date there exists no systematic design procedure.

The above discussion focuses how to shape the subspace S^e in which $S_t(s)$ and $T_t(s)$ are recovered. A practical problem of interest could be to achieve recovery of $S_t(s)$ and $T_t(s)$ in a prescribed subspace S^e. We will discuss this aspect of the problem in Sec. 6.

Remark 4.6: In general, although $S_t(s)$ and $T_t(s)$ are recoverable in a subspace such as S^e, the loop transfer function $L_t(s)$ is not necessarily recoverable in that subspace S^e as can be seen from an example given in Sec. 6. However, this may not be as important as it seems since in most of the design schemes recovery of $L_t(s)$ is only a means to recover $S_t(s)$ and $T_t(s)$.

We will next examine the asymptotic behavior of open-loop eigenvalues of the full order observer based controller $C(s, \sigma)$ and the mechanism of pole zero cancellation between the controller eigenvalues and the input or output decoupling zeros (Rosenbrock, 1970) of the system. It is important to know the eigenvalues of $C(s, \sigma)$ as they are included among the invariant zeros of the closed-loop system Σ^c (Sannuti and Saberi, 1987) and hence affect the performance of Σ^c, e.g., command following. The controller transfer function is given by (4.7) while the eigenvalues of it are

$$\lambda[A-K(\sigma)C-BF+K(\sigma)DF].$$

To study the nature of these eigenvalues, let

$$\det[sI_n-A+K(\sigma)C] = \phi^0(s)\phi^\infty(s)\phi^e(s),$$

where $\phi^0(s)$, $\phi^\infty(s)$ and $\phi^e(s)$ are polynomials in s whose zeros are the eigenvalues of $A-K(\sigma)C$ that belong to the sets $\Lambda^0(\sigma)$, $\Lambda^\infty(\sigma)$ and $\Lambda^e(\sigma)$ respectively. Also, let

$$F\tilde{\tilde{M}}^e(s) = \frac{R^e(s)}{\phi^e(s)}, \tag{4.21}$$

where $R^e(s)$ is a polynomial matrix in s. Now consider the following:

$$\begin{aligned}
&\det[sI_n-A+K(\sigma)C+BF-K(\sigma)DF] \\
&\quad = \det[sI_n-A+K(\sigma)C]\det[I_n+(sI_n-A+K(\sigma)C)^{-1}(B-K(\sigma)D)F] \\
&\quad = \phi^0(s)\phi^\infty(s)\phi^e(s)\det[I_m+F(sI_n-A+K(\sigma)C)^{-1}(B-K(\sigma)D)] \\
&\quad = \phi^0(s)\phi^\infty(s)\phi^e(s)\det[I_m+F\tilde{M}(s,\ \sigma)] \\
&\quad \to \phi^0(s)\phi^\infty(s)\phi^e(s)\det[I_m+F\tilde{\tilde{M}}^e(s)] \text{ as } \sigma\to\infty \\
&\quad = \phi^0(s)\phi^\infty(s)\phi^e(s)\det\left[I_m+\frac{R^e(s)}{\phi^e(s)}\right] \\
&\quad = \phi^0(s)\phi^\infty(s)\frac{\det[I_m\phi^e(s)+R^e(s)]}{[\phi^e(s)]^{m-1}}.
\end{aligned} \tag{4.22}$$

We note that the observer can be designed such that $\phi^0(s)$, $\phi^\infty(s)$ and $\phi^e(s)$ are coprime. Thus, the open-loop eigenvalues of the controller of (4.7) are the zeros of $\phi^0(s)$, $\phi^\infty(s)$ and $\det[I_m\phi^e(s)+R^e(s)]/[\phi^e(s)]^{m-1}$. Thus, Λ^0 and Λ^∞ are contained among the eigenvalues of the controller. Although Λ^0 and Λ^∞ are in C^-, there is no guarantee that the zeros of $\det[I_m\phi^e(s)+R^e(s)]/[\phi^e(s)]^{m-1}$ are in C^-. Hence the controller may or may not be open-loop stable. In general, the loop transfer function $C(s,\ \sigma)P(s)$ has $2n$ eigenvalues, n of them coming from the given system and the other n coming from the controller. However, there are several cancellations among the input or output decoupling zeros (Rosenbrock, 1970) of $C(s,\ \sigma)P(s)$ and the controller eigenvalues. The following Lemma 4.4 which is a slight generalization of a similar one in Goodman (1984), explores such a cancellation.

Lemma 4.4. Let λ be an eigenvalue of $A-K(\sigma)C$ and the corresponding left eigenvector V be such that $V^H[B-K(\sigma)D]=0$. Then, λ is an eigenvalue of $A-K(\sigma)C-BF+K(\sigma)DF$ with corresponding left eigenvector as V. Moreover, λ cancels an input decoupling zero of $C(s,\ \sigma)P(s)$.

Proof. See Appendix D.

Thus in view of Lemma 4.2, the above lemma implies that whatever may be the matrix F, if observer is appropriately designed, there are $n_a^-+n_b$ cancellations among the eigenvalues of the controller and the input decoupling zeros of $C(s,\ \sigma)P(s)$. As will be seen in the next section, there may be additional cancellations if F satisfies certain properties.

5. Analysis for Recoverable Target Loops

In Sec. 4, we concentrated on general loop transfer recovery analysis without taking into account any knowledge of F. It essentially involved studying the recovery error matrix $\tilde{M}(s)$ or $\tilde{M}(s, \sigma)$ to ascertain when it can or cannot be rendered zero. This section compliments the analysis of Sec. 4 by taking into account the knowledge of F. Obviously then, the analysis of this section is a study of the recovery matrix $M(s)=F\tilde{M}(s)$ or $M(s, \sigma)=F\tilde{M}(s, \sigma)$. One of the important questions that needs to be answered here is as follows. What class of target loops can be recovered exactly (or asymptotically) for the given system? Or equivalently, what are the necessary and sufficient conditions a target loop transfer function $L_t(s)$ has to satisfy so that it can exactly (or asymptotically) be recoverable for the given system? As it forms a coupling between analysis and design, characterization of $L_t(s)$ to determine whether it can be recovered either exactly or asymptotically for the given system, plays an extremely important role. Although the physical tasks of designing F and K are separable, one can benefit enormously by knowing ahead what kind of target loops are recoverable. The necessary and sufficient conditions developed here on $L_t(s)$ for its recoverability, turn out to be constraints on the finite and infinite zero structure of $L_t(s)$ as related to the corresponding structure of Σ. An interpretation of these conditions reveals that either exact or asymptotic recovery of $L_t(s)$ for general nonminimum phase systems is possible under a variety of conditions.

Another important question that arises before one undertakes formulating any target loop transfer function $L_t(s)$ for a given system Σ is as follows. What are the necessary and sufficient conditions on Σ so that it has at least one recoverable target loop? An answer to this question obviously helps a designer to remodel the given plant if necessary by appropriately modifying the number or type of inputs or outputs of the plant. To answer the question posed, we develop here an auxiliary system Σ^{ER} of Σ, and show that the set of exactly recoverable target loops $\mathbf{T}^{\mathrm{ER}}(\Sigma)$ is nonempty, if and only if Σ^{ER} is stabilizable by a static output feedback control. Similarly, another auxiliary system Σ^{R} of Σ is developed to show that the set of recoverable target loops $\mathbf{T}^{\mathrm{R}}(\Sigma)$ is nonempty, if and only if Σ^{R} is stabilizable by a static output feedback control. A close look at these conditions reveals a surprising necessary condition, namely, strong stabilizability of Σ is necessary for it to have at least one, either exactly or asymptotically, recoverable target loop.

Finally, another aspect of analysis given here shows the mechanism of pole zero cancellation between the controller eigenvalues and the input or output decoupling zeros of Σ for the case when the target loop $L_t(s)$ is known.

We proceed now to give the following results regarding the exact recoverability of a target loop transfer function $L_t(s)=F\Phi B$ for the given system Σ.

Theorem 5.1. Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily left invertible and not necessarily of minimum phase. Then, an admissible target loop transfer function $L_t(s)$ of Σ, i.e., $L_t(s)\in\mathbf{T}(\Sigma)$, is exactly recoverable by a full order observer based controller, if and only if $S^{-}(A, B, C, D)\subseteq\mathrm{Ker}(F)$. Thus, the set of recoverable target loops is characterized as $\mathbf{T}^{\mathrm{ER}}(\Sigma)=\{L_t(s)\in\mathbf{T}(\Sigma): S^{-}(A, B, C, D)\subseteq\mathrm{Ker}(F)\}$.

Proof. See Appendix E.

Remark 5.1: In view of Theorem 5.1, one needs to verify the subspace inclusion condition $S^-(A, B, C, D) \subseteq \mathrm{Ker}(F)$ in order to show that a given admissible target loop transfer function $L_t(s)$ of Σ is exactly recoverable by a full order observer based controller. It is particularly easy to do such a verification if the given system Σ is rewritten in terms of its s.c.b as given by Theorem 3.1. Indeed, the inclusion $S^-(A, B, C, D) \subseteq \mathrm{Ker}(F)$ is true, if and only if F is of the form,

$$F = \Gamma_3 \tilde{F} \Gamma_1^{-1}, \quad \tilde{F} = \begin{bmatrix} F_{a1}^- & 0 & F_{b1} & 0 & 0 \\ F_{a2}^- & 0 & F_{b2} & 0 & 0 \end{bmatrix}, \tag{5.1}$$

where Γ_3 and Γ_1 are the nonsingular transformation matrices as defined in Theorem 3.1.

Several interpretations emerge from the recoverability conditions on the target loops given in Theorem 5.1. In fact the constraints given in Theorem 5.1 are nothing more than constraints on the finite and infinite zero structure and invertibility properties of $L_t(s)$. Some interesting interpretations in this regard can easily be exemplified as follows.

1. If Σ is not left invertible, any exactly recoverable $L_t(s)$ is not left invertible. On the other hand, left invertibility of Σ does not necessarily imply that an exactly recoverable $L_t(s)$ is left invertible. That is, whenever Σ is left invertible, an exactly recoverable $L_t(s)$ could be either left invertible or not left invertible.
2. Any left invertible and exactly recoverable $L_t(s)$ must contain the unstable invertible zero structure of Σ. An exactly recoverable but not left invertible $L_t(s)$ does not necessarily contain the unstable invariant zero structure of Σ (see Example 3.2 of Saberi et al., 1991 a).

We have the following interesting corollaries of Theorem 5.1.

Corollary 5.1. Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D). Then $\mathbf{T}^{\mathrm{ER}}(\Sigma) = \mathbf{T}(\Sigma)$, i.e., any admissible target loop is exactly recoverable by a full order observer based controller, if and only if Σ is left invertible and of minimum phase with no infinite zeros.

Proof. It follows from the properties of s.c.b that $S^-(A, B, C, D) = \emptyset$, if and only if Σ is left invertible, of minimum phase, and has no infinite zeros. It is then obvious that the result of Corollary 5.1 follows from Theorem 5.1.

Corollary 5.2. Consider a single-input single-output non-strictly proper system Σ. Then a target loop transfer function $L_t(s) = F\Phi B$ is exactly recoverable by a full order observer based controller, if and only if it contains the nonminimum phase zero structure of Σ.

Proof. A single-input single-output non-strictly proper system is always invertible. Hence, the result follows from interpretation 2 given above.

Our aim next is to develop the conditions on Σ so that $\mathbf{T}^{\mathrm{ER}}(\Sigma)$ is nonempty. We have the following theorem.

Theorem 5.2. Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let $\bar{C}^{\text{ER}}$ be any full rank matrix of dimension $(n-n_e-n_f)\times n$ such that $\text{Ker}(\bar{C}^{\text{ER}})=S^-(A, B, C, D)$. Then the given system Σ has at least one exactly recoverable target loop, i.e., $\mathbf{T}^{\text{ER}}(\Sigma)$ is nonempty, if and only if an auxiliary system Σ^{ER} characterized by the matrix triple $(A, B, \bar{C}^{\text{ER}})$ is stabilizable by a static output feedback controller.

Proof. See Appendix F.

Theorems 5.1 and 5.2 deal with ELTR. Since the required conditions for ELTR in general are severe, most often in practice one is interested only in ALTR. From its definition, it is easy to see that ALTR occurs, i.e., $\bar{M}^e(s)=F\bar{\bar{M}}^e(s)=0$, if and only if $F\bar{W}^e=0$. We have the following results regarding ALTR.

Theorem 5.3. Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily left invertible and not necessarily of minimum phase. Then, an admissible target loop transfer function $L_t(s)$ of Σ, i.e., $L_t(s)\in\mathbf{T}(\Sigma)$, is asymptotically recoverable by the full order observer based controller, if and only if $V^+(A, B, C, D)\subseteq\text{Ker}(F)$. That is, $\mathbf{T}^{\text{R}}(\Sigma)=\{L_t(s)\in\mathbf{T}(\Sigma)\text{: } V^+(A, B, C, D)\subseteq\text{Ker}(F)\}$.

Proof. Following the arguments in Appendix E, it is simple to see that our problem is equivalent to the well-known almost disturbance decoupling problem with internal stability (ADDPS) for the auxiliary system Σ_{au} in (E.1). It is shown in Scherer (1992) that the above ADDPS is solvable, if and only if $V^+(A, B, C, D)\subseteq\text{Ker}(F)$. Here, we adhere to the notion of closed-loop stability by excluding those cases where, in the limits as $\sigma\to\infty$, the finite eigenvalues of the closed-loop system are on the $j\omega$ axis.

Remark 5.2: In view of Theorem 5.3, one needs to verify the subspace inclusion condition $V^+(A, B, C, D)\subseteq\text{Ker}(F)$ in order to show that a given admissible target loop transfer function $L_t(s)$ of Σ is recoverable by a full order observer based controller. Again, as in the case of Theorem 5.1 and Remark 5.1, it is particularly easy to do such a verification if the given system Σ is rewritten in terms of its s.c.b as given by Theorem 3.1. Indeed, the inclusion $V^+(A, B, C, D)\subseteq\text{Ker}(F)$ is true, if and only if F is of the form,

$$F=\Gamma_3\tilde{F}\Gamma_1^{-1},\quad \tilde{F}=\begin{bmatrix} F_{a1}^- & 0 & F_{b1} & 0 & F_{f1} \\ F_{a2}^- & 0 & F_{b2} & 0 & F_{f2}\end{bmatrix}, \tag{5.2}$$

where again Γ_3 and Γ_1 are the nonsingular transformation matrices as defined in Theorem 3.1.

As in the case of ELTR, we can interpret the constraints imposed by Theorem 5.3 in terms of the invertibility and the finite zero structures of $L_t(s)$ and Σ as follows.

1. If Σ is not left invertible, any asymptotically recoverable $L_t(s)$ is not left invertible. On the other hand, left invertibility of Σ does not necessarily imply that an asymptotically recoverable $L_t(s)$ is left invertible. That is, whenever Σ is left invertible, an asymptotically recoverable $L_t(s)$ could be either left invertible or not left invertible.

2. Any left invertible and asymptotically recoverable $L_t(s)$ must contain the unstable invariant zero structure of Σ. An asymptotically recoverable but not left invertible $L_t(s)$ does not necessarily contain the unstable invariant zero structure of Σ.

Again, we have the following interesting corollary.

Corollary 5.3. Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D). Then, $\mathbf{T}^{\mathrm{R}}(\Sigma)=\mathbf{T}(\Sigma)$, i.e., any admissible target loop is recoverable by a full order observer based controller, if and only if Σ is left invertible and of minimum phase.

Proof. It follows from the properties of s.c.b that $V^{+}(A, B, C, D)=\emptyset$, if and only if Σ is left invertible and of minimum phase. The result of Corollary 5.3 follows then obviously from Theorem 5.3.

Corollary 5.3 strengthens the result of Corollary 4.1 in the sense that it provides both the necessary and sufficient conditions for $\mathbf{T}^{\mathrm{R}}(\Sigma)=\mathbf{T}(\Sigma)$.

Analogous to Theorem 5.2, we have the following Theorem 5.4 regarding the nonemptiness of $\mathbf{T}^{\mathrm{R}}(\Sigma)$.

Theorem 5.4. Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let $\bar{C}^{\mathrm{R}}$ be any full rank matrix of dimension $(n-n_e)\times n$ such that $\mathrm{Ker}(\bar{C}^{\mathrm{R}})=V^{+}(A, B, C, D)$. Then, the given system Σ has at least one recoverable target loop, i.e., $\mathbf{T}^{\mathrm{R}}(\Sigma)$ is nonempty, if and only if an auxiliary system Σ^{R} characterized by the matrix triple $(A, B, \bar{C}^{\mathrm{R}})$ is stabilizable by a static output feedback controller.

Proof. The proof follows along the same lines as that of Theorem 5.2.

Theorems 5.2 and 5.4 respectively give the necessary and sufficient conditions under which the set of exactly recoverable target loops, $\mathbf{T}^{\mathrm{ER}}(\Sigma)$, and the set of recoverable target loops, $\mathbf{T}^{\mathrm{R}}(\Sigma)$, are nonempty. However, the conditions given there are not conducive to any intuitive feelings. The following corollary gives a necessary condition which is surprising as well as intuitively appealing.

Corollary 5.4. The strong stabilizability of the given system Σ is a necessary condition for it to have at least one, exactly or asymptotically, recoverable target loop.

Proof. See Appendix G.

We now proceed to discuss the possible cancellations between the eigenvalues of the controller and the input or output decoupling zeros of $C(s, \sigma)$ or $C(s, \sigma)P(s)$. Lemma 4.4 already discussed one such result which is a slight generalization of a similar one in Goodman (1984). The following lemma is also a slight generalization of a similar one in Goodman (1984).

Lemma 5.1. Let λ be an eigenvalue of $A-K(\sigma)C$ and the corresponding right eigenvector W be such that $FW=0$. Then λ is an eigenvalue of $A-K(\sigma)C-BF+K(\sigma)DF$ with corresponding right eigenvector as W. Moreover, λ cancels an output decoupling zero of $C(s, \sigma)$.

Proof. It follows from some simple algebra.

We have the following theorems.

Theorem 5.5. If ELTR is achieved, i.e., if $E(j\omega, \sigma)=0$ for all $0 \leq |\omega| < \infty$, then every eigenvalue of $A-K(\sigma)C-BF+K(\sigma)DF$ cancels either an output decoupling zero of $C(s, \sigma)$ or an input decoupling zero of $C(s, \sigma)P(s)$.

Proof. ELTR is achieved, if and only if either $FW_i=0$ or $V_i^H[B-K(\sigma)D]=0$ or both. Hence, the result follows from Lemmas 4.4 and 5.1.

Theorem 5.6. If ALTR is achieved, i.e., if $E(j\omega, \sigma) \to 0$ as $\sigma \to \infty$ for all $0 \leq |\omega| < \infty$, then every asymptotically finite eigenvalue of $A-K(\sigma)C-BF+K(\sigma)DF$ cancels either an output decoupling zero of $C(s, \sigma)$ or an input decoupling zero of $C(s, \sigma)P(s)$.

Proof. If ALTR is achieved, then every asymptotically finite eigenvalue of $A-K(\sigma)C$ with corresponding right and left eigenvectors W_i and V_i must be such that either $FW_i=0$ or $V_i^H[B-K(\sigma)D]=0$ or both. Hence, this result also follows from Lemmas 4.4 and 5.1.

In view of Lemmas 4.4 and 5.1, and Theorem 5.5, whenever ELTR occurs, there are n exact cancellations among the eigenvalues of the controller and the output decoupling zeros of $C(s)$ or the input decoupling zeros of $C(s)P(s)$.

6. Recovery Analysis in a Given Subspace

In the last two sections, we discussed recovery of a target loop transfer function $L_t(s)=F\Phi B$ when the recovery is required over the entire control space $\mathscr{R}^m$ and when the knowledge of state feedback gain F is either unknown or known. This traditional LTR problem as treated in the last two sections, concentrates on recovering a open-loop transfer function $L_t(s)$ which has been formed to take into account the given design specifications. Actually, design specifications are normally formulated in terms of certain required closed-loop sensitivity and complimentary sensitivity functions, $S_t(s)=[I_m+F\Phi B]^{-1}$, and $T_t(s)=I_m-S_t(s)$. In LQG/LTR design philosophy, these given specifications are reflected in formulating an open-loop transfer function called target loop transfer function. As discussed earlier, this aspect of determining a target loop transfer function is a first step in LQG/LTR design and falls in the category of loop shaping. Generating a target loop transfer function $L_t(s)$ at the present time is an engineering art and often involves the use of linear quadratic design in which the cost matrices are used as free design parameters to obtain the state feedback gain F and thus to obtain $L_t(s)=F\Phi B$ and $S_t(s)=[I_m+F\Phi B]^{-1}$. In the second step of design, the so called loop transfer recovery (LTR) design, $L_t(s)$ is recovered using a measurement feedback controller. Obviously, in the traditional LTR design where recovery is required over the entire control space $\mathscr{R}^m$, the recovery of $L_t(s)$ implies the recovery of the corresponding sensitivity function $S_t(s)$ and hence the recovery of the complimentary sensitivity function $T_t(s)$. Conversely, as will be discussed in Observation 6.1, the recovery of $S_t(s)$ or equivalently that of $T_t(s)$, implies the recovery of $L_t(s)$. In other words, when recovery is required over the entire control space $\mathscr{R}^m$, recovering a certain target loop transfer function is equivalent to recovering a certain target sensitivity function. Thus, without loss of any freedom, historically, recovery of a target loop transfer function has been sought. As seen in earlier sections, for general nonminimum phase systems, recovery of a target loop transfer function or a target sensitivity function is not possible in the entire control space $\mathscr{R}^m$. This may force a designer to seek recovery, say of a target sensitivity function, in a chosen subspace S of the control space $\mathscr{R}^m$. Recovering a target function (either it be a target loop transfer function or a target sensitivity function) in a subspace S of $\mathscr{R}^m$ means matching the projections of the target and the achieved functions onto S. As will be shown by an example in this section, when recovery is required over a specified proper subspace S of $\mathscr{R}^m$, recovering a sensitivity function $S_t(s)$ in S is not equivalent to recovering the corresponding target loop transfer function in S. Thus when one is interested in meeting the design specifications only over a specified subspace S of $\mathscr{R}^m$, the required recovery problem has to be formulated carefully. This section formulates clearly such a problem as a sensitivity recovery problem in a subspace S of $\mathscr{R}^m$. This is done in view of the fact that design specifications are normally given in terms of the required sensitivity or complimentary sensitivity functions. However, for the case when S equals $\mathscr{R}^m$, as proved in Observation 6.1, sensitivity recovery formulation of this section coincides with the conventional LTR formulation. Thus, this section can indeed be viewed as a generalization of the notion of traditional LTR to cover recovery over either the entire or any specified subspace S of the control space $\mathscr{R}^m$.

A brief outline of this section is as follows. At first, precise definitions dealing with the sensitivity recovery problem are given. Then, Lemma 6.1 is developed generalizing Lemma 4.1. It formulates the condition for the recoverability of a sensitivity function in S in terms of a matrix $M^s(s)$. Next, an example is given to demonstrate that sensitivity recovery in a proper subspace S of $\mathscr{R}^m$ does not necessarily imply the corresponding target loop transfer function recovery in S. On the other hand, Observation 6.1 shows that if $S = \mathscr{R}^m$, sensitivity recovery is equivalent to the corresponding target loop transfer function recovery. Next, Theorem 6.1 specifies the required conditions on Σ so that asymptotic sensitivity recovery in S is possible for any arbitrarily specified target sensitivity function $S_t(s)$. Similarly, Theorems 6.2 and 6.4 specify the necessary and sufficient conditions respectively for exact and asymptotic recoverability of a sensitivity function when the knowledge of F is known. In an analogous manner, Theorems 6.3 and 6.5 respectively establish the necessary and sufficient conditions so that sets of exactly or asymptotically recoverable sensitivity functions of the given system Σ for a specified subspace S, are nonempty. An important aspect of recovery analysis in a subspace is to determine the maximum possible dimension of a recoverable subspace S. Our results here in this regard show that for a left invertible nonminimum phase system, whatever may be the given target sensitivity and complimentary sensitivity functions and whatever may be the number of unstable invariant zeros, there exists at least one $m-1$ dimensional subspace S of $\mathscr{R}^m$ in which complete recovery of sensitivity and complimentary sensitivity functions is possible.

We have the following formal definitions.

Definition 6.1. The set of admissible target sensitivity functions $\mathbf{S}(\Sigma)$ for a given system Σ is defined as follows:

$$\mathbf{S}(\Sigma) \triangleq \{S_t(s) \in M^{m\times m}(R_p) \,|\, S_t(s) = [I_m + L_t(s)]^{-1},\ L_t(s) \in \mathbf{T}(\Sigma)\}.$$

Definition 6.2. Given $S_t(s)\in\mathbf{S}(\Sigma)$ and a subspace $S\in\mathscr{R}^m$, we say $S_t(s)$ is exactly recoverable in the subspace S if there exists a $C(s)\in M^{m\times p}(R_p)$ such that (i) the closed-loop system comprising of $C(s)$ and $P(s)$ as in the configuration of Fig. 2.1 is asymptotically stable, and (ii) $S_o(s)P^s=S_t(s)P^s$, where $S_o(s)$ is the achieved sensitivity function and P^s is the orthogonal projection matrix onto S.

Definition 6.3. Given $S_t(s)\in\mathbf{S}(\Sigma)$ and a subspace $S\in\mathscr{R}^m$, we say $S_t(s)$ is asymptotically recoverable in the subspace S if there exists a parameterized family of controllers $C(s,\ \sigma)\in M^{m\times p}(R_p)$, where σ is a scalar parameter taking positive values, such that (i) the closed-loop system comprising of $C(s,\ \sigma)$ and $P(s)$ as in the configuration of Fig. 2.1 is asymptotically stable for all $\sigma>\sigma^*$, where $0\leq\sigma^*<\infty$, and (ii) $S_o(s,\ \sigma)P^s=S_t(s)P^s$, as $\sigma\rightarrow\infty$. Moreover, the limits, as $\sigma\rightarrow\infty$, of the finite eigenvalues of the closed-loop system should remain in C^-.

Definition 6.4. Given $S_t(s)\in\mathbf{S}(\Sigma)$ and a subspace $S\in\mathscr{R}^m$, we say that $S_t(s)$ is recoverable in the subspace S if $S_t(s)$ is either exactly or asymptotically recoverable in S.

Definition 6.5.
1. The set of exactly recoverable $S_t(s)\in\mathbf{S}(\Sigma)$ in the given subspace S is denoted by $\mathbf{S}^{\mathrm{ER}}(\Sigma,\ S)$.
2. The set of recoverable $S_t(s)\in\mathbf{S}(\Sigma)$ in the given subspace S is denoted by $\mathbf{S}^{\mathrm{R}}(\Sigma,\ S)$.
3. The set of admissible $S_t(s)\in\mathbf{S}(\Sigma)$ which are asymptotically recoverable but not exactly recoverable in the given subspace S is denoted by $\mathbf{S}^{\mathrm{AR}}(\Sigma,\ S)$.

Obviously, $\mathbf{S}^{\mathrm{R}}(\Sigma,\ S)=\mathbf{S}^{\mathrm{ER}}(\Sigma,\ S)\cup\mathbf{S}^{\mathrm{AR}}(\Sigma,\ S)$.

The following lemma is analogous to Lemma 4.1.

Lemma 6.1. Consider any arbitrary F such that $A-BF$ is asymptotically stable. Then $E^s(s)$, the projection onto a given subspace $S\in\mathscr{R}^m$ of the error between the achieved sensitivity function $S_o(s)$ and the target sensitivity function $S_t(s)$, is given by

$$E^s(s) = [I_m+F\Phi B]^{-1}M^s(s), \tag{6.1}$$

where

$$M^s(s) = M(s)P^s \tag{6.2}$$

and where $M(s)$ is as defined in (4.3). Furthermore for all $\omega\in\Omega$,

$$E^s(j\omega) = 0, \quad \text{if and only if} \quad M^s(j\omega) = 0,$$

where Ω is the set of all $0\leq|\omega|<\infty$ for which $S_t(j\omega)$ and $S_o(j\omega)$ are well defined (i.e., all required inverses exist).

Proof. See Appendix H.

We consider next an example to demonstrate that, when S is a proper subspace of $\mathscr{R}^m$, recoverability of target sensitivity function in S does not necessarily imply the recoverability of the corresponding target loop transfer function in S.

Example 6.1. Consider a non-strictly proper system characterized by

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = C = D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is invertible with two unstable invariant zeros at $s=1$ and $s=2$. Let the target loop $L_t(s)$ and target sensitivity function $S_t(s)$ be specified by

$$F = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}.$$

Now consider a subspace S which is a span of the vector,

$$V^s = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}.$$

Let

$$K(\sigma) = \begin{bmatrix} 10 & -9 \\ 4 & -3 \end{bmatrix}.$$

Then the plots in Fig. 6.1 clearly show that $\sigma_{\max}[E^s(j\omega)] \equiv 0$, and hence $S_t(s)$ is exactly recoverable in the subspace S. On the other hand, $\sigma_{\max}[E(j\omega)P^s]$ is nonzero which implies that $L_t(s)$ is not recoverable in S.

The following observation pertains to the case when $S = \mathscr{R}^m$.

Observation 6.1. If $S = \mathscr{R}^m$, then $S_t(s) = [I_m + L_t(s)]^{-1}$ is exactly recoverable in S, if and only if the corresponding target loop transfer function $L_t(s)$ is exactly recoverable in S. Similarly, if $S = \mathscr{R}^m$, then $S_t(s)$ is asymptotically recoverable in S, if and only if $L_t(s)$ is asymptotically recoverable in S.

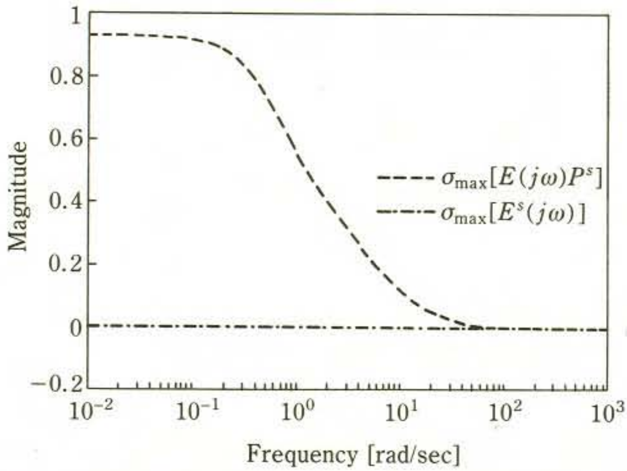

Fig. 6.1. Maximum singular values of $E(j\omega)P^s$ and $E^s(j\omega)$.

Proof. $S=\mathscr{R}^m$ implies that $P^s=I_m$ and $M^s(s)\equiv M(s)$. Hence, the result follows from Lemmas 6.1 and 4.1.

In view of the results of Observation 6.1, for the case when $S=\mathscr{R}^m$, the recoverability of any sensitivity function in $\mathscr{R}^m$ does indeed imply the recoverability of the corresponding target loop transfer function in $\mathscr{R}^m$. This implies then that when $S=\mathscr{R}^m$, Definitions 6.1 to 6.5 are equivalent to the Definitions 2.1 to 2.5 given earlier. On the other hand, Definitions 6.1 to 6.5 generalize the concept of recovery to a subspace and thus enable us to reanalyze all the results of the previous two sections to cover recovery in a given subspace S.

To proceed with the recovery analysis, let V^s be a matrix whose columns form an orthogonal basis of $S\in\mathscr{R}^m$. Assume that the columns of V^s are scaled so that the norm of each column is unity. Let $P^s=V^s(V^s)'$ be the unique orthogonal projection matrix onto S. Then, define an auxiliary system Σ^s characterized by the quadruple (A, BV^s, C, DV^s). Now treating Σ^s as the given system, one can rediscuss here mutatis mutandis all the results of Secs. 4 and 5. In particular, we have the following theorems.

Theorem 6.1. Consider a given system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $S\in\mathscr{R}^m$. Then any admissible sensitivity function $S_t(s)$ of Σ, i.e., $S_t(s)\in\mathbf{S}(\Sigma)$ is asymptotically recoverable in S if the auxiliary system Σ^s is left invertible and of minimum phase.

Proof. It is obvious.

Theorem 6.1 is concerned with the recovery analysis when F is arbitrary or unknown. As in Sec. 5, one can formulate the recovery conditions for a known F as follows.

Theorem 6.2. Consider a given system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $S\in\mathscr{R}^m$. Then an admissible sensitivity function $S_t(s)$ of Σ, i.e., $S_t(s)\in\mathbf{S}(\Sigma)$, is exactly recoverable in S by means of a full order observer based controller, if and only if $S^-(A, BV^s, C, DV^s)\subseteq\mathrm{Ker}(F)$. That is, $\mathbf{S}^{\mathrm{ER}}(\Sigma, S)=\{S_t(s)\in\mathbf{S}(\Sigma)\colon S^-(A, BV^s, C, DV^s)\subseteq\mathrm{Ker}(F)\}$.

Proof. The proof is a consequence of Theorem 5.1.

In what follows, we give a necessary and sufficient condition under which $\mathbf{S}^{\mathrm{ER}}(\Sigma, S)$ is non-empty for the given subspace $S\in\mathscr{R}^m$. We have the following theorem.

Theorem 6.3. Consider a given system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $S\in\mathscr{R}^m$. Let $\bar{C}^{se}$ be any full rank matrix such that $\mathrm{Ker}(\bar{C}^{se})=S^-(A, BV^s, C, DV^s)$. Then the given system Σ has at least one target sensitivity function that is exactly recoverable in S, i.e., $\mathbf{S}^{\mathrm{ER}}(\Sigma, S)$ is nonempty, if and only if an auxiliary system Σ^{se} characterized by the matrix triple $(A, B, \bar{C}^{se})$ is stabilizable by a static output feedback controller.

Proof. The proof is a consequence of Theorem 5.2.

The following theorem deals with asymptotic recoverability of $S_t(s)$.

Theorem 6.4. Consider a given system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $S \in \mathscr{R}^m$. Then an admissible sensitivity function $S_t(s)$ of Σ, i.e., $S_t(s) \in \mathbf{S}(\Sigma)$, is asymptotically recoverable in S by means of a full order observer based controller, if and only if $V^+(A, BV^s, C, DV^s) \subseteq \mathrm{Ker}(F)$. That is, $\mathbf{S}^R(\Sigma, S) = \{S_t(s) \in \mathbf{S}(\Sigma):\ V^+(A, BV^s, C, DV^s) \subseteq \mathrm{Ker}(F)\}$.

Proof. The proof is a consequence of Theorem 5.3.

Again, as in Theorem 6.3, we have the following theorem regarding non-emptiness of the set $\mathbf{S}^R(\Sigma, S)$.

Theorem 6.5. Consider a given system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $S \in \mathscr{R}^m$. Let $\bar{C}^{sa}$ be any full rank matrix such that $\mathrm{Ker}(\bar{C}^{sa}) = V^+(A, BV^s, C, DV^s)$. Then, the given system Σ has at least one target sensitivity function that is recoverable in S, i.e., $\mathbf{S}^R(\Sigma, S)$ is nonempty, if and only if an auxiliary system Σ^{sa} characterized by the matrix triple $(A, B, \bar{C}^{sa})$ is stabilizable by a static output feedback controller.

Proof. The proof is a consequence of Theorem 5.4.

An important aspect that arises when one is interested in recovery analysis in a subspace is to determine the maximum possible dimension of a recoverable subspace S. In this regard, our goal in what follows, as in Saberi et al. (1991 a), is to prove that whatever may be the given target loop transfer function and whatever may be the number of unstable invariant zeros, there exists at least one $m-1$ dimensional subspace S of $\mathscr{R}^m$ which is always recoverable provided that the given system is left invertible. To prove this, for simplicity of presentation, we will make a technical assumption that all the unstable invariant zeros of Σ have geometric multiplicity equal to unity. We next state two lemmas which lead to the intended result.

Lemma 6.2. Let the given system Σ be left invertible and let z, x and w be respectively an invariant zero, the associated right state and input zero directions of Σ. Then we have the following properties.

1. The auxiliary system Σ^s is left invertible.
2. Every invariant zero and the associated right state zero direction of Σ^s are also the invariant zero and the associated right state zero direction of Σ.
3. z and x are respectively an invariant zero and the associated right state zero direction of Σ^s, if and only if $w \in S$.

Proof. See Appendix I.

Now let z_i, x_i and w_i, $i=1,\cdots,n_a^+$, be respectively an unstable invariant zero and the associated right state and input zero directions of the given system Σ. Since Σ is assumed to be stabilizable and detectable, we have $w_i \neq 0$ for all $i=1,\cdots,n_a^+$. Because if $w_i=0$, then by definition,

$$(z_i I_n - A)x_i = Bw_i = 0, \quad Cx_i + Dw_i = Cx_i = 0.$$

This implies that z_i is an output decoupling zero of Σ. But this contradicts the detectability of Σ as $z_i \in C^+$. Next let us define for each $i=1,\cdots,n_a^+$,

$$N_i = \mathrm{Ker}[w_i'].$$

Since $w_i \neq 0$, each N_i is an $m-1$ dimensional subspace. We have the following lemma.

Lemma 6.3. There exists at least one nonzero vector $e \in \mathscr{R}^m$ such that

$$e \notin \bigcup_{i=1}^{n_a^+} N_i.$$

Proof. See Saberi et al. (1991 a).

Thus, in view of Lemma 6.3, there exists at least one e such that

$$e' w_i \neq 0 \quad \text{for all} \quad i = 1, \cdots, n_a^+. \tag{6.3}$$

We have the following theorem.

Theorem 6.6. Let the given system Σ be left invertible with unstable invariant zeros having geometric multiplicity equal to unity. Then, there exists at least one $m-1$ dimensional subspace S of $\mathscr{R}^m$ such that any admissible target sensitivity functions $S_t(s)$ of Σ, i.e., $S_t(s) \in \mathbf{S}(\Sigma)$, is recoverable in S.

Proof. Select e as in (6.3). Define S as

S = The orthogonal complement of the subspace spanned by e in $\mathscr{R}^m$.

Then, it is trivial to see S has a dimension of $m-1$ and that $w_i \notin S$ for all $i=1,\cdots,n_a^+$. Because if $w_i \in S$, say $w_i = V^s v_i \in S$, then $e' w_i = 0$ which is a contradiction. In view of Lemma 6.2, this implies that Σ^s is left invertible and of minimum phase. This in turn implies the results of Theorem 6.6.

7. Conclusions

Here we deal with issues concerning the analysis of loop transfer recovery problem using full order observer based controllers for general non-strictly proper systems. As in our earlier work, all the analysis given here is independent of the methodology by which observers are designed. There are several fundamental results given here. Based on the structural properties of the given system, we decompose the recovery error between the target loop transfer function and that which can be achieved by the observer based controllers, into three distinct parts for any arbitrarily specified target loop transfer function. The first part of recovery error can be rendered exactly zero by an appropriate finite eigenstructure assignment of the observer dynamic matrix, while the second part can be rendered arbitrarily close to zero by an appropriate asymptotically infinite eigenstructure assignment. The third part in general cannot be rendered zero, either exactly or asymptotically, by any means although there exists a multitude of ways to shape it. Such a decomposition of loop transfer function recovery mechanism helps us to discover the subspace of the control space in which target sensitivity and complimentary sensitivity functions can either exactly or asymptotically be recovered. Moreover, it helps to formulate explicit singular value bounds on the recovery error. All this analysis is given for an arbitrarily specified target loop transfer function. Thus, it shows the limitations of the given system in recovering the target loop transfer functions as a consequence of its structural properties, namely finite and infinite zero structure and invertibility. On the other hand, the next issue of our analysis concentrates on characterizing the required necessary and sufficient conditions on the target loop transfer functions so that they are either exactly or asymptotically recoverable by means of observer based controllers for the given system. The conditions developed here on a target loop transfer function for its recoverability, turn out to be constraints on its finite and infinite zero structure as related to the corresponding structure of the given system. We next move on to find the necessary and sufficient conditions on the given system such that it has at least one recoverable target loop. In this regard, we show that strong stabilizability of the given system is necessary for it to have at least one recoverable target loop. Since recovery in all control loops in general is not feasible, we concentrate next in developing the necessary and/or sufficient conditions under which either exact or asymptotic recovery of target sensitivity and complimentary sensitivity functions is possible in any specified subspace of the control space. This generalizes the traditional notion of LTR to cover recoverability in a subspace. We prove next that for left invertible non-strictly proper systems irrespective of the number of unstable invariant zeros and irrespective of the nature of the target loop transfer function, there exists at least one $m-1$ dimensional subspace of m dimensional control space, in which the target sensitivity and complimentary sensitivity functions can always be recovered by an appropriate design of the controller. Inherent in all the issues discussed here is the characterization of the resulting controller eigenvalues and possible pole zero cancellations. Such an investigation is important in view of the fact, controller eigenvalues become the invariant zeros of the closed-loop system and thus affect the performance with respect to command following and other design objectives.

To summarize, the analysis presented here adds a considerable amount of flexibility to the process of design and helps a designer to set meaningful goals at the onset of design. In other words, although the actual physical tasks of first designing a target loop and then designing an observer based controller are separable, one can link these two tasks philosophically by knowing ahead what is feasible and how. In a sequel to this paper, we will present design methodologies which are capable of utilizing the complete freedom a design can have as is discovered here.

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Appendix A: Proof of Lemma 4.1

We have the following obvious reductions:

$$\begin{aligned} L_o(s) &= C(s)P(s) \\ &= F[\Phi^{-1}+BF+KC-KDF]^{-1}K[C\Phi B+D] \\ &= F[I_n+(\Phi^{-1}+KC)^{-1}(B-KD)F]^{-1}(\Phi^{-1}+KC)^{-1}K[C\Phi B+D] \\ &= [I_n+F(\Phi^{-1}+KC)^{-1}(B-KD)]^{-1}F(\Phi^{-1}+KC)^{-1}K[C\Phi B+D] \\ &= [I_m+M(s)]^{-1}[F\Phi B-F(\Phi^{-1}+KC)^{-1}(B-KD)] \\ &= [I_m+M(s)]^{-1}[F\Phi B-M(s)]. \end{aligned}$$

Hence,

$$\begin{aligned} E(s) &= L_t(s) - L_o(s) \\ &= M(s)[I_m+M(s)]^{-1}(I_m+F\Phi B). \end{aligned}$$

Appendix B: Proof of Lemma 4.2

Let λ_i and V_i be an eigenvalue and the corresponding left eigenvector of $A-KC$ for any gain K. To show that there are at most $n_a^-+n_b$ left eigenvectors of $A-KC$ for any gain K such that the corresponding $\lambda_i \in C^-$ and that $V_i^H(B-KD)=0$, consider the dual system Σ_d characterized by (A_d, B_d, C_d, D_d) where

$$A_d = A', \quad B_d = C', \quad C_d = B', \quad D_d = D'.$$

Let V_d be the subspace of all right eigenvectors V_d of $(A_d-B_dK_d)$ for some K_d such that $(C_d-D_dK_d)V_d=0$. Observe that V_d is a stable (A_d, B_d)-invariant subspace. Furthermore, V_d is in the kernel of $(C_d-D_dK_d)$. Hence, V_d is a subset of $V^-(A_d, B_d, C_d, D_d)$. The largest possible dimension of $V^-(A_d, B_d, C_d, D_d)$ is $n_a^-+n_b$. Hence, there are at most $n_a^-+n_b$ left eigenvectors of $A-KC$ for any gain K such that the corresponding $\lambda_i \in C^-$ and that $V_i^H(B-KD)=0$.

We now proceed to determine the necessary gain K to assign such eigenvalues. Without loss of generality we can assume that the given system is represented by the s.c.b as given in Theorem 3.1. Then, consider a gain K of the form,

$$K = \begin{bmatrix} B_{0a}^{-} & L_{af}^{-} & L_{ab}^{-} \\ B_{0a}^{+} & 0 & 0 \\ B_{0b} & L_{bf} & K_{bb} \\ B_{0c} & 0 & 0 \\ B_{0f} & 0 & 0 \end{bmatrix},$$

where K_{bb} is selected such that $\lambda(A_{bb}-K_{bb}C_b)$ are in C^-. Let V_{a-} and V_b respectively be any left eigenvectors of A_{aa}^- and $A_{bb}-K_{bb}C_b$. It can easily be verified that $\lambda(A_{aa}^-)$ and $\lambda(A_{bb}-K_{bb}C_b)$ are among the eigenvalues of $A-KC$ and that $[V_{a-}^H, 0, 0, 0, 0]^H$ and $[0, 0, V_b^H, 0, 0]^H$ are the associated left eigenvectors of $A-KC$. Furthermore, it is easy to verify that

$$[V_{a-}^H, 0, 0, 0, 0](B-KD) = [V_{a-}^H, 0, 0, 0, 0] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_c \\ 0 & B_f & 0 \end{bmatrix} = 0$$

and similarly

$$[0, 0, V_b^H, 0, 0](B-KD) = 0.$$

Finally, in view of the properties of s.c.b, it is straightforward to see that such vectors $[V_{a-}^H, 0, 0, 0, 0]^H$ and $[0, 0, V_b^H, 0, 0]^H$ respectively span the subspaces x_a^- and x_b. Moreover, x_a^- spans $V^*(A, B, C, D)/V^+(A, B, C, D)$ and hence the result.

Appendix C: Proof of Theorem 4.1

Expression (4.13) follows directly from the definition of $K^*(\Sigma, \sigma)$. To prove (4.14) and (4.15), let us consider the following. From (4.1), we have

$$\begin{aligned} E(s, \sigma) &= F\Phi B - C(s, \sigma)P(s) \\ &= M(s, \sigma)[I+M(s, \sigma)]^{-1}(I+F\Phi B), \end{aligned}$$

and hence

$$\begin{aligned} I + C(s, \sigma)P(s) &= I + F\Phi B - E(s, \sigma) \\ &= I + F\Phi B - M(s, \sigma)[I+M(s, \sigma)]^{-1}(I+F\Phi B) \\ &= [I+M(s, \sigma)]^{-1}(I+F\Phi B). \end{aligned}$$

Thus, we obtain

$$S_o(s, \sigma) = S_t(s)[I+M(s, \sigma)] \tag{C.1}$$

and

$$T_o(s, \sigma) = T_t(s) - S_t(s)M(s, \sigma). \tag{C.2}$$

It is simple to see that (4.14) and (4.15) follow from the definition of $K^*(\Sigma, \sigma)$.

We now proceed to show (4.16) and (4.17). Applying singular value inequalities to (C.1), we have for each $i=1,\cdots,m$,

$$\sigma_i[S_o(j\omega, \sigma)] \leq \sigma_i[S_t(j\omega)] + \sigma_{\max}[S_t(j\omega)M(j\omega, \sigma)],$$

and thus,

$$\sigma_i[S_o(j\omega, \sigma)] - \sigma_i[S_t(j\omega)] \leq \sigma_{\max}[S_t(j\omega)]\sigma_{\max}[M(j\omega, \sigma)]. \tag{C.3}$$

Now rewriting (C.1) as,

$$S_t(s) = S_o(s, \sigma) - S_t(s)M(s, \sigma),$$

we have for each $i=1,\cdots,m$

$$\sigma_i[S_t(j\omega)] - \sigma_i[S_o(j\omega, \sigma)] \leq \sigma_{\max}[S_t(j\omega)]\sigma_{\max}[M(j\omega, \sigma)]. \tag{C.4}$$

Then, in view of (C.3) and (C.4), we get

$$\frac{|\sigma_i[S_o(j\omega, \sigma)] - \sigma_i[S_t(j\omega)]|}{\sigma_{\max}[S_t(j\omega)]} \leq \sigma_{\max}[M(j\omega, \sigma)].$$

Next using singular value inequalities and proceeding as above, we get

$$\frac{|\sigma_i[T_o(j\omega, \sigma)] - \sigma_i[T_t(j\omega)]|}{\sigma_{\max}[S_t(j\omega)]} \leq \sigma_{\max}[M(j\omega, \sigma)].$$

Then, (4.16) and (4.17) follow trivially. This completes the proof of Theorem 4.1.

Appendix D: Proof of Lemma 4.4

For economy of notations, we drop the dependency on σ throughout this proof. Noting from Lemma 4.1 that

$$E(s) \triangleq L_t(s) - C(s)P(s) = M(s)[I_m+M(s)]^{-1}[I_m+L_t(s)],$$

we obtain,

$$\begin{aligned} C(s)P(s) &= L_t(s) - M(s)[I_m+M(s)]^{-1}[I_m+L_t(s)] \\ &= [I_m+M(s)]^{-1}[L_t(s)-M(s)]. \end{aligned}$$

The above expression facilitates the interpretation of $C(s)P(s)$ in terms of a block diagram given below (Fig. D.1).

In view of the above block diagram, it is straightforward to write a state-space realization of $C(s)P(s)$ as

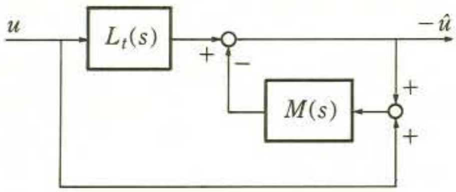

Fig. D.1. Interpretation of $C(s)P(s)$.

$$\begin{cases} \dot{\tilde{x}} = \begin{bmatrix} A & 0 \\ (B-KD)F & A-KC-BF+KDF \end{bmatrix}\tilde{x} + \begin{bmatrix} B \\ B-KD \end{bmatrix}u, \\ -\hat{u} = [F,\ -F]\tilde{x}. \end{cases}$$

Let λ be an eigenvalue of $A-KC$ and the corresponding left eigenvector V be such that $V^H(B-KD)=0$. It is simple then to verify that

$$[0,\ V^H]\begin{bmatrix} \lambda I-A & 0 \\ -(B-KD)F & \lambda I-A+KC+(B-KD)F \end{bmatrix} = 0$$

and

$$[0,\ V^H]\begin{bmatrix} B \\ B-KD \end{bmatrix} = 0.$$

This shows that λ is an input decoupling zero of $C(s)P(s)$ and thus the result follows.

Appendix E: Proof of Theorem 5.1

Consider an auxiliary system characterized by

$$\Sigma_{au}: \begin{cases} \dot{x} = A'x + C'u + F'w, \\ z = B'x + D'u. \end{cases} \tag{E.1}$$

Then, with a state feedback law

$$u = -K'x,$$

the closed-loop transfer function from w to z, denoted here by $T^{au}_{zw}(s)$, is simply

$$T^{au}_{zw}(s) = M'(s).$$

Hence, the problem of finding an observer gain matrix such that $A-KC$ is asymptotically stable and that $M(s)=0$ is equivalent to the well-known disturbance decoupling problem with internal stability when the plant considered is Σ_{au} as given in (E.1). Then, it follows from Stoorvogel (1990) that the above disturbance decoupling problem with internal stability is solvable, if and only if $S^-(A,\ B,\ C,\ D) \subseteq \mathrm{Ker}(F)$.

Appendix F: Proof of Theorem 5.2

Without loss of generality we assume that the given system Σ is in the form of s.c.b as in Theorem 3.1. Now in view of Theorem 5.1, an exactly recoverable $L_t(s)=F\Phi B$ must satisfy $S^-(A, B, C, D)\subseteq \text{Ker}(F)$. This implies that $L_t(s)$ is recoverable, if and only if F is of the form,

$$F = \begin{bmatrix} F_{a1}^- & 0 & F_{b1} & 0 & 0 \\ F_{a2}^- & 0 & F_{b2} & 0 & 0 \end{bmatrix}. \tag{F.1}$$

Thus, the fact that the given system has at least one exactly recoverable target loop is equivalent to the existence of some appropriate matrices F_{a1}^-, F_{b1}, F_{a2}^- and F_{b2} such that $A-BF$ is asymptotically stable. Next, in view of the fact that $x_a^+ \oplus x_c \oplus x_f$ spans $S^-(A, B, C, D)$, we note that $\bar{C}^{\text{ER}}$ as defined in Theorem 5.2 is of the form,

$$\bar{C}^{\text{ER}} = \Gamma\begin{bmatrix} I_{n_a^-} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_b} & 0 & 0 \end{bmatrix},$$

where Γ is any nonsingular matrix of dimension $(n-n_e-n_f)\times(n-n_e-n_f)$. It is now trivial to verify that the existence of a matrix F of the form in (F.1) such that $A-BF$ is asymptotically stable, is equivalent to the existence of a matrix G of dimension $m\times(n-n_e-n_f)$ such that $A-BG\bar{C}^{\text{ER}}$ is asymptotically stable. This is simply due to the fact that $G\bar{C}^{\text{ER}}$ has the same structure as F in (F.1). This completes the proof of Theorem 5.2.

Appendix G: Proof of Corollary 5.4

It is well known that any stabilizable and detectable system Σ can be stabilized by using an observer based controller. We next prove that Σ is strongly stabilizable whenever it has at least one asymptotically recoverable target loop transfer function $L_t(s)=F\Phi B$. The fact that there exists at least one asymptotically recoverable $L_t(s)$ implies that there exists a gain F such that $A-BF$ is asymptotically stable and $V^+(A, B, C, D)\subseteq\text{Ker}(F)$. Moreover, asymptotic recoverability of $L_t(s)$ implies that there exists an observer gain, $K(\sigma)$, such that $A-K(\sigma)C$ is asymptotically stable and

$$M(s, \sigma) = F[\Phi^{-1}+K(\sigma)C]^{-1}[B-K(\sigma)D] \to 0$$
$$\text{pointwise in} \quad s \quad \text{as } \sigma\to\infty. \tag{G.1}$$

Next, we examine the eigenvalues of the full order observer based controller. In view of Eqs. (4.22) and (G.1), we have

$$\det[sI_n-A+K(\sigma)C+BF-K(\sigma)DF] \to \phi^0(s)\phi^\infty(s)\phi^e(s),$$

as $\sigma\to\infty$, where the roots of $\phi^0(s)\phi^\infty(s)\phi^e(s)=0$ are the eigenvalues of $A-K(\sigma)C$. Thus, the full order observer based controller is open-loop stable for sufficiently large σ. We conclude then that the given system Σ is strongly stabilizable as it can be stabilized by an open-loop stable controller.

Appendix H: Proof of Lemma 6.1

Let us recall that

$$E(s) = L_t(s) - L_o(s) = F\Phi B - C(s)P(s) = M(s)[I_m+M(s)]^{-1}(I_m+F\Phi B).$$

Hence,

$$I_m + C(s)P(s) = I_m + F\Phi B - E(s) = I_m + F\Phi B - M(s)[I_m+M(s)]^{-1}(I_m+F\Phi B) = [I_m+M(s)]^{-1}(I_m+F\Phi B).$$

Since $S_t(s)=[I_m+F\Phi B]^{-1}$ and $S_o(s)=[I_m+C(s)P(s)]^{-1}$, we have

$$S_o(s) = S_t(s)[I_m+M(s)].$$

Thus,

$$E^s(s) = S_o(s)P^s - S_t(s)P^s = S_t(s)M(s)P^s = S_t(s)M^s(s).$$

Appendix I: Proof of Lemma 6.2

Assume that Σ^s is not left invertible. Then, it is well known that for any complex number z_1, there exist $0\neq x_1\in\mathscr{R}^n$ and $v_1\in\mathscr{R}^m$ such that

$$\begin{bmatrix} z_1I_n-A & -BV^s \\ C & DV^s \end{bmatrix}\begin{bmatrix} x_1 \\ v_1 \end{bmatrix} = 0.$$

This implies that

$$\begin{bmatrix} z_1I_n-A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x_1 \\ V^sv_1 \end{bmatrix} = 0.$$

Since Σ is left invertible, this then in turn implies that z_1 is an invariant zero of Σ. This is a contradiction and hence Σ^s is left invertible. To prove the second property of the lemma, let z^s, x^s and w^s be respectively an invariant zero, the associated right state and input zero directions of Σ^s. Then, by definition, we have

$$\begin{bmatrix} z^sI_n-A & -BV^s \\ C & DV^s \end{bmatrix}\begin{bmatrix} x^s \\ w^s \end{bmatrix} = 0.$$

Thus, we note that

$$\begin{bmatrix} z^sI_n-A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x^s \\ V^sw^s \end{bmatrix} = 0.$$

This proves the second property of the lemma. Let us next prove the sufficiency part of Property 3. Let $w = V^s v$, then

$$\begin{bmatrix} zI_n - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0$$

implies that

$$\begin{bmatrix} zI_n - A & -BV^s \\ C & DV^s \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = 0.$$

As Σ^s is left invertible, the above implies that z and x are an invariant zero and the associated right state zero direction of Σ^s. To prove necessity, assume that z and x are an invariant zero and the associated right state zero direction of Σ^s. Then, there exists a w^s such that

$$(zI_n - A)x = BV^s w^s, \quad Cx + DV^s w^s = 0.$$

In view of this and by the definition of z, x and w, we have

$$BV^s w^s = Bw, \quad DV^s w^s = Dw.$$

Since $[B', D']'$ is of full rank, it implies then that $w \in S$.

Ben M. Chen see p.35 in this issue.

Ali Saberi see p.35 in this issue.

Peddapullaiah Sannuti was born in Rajupalem, Proddatur (Taluk), India. He received the B.E. degree (with honors) in electrical engineering from the College of Engineering, Anantapur, India, in 1963, the M. Tech. degree in control systems engineering from the Indian Institute of Technology, Kharagpur, India, in 1965, and the Ph.D. degree in electrical engineering from the University of Illinois, Urbana-Champaign, in 1968.

From 1965 to 1968, he was first a Teaching Assistant and then a Research Assistant in the Department of Electrical Engineering and the Coordinated Science Laboratory, University of Illinois. Since 1968 he has been with the Department of Electrical Engineering, Rutgers University, Piscataway, N.J., where he is currently a Professor. His research interests include singular perturbation theory, computational methods, and communication theory.