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Necessary and Sufficient Conditions for a Nonminimum Phase Plant to have a Recoverable Target Loop—A Stable Compensator Design for LTR*

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All possible recoverable target loop transfer functions for a given plant are characterized, necessary and sufficient conditions for a given plant to have at least one recoverable target loop are established, a compensator structure which has definite advantages over a conventional observer based controller for LTR is developed.

Key Words—Loop transfer recovery; robust control; linear quadratic Gaussian theory.

Abstract—In connection with loop transfer recovery of nonminimum phase systems, the purpose of this paper is two-fold: (1) to study the set of recoverable target loops and to establish necessary or/and sufficient conditions for a given plant to have at least one recoverable target loop, and (2) to show that the compensator structure developed earlier in Chen, Saberi and Sannuti [(1991) *Automatica*, **27**, 257–280] for minimum phase systems, can also recover any recoverable target loop for nonminimum phase systems as well while retaining all its advantages over conventional observer based controllers.

1. INTRODUCTION

LOOP TRANSFER RECOVERY (LTR) has recently been studied with a vigorous interest by a number of authors including Athans (1986), Chen *et al.* (1991), Doyle and Stein (1979), Goodman (1984), Kwakernaak (1969), Niemann and Jannerup (1990), Niemann *et al.* (1991), Ridgely and Banda (1986), Stein and Athans (1987), Saberi *et al.* (1991a), Saberi *et al.* (1991b), Sogaard-Andersen and Neimann (1989), Sogaard-Andersen (1989), Saberi and Sannuti (1990), Zhang and Freudenberg (1987) and Zhang and Freudenberg (1990). The problem of LTR is to design a measurement feedback controller such that the resulting loop transfer function is either exactly or approximately equal to a target loop transfer function $L(s)$ which meets the given specifications on sensitivity and complementary sensitivity functions. The original work of Doyle and Stein (1979) on LTR has two attributes: (1) it considered only left invertible and minimum phase plants, and (2) it used observer based measurement feedback controllers to recover a given target loop transfer function. In view of these two attributes, let us next review briefly the direction in which the research on LTR proceeded since the seminal work of Kwakernaak (1969) and Doyle and Stein (1979). As shown in Doyle and Stein (1979) as well as in Saberi and Sannuti (1990), for left invertible and minimum phase plants, any given target loop transfer function designed via a state feedback control is recoverable asymptotically by an observer based controller employing an asymptotically infinite gain. Although Doyle and Stein (1979) considered only full order observer based structures, subsequent work by Saberi and Sannuti (1990) revealed that a reduced order observer can be used in place of a full order observer. To relieve the designer from attribute (1), recent work, especially that of Niemann and Jannerup (1990), Zhang and Freudenberg (1987), Zhang and Freudenberg (1990), Saberi *et al.* (1991a) and Saberi *et al.* (1991b) concentrated on general nonminimum phase plants. It turns out that not all target loops can be recovered by observer based controllers for

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general plants. To understand thoroughly the mechanism of LTR, Saberi *et al.* (1991a) examined it in detail using observer based structures for controllers. The analysis there, while showing that neither exact LTR (ELTR) nor asymptotic LTR (ALTR) can in general be achieved, focused on three fundamental issues. The first issue was concerned with what can and what cannot be achieved for a given system and for an arbitrarily specified target loop transfer function, while the second issue was concerned with the development of necessary or/and sufficient conditions a target loop has to satisfy so that it can either exactly or asymptotically be recovered for the given system. The third issue dealt with the development of method(s) to test whether recovery is possible in a given subspace of the control space or not, i.e. to test whether projections of target and achievable sensitivity and complementary sensitivity functions onto a given subspace match each other or not. This analysis clearly shows that if there exists a target loop that can be recovered, it must satisfy some geometric conditions. However, Saberi *et al.* (1991a) considers only full order observer based structure for the controller, and it does not make any attempt to find out the conditions on the plant such that the plant has at least one target loop which is either exactly or asymptotically recoverable.

One of the goals of this paper is to generalize as well as to complement the results of Saberi *et al.* (1991a), that is (1) to characterize the set of recoverable target loops when the controller structure is general and is not necessarily observer based, and (2) to establish the necessary or/and sufficient conditions on the plant so that it has at least one recoverable target loop. In fact, given a general not necessarily minimum phase and not necessarily left invertible plant, we construct here an auxiliary system from it and show that the set of recoverable target loops for the given plant is nonempty if and only if the auxiliary system is stabilizable by a static output feedback controller. This then leads to a simple and surprising necessary condition on the given plant, namely, *strong stabilizability*† *of the given nonminimum phase plant is a necessary condition for the plant to have at least one recoverable target loop.* However, the fact that the given plant is strongly stabilizable itself does not guarantee that there exists at least one recoverable target loop. Moreover, we show that any recoverable target loop, whenever it exists, can always be recovered by a stable controller.

Regarding the structure of controllers used for LTR, the entire literature on LTR with the exception of Chen *et al.* (1991) and Niemann *et al.* (1991), uses only observer based controllers. A question then arises of whether there are any advantages if one uses any arbitrarily structured controller. (Here we mean by any arbitrarily structured controller, any stabilizing controller which does not have any specific structure such as observer based, and has any arbitrary finite dimension.) In this connection, we show that the set of recoverable target loops for a given plant cannot be expanded by using any arbitrarily structured controller instead of an observer based one. That is, the set of recoverable target loops obtainable via observer based controllers is the same one as that obtainable by any arbitrarily structured controller. However, it turns out that an observer based controller is not the best one in view of the required controller gain and band-width. Let us expand on this. As the conditions for ELTR are severe, one usually attempts to obtain ALTR whenever it can be done. ALTR invariably results in the use of high values of gain for the observer. However, use of high gain is not always practical as it brings with it the woes of high controller band-width and signal saturation. To liberate the designer from such woes, Chen *et al.* (1991) introduced earlier a compensator structure and studied the LTR for minimum phase and left invertible plants. The compensator structure introduced in Chen *et al.* (1991) has several distinct advantages over conventional observer based controllers of either full or reduced order type, i.e. the compensator is (a) open-loop stable, (b) guarantees closed-loop stability and above all (c) requires much smaller values of gain than the conventional observer based controller for the same degree of loop transfer recovery. The fact that the compensator requires a much smaller value of gain than the observer based controller, implies that the compensator band-width is much smaller than that of the conventional controller and thus one gains the freedom from the woes of saturation as well as insensitivity to noise or other high-frequency disturbances. We will show here that the compensator structure of Chen *et al.* (1991) along with its advantages is applicable for LTR of general nonminimum phase plants. In fact, we show that *any target loop recoverable by any arbitrarily structured controller can also be recovered via an open-loop stable compensator of either full order* (*dimension* n) *or reduced*

† A plant is said to be strongly stabilizable if there exists a stable and proper compensator which stabilizes the plant (Vidyasagar, 1985).

order (dimension n − p) *while accruing all the advantages quoted in* Chen *et al.* (1991).

Some of the conditions we develop here (see in particular Theorem 3.1) involve subspace inclusions. The subspaces involved are well known invariant subspaces of a linear system. Such subspaces can be constructed easily following the special coordinate basis of a linear system (Sannuti and Saberi, 1987). A software package has been developed to construct the special coordinate basis, and hence the required subspaces of a given linear system (Lin *et al.*, 1991).

The paper is organized as follows. Section 2 defines the LTR problem in precise terms. Section 3 deals with all the fundamental analysis. It develops the necessary and sufficient conditions such that the given nonminimum phase plant has at least one recoverable target loop. A clear interpretation of these conditions is also given. Also, we show here that the set of recoverable target loops obtainable by using observer based controllers cannot be expanded by using any arbitrarily structured controller. However, since one normally attempts to have asymptotic LTR, for a certain degree of recovery, one controller structure may have advantages over the other in view of the required controller gain and band-width. In Section 4, we advocate using the compensator structure of Chen *et al.* (1991) for a controller and show that it can also recover any recoverable target loop. We next move on to show the advantages of using the compensator structure for the controller over the conventional observer based controller structure. Numerical examples of Section 5 illustrate several aspects of the theory developed while Section 6 draws conclusions of our work.

Throughout the paper, A' denotes the transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. $\lambda(A)$ and $\mathrm{Re}\,[\lambda(A)]$, respectively denote the set of eigenvalues and real parts of eigenvalues of A. Similarly, $\sigma_{\max}[A]$ and $\sigma_{\min}[A]$, respectively denote the maximum and minimum singular values of A. $\mathrm{Ker}\,[V]$ and $\mathrm{Im}\,[V]$ denote, respectively, the kernel and the image of V. The open left and closed right half s-planes are, respectively, denoted by C^- and C^+. Also, $\mathcal{R}_p$ denotes the subring of all proper rational functions of s while the set of matrices of dimension $l \times q$ whose elements belong to $\mathcal{R}_p$ is denoted by $\mathcal{M}^{l\times q}(\mathcal{R}_p)$. Also, we define a geometric subspace $v^+(A, B, C)$ for the system $\Sigma(A, B, C)$, as the maximal subspace of $\Re^n$ which is $(A + BF)$-invariant and contained in $\mathrm{Ker}\,(C)$ such that the eigenvalues of $(A + BF) \mid v^+$ are contained in C^+ for some F.

2. PROBLEM STATEMENT

Let us consider a general nonminimum phase plant Σ,

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{2.1}$$

where the state vector $x \in \Re^n$, output vector $y \in \Re^p$ and input vector $u \in \Re^m$. Without loss of generality, assume that B and C are of maximal rank. Let us also assume that Σ is stabilizable and detectable. Let F be a full state feedback gain matrix such that (a) the closed-loop system is asymptotically stable, i.e. the eigenvalues of $A - BF$ lie in the left half s-plane, and (b) the open-loop transfer function when the loop is broken at the input point of the plant† meets the given frequency dependent specifications. The state feedback control is

$$u = -Fx, \tag{2.2}$$

and the loop transfer function evaluated when the loop is broken at the input point of the plant, the so called target loop transfer function, is

$$L(s) = F\Phi B, \tag{2.3}$$

where $\Phi = (sI - A)^{-1}$. Arriving at an appropriate value for F is concerned with the issue of loop shaping which is an engineering art and often includes the use of linear quadratic regulator (LQR) design in which the cost matrices are used as free design parameters to generate the target loop transfer function $L(s)$ and thus the desired sensitivity and complementary sensitivity functions. The next step of design is to recover the target loop using only a measurement feedback controller. This is the problem of loop transfer recovery (LTR) and is the focus of this paper.

To explain it clearly, consider the configuration of Fig. 1 where $C(s)$ and $P(s) = C\Phi B$ are, respectively, the transfer functions of a controller and of the given plant. Given $P(s)$ and a target loop transfer function $L(s)$, one seeks then to design a $C(s)$ such that the recovery error,

$$E(s) \equiv L(s) - C(s)P(s),$$

is either exactly or approximately equal to zero in the frequency region of interest while

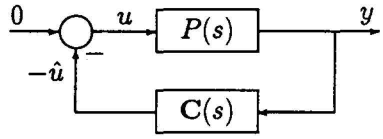

FIG. 1. Plant-controller closed-loop configuration.

† The loop can be broken at the output or at any other point. However, here without loss of generality, we assume that it is broken at the input point.

guaranteeing the stability of the resulting closed-loop system. Achieving exact LTR (ELTR) is in general not possible even for left invertible and minimum phase systems. One seeks then approximate LTR. The notion of "approximate" LTR has to be defined a little carefully. Here we seek achieving LTR to any arbitrarily desired accuracy. In an attempt to make this feasible, one normally parameterizes $C(s)$ as a function of a scalar parameter σ and thus obtains a family of controllers $C(s, \sigma)$. We say asymptotic LTR (ALTR) is achieved if $C(s, \sigma)P(s) \to L(s)$ pointwise in s as $\sigma \to \infty$, i.e. $E(s, \sigma) \to 0$ pointwise in s as $\sigma \to \infty$. Achievability of ALTR enables the designer to choose a member of the family of controllers that corresponds to a particular value of σ which achieves a desired level of recovery. Traditionally in observer based controllers, such a parameterization is done by adding a fictitious process noise of intensity proportional to σ which is injected into the system through the input into the plant. Then the observer gain is calculated by solving the resulting filter algebraic Riccati equations (AREs). In an asymptotic and time-scale structure assignment (ATEA) procedure of Saberi and Sannuti (1990) and Saberi *et al.* (1991b), appropriate parameterization of a controller assigns a chosen time-scale structure to the resulting closed-loop system. The relative fastness of fast time-scales is then adjusted as desired by tuning the parameter σ. We now consider the following definitions in order to impart precise meanings to ELTR and ALTR:

Definition 2.1. The set of admissible target loops $T(\Sigma)$ for the plant Σ is defined by

$$T(\Sigma) = \{L(s) \in \mathcal{M}^{m \times m}(\mathcal{R}_p) \mid L(s) = F\Phi B, \text{ and } \lambda(A - BF) \in C^-\}.$$

Definition 2.2. $L(s) \in T(\Sigma)$ is said to be exactly recoverable (ELTR) if there exists a $C(s) \in \mathcal{M}^{m \times p}(\mathcal{R}_p)$ such that (i) the closed-loop system comprising of $C(s)$ and $P(s)$ as in the configuration of Fig. 1 is asymptotically stable, and (ii) $C(s)P(s) = L(s)$.

Definition 2.3. $L(s) \in T(\Sigma)$ is said to be asymptotically recoverable (ALTR) if there exists a parameterized family of controllers $C(s, \sigma) \in \mathcal{M}^{m \times p}(\mathcal{R}_p)$, where σ is a scalar parameter taking positive values, such that (i) the closed-loop system comprising of $C(s, \sigma)$ and $P(s)$ as in the configuration of Fig. 1 is asymptotically stable for all $\sigma > \sigma^*$, where $0 \le \sigma^* < \infty$, and (ii) $C(s, \sigma)P(s) \to L(s)$ pointwise in s as $\sigma \to \infty$. Moreover, the limits, as $\sigma \to \infty$, of the finite eigenvalues of the closed-loop system should remain in C^-.†

Definition 2.4. $L(s)$ belonging to $T(\Sigma)$ is said to be recoverable if $L(s)$ is either exactly or asymptotically recoverable.

Definition 2.5. The set of recoverable target loops for the plant Σ is denoted by $T_{\mathfrak{R}}(\Sigma)$.

These definitions differ from the conventional ones of LTR as there is no prior structure assumed for either $C(s)$ or $C(s, \sigma)$.

It is well known that for left invertible and minimum phase plants, any arbitrary admissible target loop is asymptotically recoverable and hence $T_{\mathfrak{R}}(\Sigma)$ is equal to $T(\Sigma)$. On the other hand, if the given plant Σ is not left invertible or/and of nonminimum phase, not all target loops are recoverable, i.e. $T_{\mathfrak{R}}(\Sigma)$ is not equal to $T(\Sigma)$. In fact, $T_{\mathfrak{R}}(\Sigma)$ might be an empty set. As mentioned in the Introduction, this paper has two goals, (1) to characterize $T_{\mathfrak{R}}(\Sigma)$ and to examine it carefully in order to establish the necessary or/and sufficient conditions on the given plant Σ so that $T_{\mathfrak{R}}(\Sigma)$ is nonempty, and (2) to show that whenever a given target loop is recoverable, the open-loop compensator structure of Chen *et al.* (1991) for the controller can always be used to recover the target loop. The design of a compensator and the advantages of using a compensator over the conventional observer based controller are also studied both theoretically as well as numerically by means of examples.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR $T_{\mathfrak{R}}(\Sigma)$ TO BE NONEMPTY

This section deals with all the fundamental analysis on general nonminimum phase plants. At first, assuming that the set of recoverable target loops for Σ, namely $T_{\mathfrak{R}}(\Sigma)$, is nonempty, we establish a condition in terms of a geometric subspace for a given $L(s)$ to be an element of $T_{\mathfrak{R}}(\Sigma)$. This condition on the geometric subspace leads to an interesting result, namely, $T_{\mathfrak{R}}(\Sigma)$ is nonempty if and only if an auxiliary system, derived from the given plant, is stabilizable by a static output feedback controller. Next, it is shown that strong stabilizability of the given plant is a necessary condition for $T_{\mathfrak{R}}(\Sigma)$ to be nonempty. However, strong stabilizability of the given plant alone does not

† Here we have strengthened the notion of the closed-loop stability in order to exclude those cases having the limits, as $\sigma \to \infty$, of some finite eigenvalues of the closed-loop system being on the $j\omega$ axis. This avoids having an almost unstable behavior of the closed loop system for large σ.

guarantee that it has at least one recoverable target loop. We prove next that any recoverable target loop for a given system Σ can be recovered using an open-loop stable controller.

We first have the following result.

Theorem 3.1. Consider a stabilizable and detectable system Σ characterized by the triple (A, B, C), as in (2.1), which is not necessarily of minimum phase and which is not necessarily left invertible. Let $L(s)$ be any admissible target loop transfer function of Σ, i.e. $L(s) \in T(\Sigma)$, then $L(s)$ is recoverable, i.e. $L(s) \in T_{\Re}(\Sigma)$, if and only if $v^+(A, B, C) \subseteq \mathrm{Ker}(F)$.

Proof. It is well known that any proper internally stabilizing controller of the given plant can be parameterized as a full order observer plus an additional stable compensator $Q(s)$. The state space interpretation of such a controller can be written as follows,

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}),$$
$$\dot{x}_Q = A_Q x_Q + B_Q(y - C\hat{x}),$$
$$y_Q = C_Q x_Q + D_Q(y - C\hat{x}),$$
$$u = \hat{u} = -F\hat{x} - y_Q,$$

where F and K are any fixed matrices such that $A - BF$ and $A - KC$ are asymptotically stable while in our case F is chosen to be the one that specifies the target loop $F\Phi B$. Then it is shown in Niemann *et al.* (1991) that the recovery error $E_Q(s)$ for such a controller is given by

$$E_Q(s) = M_Q(s)[I_m + M_Q(s)]^{-1}(I_m + F\Phi B),$$

where

$$M_Q(s) = F(\Phi^{-1} + KC)^{-1}B - Q(s)C(\Phi^{-1} + KC)^{-1}B, \qquad (3.1)$$

and where

$$Q(s) = C_Q(sI - A_Q)^{-1}B_Q + D_Q.$$

It is now trivial to see that ALTR is achievable using the general internally stabilizing controller, if and only if $\sigma_{max}[M_Q(j\omega)]$ can be made arbitrarily small for all $\omega, 0 \leq \omega < \infty$. One can parameterize $Q(s)$ with a tuning parameter σ as $Q(s, \sigma)$ and consequently $E_Q(s)$ and $M_Q(s)$ are also parameterized as $E_Q(s, \sigma)$ and $M_Q(s, \sigma)$, respectively. Noting that the transfer function matrices $F(\Phi^{-1} + KC)^{-1}B$ and $C(\Phi^{-1} + KC)^{-1}B$ are fixed and generally nonzero, then the problem of ALTR is reduced to the problem of finding the conditions for the existence of $Q(s, \sigma)$ such that

$$M_Q(s, \sigma) \to 0 \text{ pointwise in } s \text{ as } \sigma \to \infty.$$

This problem can be reformulated as a H_∞ optimization problem. Consider the following auxiliary system Σ_{au},

$$\Sigma_{au}: \begin{cases} \dot{x} = (A - KC)'x + C'u + F'\omega, \\ y = 0x + I_m\omega, \\ z = B'x. \end{cases}$$

Also, let the output dynamic feedback be

$$u = -Q'(s, \sigma)y. \qquad (3.2)$$

Then it is simple to verify that the transfer function from the controlled output z to the disturbance w, denoted by $T_{zw}(s, \sigma)$, is in fact given by

$$T_{zw}(s, \sigma) = M'_Q(s, \sigma).$$

Also, note that the stability of $Q(s, \sigma)$ is necessary for the internal stability of the closed-loop system consisting of Σ_{au} and (3.2). Hence, the problem of finding the conditions for the solution of the original ALTR problem, i.e. the problem of finding the conditions for the existence of $Q(s, \sigma)$ such that

$$M_Q(s, \sigma) \to 0 \text{ pointwise in } s \text{ as } \sigma \to \infty,$$

can equivalently be formulated as the problem of finding the conditions under which the infimum of a H_∞ optimization problem for Σ_{au} is equal to zero, namely, as the problem of finding the conditions under which there exists a measurement output feedback controller as in (3.2) for Σ_{au} such that the H_∞-norm of the transfer function from the controlled output z to the disturbance w can be made arbitrarily small as $\sigma \to \infty$. It is shown in Khargonekar *et al.* (1988) that γ_s, the infimum H_∞-norm of the closed-loop transfer function from z to w under static state feedback is equal to the infimum H_∞-norm of the closed-loop transfer function from z to w under dynamic state feedback. Now, let us define γ_0 as the infimum of the H_∞-norm of the transfer function $T_{zw}(s, \sigma)$ under the dynamic output feedback. Then we have $\gamma_0 \geq \gamma_s$ since the output dynamic feedback is a subset of a full state dynamic feedback. However, it follows from the results of Fujita *et al.* (1990) that the achievement of any arbitrary static state feedback can be exactly recovered using an output dynamic feedback since the system characterized by the quadruple $(A' - C'K', F', 0, I_m)$ is square invertible and of minimum phase. Thus, $\gamma_0 = \gamma_s$ and the problem of finding an internally stabilizing $Q'(s, \sigma)$ such that the H_∞-norm of $T_{zw}(s, \sigma)$ is arbitrarily small is reduced to the problem of finding a state feedback gain $K'_s(\sigma)$ such that (i) $\mathrm{Re}[\lambda(A' - C'K'_s(\sigma) - C'K')] < 0$ for all $\sigma > \sigma_1^*$ for some

$\sigma_1^* \geq 0$, and (ii)

$$\|B'[sI_n - A' + C'K' + C'K_s'(\sigma)]^{-1}F'\|_\infty \to 0 \quad \text{as} \quad \sigma \to \infty,$$

which is equivalent to

$$F[\Phi^{-1} + (K_s(\sigma) + K)C]^{-1}B \to 0$$

pointwise in s as $\sigma \to \infty$.

Then, it follows from the results of Saberi *et al.* (1991a) that there exists such a gain $K_s(\sigma)$ satisfying the conditions (i) and (ii), if and only if $v^+(A, B, C) \subseteq \text{Ker}(F)$.

It is worth noting that in this case, the transfer function from z to w in the auxiliary system Σ_{au} under the state feedback gain $K_s'(\sigma)$, can be exactly recovered by using $Q(s, \sigma)$ as

$$Q(s, \sigma) = F[sI_n - A + (K_s(\sigma) + K)C]^{-1}K_s(\sigma).$$

Moreover, in this case, $M_Q(s, \sigma)$ of (3.1) can be simplified as follows,

$$\begin{aligned} M_Q(s, \sigma) &= F(\Phi^{-1} + KC)^{-1}B \\ &\quad - Q(s)C(\Phi^{-1} + KC)^{-1}B \\ &= F(\Phi^{-1} + KC)^{-1}B \\ &\quad - F[\Phi^{-1} + (K_s(\sigma) + K)C]^{-1} \\ &\quad \times K_s(\sigma)C(\Phi^{-1} + KC)^{-1}B. \end{aligned}$$

Now using the matrix identity,

$$\begin{aligned} &[\Phi^{-1} + KC + K_s(\sigma)C]^{-1}K_s(\sigma)C \\ &\quad = I_n - [\Phi^{-1} + (K_s(\sigma) + K)C]^{-1}(\Phi^{-1} + KC), \end{aligned}$$

we have

$$\begin{aligned} M_Q(s, \sigma) &= F(\Phi^{-1} + KC)^{-1}B \\ &\quad - F\{I_n - [\Phi^{-1} + (K_s(\sigma) + K)C]^{-1} \\ &\quad \times (\Phi^{-1} + KC)\}(\Phi^{-1} + KC)^{-1}B \\ &= F[\Phi^{-1} + (K_s(\sigma) + K)C]^{-1}B. \end{aligned}$$

Hence,

$$M_Q(s, \sigma) \to 0 \text{ pointwise in } s \text{ as } \sigma \to \infty.$$

This completes the proof of Theorem 3.1.

Let us note that a finite step algorithm to construct the geometric space $v^+(A, B, C)$ of any given linear system can easily be given via the special coordinate basis (Sannuti and Saberi, 1987). As such the necessary and sufficient condition of Theorem 3.1 can easily be used to come up with a simple and verifiable finite step algorithm which constructs the set of all recoverable target loops. Such a set obviously helps a designer at the onset of design to formulate a meaningful target loop transfer function.

Theorem 3.1 leads to the next theorem which says that any $L(s)$ that is recoverable, can be recovered by using only a controller having an observer based structure which could either be full or reduced order type depending on the designer's choice. In other words, $T_{\Re}(\Sigma)$, the set of all recoverable target loops without any restriction on the type of controller one uses, is the same one as the set of all recoverable target loops obtainable by using only either full or reduced order type of observer based controllers.

Theorem 3.2. Consider a stabilizable and detectable system Σ characterized by the triple (A, B, C), as in (2.1), which is not necessarily of minimum phase and which is not necessarily left invertible. Let $L(s)$ be any recoverable target loop transfer function of Σ, i.e. $L(s) \in T_{\Re}(\Sigma)$, then $L(s)$ can be recovered by both a full and a reduced order observer based controller.

Proof. It is proven in Theorem 3.1 that any recoverable target loop of Σ must satisfy the condition $v^+(A, B, C) \subseteq \text{Ker}(F)$. Then it follows from the results of Saberi *et al.* (1991a) and Saberi *et al.* (1990) that any target loop which satisfies this geometric condition, i.e. $v^+(A, B, C) \subseteq \text{Ker}(F)$, can be recovered by both full and reduced order type of observer based controllers.

Theorem 3.1 characterizes a recoverable target loop $L(s)$ in terms of a geometric subspace of the given system Σ. Perhaps, a more fundamental question that one needs to answer at this stage is "What are the necessary or/and sufficient conditions on the system Σ so that Σ has at least one recoverable target loop?". This is pursued in Theorem 3.3.

Theorem 3.3. Consider a stabilizable and detectable system Σ characterized by the triple (A, B, C), as in (2.1), which is not necessarily of minimum phase and which is not necessarily left invertible. Let n_0 be the dimension of $v^+(A, B, C)$. Also, let $\bar{C}$ be any full rank matrix of dimension $(n - n_0) \times n$ such that $\text{Ker}(\bar{C}) = v^+(A, B, C)$. Then the given system Σ has at least one recoverable target loop, i.e. $T_{\Re}(\Sigma)$ is nonempty, if and only if an auxiliary system Σ_a characterized by the matrix triple $(A, B, \bar{C})$ is stabilizable by a static output feedback controller.

Proof. It follows from Theorem 3.1 that any admissible target loop $L(s) = F\Phi B$ is recoverable iff $v^+(A, B, C) \subseteq \text{Ker}(F)$. Hence, $\text{Ker}(\bar{C}) = v^+(A, B, C)$ implies $\text{Ker}(\bar{C}) \subseteq \text{Ker}(F)$ and $\text{Im}(F') \subseteq \text{Im}(\bar{C}')$. Then we have

$F = G\bar{C}$ for some constant matrix G. It is trivial to verify that the existence of a recoverable target loop $L(s) = F\Phi B$ such that $A - BF$ is asymptotically stable is equivalent to the existence of a matrix G such that $A - BG\bar{C}$ is asymptotically stable, i.e. the matrix triple $(A, B, \bar{C})$ is stabilizable by a static output feedback law. This completes the proof of Theorem 3.3.

Theorem 3.3 gives necessary and sufficient conditions under which the set of recoverable target loops, $T_{\Re}(\Sigma)$, is nonempty. However, the condition given there is not conducive to any intuitive feelings. The following corollary gives a necessary condition which is surprising, as well as intuitively appealing.

Corollary 3.1. The strong stabilizability of a plant Σ is a necessary condition for it to have at least one recoverable target loop.

Proof. It is well known that any stabilizable and detectable system Σ can be stabilized by using an observer based controller. Now if the auxiliary system Σ_a is stabilizable by a static output feedback controller, then there exists a gain F such that $A - BF$ is asymptotically stable and $v^+(A, B, C) \subseteq \text{Ker}(F)$. Moreover, it follows from the results of Saberi *et al.* (1991a) that there exists an observer gain, $K(\sigma)$, such that $A - K(\sigma)C$ is asymptotically stable and

$$M(s, \sigma) = F[\Phi^{-1} + K(\sigma)C]^{-1}B \to 0 \quad \text{pointwise in } s \text{ as } \sigma \to \infty.$$

Next, considering a stabilizing full order observer based controller,

$$C(s, \sigma) = F[sI_n - A + K(\sigma)C + BF]^{-1}K(\sigma),$$

we examine its eigenvalues as $\sigma \to \infty$. We have

$$\begin{aligned}
\det[sI_n - A + K(\sigma)C + BF] &= \det[\Phi^{-1} + K(\sigma)C] \\
&\quad \times \det[I_n + (\Phi^{-1} + K(\sigma)C)^{-1}BF] \\
&= \det[\Phi^{-1} + K(\sigma)C] \\
&\quad \times \det[I_m + F(\Phi^{-1} + K(\sigma)C)^{-1}B] \\
&= \det[\Phi^{-1} + K(\sigma)C] \\
&\quad \times \det[I_m + M(s, \sigma)] \\
&\to \det[sI_n - A + K(\sigma)C] \quad \text{as } \sigma \to \infty.
\end{aligned}$$

Thus the full order observer based controller is open-loop stable for sufficiently large σ. We conclude then that the given system Σ is strongly stabilizable as it can be stabilized by an open-loop stable controller.

Corollary 3.1 tells us that any given system Σ must be strongly stabilizable in order to have at least one recoverable target loop. On the other hand, as seen from Theorem 3.3, strong stabilizability of Σ alone is not sufficient for $T_{\Re}(\Sigma)$ to be nonempty. The following example illustrates this.

Example 1. Consider a system Σ characterized by

$$\dot{x} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = [0 \quad 1]x,$$

which is of nonminimum phase with an invariant zero at $s = 1$. It is simple to verify that the given system is strongly stabilizable and $v^+(A, B, C)$ is spanned by $[1, 0]'$. Hence, it follows from Theorem 3.1 that any recoverable target loop $F\Phi B$ must have the following form of F,

$$F = [0 \quad \alpha],$$

where α is any constant. It is now straightforward to verify that

$$\det[sI_n - A + BF] = s^2 + (\alpha + 1)s - (\alpha + 1),$$

which cannot be a Hurwitz polynomial for any α. Hence, this system has no recoverable target loop although it is strongly stabilizable.

The fact that a nonminimum phase system must at least be strongly stabilizable for $T_{\Re}(\Sigma)$ to be nonempty, raises the question whether it is possible to recover any recoverable target loop of Σ by using an open-loop stable controller. The following lemma answers the question affirmatively.

Lemma 3.1. Any recoverable target loop for a given system Σ can be recovered using an open-loop stable controller.

Proof. In view of Theorem 3.3, we note that one can recover any recoverable target loop using only observer based controllers. Now, it can be easily seen, from the proof of Corollary 3.1, that such a controller is also asymptotically stable for sufficiently large σ. Hence the result of Lemma 3.1.

4. A STABLE COMPENSATOR DESIGN FOR LTR

In the previous section, we showed that the set of recoverable target loops obtainable by using only an observer based controller is the same one as that obtainable by using any arbitrarily structured controller. Then one wonders whether any advantages can be gained by using an arbitrarily structured controller instead of an observer based controller. To investigate along these lines, let us first note that since ELTR in general is not possible even for left invertible and minimum phase systems, one seeks only to achieve ALTR whenever it can be done. Thus we focus in this section only on ALTR. ALTR invariably results in the use of high gain for the observer. Use of high gain is not always practical as it brings with it the woes of high controller band-width and signal saturation. To liberate the designer from such woes, Chen *et al.* (1991) introduced earlier a compensator structure and studied the LTR for minimum phase and left invertible plants. The compensator structure introduced in Chen *et al.* (1991) has several distinct advantages over conventional observer based controllers of either full or reduced order type, i.e. the compensator is (a) open-loop stable, (b) guarantees closed-loop stability and above all (c) requires much smaller values of gain than the conventional observer based controller for the same degree of loop transfer recovery. The fact that the compensator requires much smaller value of gain than the observer based controller implies that the compensator band-width is much smaller than that of the conventional controller and thus one gains some freedom from the woes of saturation as well as insensitivity to noise or other high-frequency disturbances. The intention of this section is to show that the compensator structure of Chen *et al.* (1991) along with its advantages is applicable for LTR of general nonminimum phase plants. As is true for the observer based controllers, we show that any recoverable target loop can be recovered by using the compensator structure for the controller. We will next compare the compensator structure with that of the conventional observer based structure, and show that for the same value of gain, compensator has a much better recovery than the observer has. We will also establish bounds on the sensitivity and complementary sensitivity functions for both observer based and compensator based structures. These bounds again ascertain that the compensator structure has significant advantages over the observer based structure. In short, the message we want to convey in this section is that whenever one has a recoverable target loop, the use of compensator structure of Chen *et al.* (1991) for the controller is much preferable over a conventional observer based structure. Of course, whenever the given target loop is not recoverable, one can perhaps recover it partially. Determination of advantages of one controller structure over another for partial recovery, is still an open research problem and is a topic of our future research.

Analogous to observer based controllers, there are two compensator structures, one full order type of dynamic order n and another reduced order type of dynamic order $n-p$. First, let us recall the full order compensator having the transfer function,

$$C_c(s,\sigma)=F[\Phi^{-1}+K(\sigma)C]^{-1}K(\sigma). \quad (4.1)$$

In the parameterized family of controllers given in (4.1), the only free design variable is the parameterized gain $K(\sigma)$. We need to parameterize $K(\sigma)$ in such a way that there exists a σ_1^* so that for all $\sigma>\sigma_1^*$, the controller $C_c(s,\sigma)$ is open-loop stable while capable of achieving ALTR. That is, the design of $K(\sigma)$ is to be done to meet the following goals:

(1) (Stability of the closed-loop system.) The closed-loop system comprising the given system and the full order compensator is asymptotically stable, i.e. there exists a σ_2^* such that for all $\sigma>\sigma_2^*$, we have

$$\mathrm{Re}\,[\lambda(A_{cl}(\sigma))]<0,$$

where

$$A_{cl}(\sigma)=\begin{bmatrix}A-K(\sigma)C & K(\sigma)C\\ -BF & A\end{bmatrix}. \quad (4.2)$$

Moreover, the limits of all finite eigenvalues of $A_{cl}(\sigma)$ remain in C^-.

(2) (Recovery.) The achieved loop transfer function $L_c(j\omega,\sigma)$,

$$L_c(j\omega,\sigma)=C_c(j\omega,\sigma)P(j\omega),$$

is asymptotically equal to the target loop $L(j\omega)$ as $\sigma\to\infty$, i.e. $C_c(j\omega,\sigma)P(j\omega)\to L(j\omega)$ pointwise in ω as $\sigma\to\infty$.

(3) (Open-loop stability of the compensator.) The compensator is open-loop asymptotically stable, i.e. there exists a σ_1^* such that for all $\sigma>\sigma_1^*$, we have

$$\mathrm{Re}\,[\lambda(A-K(\sigma)C)]<0.$$

Next, let us recall a reduced order compensator. Without loss of generality, let us assume that

$$C=[I_p,0],$$

and hence the plant (2.1) can be written in the form,

$$\dot{x}_1=A_{11}x_1+A_{12}x_2+B_1u,\qquad \dot{x}_2=A_{21}x_1+A_{22}x_2+B_2u,\quad y=x_1. \quad (4.3)$$

Also, let the state feedback gain matrix F which achieves the target loop transfer function $L(s)$ be partitioned in conformity with (4.3) as

$$F=[F_1,F_2].$$

The transfer function of the reduced order compensator is given by,

$$C_{cr}(s,\sigma) = F_2[sI_{n-p} - A_r(\sigma)]^{-1} \times G_r(\sigma) + F_1 + F_2K_r(\sigma), \quad (4.4)$$

where

$$A_r(\sigma) = A_{22} - K_r(\sigma)A_{12},$$

and

$$G_r(\sigma) = A_{22}K_r(\sigma) - K_r(\sigma)A_{12}K_r(\sigma) + A_{21} - K_r(\sigma)A_{11}.$$

In the parameterized family of controllers given in (4.4), the only free design variable is the parameterized gain $K_r(\sigma)$. We need to parameterize $K_r(\sigma)$ in such a way that there exists a σ_{1r}^* so that for all $\sigma > \sigma_{1r}^*$, the controller $C_{cr}(s, \sigma)$ is open-loop stable while capable of achieving ALTR. That is, the design of $K_r(\sigma)$ is to be done to meet the following goals:

(1) (Stability of the closed-loop system.) The closed-loop system comprising the given system and the reduced order compensator is asymptotically stable, i.e. there exists a σ_{2r}^* such that for all $\sigma > \sigma_{2r}^*$, we have

$$\mathrm{Re}\,[\lambda(A_{clr}(\sigma))] < 0,$$

where

$$A_{clr}(\sigma) = \begin{bmatrix} A_{22} - K_r(\sigma)A_{12} - K_r(\sigma)B_1F_2 & A_{21} - K_r(\sigma)B_1F_1 & K_r(\sigma)A_{12} \\ -B_1F_2 & A_{11} - B_1F_1 & A_{12} \\ -B_2F_2 & A_{21} - B_2F_1 & A_{22} \end{bmatrix}.$$

Moreover, the limits of all finite eigenvalues of $A_{clr}(\sigma)$ remain in C^-.

(2) (Recovery.) The achieved loop transfer function $L_{cr}(j\omega, \sigma)$,

$$L_{cr}(j\omega, \sigma) = C_{cr}(j\omega, \sigma)P(j\omega),$$

is asymptotically equal to the target loop $L(j\omega)$ as $\sigma \to \infty$, i.e. $C_{cr}(j\omega, \sigma)P(j\omega) \to L(j\omega)$ pointwise in ω as $\sigma \to \infty$.

(3) (Open-loop stability of the compensator.) The compensator is open-loop asymptotically stable, i.e. there exists a σ_{1r}^* such that for all $\sigma > \sigma_{1r}^*$, we have

$$\mathrm{Re}\,[\lambda(A_r(\sigma))] < 0.$$

The following lemma characterizes the recovery error between the target and the achieved loop transfer functions.

Lemma 4.1. The error $E_c(s, \sigma)$ between the target loop transfer function $L(s)$ and $L_c(s, \sigma)$, the one realized by the full order compensator, is given by

$$E_c(s, \sigma) = M(s, \sigma),$$

where

$$M(s, \sigma) = F[\Phi^{-1} + K(\sigma)C]^{-1}B. \quad (4.5)$$

Similarly, the error $E_{cr}(s, \sigma)$ between the target loop transfer function $L(s)$ and $L_{cr}(s, \sigma)$, the one realized by the reduced order compensator, is given by

$$E_{cr}(s, \sigma) = M_r(s, \sigma),$$

where

$$M_r(s, \sigma) = F_2[\Phi_{22}^{-1} + K_r(\sigma)A_{12}]^{-1}[B_2 - K_r(\sigma)B_1], \quad (4.6)$$

and where $\Phi_{22} = (sI_{n-p} - A_{22})^{-1}$.

Proof. The proof of this lemma for general nonminimum phase systems is exactly the same as that of a similar result for left invertible and minimum phase systems given in Chen *et al.* (1991).

Theorem 4.1. Consider a stabilizable and detectable system Σ characterized by the triple (A, B, C), as in (2.1), which is not necessarily of minimum phase and which is not necessarily left invertible. Let $L(s)$ be any recoverable target loop transfer function of Σ, i.e. $L(s) \in T_{\Re}(\Sigma)$, then $L(s)$ can be recovered via either a full or a reduced order type of compensator.

Proof. It is shown in Saberi *et al.* (1991a) and Saberi *et al.* (1990) that whenever a target loop is recoverable, there exist gain matrices $K(\sigma)$ and $K_r(\sigma)$ such that $A - K(\sigma)C$ and $A_{22} - K_r(\sigma)A_{12}$ are stable matrices for all $\sigma > \sigma^*$, where $0 \leq \sigma^* < \infty$ while the limits of the finite eigenvalues of these matrices belong to C^-. Also, $K(\sigma)$ and $K_r(\sigma)$ guarantee that

$$E_c(s, \sigma) = M(s, \sigma) \to 0 \text{ pointwise in } s \text{ as } \sigma \to \infty, \quad (4.7)$$

and

$$E_{cr}(s, \sigma) = M_r(s, \sigma) \to 0 \quad \text{pointwise in } s \text{ as } \sigma \to \infty. \quad (4.8)$$

Hence, such $K(\sigma)$ and $K_r(\sigma)$ achieve loop transfer recovery and yield open-loop stability of either a full or a reduced order type of compensator. Next, the stability of the closed-loop systems comprising the given system and the compensator, either full or reduced order, can be proven following the same line as in Theorems 5 and 10 of Chen *et al.* (1991).

Remark 4.1. We point out that the exact structure and values of $K(\sigma)$ and $K_r(\sigma)$ required to satisfy, respectively, (4.7) and (4.8), and to render $A - K(\sigma)C$ and $A_{22} - K_r(\sigma)A_{12}$ stable, are not in general unique. Also, there are two methods of design available to find $K(\sigma)$ and $K_r(\sigma)$. The classical method introduced by Doyle and Stein (1979) is based on solving parameterized algebraic Riccati equations (AREs). Recently, a method based on asymptotic time-scale and eigenstructure assignment (ATEA) has been introduced by Saberi and Sannuti (1990). For detailed comparison of these two methods, see Saberi et al. (1991b).

We should note that in the proof of Theorem 4.1, no explicit requirement is made on the given plant. The only condition we need is that the given target loop transfer function $F\Phi B$ is recoverable. In fact, this is also true of all the results of Chen et al. (1991) although there we considered only left invertible and minimum phase systems. Hence, all of the following results follow from Chen et al. (1991) even though the given system we are considering now is not necessarily left invertible and not necessarily of minimum phase.

We now pursue the advantages of the compensator structure over the conventional observer based structure. For an appropriate comparison, we recall next the following lemma which is analogous to Lemma 4.1, however it deals with observer based controllers.

Lemma 4.2. The error $E_o(s, \sigma)$ between the target loop transfer function $L(s)$ and $L_o(s, \sigma)$, the one realized by the full order observer based controller, is given by

$$E_o(s, \sigma) = M(s, \sigma)[I_m + M(s, \sigma)]^{-1}(I_m + F\Phi B),$$

where $M(s, \sigma)$ is as in (4.5). Similarly, the error $E_{or}(s, \sigma)$ between the target loop transfer function $L(s)$ and $L_{or}(s, \sigma)$, the one realized by the reduced order observer based controller, is given by

$$E_{or}(s, \sigma) = M_r(s, \sigma)[I_m + M_r(s, \sigma)]^{-1}(I_m + F\Phi B),$$

where $M_r(s, \sigma)$ is as in (4.6).

Proof. See Saberi et al. (1990, 1991a).

We are now ready to show the advantages of compensator structure. We have the following theorem.

Theorem 4.2. Consider a general stabilizable and detectable nonminimum phase plant. Assume that the same gain $K(\sigma)$ is used for both the full order observer based controller and the full order compensator. Let σ be such that $\sigma_{max}[M(j\omega, \sigma)]$ is small (say, $\ll 1$ but nonzero) for all ω. Furthermore, assume that

$$\sigma_{min}[L(j\omega)] = \sigma_{min}[F(j\omega - A)^{-1}B] \gg 1 \quad \text{for all} \quad \omega \in D_c, \qquad (4.9)$$

for some frequency region of interest, D_c. Then for all $\omega \in D_c$, the mismatch between the target loop transfer function and the one achieved by the full order compensator is always less than the corresponding one achieved by the full order observer based controller. More specifically, we have

$$\sigma_{max}[E_o(j\omega, \sigma)] \gg \sigma_{max}[E_c(j\omega, \sigma)] \quad \text{for all} \quad \omega \in D_c. \qquad (4.10)$$

Similarly, assume that the same gain $K_r(\sigma)$ is used for both the reduced order observer based controller and the reduced order compensator. Let σ be such that $\sigma_{max}[M_r(j\omega, \sigma)]$ is small (say, $\ll 1$ but nonzero) for all ω. Furthermore, assume that (4.9) is true. Then for all $\omega \in D_c$, the mismatch between the target loop transfer function and the one achieved by the reduced order compensator is always less than the corresponding one achieved by the reduced order observer based controller. More specifically, we have

$$\sigma_{max}[E_{or}(j\omega, \sigma)] \gg \sigma_{max}[E_{cr}(j\omega, \sigma)] \quad \text{for all} \quad \omega \in D_c. \qquad (4.11)$$

Proof. The proof follows along the same lines as that of a similar result for left invertible and minimum phase systems given in Chen et al. (1991).

Remark 4.2. It is well known (Doyle and Stein, 1981) that in order to have good command following and disturbance rejection properties, the target loop transfer function $L(j\omega)$ has to be large and consequently, the minimum singular value $\sigma_{min}[L(j\omega)]$ should be large in the appropriate frequency region. Thus the condition (4.9) is always satisfied in all practical situations.

Remark 4.3. Due to the sign $\gg$ in (4.10) and (4.11), Theorem 4.2 clearly shows that the compensator structure requires much smaller value of gain and hence the controller bandwidth than that of the observer based structure for the same degree of recovery.

We move on next to compare the sensitivity and complementary sensitivity functions achievable by full and reduced order compensators, with those achievable by full and reduced order observer based controllers. Let $S_F(s)$ and $T_F(s)$

be the sensitivity and complementary sensitivity functions corresponding to the target loop transfer function $L(s)$,

$$S_F(s) = [I_m + L(s)]^{-1} \quad \text{and} \quad T_F(s) = I_m - S_F(s).$$

Similarly, let us define the sensitivity and complementary sensitivity functions achievable by a particular controller as,

$$S_*(s, \sigma) = [I_m + L_*(s, \sigma)]^{-1},$$

and

$$T_*(s, \sigma) = I_m - S_*(s, \sigma),$$

where $L_*(s, \sigma)$ is the correspondingly obtained loop transfer function. Here, the subscript * will be replaced, respectively, by c, cr, o and or, when the controller used is full order compensator, reduced order compensator, full order observer based and reduced order observer based.

We have the following result.

Theorem 4.3. Consider a general, stabilizable and detectable, nonminimum phase plant. Assume that the same gain $K(\sigma)$ is used for both the full order observer based controller and the full order compensator. Let σ be such that $\sigma_{max}[M(j\omega, \sigma)]$ is small (say, $\ll 1$ but nonzero) for all ω. Furthermore, assume that (4.9) is true. Then for all $\omega \in D_c$, we have

$$\sigma_{max}[S_o(j\omega, \sigma) - S_F(j\omega)] \gg \sigma_{max}[S_c(j\omega, \sigma) - S_F(j\omega)], \quad (4.12)$$

and

$$\sigma_{max}[T_o(j\omega, \sigma) - T_F(j\omega)] \gg \sigma_{max}[T_c(j\omega, \sigma) - T_F(j\omega)]. \quad (4.13)$$

Similarly, assume that the same gain $K_r(\sigma)$ is used for both the reduced order observer based controller and the reduced order compensator. Let σ be such that $\sigma_{max}[M_r(j\omega, \sigma)]$ is small (say, $\ll 1$ but nonzero) for all ω. Furthermore, assume that (4.9) is true. Then for all $\omega \in D_c$, we have

$$\sigma_{max}[S_{or}(j\omega, \sigma) - S_F(j\omega)] \gg \sigma_{max}[S_{cr}(j\omega, \sigma) - S_F(j\omega)], \quad (4.14)$$

and

$$\sigma_{max}[T_{or}(j\omega, \sigma) - T_F(j\omega)] \gg \sigma_{max}[T_{cr}(j\omega, \sigma) - T_F(j\omega)]. \quad (4.15)$$

Proof. We first note the following:

$$\begin{aligned} I_m + L_o(s, \sigma) &= I_m + L(s) - E_o(s, \sigma) \\ &= I_m + L(s) - M(s, \sigma) \\ &\quad \times [I_m + M(s, \sigma)]^{-1}[I_m + L(s)] \\ &= [I_m + M(s, \sigma)]^{-1}[I_m + L(s)], \end{aligned}$$

and hence

$$S_o(s, \sigma) - S_F(s) = S_F(s)M(s, \sigma). \quad (4.16)$$

Similarly, we note that

$$\begin{aligned} I_m + L_c(s, \sigma) &= I_m + L(s) - M(s, \sigma) \\ &= \{I_m - M(s, \sigma)[I_m + L(s)]^{-1}\} \\ &\quad \times [I_m + L(s)], \end{aligned}$$

and hence

$$S_c(s, \sigma) - S_F(s) = S_F(s)M(s, \sigma)[I_m + L(s) - M(s, \sigma)]^{-1}. \quad (4.17)$$

From (4.16) and (4.17), we obtain

$$S_o(s, \sigma) - S_F(s) = [S_c(s, \sigma) - S_F(s)][I_m + L(s) - M(s, \sigma)].$$

Now it is simple to see that under the assumptions of Theorem 4.3,

$$\sigma_{max}[S_o(j\omega, \sigma) - S_F(j\omega)] \gg \sigma_{max}[S_c(j\omega, \sigma) - S_F(j\omega)], \quad \forall \omega \in D_c.$$

This proves (4.12). Also, (4.13) to (4.15) follow along similar arguments.

The above theorem shows once again that the compensator structure is much better than the conventional observer based structure.

5. EXAMPLES

In what follows we consider two examples to illustrate the theoretical results of Sections 3 and 4. In these two examples, we illustrate the advantages of the compensator structure in two different ways. At first, we select the same gain $K(\sigma)$ or $K_r(\sigma)$ for both the observer based structure and the compensator structure, and then for several values of σ, we compare the performance of these two controller structures by plotting with respect to frequency for a given frequency range, (i) the target and achieved loop transfer functions and (ii) the maximum singular value of the recovery error. In another type of comparison, we fix, *a priori*, the required degree of recovery by specifying a highest tolerable value for the maximum singular value of the recovery error. Then, we obtain for both the controller structures the norm of the gain which meets the given specification. We also obtain the resulting 0 dB bandwidth as well as the eigenvalues of the controller. The comparisons by both the methods show explicitly that the compensator structure for the controller has much better recovery properties than the observer based structure.

Example 2. Consider the example given in

Zhang and Freudenberg (1990),

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.2 \end{bmatrix} x + \begin{bmatrix} -0.5 & 1.25 \\ -2.5 & -2.5 \\ 0.3 & -1.25 \\ 1.5 & 3.5 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x,$$

which is square, invertible and of nonminimum phase with two invariant zeros at $s = -8$ and $s = 1$. The geometric subspace $v^+(A, B, C)$ for this example is the span of $[-0.138675\ -0.693375\ 0.138675\ 0.693375]'$. Now let a target loop $L(s)$ be specified by the gain matrix,

$$F = \begin{bmatrix} -16.8910 & 0.5782 & -19.1586 & 1.0317 \\ -290.0338 & 7.0068 & -295.0560 & 8.0112 \end{bmatrix}.$$

It is straightforward to verify that $L(s) \in T_{\Re}(\Sigma)$, i.e. $L(s)$ is recoverable. Here, we used the ATEA algorithm of Saberi *et al.* (1991b) to obtain the following gain $K(\sigma)$,

$$K(\sigma) = \begin{bmatrix} 3.75(1-\sigma) & 0.75\sigma - 1.25 \\ 25(\sigma - 1) & 2.5(1 - 2\sigma) \\ 0.95(5\sigma - 1) & 0.25(1 - 3\sigma) \\ 5(1 - 5\sigma) & 6\sigma - 0.7 \end{bmatrix}.$$

This gain places the eigenvalues of $A - K(\sigma)C$, one precisely at -8 and others asymptotically at -1, $-\sigma$ and $-\sigma$. Figure 2 (A) and Table 1 (A) give the maximum and minimum singular values of the target loop transfer function as well as those of the two recovered loop transfer functions, one for the full order compensator and another for the full order observer based controller, for several values of the tuning parameter σ. On the other hand, for the same degree of recovery, Fig. 2 (B) and Table 1 (B) show (1) the maximum singular value graphs of the two different controller transfer functions, and (2) the required values of gains and the eigenvalues of the controllers. Again, these numerical results show clearly that the compensator structure has much better recovery properties than the observer based controller.

Example 3. Consider the following system Σ characterized by

$$\dot{x} = \begin{bmatrix} -25 & -25 & 1 & -25 \\ 0 & 0 & 0 & 1 \\ -6 & 1 & 0.3 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x,$$

which is square and invertible with one nonminimum phase invariant zero at $s = 0.3$. The geometric subspace $v^+(A, B, C)$ for this example is the span of $[0\ 0\ 1\ 0]'$. Now let a target loop, $L(s) = F\Phi B$, be specified by the following gain matrix,

$$F = \begin{bmatrix} -13 & 50 & 0 & 10 \\ 11 & 250 & 0 & 50 \end{bmatrix}.$$

It is trivial to see that $v^+(A, B, C) \subseteq \mathrm{Ker}\,(F)$ and hence $L(s) \in T_{\Re}(\Sigma)$, i.e. $L(s)$ is recoverable. Here, we used ATEA algorithm of Saberi *et al.* (1990) to obtain the following gain matrix,

$$K_r(\sigma) = \begin{bmatrix} 2.3 & 0 \\ 0 & \sigma \end{bmatrix},$$

for both the reduced order observer based controller and the reduced order stable compensator. This gain matrix $K_r(\sigma)$ places the eigenvalues of $A_{22} - K_r(\sigma)A_{12}$ precisely at -2 and $-(\sigma + 2)$. Figure 3 (A) and Table 2 (A) give the maximum and minimum singular values of the target loop transfer function as well as those of the two recovered loop transfer functions, one for the reduced order compensator and another for the reduced order observer based controller, for several values of the tuning parameter σ. Once again, these numerical results show clearly that the compensator structure has much better recovery properties than the observer based controller.

6. CONCLUSIONS

In this paper, we first focused our attention on theoretical analysis and characterized the set of recoverable target loops for a given plant. We showed that the set of recoverable target loops for any stabilizable and detectable nonminimum phase system using an observer based controller is the same one as that obtainable using any arbitrarily structured controller. Moreover, we established the necessary and sufficient conditions on the plant such that it has at least one recoverable target loop. That is, the set of recoverable target loops for a given plant is nonempty if and only if a particular auxiliary system constructed from the given plant is stabilizable by a static output feedback controller. This result leads to a surprising necessary condition, namely the strong stabilizability of the given plant is a necessary condition for the plant to have at least one recoverable target loop. However, strong stabilizability of the given plant alone does not imply that the plant has at least one recoverable target loop. This is illustrated by an example. We also proved that

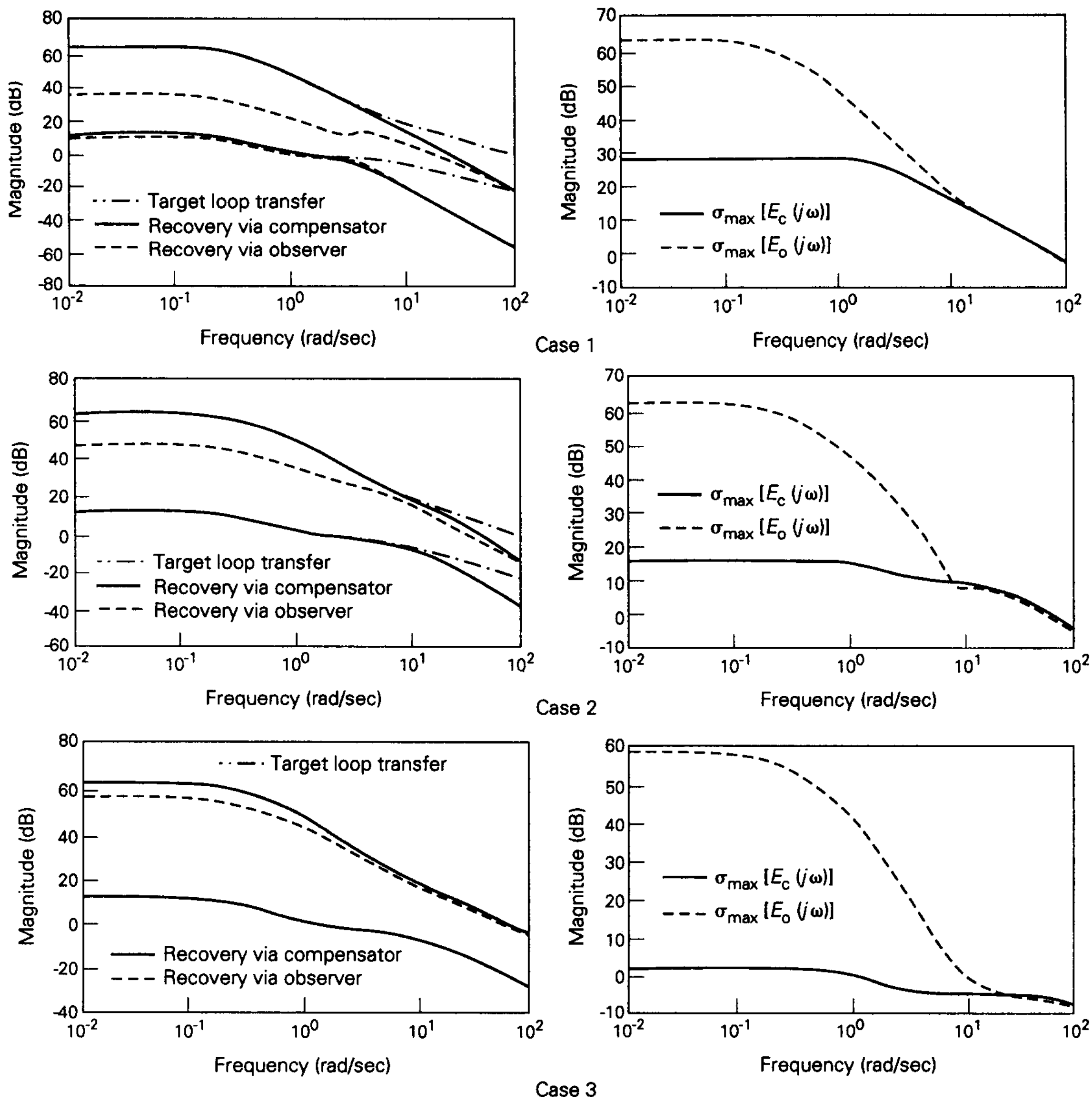

Fig. 2 (A)

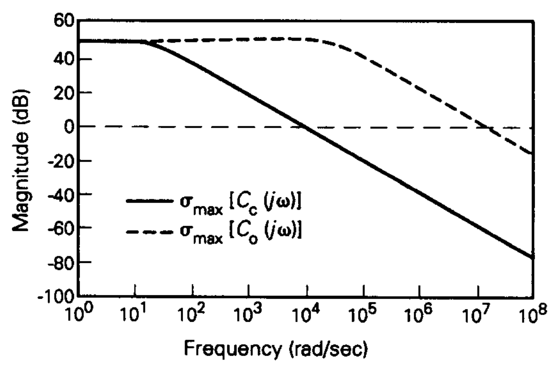

Fig. 2 (B). Maximum singular values of observer based controller and stable compensator.

TABLE 1 (A). SUPREMUM OF MAXIMUM SINGULAR VALUES OF MISMATCH FUNCTIONS, FREQUENCY RANGE: 0.01–100 RAD/SEC

Tuning parameter		$\sup\{\sigma_{max}[E_0(j\omega)]\}$	$\sup\{\sigma_{max}[E_c(j\omega)]\}$
Case 1	$\sigma = 5$	1570.7	25.9476
Case 2	$\sigma = 20$	1397.7	6.4868
Case 3	$\sigma = 100$	876.3	1.2975

TABLE 1 (B). COMPARISON OF OBSERVER BASED CONTROLLER VS STABLE COMPENSATOR FOR THE SAME DEGREE OF RECOVERY

Degree of recovery
$\sup\{\sigma_{max}[E_0(j\omega)]\} \cong \sup\{\sigma_{max}[E_c(j\omega)]\} \cong 6.4868$
for $0.01 \le \omega \le \infty$ rad/sec

	Observer based controller	Stable compensator
Gain norm-2:	1.0844×10^6	713.11
Eigenvalues:	−29563	−20
	−29522	−17.8815
	−8	−8
	−1.0034	−1.1185
(0 dB) Band-width:	1.0092×10^7 rad/sec	7794 rad/sec

any recoverable target loop can be recovered by using an open-loop stable controller.

The second aspect of the paper deals with the design of practical controllers. Although observer based controllers can recover all recoverable target loops, in connection with ALTR which is the goal in practice, there are some inherent problems in using observer based controllers. More specifically, observer based controllers require high values of gain. The use of high gain brings with it the problems

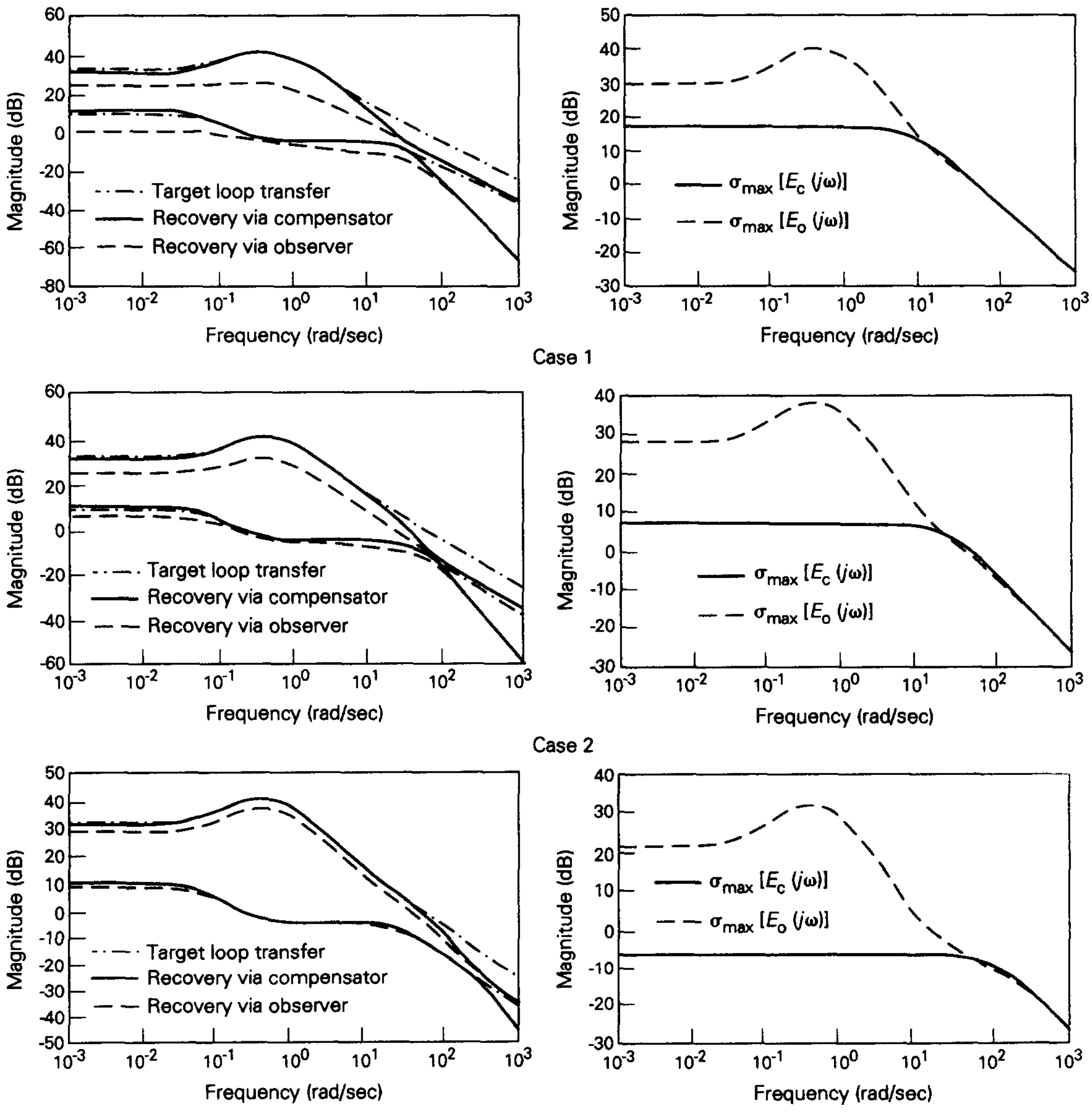

FIG. 3

TABLE 2 (A). SUPREMUM OF MAXIMUM SINGULAR VALUES OF MISMATCH FUNCTIONS, FREQUENCY RANGE: 0.001–1000 RAD/SEC

Tuning parameter		$\sup\{\sigma_{max}[E_0(j\omega)]\}$	$\sup\{\sigma_{max}[E_c(j\omega)]\}$
Case 1	$\sigma=5$	106.2351	7.2843
Case 2	$\sigma=20$	84.1034	2.3177
Case 3	$\sigma=100$	39.8388	0.4999

associated with high controller band-width and woes of signal saturation. To liberate the designer from these difficulties, we advocate the use of compensator structure of Chen *et al.* (1991) for the controller. As is the case with the observer based controllers, the compensator structure of Chen *et al.* (1991) can also recover any recoverable target loop. Moreover, the compensator structure uses values of gains orders of magnitude less than what the observer based controller does for the same degree of recovery. This is shown here both theoretically as well as by several numerical examples. Also, the theoretical bounds on sensitivity and complementary sensitivity functions obtained here confirm the advantages of using the compensator structure over the observer based controller structure. In short, we believe that the use of compensator structure for the controller brings the design procedure of LTR into practical domain.

Whenever a target loop is not recoverable, it may be partially recoverable. When one is interested in partial recovery, investigation of different types of structures for controllers (which perhaps need not be open-loop stable) and their relative merits, is still an open research problem.

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