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Brief Paper

On Blocking Zeros and Strong Stabilizability of Linear Multivariable Systems*†

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Key Words—Linear multivariable systems; system theory; blocking zeros; strong stabilizability.

Abstract—Motivated by a crucial role blocking zeros play in deciphering the strong stabilizability of a given system, a careful study of blocking zeros is undertaken here. After developing certain properties of blocking zeros and based on the multiplicity structure of invariant zeros, we identify what kind of invariant zeros are blocking zeros. For controllable and observable systems, an invariant zero is a blocking zero if and only if its geometric multiplicity is equal to the normal rank of the transfer function of the given system. This result leads to delineation of the class of controllable and observable time-invariant linear systems into two subclasses, (1) "simply SISO" systems whose normal rank is unity, and (2) "truly MIMO" systems whose normal rank is greater than unity. In a "simply SISO" system, every invariant zero is a blocking zero and hence a "simply SISO" system is not necessarily strongly stabilizable. On the other hand, a "truly MIMO" system with distinct invariant zeros does not have any blocking zeros and hence is always strongly stabilizable. Also, given any "truly MIMO" system, there always exists an arbitrarily small perturbation of its dynamic matrix such that the perturbed system has no blocking zeros and hence is strongly stabilizable. In this sense, one can say that a MIMO system "almost always" has no blocking zeros and hence is "almost always" strongly stabilizable.

1. *Introduction*

It has been acknowledged since late 1970s and early 1980s, that zeros of a multivariable dynamic system come into picture at the core of every design philosophy. Several types of zeros have been defined in the literature. For a comprehensive review of the existing definitions of zeros and their properties, see a well written paper by Schrader and Sain (1989). Among various zeros, system zeros, invariant zeros, transmission zeros, decoupling zeros and infinite zeros are by now well defined and studied by a number of researchers. In this note, we focus our attention on blocking zeros which form a subset of transmission zeros which themselves form a subset of invariant zeros. Our primary objective is to identify clearly what kind of invariant zeros (or transmission zeros) are blocking zeros. Our interest in blocking zeros arises because of the critical role played by them in characterizing strongly stabilizable systems; namely, blocking zeros are the relevant zeros in the parity interlacing property (Youla *et al.*, 1974; Vidyasagar, 1985) that defines the class of plants that are stabilizable by a stable compensator using measurement feedback. Strong stabilizability plays a significant role in a variety of control issues, e.g. study of robust stabilization and simultaneous stabilization of uncertain systems. Also, recently, it has been shown that strong stabilizability of a system is a necessary condition for the system to have at least one target loop transfer function which is defined in terms of a state feedback gain, and which is recoverable by using only measurement feedback controllers (Chen *et al.*, 1992). Besides the role played by blocking zeros in characterizing strongly stabilizable systems, as shown in Ferreira (1976), they also serve to characterize the essential feature that the error transfer function matrix of an asymptotic tracking control system must possess.

Here, at first we review and discuss the differences between various definitions of blocking zeros that exist in the literature. Choosing a particular definition of a blocking zero, we characterize blocking zeros in terms of invariant zeros (or transmission zeros) and their multiplicity structure. This process of characterization reveals several important properties of blocking zeros. In particular, it leads to a clear division of all controllable and observable linear time-invariant systems into two distinctly different families, one consisting of "simply SISO" systems and another consisting of "truly MIMO" systems. A "simply SISO" system is a system, the normal rank‖ of whose transfer function is unity; while a "truly MIMO" system is a system having the normal rank of its transfer function greater than unity. Our results show that (1) a "simply SISO" system has all its invariant zeros as blocking zeros, and (2) a "truly MIMO" system does not have any blocking zeros whenever all its invariant zeros are distinct. Since all its invariant zeros are blocking zeros, a "simply SISO" system is strongly stabilizable if and only if it satisfies a certain interlacing property (Youla *et al.*, 1974; Vidyasagar, 1985) among its invariant zeros and poles. In contrast to this, since a "truly MIMO" system does not have any blocking zeros whenever all its invariant zeros are distinct, the distinctness of its invariant zeros alone guarantees that a "truly MIMO" system is strongly stabilizable. That is, no inter-relationship whatsoever among its invariant zeros and poles of a "truly MIMO" system is required for its strong stabilizability whenever all its invariant zeros are distinct. Our next result shows that given any "truly MIMO" system characterized by a matrix quadruple (A, B, C, D), there always exists an arbitrarily small perturbation of its dynamic matrix A such that the perturbed system has no blocking zeros and hence is strongly stabilizable. In this sense, we can say that a MIMO system "almost always" has no blocking zeros and hence is "almost always" strongly stabilizable. Note that the word "almost always" is well defined in the context of this paper as it refers to arbitrarily small perturbations in the matrix A, but not

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‖ The normal rank of a rational transfer function is its rank over the field of rational functions.

necessarily in B and not necessarily in C. This we believe is a practically significant and exploitable result.

Throughout the paper, A' denotes the transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. $\lambda(A)$ denotes the set of eigenvalues of A. The open left and closed right half s-planes are, respectively denoted by $\mathscr{C}^-$ and $\mathscr{C}^+$ while $\mathscr{C}$ denotes the entire s-plane. With an abuse of notation, we use 0 to indicate a scalar zero as well as a vector zero or a matrix zero of appropriate dimension understood from the context. Ker $[V]$ denotes the kernel of V. We denote by $\mathscr{V}^g$ the maximal subspace of $\Re^n$ which is $(A+BF)$-invariant and contained in Ker $(C+DF)$ such that $\lambda[(A+BF) \mid \mathscr{V}^g]$ are contained in $\mathscr{C}_g \subseteq \mathscr{C}$ for some F. For the cases that $\mathscr{C}_g = \mathscr{C}$, $\mathscr{C}_g = \mathscr{C}^-$ and $\mathscr{C}_g = \mathscr{C}^+$, we replace the superscript g in $\mathscr{V}^g$, respectively by $*$, $-$ and $+$.

2. *Preliminaries*

Let us consider a linear time-invariant multivariable system Σ described by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{2.1}$$

where $x \in \Re^n$, $u \in \Re^m$ and $y \in \Re^p$. Without loss of generality, we assume that both $[B', D']'$ and $[C, D]$ are of maximal rank. Let $P(s) = C(sI_n - A)^{-1}B + D$ be the transfer function of Σ. Also, let the irreducible transfer function of Σ be denoted by $G(s)$, i.e. there are no cancellable factors in the elements of $G(s)$. Moreover, the normal rank of $P(s)$ is denoted by m_u. In what follows, we give various definitions of blocking zeros that exist in the literature.

Definition 2.1 (*Youla et al.*, 1974). For the plant with transfer function $P(s)$ which has no unstable hidden mode, the point $s = s_0$ is a zero† (blocking zero) of $P(s)$ if it is a zero of every entry in $P(s)$, i.e. if $P(s_0) = 0$.

Definition 2.2 (Ferreira, 1976; Ferreira and Bhattacharyya, 1977). The unique monic polynomial $\beta(s)$ which is the greatest common divisor of the numerators of the elements of $G(s)$ is the blocking polynomial of $G(s)$. The roots of $\beta(s) = 0$, counting multiplicities, are the blocking zeros of $G(s)$.

Definition 2.3 (Vidyasagar, 1985). Assume that Σ is detectable and stabilizable. An $s \in \mathscr{C} \cup \{\infty\}$ where $P(s) = 0$ is called a right half plane blocking zero of $P(s)$.

Definition 2.4 (Patel, 1986). Given a system Σ with $D \equiv 0$, a scalar $\lambda \in \mathscr{C}$ is a blocking zero of Σ if $C \cdot \text{adj}\,(\lambda I_n - A) \cdot B = 0$, where adj $(\cdot)$ denotes the adjoint of matrix $(\cdot)$.

The blocking zeros defined by Patel include the decoupling zeros (uncontrollable and unobservable modes) of Σ while the blocking zeros defined by others exclude decoupling zeros. The rationale for the exclusion of decoupling zeros has been argued by Ferreira (1986). Also, Vidyasagar (1985) has defined only right hand plane blocking zeros including those at infinity. The definition of Ferreira and Bhattacharyya (1977) is the most general and desirable one. It can be reformulated in terms of $P(s)$ rather than $G(s)$ as follows. any finite point $s = s_0 \in \mathscr{C}$ is a blocking zero if $P(s)$ tends to a zero matrix as s tends to s_0. This is the definition we use throughout this paper.

Our goal here is the study of blocking zeros as related to invariant zeros and their multiplicity structure. Invariant zeros can equivalently be defined in several ways (see e.g. Macfarlane and Karcanias, 1976). We recall the following geometric definition of invariant zeros (see Wonham, 1985).

Definition 2.5. Let $\mathbf{F}\{(\mathscr{V}^*/\mathscr{V}^+) \oplus (\mathscr{V}^*/\mathscr{V}^-)\}$ denote the class of maps $F:(\Re^n \to \Re^m)$ such that $(A+BF)\{(\mathscr{V}^*/\mathscr{V}^+) \oplus (\mathscr{V}^*/\mathscr{V}^-)\} \subset (\mathscr{V}^*/\mathscr{V}^+) \oplus (\mathscr{V}^*/\mathscr{V}^-)$. Let $A_F = A + BF$ and $\bar{A}_F$ be the map induced by A_F in $(\mathscr{V}^* + \mathscr{V}^+) \oplus (\mathscr{V}^*/\mathscr{V}^-)$. Then the eigenvalues of $\bar{A}_F$ are said to be the invariant zeros of Σ.

We now proceed to recall the multiplicity structure of invariant zeros as is defined in Saberi *et al.* (1991). We first note that $\bar{A}_F$ given in Definition 2.5 is independent of $F \in \mathbf{F}\{(\mathscr{V}^*/\mathscr{V}^+) \oplus (\mathscr{V}^*/\mathscr{V}^-)\}$ and that $\lambda(\bar{A}_F) \subset \lambda(A_F)$. Let X be a nonsingular transformation matrix such that

$$X^{-1}\bar{A}_F X = J = \text{Block Diag}\,[J_1, J_2, \ldots, J_k], \tag{2.2}$$

where J_l, $l = 1$ to k are some $n_l \times n_l$ Jordan blocks,

$$J_l = \text{Diag}\,[z_l, z_l, \ldots, z_l] + \begin{bmatrix} 0 & I_{n_l - 1} \\ 0 & 0 \end{bmatrix}. \tag{2.3}$$

Definition 2.6. For any given $z \in \lambda(\bar{A}_F)$, let there be σ_z Jordan blocks of $\bar{A}_F$ as in (2.2) and (2.3) associated with z. Let $n_{z,1}, n_{z,2}, \ldots, n_{z,\sigma_z}$ be the dimensions of the corresponding Jordan blocks. Then we say z is an invariant zero of Σ with multiplicity structure $\mathbf{S}_z^*$,

$$\mathbf{S}_z^* = \{n_{z,1}, n_{z,2}, \ldots, n_{z,\sigma_z}\}. \tag{2.4}$$

If $n_{z,1} = n_{z,2} = \cdots = n_{z,\sigma_z} = 1$, then we say z is a simple invariant zero of Σ. The geometric multiplicity of z is σ_z and the algebraic multiplicity of z is ρ_z where $\rho_z = n_{z,1} + n_{z,2} + \cdots + n_{z,\sigma_z}$.

The geometric and algebraic multiplicities of an invariant zero as defined here coincide with those defined in MacFarlane and Karcanias (1976). However, just the knowledge of algebraic multiplicities of an invariant zero, is not sufficient enough to define pseudo-state and input zero directions associated with an invariant zero. As pointed out by Saberi *et al.* (1991), one needs the above defined multiplicity structure of an invariant zero to define appropriate chains of its state and input zero directions. For further details in this regard, see Saberi *et al.* (1991).

3. *Main results*

This section gives our main results. We first have the following proposition which states the effect of static state feedback and static output feedback on blocking zeros.

Proposition 2.1. Consider a system Σ with transfer function $P(s) = C(sI_n - A)^{-1}B + D$. Let z be a blocking zero of Σ, i.e. $\lim_{s \to z} P(s) = 0$. Then we have the following.

(1) There exists a state feedback law, $u = Fx + v$, $F \in \Re^{m \times n}$, such that $\lim_{s \to z} P_F(s) \neq 0$, where $P_F(s) = (C+DF)(sI_n - A - BF)^{-1}B + D$, i.e. z is not a blocking zero of $(A+BF, B, C+DF, D)$. That is, blocking zeros are not necessarily invariant with respect to static state feedback.

(2) For any static output feedback law, $u = Ky + v$, $K \in \Re^{m \times p}$ such that $I_m - KD$ is nonsingular (i.e. the closed-loop system is well-posed), $\lim_{s \to z} P_K(s) = 0$ holds, where $P_K(s)$ is the closed-loop transfer function, and thus z remains as a blocking zero of $P_K(s)$. That is, blocking zeros are invariant with respect to static output feedback.

Proof. Part 1. Let F be chosen such that z is completely unobservable in the pair $(A+BF, C+DF)$. In this case, the closed loop system $(A+BF, B, C+DF, D)$ does not have z as its transmission zero. Hence, z is not a blocking zero of $(A+BF, B, C+DF, D)$.

Part 2. It is simple to obtain the state space equations of the closed-loop system under the static output feedback $u = Ky + v$ as follows,

$$\dot{x} = [A + B(I_m - KD)^{-1}KC]x + B(I_m - KD)^{-1}v,$$
$$y = [C + D(I_m - KD)^{-1}KC]x + D(I_m - KD)^{-1}v.$$

† Youla *et al.* (1974) define it as a system zero. However the name system zero conflicts with that given by Rosenbrock (1970). It is evident from other definitions, what Youla *et al.* defined is a 'blocking zero'.

Let $\Phi = (sI_n - A)^{-1}$. Then we have,

$$P_K(s) = \{[C + D(I_m - KD)^{-1}KC] \times [\Phi^{-1} - B(I_m - KD)^{-1}KC]^{-1}B + D\}(I_m - KD)^{-1}. \quad (3.1)$$

Next, recalling the matrix identity,

$$(W + XYZ)^{-1} = W^{-1} - W^{-1}X(ZW^{-1}X + Y^{-1})^{-1}ZW^{-1},$$

where W and Y are, respectively any nonsingular $k \times k$ and $n \times n$ matrices, we obtain

$$\begin{aligned}[\Phi^{-1} - B(I_m - KD)^{-1}KC]^{-1} &= \Phi + \Phi B[I_m - K(C\Phi B + D)]^{-1}KC\Phi \\ &= \Phi(I_n + B[I_m - KP(s)]^{-1}KC\Phi). \quad (3.3)\end{aligned}$$

Also, since z is a blocking zero of Σ, let us note that

$$\lim_{s \to z} P(s) = \lim_{s \to z} (C\Phi B + D) = 0. \quad (3.3)$$

Then in view of equations (3.1) to (3.3), we have the following reductions as $s \to z$:

$$\begin{aligned}&P_K(s)(I_m - KD) \\ &= [C + D(I_m - KD)^{-1}KC]\Phi \\ &\quad \times (I_n + B[I_m - KP(s)]^{-1}KC\Phi)B + D \\ &\to [C + D(I_m - KD)^{-1}KC]\Phi(I_n + BKC\Phi)B + D \\ &= (C\Phi B + D) + C\Phi BKC\Phi B \\ &\quad + D(I_m - DK)^{-1}KC\Phi(I_n + BKC\Phi)B \\ &\to D(I_m - KD)^{-1}(I_m + KC\Phi B)KC\Phi B - DKC\Phi B \\ &= D(I_m - KD)^{-1}K(C\Phi B + D)K(C\Phi B + D - D) \\ &\to 0.\end{aligned}$$

Thus $\lim_{s \to z} P_K(s) = 0$ and hence z is a blocking zero of the closed-loop system. ∎

We now state a theorem which identifies the necessary and sufficient conditions under which an invariant (or transmission) zero becomes a blocking zero.

Theorem 3.1. Consider a controllable and observable system Σ. Let z be an invariant zero of Σ with multiplicity structure $\mathbf{S}_z^*$ as in (2.4). Then z is a blocking zero of Σ if and only if $\sigma_z = m_u$ where m_u is the normal rank of the transfer function matrix $P(s)$.

Proof.† The fact that z is a blocking zero of Σ implies that $P(z) = C(zI_n - A)^{-1}B + D = 0$. Let us define

$$[\omega_1, \omega_2, \ldots, \omega_m] = I_m,$$

and

$$x_i = (zI_n - A)^{-1}B\omega_i, \quad i = 1, \ldots, m.$$

Now it is trivial to see that $[x_i', \omega_i']'$, $i = 1$ to m, are linearly independent and satisfy

$$\begin{bmatrix} zI_n - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x_i \\ \omega_i \end{bmatrix} = 0.$$

Thus z is an invariant zero of Σ with its geometric multiplicity σ_z satisfying the condition, $\sigma_z \geq m_u$. But if $\sigma_z > m_u$, it can easily be shown that Σ is neither completely controllable nor completely observable, which is a contradiction to the assumption that Σ is controllable and observable. Hence, $\sigma_z = m_u$.

To prove the sufficiency, we consider the following. If $\sigma_z = m_u$, then it is simple to verify using the result of Saberi *et al.* (1991) that there must exist x_i and ω_i, $i = 1$ to $m - m_u + \sigma_z = m$, such that

$$\begin{bmatrix} zI_n - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x_i \\ \omega_i \end{bmatrix} = 0,$$

where x_i, $i = 1$ to m, are linearly independent. In what follows, we will show that w_i, $i = 1$ to m, are also linearly independent. First assume that ω_i, $i = 1$ to m, are linearly dependent. Then there exist constants c_i, $i = 1$ to m, such that

$$x_0 = \sum_{i=1}^{m} c_i x_i \neq 0 \quad \text{and} \quad \omega_0 = \sum_{i=1}^{m} c_i \omega_i = 0.$$

This implies that

$$(zI_n - A)x_0 = B\omega_0 = 0 \quad \text{and} \quad Cx_0 + D\omega_0 = Cx_0 = 0.$$

Hence, z is an output decoupling zero of Σ contradicting the assumption that Σ is controllable and observable. This shows that ω_i, $i = 1$ to m, are linearly independent. We next consider,

$$\begin{aligned}&P(z)[\omega_1, \omega_2, \ldots, \omega_m] \\ &\quad = [C(zI_n - A)^{-1}B + D][\omega_1, \omega_2, \ldots, \omega_m] = 0.\end{aligned}$$

The above equation implies that $P(z) = 0$. Thus z is blocking zero of Σ. ∎

Remark 3.1. The multiplicity of a blocking zero z as defined by Ferreira and Bhattacharyya (1977), is the multiplicity of the root z of $\beta(s)$, the greatest common divisor of the numerators of the elements of $G(s)$. It is straightforward to show that multiplicity of z is equal to $\alpha_z = \min\{n_{z,1}, n_{z,2}, \ldots, n_{z,\sigma_z}\}$.

The following Proposition is identical to Proposition 2 of the Appendix in Ferreira (1976).

Proposition 3.2. Consider a controllable and observable system Σ. Let z be a blocking zero of Σ with its multiplicity equal to α_z. Then z is an invariant zero of Σ with algebraic multiplicity greater than or equal to $m_u\alpha_z$.

Proof. In view of Theorem 3.1, the fact that z is a blocking zero of Σ implies that $\sigma_z = m_u$ and $\alpha_z = \min\{n_{z,1}, n_{z,2}, \ldots, n_{z,\sigma_z}\}$. Then by definition, z is an invariant zero with algebraic multiplicity $\rho_z = n_{z,1} + n_{z,2} + \cdots + n_{z,\sigma_z} \geq \sigma_z\alpha_z = m_u\alpha_z$. ∎

The following example illustrates the results of Theorem 3.1.

Example. Consider a controllable and observable system Σ characterized by

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then it is simple to verify that $m_u = 2$ and

$$\bar{A}_F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Obviously, $z = 1$ is an invariant zero of Σ with $\mathbf{S}_z^* = \{2, 2\}$. Hence, by Theorem 3.1, we can conclude that $z = 1$ is a blocking zero of Σ with multiplicity $\alpha_z = 2$. This can be easily verified from the transfer function $P(s)$ of Σ, where

$$P(s) = \begin{bmatrix} \dfrac{2(s-1)^2}{s(s-2)} & \dfrac{(s-1)^2}{s(s-2)} \\ \dfrac{(s-1)^2}{s(s-2)} & \dfrac{(s-1)^2}{s(s-2)} \end{bmatrix}.$$

Theorem 3.1 states clearly that for controllable and observable systems, an invariant zero is a blocking zero if and only if its geometric multiplicity is equal to the normal rank of the transfer function of the given system. To exploit

† An alternate proof of this theorem can be generated by using the Smich–McMillan form of $G(s)$ as suggested by an anonymous reviewer.

this result, we like to study next two major disjoint subclasses of linear time-invariant systems, (1) "simply SISO" systems whose transfer functions have a normal rank of unity, and (2) "truly MIMO" systems whose transfer functions have a normal rank greater than unity. The labeling and the distinction between "simply SISO" and truly MIMO" systems are obvious. Note that "simply SISO" systems include all single-input-single-output (SISO) systems, single-input-multiple-output (SIMO) systems, multiple-input-single-output (MISO) systems and some very special class of multiple-input-multiple-output (MIMO) systems. It turns out "simply SISO" and "truly MIMO" systems have radically different properties as to their blocking zero structure and strong stabilizability. The following propositions explore these differences.

Proposition 3.3. For a controllable and observable "simply SISO" system Σ, all its invariant zeros are also blocking zeros.

Proof. For a "simply SISO" system Σ, by definition, the normal rank m_u is unity. Also, trivially, all invariant zeros of Σ have geometric multiplicity of unity. Hence, the result follows directly from Theorem 3.1. ■

Proposition 3.4. Consider a controllable and observable "truly MIMO" system Σ. Also, let all the invariant zeros of Σ be distinct. Then the given system Σ has no blocking zeros and hence is strongly stabilizable.

Proof. The fact that all the invariant zeros of Σ are distinct implies that for any invariant zero z of Σ, we have $\sigma_z = 1$. It follows then from Theorem 3.1 that Σ does not have any blocking zeros since by assumption $m_u > 1$. Thus, Σ is strongly stabilizable. ■

One can deduce from Proposition 3.4 that a "truly MIMO" system may not be strongly stabilizable if it has one or more invariant zeros with geometric multiplicity greater than unity. However, the following proposition indicates that an arbitrarily small regular perturbation in the dynamic matrix A of the given MIMO system can render the perturbed system free of blocking zeros and hence the perturbed system is strongly stabilizable.

Proposition 3.5. Given a controllable and observable "truly MIMO" system Σ characterized by a matrix quadruple (A, B, C, D), and any positive ϵ, there exists a regular perturbation δA with $\|\delta A\| < \epsilon$ such that the system characterized by the matrix quadruple $(A + \delta A, B, C, D)$ does not have any blocking zeros and hence is strongly stabilizable.

Proof. Given a system Σ characterized by (A, B, C, D), it is simple to verify that there exist nonsingular transformations U and V such that

$$UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.4}$$

where m_0 is the rank of matrix D. Hence without loss of generality, we assume that matrix D has the form given on the right hand side of (3.4). One can then rewrite the system of (2.1) as,

$$\begin{cases} \dot{x} = Ax + [B_0 \quad B_1]\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \end{cases}$$

where the matrices B_0, B_1, C_0 and C_1 have appropriate dimensions. Then a theorem of Sannuti and Saberi (1987), as well as Saberi and Sannuti (1990), implies that there exist nonsingular transformations Γ_1, Γ_2 and Γ_3 such that

$$\Gamma_1^{-1}(A - B_0C_0)\Gamma_1 = \begin{bmatrix} A_{aa} & 0 & L_{ab}C_b & L_{af}C_f \\ B_cE_{ca} & A_{cc} & L_{cb}C_b & L_{cf}C_f \\ 0 & 0 & A_{bb} & L_{bf}C_f \\ B_fE_a & B_fE_c & B_fE_b & A_f \end{bmatrix},$$

$$\Gamma_1^{-1}[B_0 \quad B_1]\Gamma_3 = \begin{bmatrix} B_{0a} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0b} & 0 & 0 \\ B_{0f} & B_f & 0 \end{bmatrix},$$

$$\Gamma_2^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_1 = \begin{bmatrix} C_{0a} & C_{0c} & C_{0b} & C_{of} \\ 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 \end{bmatrix},$$

$$\Gamma_2^{-1}D\Gamma_3 = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here we note that $\text{rank}(B_f) + m_0 = \text{rank}(C_f) + m_0 = m_u$, $\text{rank}(B_c) = m - m_u$. Also, (A_f, B_f, C_f) is square invertible with no invariant zeros. (A_{bb}, C_b) is observable, (A_{cc}, B_c) is controllable. Moreover, A_{aa} and $\bar{A}_F$ are related by a similarity transformation, i.e. the eigenvalues of A_{aa} are the invariant zeros of Σ. Then it is simple to see that there exists an arbitrarily small regular perturbation in the sub-matrix A_{aa}, say δA_{aa}, such that $A_{aa} + \delta A_{aa}$ has distinct eigenvalues. Therefore, the invariant zeros of the perturbed system $(A + \delta A, B, C, D)$ are also distinct. It follows then from Theorem 3.1 and Proposition 3.4 that $(A + \delta A, B, C, D)$ does not have any blocking and thus is strongly stabilizable. ■

We emphasize that the results of Proposition 3.5 cannot however be obtained for "simply SISO" systems. The importance of Propositions 3.3 and 3.4 and Proposition 3.5 cannot be overemphasized in view of the role they play in characterizing the stong stabilizability (or a lack of it) of a given system. These propositions explicitly point out that "simply SISO" and "truly MIMO" systems have radically different properties. For a "simply SISO" system, as stated in Proposition 3.3, all invariant zeros are blocking zeros. Thus a "simply SISO" system is strongly stabilizable if and only if a certain interlacing property (Youla *et al.*, 1974; Vidyasagar, 1985) among its invariant zeros and poles is satisfied. In contrast to this, since a "truly MIMO" system does not have any blocking zeros whenever all its invariant zeros are distinct, the distinctness of all its invariant zeros alone guarantees that a "truly MIMO" system is strongly stabilizable. That is, no inter-relationship whatsoever among its invariant zeros and poles of a "truly MIMO" system is required for its strong stabilizability whenever it has only distinct invariant zeros. When a "truly MIMO" system does not have distinct invariant zeros, it could have blocking zeros; hence it may or may not be strongly stabilizable. Apparently then, the lack of strong stabilizability of a "truly MIMO" system is deeply rooted in the multiplicity structure of its invariant zeros. However, in view of Proposition 3.5, let us note that there exists an arbitrarily small regular perturbation which can destroy the multiplicity structure of invariant zeros and there by rendering a "truly MIMO" system free of blocking zeros. Thus we can infer that a "truly MIMO" system almost always has no blocking zeros and hence is almost always strongly stabilizable. On the other hand, as mentioned earlier, a "simply SISO" system must satisfy a certain interlacing property among its invariant zeros and poles in order to be strongly stabilizable; a small perturbation of its parameters will not be enough to render it strongly stabilizable if it is not already so. This then is the essential difference between "truly MIMO" systems and "simply SISO" systems.

In most control system design considerations, the interest normally is in blocking zeros which lie in the closed right-half complex plane, i.e. in right-half plane blocking zeros. In fact, in Definition 2.3, Vidyasagar had cleverly defined the blocking zeros only in $\mathscr{C}^+$ for detectable and stabilizable systems. It is easy to show that all the results obtained so far in this section for controllable and observable systems, are applicable to detectable and stabilizable systems as well if we consider only the right-half plane blocking zeros. In particular, we have the following theorem.

Theorem 3.2. Consider a detectable and stabilizable system Σ. Let $z \in \mathscr{C}^+$ be an invariant zero of Σ with multiplicity structure $\mathbf{S}_z^*$ as in (2.4). Then z is a right-half plane blocking zero of Σ if and only if $\sigma_z = m_u$. Moreover, the multiplicity of blocking zero z is equal to $\alpha_z = \min\{n_{z,1}, n_{z,2}, \ldots, n_{z,\sigma_z}\}$.

Proof. It follows along the same lines as Theorem 3.1. ∎

We emphasize that all the Propositions 3.2 to 3.5 also carry over to detectable and stabilizable systems for the case of right-half plane blocking zeros. That is, in these Propositions, when detectable and stabilizable systems are considered, the words "invariant zero" and "blocking zero" must, respectively be replaced by "right-half plane invariant zero" and "right-half plane blocking zero". Also, we have the following proposition.

Proposition 3.6. Consider a detectable and stabilizable system Σ with $[B', D']'$ and $[C, D]$ being of maximal rank. If either B or C is not of maximal rank, then Σ does not have any blocking zeros and hence it is strongly stabilizable.

Proof. It is well known that there exist non-singular state, input and output transformations Γ_s, Γ_i and Γ_o such that

$$A_s = \Gamma_s^{-1} A \Gamma_s, \quad B_s = \Gamma_s^{-1} B \Gamma_i,$$
$$C_s = \Gamma_o C \Gamma_s, \quad D_s = \Gamma_o^{-1} D \Gamma_i,$$

where

$$C_s = \begin{bmatrix} 0 & C_0 \\ I_{p-r} & 0 \end{bmatrix}, \quad D_s = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.5}$$

Here we note that $r \geq 0$ is the rank of matrix D. It is simple to see from the special structures of C_s and D_s given in (3.5) that if C_s is not of maximal rank, i.e. C_0 is not of maximal rank, $P_s(z) = C_s(zI_{mn} - A_s)^{-1}B_s + D_s \neq 0$ and hence $P(z)$ cannot be identically zero for any $z \in \mathscr{C}$. Thus, Σ does not have any blocking zeros and hence it is strongly stabilizable. By similar arguments, one can see that Σ is strongly stabilizable when B is not of maximal rank. ∎

As shown in Proposition 3.1 blocking zeros can be removed by some static state feedback. However, the following proposition shows that the right-half plane blocking zeros are invariant under a state feedback which preserves the detectability of the closed-loop system.

Proposition 3.7. Consider a detectable and stabilizable system Σ. Let

$$\mathscr{F} := \{F \in \Re^{m \times n} \mid (A + BF, C + DF) \text{ is detectable}\}.$$

Also, let $z \in \mathscr{C}^+$ be a blocking zero of Σ. Then z is invariant under a state feedback law, $u = Fx$, $F \in \mathscr{F}$, i.e. z is also a blocking zero of the closed-loop system $(A + BF, B, C + DF, D)$.

Proof. It is well known that the invariant zero structure of Σ is preserved in $(A + BF, B, C + DF, D)$. Also, the normal rank over the field of rational functions of $(C + DF)(sI_n - A - BF)^{-1}B + D$ is equal to m_u. Hence, the result follows from Theorem 3.2 due to the fact that $(A + BF, B, C + DF, D)$ is detectable and stabilizable. ∎

4. *Conclusions*

As they play a dominant role in characterizing strong stabilizability of a system, we focus our attention here on an in depth study of blocking zeros and their properties. We identify what kind of invariant zeros turn our to be blocking zeros. In the course of this identification. we show that all controllable and observable linear time invariant multivariable systems can be divided into two categories; "simply SISO" systems having normal rank of their transfer functions as unity, and "truly MIMO" systems having the normal rank of their transfer functions greater than unity. Our results show that a simply SISO system has all its invariant zeros as blocking zeros, and a truly MIMO system does not have any blocking zeros whenever all its invariant zeros are distinct. Moreover, we show that the distinctness of its invariant zeros alone guarantees that a truly MIMO system is strongly stabilizable. That is, no inter-relationship whatsoever among its invariant zeros and poles of a truly MIMO system is required for its strong stabilizability whenever all its invariant zeros are distinct. Thus, since the multiplicity structure of invariant zeros of a truly MIMO can be changed by an arbitrarily small regular perturbation of its parameters, any truly MIMO system is almost always strongly stabilizable. In contrast to this, a simply SISO system is strongly stabilizable if and only if it satisfies a certain interlacing property among its invariant zeros and poles. Arbitrarily small perturbations in its parameters cannot render a simply SISO system strongly stabilizable if it is not already so.

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