THEORY OF LOOP TRANSFER RECOVERY FOR MULTIVARIABLE LINEAR SYSTEMS

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ABSTRACT

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In this thesis, we develop a fairly complete theory of loop transfer recovery (LTR) using observer based controller and stable compensator structures for general nonstrictly proper continuous-time multivariable linear systems. The given system need not be left invertible and of minimum phase. The thesis can be divided into the following three parts.

The first part deals with the analysis of LTR using full and reduced order observer based controllers. The analysis of this part focuses on four fundamental issues. The first issue is concerned with what can and what cannot be achieved for a given system and for an arbitrarily specified target loop transfer function. The second issue is concerned with the development of necessary and sufficient conditions a target loop has to satisfy so that it can be either exactly or asymptotically recovered for a given system while the third issue is concerned with the development of necessary and sufficient conditions on a given system such that it has at least one, either exactly or asymptotically, recoverable target loop. The fourth issue deals with generalizing the traditional LTR concept to sensitivity recovery over a control subspace. All the results for the full and reduced order observer based controllers are unified in the same framework.

Three types of design schemes are developed in detail in the second part. The first one is an asymptotic time-scale and eigenstructure assignment (ATEA) scheme, and the other two are optimization based designs; one deals with the minimization of the H_2 norm of

a so called 'recovery matrix' while the other deals with the minimization of H_∞ norm of the same. Relative advantages and disadvantages of both ATEA and optimization based design schemes are discussed. A helicopter attitude and rate command system design is also presented to illustrate these algorithms.

In the third part, a new compensator structure is proposed for LTR design for the recoverable target loops. The proposed compensator (i) is open-loop stable, (ii) guarantees closed-loop stability, and above all (iii) requires much smaller values of gain than the conventional observer based controller for the same degree of recovery. This is shown both theoretically as well as by a bank of numerical examples including the helicopter control system design. Also, the theoretical bounds on sensitivity and complementary sensitivity functions obtained here demonstrate the advantages of using the compensator structure over the observer based controller structure in loop transfer recovery.

Contents

ACKNOWLEDGMENT iii

ABSTRACT iv

TABLE OF CONTENTS vi

DEDICATION ix

1 INTRODUCTION AND PRELIMINARIES 1

1.1 Introduction 1

1.2 Problem formulation 6

1.2.1 LTR design for input break point 7

1.2.2 LTR design for output break point 10

1.3 Preliminaries 12

2 OBSERVER BASED CONTROLLERS 18

2.1 Full and reduced order observer based controllers 18

2.1.1 Full order observer based controller 19

2.1.2 Reduced order observer based controller 19

2.2 Preliminary analysis 21

2.A Appendix 2.A — Proof of Lemma 2.2.1 24

2.B Appendix 2.B — Proof of Proposition 2.2.1 26

3 GENERAL LTR ANALYSIS 29
3.1 Introduction 29
3.2 Recovery analysis for an arbitrary target loop 31
3.3 Analysis for recoverable target loops 47
3.4 Recovery analysis in a given subspace 59
3.A Appendix 3.A — Proof of Lemma 3.2.1 70
3.B Appendix 3.B — Proof of Lemma 3.2.3 72
3.C Appendix 3.C — Proof of Corollary 3.2.2 73
3.D Appendix 3.D — Proof of Lemma 3.4.2 75

4 GENERAL LTR DESIGN 77
4.1 Introduction 77
4.2 Design constraints and available freedom 80
4.3 Observer design by 'ATEA' 83
4.3.1 General 'ATEA' design 84
4.3.2 A helicopter control system design 92
4.3.3 Design for exactly recoverable target loops 96
4.3.4 A helicopter control system design (cont.) 100
4.4 Optimization based design methods 100
4.4.1 H_2-optimization design algorithm 103
4.4.2 A helicopter control system design (cont.) 107
4.4.3 H_∞-optimization design algorithm 108
4.4.4 A helicopter control system design (cont.) 110
4.5 Design for recovery over a specified subspace 110
4.6 Comparison of 'ATEA' and 'ARE' based algorithms 113
4.A Appendix 4.A — Proof of Theorem 4.3.1 116
4.B Appendix 4.B — Proof of Theorem 4.4.1 129

5 A STABLE COMPENSATOR DESIGN FOR ALTR 130

5.1 Introduction 130

5.2 Compensator structure 133

5.2.1 Full order stable compensator 133

5.2.2 Reduced order stable compensator 134

5.3 Properties of compensators 136

5.4 Examples 140

5.A Appendix 5.A — Proof of Lemma 5.2.1 152

5.B Appendix 5.B — Proof of Theorem 5.2.1 153

6 CONCLUDING REMARKS 157

6.1 Summary of the thesis 157

6.2 Future research topics 160

BIBLIOGRAPHY 162

To

my grandmas, my parents,

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and Andy, my son

Chapter 1

INTRODUCTION AND PRELIMINARIES

1.1. Introduction

In feedback control system design, specifications such as stability of the closed-loop system, performance objectives of command following, fast response and disturbance rejection, and robustness objectives of maintaining stability and insensitivity to plant parameter variations and other uncertainties, are common. In single-input single-output (SISO) systems, as is well known from the days of Bode and Nyquist, such specifications can easily be cast in terms of the magnitude of a sensitivity function (the inverse of return difference) or equivalently in terms of the magnitude of loop gain (loop transfer function) obtained by breaking the loop at any point of the feedback loop. In fact, it is intuitively clear in SISO design that large loop gains or so called tight loops, yield good performance. However, as is known, loop gains cannot be made arbitrarily high over arbitrarily large frequency ranges. Rather they must satisfy certain tradeoffs and design limitations. A major tradeoff, for example, concerns command and disturbance error reduction versus output measurement noise error reduction. Large loop gain values over a large frequency range make errors due to command and disturbances small; but they also make errors due to output measurement noise large. There is yet another and perhaps more important limitation – namely the robustness requirement of tolerance to model uncertainties. Physical plants, in particular the way they deviate from finite dimensional linear models, put strict limitations on the frequency range over which the loop gains may be large. This is to say that the open-loop transfer gain of a feedback system must appropriately be 'shaped' in the frequency domain in order to achieve the desired stability and performance objectives in the face of model uncertainties. This is the essence of classical SISO frequency domain design philosophy. Recently, the efforts of many researchers, notably those of Doyle and Stein [18], have shown that multi-input multi-output (MIMO) design problems do not differ fundamentally from their SISO counter parts. What distinguishes MIMO from SISO design is that instead of a single scalar sensitivity function or equivalently a loop gain in SISO design, in MIMO case one has to deal with a sensitivity or complementary sensitivity matrix function, or equivalently a loop transfer matrix obtained by breaking the loop at an appropriate point of the feedback loop. Consequently, some norm of loop transfer matrix replaces the absolute value or magnitude of loop gain. Also, since matrix multiplication is not commutative, in the MIMO case the point at which the loop is broken to calculate the loop transfer matrix depends on where the expected unstructured model uncertainties can most appropriately be modeled. Thus in analogy with classical frequency domain concepts, the work of Doyle and Stein [19] has shown that in the MIMO feedback system design as well, the objectives of stability, performance and robustness in the face of uncertainties, can be cast in terms of bounds on maximum and minimum singular values of some sensitivity and complementary sensitivity matrix functions, or equivalently in terms of bounds on maximum and minimum singular values of appropriate loop transfer matrices. In short, the work of Doyle and Stein revealed that the classical concept of 'loop shaping' is still a viable key to design MIMO feedback loops as well.

A prominent design methodology for MIMO systems which is based on the 'loop shaping' concept is LQG/LTR. It involves two separate designs of a state feedback controller and an observer. The exact design procedure depends on the point where the unstructured uncertainties are modeled and where the loop is broken to evaluate the open-loop transfer matrices. Commonly either an input point or output point of the plant is taken as such a point. We will concentrate our discussion on the case when the loop is broken at the input point of the plant. The required results for the output point are obtained via duality. Thus in the two step procedure of LQG/LTR, the first step of design involves loop shaping by state feedback design to obtain an appropriate loop transfer function, called the target loop transfer function. Such a loop shaping is an engineering art and often involves the use of linear quadratic regulator (LQR) design in which the cost matrices are used as free design parameters to generate the target loop transfer function, and thus the desired sensitivity and complementary sensitivity functions. However, when such a feedback design is implemented via an observer based controller (or Kalman filter) that uses only the measurement feedback, the obtained loop transfer function in general is not same as the target loop transfer function, unless proper care is taken in designing the observers. This is when the second step of LQG/LTR design philosophy comes into picture. In this step, the required observer is designed so as to recover either exactly or approximately the loop transfer function of the full state feedback controller. This second step has come to be known as LTR.

The topic of LTR has been the subject of a number of authors including [2], [4], [8], [9], [10], [11], [12], [13], [16], [18], [19], [22], [23], [25], [27], [28], [29], [31], [32], [33], [34], [37], [56], [39], [40], [43], [50], [53], [58] and [59]. Both continuous and discrete systems have been considered. During the last ten years the subject has achieved a certain amount of maturity. My own research efforts for the last two years have uncovered a number of aspects of LTR analysis and design [4], [8], [9], [10], [11], [12], [13], [14], [15], [39] and [40]. The purpose of this thesis is to discuss various aspects of LTR for continuous systems under a single cover.

Throughout this thesis we shall adopt the following conventions and notations:

$A' :=$ transpose of A,

$A^H :=$ complex conjugate transpose of A,

$I :=$ an identity matrix,

$I_k :=$ an identity matrix of dimension $k \times k$,

$\lambda(A) :=$ the set of eigenvalues of A,

$\mathrm{Re}\,[\lambda(A)] :=$ the set of real parts of eigenvalues of A,

$\sigma_{max}[A] :=$ the maximum singular value of A,

$\sigma_{min}[A] :=$ the minimum singular value of A,

$\mathrm{Ker}\,[V] :=$ the kernel of V,

$\mathrm{Im}\,[V] :=$ the image of V,

$\Re :=$ the set of real numbers,

$\mathcal{C} :=$ the whole complex plane,

$\mathcal{C}^- :=$ the open left-half complex plane,

$\mathcal{C}^+ :=$ the closed left-half complex plane,

$\mathcal{C}^o :=$ the $j\omega$ axis of complex plane,

$G(j\omega) := G(s)\,|_{s=j\omega}$,

$\mathcal{R}_p :=$ the subring of all proper rational functions of s,

$\mathcal{M}^{\ell \times q}(\mathcal{R}_p) :=$ the set of matrices of dimension $\ell \times q$ whose elements belong to $\mathcal{R}_p$.

The subject matter of this thesis is organized as follows. In the next section, we introduce the problem formulation of loop transfer recovery in precise mathematical terms for the cases when plant uncertainties are respectively modeled at the plant input and output points. The section on preliminaries recalls a special coordinate basis (s.c.b) of multivariable linear systems. As will be seen throughout the thesis, the finite and infinite zero structure of Σ plays a dominant role in LTR analysis and design, and the s.c.b given here displays explicitly the required zero structure and there-by helps the readers to understand clearly the various results presented. Chapter 2 describes the structural details of both full and reduced order observer based controllers. It also develops some preliminary analysis and introduces what is called a recovery matrix $M(s)$. It turns out that the recovery error $E(s)$ can be rendered zero if and only if $M(s)$ can be rendered zero. Thus the study of the recovery matrix $M(s)$ is the key to both the analysis as well as design of controllers for LTR. Chapter 2 also shows how the study of LTR utilizing either a full or reduced order observer based controller can be unified into a single mathematical framework. Chapter 3 considers the detailed analysis of the LTR problem. The analysis of the LTR mechanism focuses on four fundamental issues. The first issue is concerned with what can and what cannot be achieved for a given system and for an arbitrarily specified target loop transfer function. On the other hand, the second issue is concerned with the development of necessary or/and sufficient conditions a target loop has to satisfy so that it can be either exactly or asymptotically recovered for a given system, while the third issue is concerned with the development of necessary or/and sufficient conditions on a given system such that it has at least one recoverable (either exactly or asymptotically) target loop. The fourth issue deals with a generalization of all the above three issues when recovery is required over a subspace of the control space. It is concerned with generalizing the traditional LTR concept to sensitivity recovery over a subspace and deals with method(s) to test whether projections of target and achievable sensitivity and complementary sensitivity functions onto a given subspace match each other or not. All this analysis shows some fundamental limitations of the given system as a consequence of its structural properties, namely finite and infinite zero structure and invertibility. Consequently, for general systems, LTR is not completely feasible although there exists considerable amount of freedom to shape the inevitable recovery error. Thus the analysis given here helps to set meaningful design goals at the onset of design. Actual design of controllers for LTR is considered in Chapter 4. In view of the necessary design constraints and the available design freedom, possible specifications on the time-scale and/or eigenstructure of the observer dynamic matrix are formulated there at first. Then three types of design schemes are developed in detail. The first one is an asymptotic time-scale and eigenstructure assignment (ATEA) scheme, and the other two are optimization based designs; one dealing with the minimization of the H_2 norm of the so called 'recovery matrix' while the other dealing with the minimization of the H_∞ norm of the same. Relative advantages and disadvantages of both ATEA and optimization based design schemes are discussed. The traditional observer design based on Kalman filter formalism [18] belongs to the class of H_2 norm minimization schemes. Besides the conventional LTR problem which is concerned with recovery over the entire control space, another generalized recovery problem where the concern is with recovery over a specified subspace of the control space is also considered in Chapter 4. In Chapter 5, a new stable compensator structure for LTR design for recoverable target loop is proposed. The proposed compensator (a) is open-loop stable, (b) guarantees closed-loop stability and above all (c) requires much smaller values of gain than the conventional observer based controller for the same degree of recovery. Finally, Chapter 6 makes relevant conclusions and points out other aspects of LTR which are not discussed in this thesis.

1.2. Problem formulation

Let us consider continuous-time systems and formulate the LTR problem in precise mathematical terms. Let a model of the given plant be described by a system Σ,

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{1.2.1}$$

where the state vector $x \in \Re^n$, output vector $y \in \Re^p$ and input vector $u \in \Re^m$. Without loss of generality, throughout the thesis, we assume that $[B', D']'$ and $[C, D]$ are of maximal rank, and that Σ is stabilizable and detectable. As mentioned earlier, let us first concentrate on a case when plant uncertainties are modeled at the input point of a nominal plant model and hence the required loop transfer function is specified at the plant input point. However, our results can be generalized easily for the case when the required loop transfer function

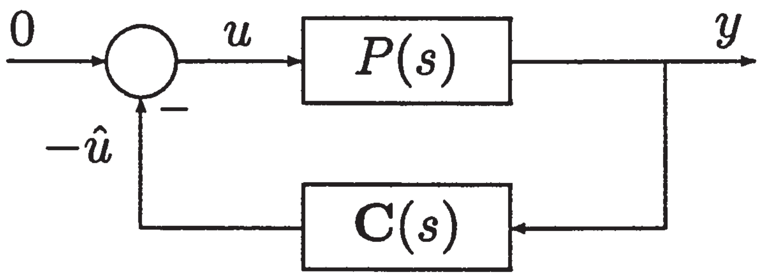

Figure 1.2.1: Plant — Controller closed-loop configuration.

is specified at any arbitrary point. In fact, for the case when the required loop transfer function is specified at the plant output point, our results can easily be dualized. We discuss these issues in the next subsections.

1.2.1. LTR design for input break point

Let F be a full state feedback gain matrix such that (a) the closed-loop system is asymptotically stable, i.e. eigenvalues of $A-BF$ lie in the left half s-plane, and (b) the open-loop transfer function when the loop is broken at the input point of the given system meets some given frequency dependent specifications. The state feedback control is

$$u = -Fx \tag{1.2.2}$$

and the loop transfer function evaluated when the loop is broken at the input point of the given system, the so called target loop transfer function, is

$$L_t(s) = F\Phi B \tag{1.2.3}$$

where $\Phi = (sI_n - A)^{-1}$. The corresponding target sensitivity and complementary sensitivity functions are

$$S_t(s) = [I_m + F\Phi B]^{-1} \quad \text{and} \quad T_t(s) = I_m - S_t(s).$$

Arriving at an appropriate value for F is concerned with the issue of loop shaping which, as discussed earlier, often includes the use of linear quadratic regulator (LQR) design in which the cost matrices are used as free design parameters to generate $L_t(s)$ that satisfies the given specifications, and thus yields the desired sensitivity and complementary sensitivity functions. The next step of design is to recover the target loop using only a measurement feedback controller. This is the problem of loop transfer recovery (LTR). To explain it clearly, consider the configuration of Fig. 1.2.1 where $\mathbf{C}(s)$ and $P(s)$ are respectively the transfer functions of a controller and of the given system. Given a

$$P(s) = C\Phi B + D, \tag{1.2.4}$$

and a target loop transfer function $L_t(s)$, one seeks then to design a $\mathbf{C}(s)$ such that the recovery error $E(s)$,

$$E(s) \equiv L_t(s) - \mathbf{C}(s)P(s), \tag{1.2.5}$$

is either exactly or approximately equal to zero in the frequency region of interest while guaranteeing the stability of the resulting closed-loop system. Achieving exact LTR (ELTR) is in general not possible. One seeks then approximate LTR. The notion of 'approximate' LTR has to be defined a little carefully. Here we seek achieving LTR to any arbitrarily desired accuracy. In an attempt to make this feasible, one normally parameterizes $\mathbf{C}(s)$ as a function of a scalar parameter σ and thus obtains a family of controllers $\mathbf{C}(s,\sigma)$. We say asymptotic LTR (ALTR) is achieved if $\mathbf{C}(s,\sigma)P(s) \to L_t(s)$ pointwise in s as $\sigma \to \infty$, i.e., $E(s,\sigma) \to 0$ pointwise in s as $\sigma \to \infty$. Achievability of ALTR enables the designer to choose a member of the family of controllers that corresponds to a particular value of σ which achieves a desired level of recovery. In order to impart precise meanings to ELTR and ALTR, let us next consider the following definitions:

Definition 1.2.1. *The set of admissible target loops* $\mathbf{T}(\Sigma)$ *for the given system* Σ *is defined by*

$$\mathbf{T}(\Sigma) = \{L_t(s) \in \mathcal{M}^{m\times m}(\mathcal{R}_p) \mid L_t(s) = F\Phi B \text{ and } \lambda(A - BF) \in \mathcal{C}^-\}.$$

Definition 1.2.2. *$L_t(s) \in \mathbf{T}(\Sigma)$ is said to be exactly recoverable (ELTR) if there exists a $\mathbf{C}(s) \in \mathcal{M}^{m \times p}(\mathcal{R}_p)$ such that*

(i) the closed-loop system comprising of $\mathbf{C}(s)$ and $P(s)$ as in the configuration of Fig. 1.2.1 is asymptotically stable, and

(ii) $\mathbf{C}(s)P(s) = L_t(s)$.

Definition 1.2.3. *$L_t(s) \in \mathbf{T}(\Sigma)$ is said to be asymptotically recoverable (ALTR) if there exists a parameterized family of controllers $\mathbf{C}(s,\sigma) \in \mathcal{M}^{m \times p}(\mathcal{R}_p)$, where σ is a scalar parameter taking positive values, such that*

(i) the closed-loop system comprising of $\mathbf{C}(s,\sigma)$ and $P(s)$ as in the configuration of Fig. 1.2.1 is asymptotically stable for all $\sigma > \sigma^*$, where $0 \leq \sigma^* < \infty$, and

(ii) $\mathbf{C}(s,\sigma)P(s) \to L_t(s)$ pointwise in s as $\sigma \to \infty$. Moreover, the limits, as $\sigma \to \infty$, of the finite eigenvalues of the closed-loop system should remain in $\mathcal{C}^-$.†

Definition 1.2.4. *$L_t(s)$ belonging to $\mathbf{T}(\Sigma)$ is said to be recoverable if $L_t(s)$ is either exactly or asymptotically recoverable.*

Definition 1.2.5.

1. *The set of exactly recoverable target loops for the given system Σ is denoted by $\mathbf{T}^{ER}(\Sigma)$.*

2. *The set of recoverable target loops for the given system Σ is denoted by $\mathbf{T}^{R}(\Sigma)$.*

3. *The set of target loops which are asymptotically recoverable but not exactly recoverable for the given system Σ is denoted by $\mathbf{T}^{AR}(\Sigma)$.*

Obviously, $\mathbf{T}^{R}(\Sigma) = \mathbf{T}^{ER}(\Sigma) \cup \mathbf{T}^{AR}(\Sigma)$.

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†Here we have strengthened the notion of the closed-loop stability in order to exclude those cases having the limits, as $\sigma \to \infty$, of some finite eigenvalues of the closed-loop system being on the $j\omega$ axis. This avoids having an almost unstable behavior of the closed-loop system for large σ.

1.2.2. LTR design for output break point

We will show next in this subsection, how duality arises for LTR between the two cases when the loop is broken at the input point or at the output point of the plant. The problem of LTR when the loop is broken at the input point of the plant is dual to that when the loop is broken at the output point of the plant. In order to avoid any confusion, we give below a formal step by step algorithm to show how duality arises for LTR at the input and output points.

1. Let the given plant Σ be characterized by the quadruple (A, B, C, D) where A, B, C and D are respectively $n \times n$, $n \times m$, $p \times n$ and $p \times m$ matrices. Also, let $P(s)$ be the transfer function of Σ,

$$P(s) = C(sI_n - A)^{-1}B + D.$$

Let $L_t(s) = C(sI_n - A)^{-1}K$ be an admissible target open-loop transfer function, i.e. $\lambda(A - KC) \in C^-$, when the loop is broken at the output point of the given plant. Then, in the configuration of Figure 1.2.1, we are seeking a controller $\mathbf{C}(s)$ such that the closed loop system is asymptotically stable and

$$E^o(s) := L_t(s) - P(s)\mathbf{C}(s) \equiv 0 \quad \text{for all } s$$

in the case of exact loop transfer recovery when the loop is broken at the output point (ELTRO) of Σ, or we are seeking a controller $\mathbf{C}(s, \sigma)$ such that the closed-loop system is asymptotically stable for all $\sigma > \sigma^*$, where $0 \leq \sigma^* < \infty$, and

$$E^o(s, \sigma) := L_t(s) - P(s)\mathbf{C}(s, \sigma) \to 0 \quad \text{pointwise in } s \text{ as } \sigma \to \infty$$

in the case of asymptotic loop transfer recovery.

2. Define a dual system Σ_d characterized by the quadruple (A_d, B_d, C_d, D_d) where

$$A_d := A', \quad B_d := C', \quad C_d := B', \quad D_d := D'.$$

Note that $P_d(s)$, the transfer function of the dual plant Σ_d is $P'(s)$. Let $L_d(s)$ be defined as

$$L_d(s) := L_t'(s) = F_d(sI_n - A_d)^{-1}B_d,$$

where $F_d := K'$. It is simple to verify that the loop recovery error $E^d(s)$ is

$$E^d(s) := L_d(s) - \mathbf{C}_d(s)P_d(s) = [E^o(s)]'.$$

Then, it follows that $L_t(s)$ is either exactly or asymptotically recoverable at the output point iff $L_d(s)$ is respectively either exactly or asymptotically recoverable for Σ_d at the input point.

3. For the purpose of design alone, consider the fictitious plant Σ_d as given in step 2. Then design a controller $\mathbf{C}_d(s)$ such that the closed-loop system is asymptotically stable and

$$E^i(s) := L_d(s) - \mathbf{C}_d(s)P_d(s) \equiv 0 \quad \text{for all } s$$

in the case of exact loop transfer recovery when the loop is broken at the input point (ELTRI) of Σ_d, or a controller $\mathbf{C}_d(s,\sigma)$ such that the closed-loop system is asymptotically stable for all $\sigma > \sigma^*$, where $0 \leq \sigma^* < \infty$, and

$$E^i(s,\sigma) := L_d(s) - \mathbf{C}_d(s,\sigma)P_d(s) \to 0 \quad \text{pointwise in } s \text{ as } \sigma \to \infty$$

in the case of asymptotic loop transfer recovery when the loop is broken at the input point (ALTRI) of Σ_d.

4. Define a controller $\mathbf{C}(s)$ or $\mathbf{C}(s,\sigma)$:

$$\mathbf{C}(s) := \mathbf{C}_d'(s)$$

or

$$\mathbf{C}(s,\sigma) := \mathbf{C}_d'(s,\sigma).$$

Then it can be verified trivially that the controller $\mathbf{C}(s)$ or $\mathbf{C}(s,\sigma)$ designed above and implemented as in figure 1.2.1 achieves either ELTRO or ALTRO. In fact, by the same reasoning, any LTR analysis or design required at the output point for the plant Σ can equivalently be done by considering the LTR analysis or design for Σ_d at its input point. As such, throughout this thesis, we consider only LTR analysis and design at the input point for any given system Σ.

1.3. Preliminaries

As mentioned earlier, finite and infinite zero structures of both the given system and the target loop transfer function play dominant roles in the recovery analysis as well as design. In fact, the whole subject of LTR can be viewed as a study of assigning an appropriate zero structure to the closed-loop system within the constraints imposed by the zero structures of the given open-loop system and the target loop transfer function. Thus a good nonambiguous understanding of zero structure is essential for our study. Keeping this in mind, we recall in this section a special coordinate basis (s.c.b) of a linear time invariant system [41], [42]. Such a s.c.b has a distinct feature of explicitly displaying the finite and infinite zero structure of a given system. Connections between the s.c.b and the various invariant and almost invariant subspaces of geometric theory as needed for our development are also given. In fact, it is easy to understand all the structural properties of a given system via its s.c.b presented here. The s.c.b described here forms an integral part of all our analysis and design methods throughout this thesis.

Consider a system Σ characterized by the quadruple (A, B, C, D) as in (1.2.1). It is simple to verify that there exist non-singular transformations U and V such that

$$UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{1.3.1}$$

where m_0 is the rank of matrix D. Hence hereafter, without loss of generality, it is assumed that the matrix D has the form given on the right hand side of (1.3.1). One can now rewrite the system of (1.2.1) as,

$$\begin{cases} \dot{x} = Ax + \begin{bmatrix} B_0 & B_1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \\ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \end{cases} \tag{1.3.2}$$

where the matrices B_0, B_1, C_0 and C_1 have appropriate dimensions. We have the following theorem.

Theorem 1.3.1 (s.c.b). *Consider the system Σ characterized by (A, B, C, D). There exist nonsingular transformations Γ_1, Γ_2 and Γ_3, an integer $m_f \leq m - m_0$, and integer indexes q_i, $i = 1$ to m_f, such that*

$$x = \Gamma_1 \tilde{x} \ , \ y = \Gamma_2 \tilde{y} \ , \ u = \Gamma_3 \tilde{u}$$

$$\tilde{x} = [x_a', \ x_b', \ x_c', \ x_f']' \ , \ \tilde{x}_a = [(x_a^-)', (x_a^+)']'$$

$$\tilde{x}_f = [x_1', \ x_2', \ \cdots, x_{m_f}']'$$

$$\tilde{y} = [y_0', \ y_f', \ y_b']' \ , \ y_f = [y_1, \ y_2, \ \cdots, y_{m_f}]'$$

$$\tilde{u} = [u_0', u_f', \ u_c']' \ , \ u_f = [u_1, \ u_2, \ \cdots, u_{m_f}]',$$

and

$$\dot{x}_a^- = A_{aa}^- x_a^- + B_{0a}^- y_0 + L_{af}^- y_f + L_{ab}^- y_b \tag{1.3.3}$$

$$\dot{x}_a^+ = A_{aa}^+ x_a^+ + B_{0a}^+ y_0 + L_{af}^+ y_f + L_{ab}^+ y_b \tag{1.3.4}$$

$$\dot{x}_b = A_{bb} x_b + B_{0b} y_0 + L_{bf} y_f \ , \ y_b = C_b x_b \tag{1.3.5}$$

$$\dot{x}_c = A_{cc} x_c + B_{0c} y_0 + L_{cb} y_b + L_{cf} y_f + B_c [E_{ca}^- x_a^- + E_{ca}^+ x_a^+] + B_c u_c \tag{1.3.6}$$

$$y_0 = C_{0a}^- x_a^- + C_{0a}^+ x_a^+ + C_{0b} x_b + C_{0c} x_c + C_{0f} x_f + u_0 \tag{1.3.7}$$

and for each $i = 1$ to m_f,

$$\dot{x}_i = A_{q_i} x_i + L_{i0} y_0 + L_{if} y_f + B_{q_i} \left[u_i + E_{ia} x_a + E_{ib} x_b + E_{ic} x_c + \sum_{j=1}^{m_f} E_{ij} x_j \right] \tag{1.3.8}$$

$$y_i = C_{q_i} x_i \ , \ y_f = C_f x_f. \tag{1.3.9}$$

Here the states x_a^-, x_a^+, x_b, x_c and x_f are respectively of dimension n_a^-, n_a^+, n_b, n_c and $n_f = \sum_{i=1}^{m_f} q_i$ while x_i is of dimension q_i for each $i = 1$ to m_f. The control vectors u_0, u_f and u_c are respectively of dimension m_0, m_f and $m_c = m - m_0 - m_f$ while the output vectors y_0, y_f and y_b are respectively of dimension $p_0 = m_0$, $p_f = m_f$ and $p_b = p - p_0 - p_f$. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the following form:

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix} \ , \ B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \ C_{q_i} = [1, \ 0, \ \cdots, \ 0]. \tag{1.3.10}$$

(Obviously for the case when $q_i = 1$, we have $A_{q_i} = 0$, $B_{q_i} = 1$ and $C_{q_i} = 1$.) Furthermore, we have $\lambda(A_{aa}^-) \in \mathcal{C}^-$, $\lambda(A_{aa}^+) \in \mathcal{C}^+$, the pair (A_{cc}, B_c) is controllable and the pair (A_{bb}, C_b) is observable. Also, assuming that x_i are arranged such that $q_i \leq q_{i+1}$, the matrix L_{if} has the particular form,

$$L_{if} = [L_{i1}, \ L_{i2}, \ \cdots, \ L_{i\,i-1}, \ 0, \ 0, \ \cdots, \ 0].$$

Also, the last row of each L_{if} is identically zero.

Proof : For strictly proper systems, using a modified structural algorithm of Silverman [49], an explicit procedure of constructing the above s.c.b is given in [41]. The required modifications for nonstrictly proper systems are given in [42]. Here by an obvious change of basis, the variable x_a is further decomposed into x_a^- and x_a^+. Also, a software package to generate the s.c.b of any given system is given by [26]. ∎

Remark 1.3.1. *Given a specified region $\mathcal{C}_g$ of complex plane, one can easily decompose x_a into x_a^- and x_a^+ such that $\lambda(A_{aa}^-) \in \mathcal{C}_g$ and $\lambda(A_{aa}^+) \notin \mathcal{C}_g$.*

We can rewrite the s.c.b given by theorem 1.3.1 in a more compact form,

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \ , \ \tilde{y} = \tilde{C}\tilde{x} + \tilde{D}\tilde{u}, \tag{1.3.11}$$

where

$$\tilde{A} := \Gamma_1^{-1}(A - B_0C_0)\Gamma_1 = \begin{bmatrix} A_{aa}^- & 0 & L_{ab}^-C_b & 0 & L_{af}^-C_f \\ 0 & A_{aa}^+ & L_{ab}^+C_b & 0 & L_{af}^+C_f \\ 0 & 0 & A_{bb} & 0 & L_{bf}C_f \\ B_cE_{ca}^- & B_cE_{ca}^+ & L_{cb}C_b & A_{cc} & L_{cf}C_f \\ B_fE_a^- & B_fE_a^+ & B_fE_b & B_fE_c & A_f \end{bmatrix},$$

$$\tilde{B} := \Gamma_1^{-1}[B_0 \quad B_1]\Gamma_3 = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0f} & B_f & 0 \end{bmatrix},$$

$$\tilde{C} := \Gamma_2^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_1 = \begin{bmatrix} C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0c} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix},$$

and

$$\tilde{D} := \Gamma_2^{-1}D\Gamma_3 = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can also define a dual system Σ_d characterized by the quadruple (A_d, B_d, C_d, D_d) where

$$A_d := A', \quad B_d := C', \quad C_d := B', \quad D_d := D'. \tag{1.3.12}$$

In what follows, we state some important properties of the s.c.b which are pertinent to our present work. These properties are stated without proofs, however the proofs are straightforward and simple.

Property 1.3.1. *The given system Σ is right invertible iff x_b and hence y_b are nonexistent ($n_b = 0$, $p_b = 0$), left invertible iff x_c and hence u_c are nonexistent ($n_c = 0$, $m_c = 0$), invertible iff both x_b and x_c are nonexistent. Moreover, Σ is degenerate iff it is neither left nor right invertible.*

Property 1.3.2. *We note that (A_{bb}, C_b) and (A_{q_i}, C_{q_i}) form observable pairs. Unobservability could arise only in the variables x_a and x_c. In fact, the system Σ is observable (detectable) iff (A_{obs}, C_{obs}) is an observable (detectable) pair, where*

$$A_{obs} = \begin{bmatrix} A_{aa} & 0 \\ B_cE_{ca} & A_{cc} \end{bmatrix}, \quad A_{aa} = \begin{bmatrix} A_{aa}^- & 0 \\ 0 & A_{aa}^+ \end{bmatrix}, \quad C_{obs} = \begin{bmatrix} C_{0a} & C_{0c} \\ E_a & E_c \end{bmatrix},$$

$$C_{0a} = [C_{0a}^-, C_{0a}^+], \quad E_a = [E_a^-, E_a^+], \quad E_{ca} = [E_{ca}^-, E_{ca}^+].$$

Similarly, (A_{cc}, B_c) and (A_{q_i}, B_{q_i}) form controllable pairs. Uncontrollability could arise only in the variables x_a and x_b. In fact, Σ is controllable (stabilizable) iff (A_{con}, B_{con}) is a controllable (stabilizable) pair, where

$$A_{con} = \begin{bmatrix} A_{aa} & L_{ab}C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{con} = \begin{bmatrix} B_{0a} & L_{af} \\ B_{0b} & L_{bf} \end{bmatrix},$$

$$B_{0a} = \begin{bmatrix} B_{0a}^- \\ B_{0a}^+ \end{bmatrix}, \quad L_{ab} = \begin{bmatrix} L_{ab}^- \\ L_{ab}^+ \end{bmatrix}, \quad L_{af} = \begin{bmatrix} L_{af}^- \\ L_{af}^+ \end{bmatrix}.$$

Property 1.3.3. *Invariant zeros of Σ are the eigenvalues of A_{aa}. Moreover, the stable and the unstable invariant zeros of Σ are the eigenvalues of A_{aa}^- and A_{aa}^+, respectively. Thus, the system Σ is of minimum phase iff $n_a^+ = 0$ and hence A_{aa}^+ is nonexistent. On the other hand, Σ is of nonminimum phase iff $n_a^+ > 0$.*

There are interconnections between the s.c.b and the invariant and almost invariant geometric subspaces. To show these interconnections, we define the following geometric subspaces of a linear system.

Definition 1.3.1.

1. *$\mathcal{V}^g(\Sigma)$ — the maximal subspace of $\Re^n$ which is $(A + BF)$-invariant and contained in* Ker $(C + DF)$ *such that the eigenvalues of $(A + BF)|\mathcal{V}^g$ are contained in $\mathcal{C}_g \subseteq \mathcal{C}$ for some F.*

2. *$\mathcal{S}^g(\Sigma)$ — the minimal $(A + KC)$-invariant subspace of $\Re^n$ containing in* Im $(B + KD)$ *such that the eigenvalues of the map which is induced by $(A + KC)$ on the factor space $\Re^n/\mathcal{S}^g$ are contained in $\mathcal{C}_g \subseteq \mathcal{C}$ for some K.*

For the cases that $\mathcal{C}_g = \mathcal{C}$, $\mathcal{C}_g = \mathcal{C}^-$ and $\mathcal{C}_g = \mathcal{C}^+$, we replace the index ^{g} in $\mathcal{V}^g$ and $\mathcal{S}^g$ by *, $^-$ and $^+$, respectively.

Various components of the state vector of s.c.b have the following geometrical interpretations.

Property 1.3.4.

1. $x_a^- \oplus x_a^+ \oplus x_c$ *spans* $\mathcal{V}^*(\Sigma)$.
2. $x_a^- \oplus x_c$ *spans* $\mathcal{V}^-(\Sigma)$.
3. $x_a^+ \oplus x_c$ *spans* $\mathcal{V}^+(\Sigma)$.
4. $x_c \oplus x_f$ *spans* $\mathcal{S}^*(\Sigma)$.
5. $x_a^- \oplus x_c \oplus x_f$ *spans* $\mathcal{S}^+(\Sigma)$.
6. $x_a^+ \oplus x_c \oplus x_f$ *spans* $\mathcal{S}^-(\Sigma)$.

Chapter 2

OBSERVER BASED CONTROLLERS

2.1. Full and reduced order observer based controllers

In this chapter, we give the structural details of both full and reduced order observer based controllers commonly used in LTR. Other controller structures for LTR are possible and one such structure, called compensator structure [10], is discussed in chapter 5. As might be expected, the two controllers considered here might have different capabilities regarding LTR; but as will be seen shortly there exists a common mathematical machinery to analyze them under a single frame work. In subsequent chapters, we will systematically do LTR analysis using a generic controller which could be either of the two controllers. In such an analysis, we will use the following notation :

$\mathrm{C}(s) :=$ transfer function of the controller,

$L(s) := \mathrm{C}(s)P(s) =$ achieved loop transfer function,

$S(s) := [I_m + L(s)]^{-1} =$ achieved sensitivity function,

$T(s) := I_m - S(s) =$ achieved complementary sensitivity function,

$E(s) := L_t(s) - L(s) =$ recovery error,

$M(s) :=$ The recovery matrix (to be defined later on),

$\overline{M}^e(s) =$ *recovery error matrix*, which is the limit of the recovery matrix,

$$\mathbf{T}^{\mathrm{ER}}(\Sigma) = \text{ set of exactly recoverable target loops for } \Sigma,$$

$$\mathbf{T}^{\mathrm{R}}(\Sigma) = \text{ set of recoverable target loops for } \Sigma,$$

$$\mathbf{T}^{\mathrm{AR}}(\Sigma) = \text{ set of asymptotically but not exactly recoverable target loops for } \Sigma.$$

The above notation applies to a generic controller, however, whenever we refer to a particular type of controller, we shall use appropriate subscripts to identify it. Subscripts f and r are used respectively to represent full and reduced order observer based controllers. For example, $\overline{M}_f^e(s)$ and $\mathbf{T}_r^{\mathrm{R}}(\Sigma)$ denote respectively the recovery error matrix when a full order observer based controller is used, and the set of recoverable target loops for Σ when a reduced order observer based controller is used.

We now proceed to give the structural details of the controllers considered here.

2.1.1. Full order observer based controller

The dynamic equations of a full order observer based controller are

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x} - Du), \\ u = \hat{u} = -F\hat{x}, \end{cases} \tag{2.1.1}$$

where K_f is the observer gain chosen so that $A - K_fC$ is asymptotically stable, and F is the state feedback gain that prescribes the target loop transfer function $L_t(s) = F\Phi B$. The transfer function of the controller is

$$\mathbf{C}_f(s) = F[sI_n - A + BF + K_fC - K_fDF]^{-1}K_f. \tag{2.1.2}$$

2.1.2. Reduced order observer based controller

In this case, without any loss of generality but for simplicity of presentation, it is assumed that the matrices C and D have been transformed into the form,

$$C = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} D_0 \\ 0 \end{bmatrix}.$$

Then the dynamic equations of Σ can be partitioned as follows,

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u, \\ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_0 \\ 0 \end{bmatrix} u. \end{cases} \tag{2.1.3}$$

We note that $y_1 = x_1$ is already available and need not be estimated. Hence one needs only the estimate of x_2. To proceed further, let us rewrite the state equation for x_1 in terms of the output y_1 and state x_2 as,

$$\dot{y}_1 = A_{11}y_1 + A_{12}x_2 + B_{11}u. \tag{2.1.4}$$

Since $\dot{y}_1$ and y_1 are known, (2.1.4) can be rewritten as

$$\tilde{y}_1 = A_{12}x_2 + B_{11}u = \dot{y}_1 - A_{11}y_1. \tag{2.1.5}$$

Thus, observation of x_2 is made via (2.1.5) as well as by

$$y_0 = C_{02}x_2 + D_0u.$$

Now, a reduced order system suitable for estimating the state x_2 is given by

$$\begin{cases} \dot{x}_2 = A_r\, x_2 + B_r\, u + A_{21}\, y_1, \\ \begin{bmatrix} y_0 \\ \tilde{y}_1 \end{bmatrix} = y_r = C_r x_2 + D_r u, \end{cases} \tag{2.1.6}$$

where

$$A_r = A_{22}, \quad B_r = B_{22}, \quad C_r = \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix}, \quad D_r = \begin{bmatrix} D_0 \\ B_{11} \end{bmatrix}. \tag{2.1.7}$$

Based on equation (2.1.6), we can construct a reduced order estimate of the state x_2 as,

$$\dot{\hat{x}}_2 = A_r\hat{x}_2 + B_r u + A_{21}y_1 + K_r\left(y_r - C_r\hat{x}_2 - D_r u\right), \tag{2.1.8}$$

where K_r is the reduced order estimator gain matrix and is chosen such that $A_r - K_rC_r$ is asymptotically stable. For the purpose of implementing it, (2.1.8) can be rewritten by partitioning $K_r = [K_{r0},\ K_{r1}]$ in conformity with y_0 and $\tilde{y}_1$ and by defining the following variable v,

$$v = \hat{x}_2 - K_{r1}y_1. \tag{2.1.9}$$

Then the reduced order observer based controller becomes

$$\begin{cases} \dot{v} = (A_r - K_r C_r)v + (B_r - K_r D_r)u + G_r y, \\ u = \hat{u} = -F_1 x_1 - F_2 \hat{x}_2 = -F_2 v - [0, \ F_1 + F_2 K_{r1}]y, \end{cases} \tag{2.1.10}$$

where

$$F = [\,F_1, \quad F_2\,], \qquad G_r = [K_{r0}, \ A_{21} - K_{r1}A_{11} + (A_r - K_r C_r)K_{r1}]. \tag{2.1.11}$$

The transfer function from $-u$ to y that results in using the reduced order estimator is then given by

$$\mathbf{C}_r(s) = F_2(sI - A_r + K_r C_r + B_r F_2 - K_r D_r F_2)^{-1} \cdot \Big(G_r - (B_r - K_r D_r)[0, \ F_1 + F_2 K_{r1}]\Big) + [0, \ F_1 + F_2 K_{r1}]. \tag{2.1.12}$$

Proposition 2.1.1. *For the case when Σ is right invertible and the matrix D is of maximal rank, full and reduced order observer based controllers coalesce into one and the same.*

Proof : When Σ is right invertible and matrix D is of maximal rank, we have

$$A_r = A_{22} = A, \quad B_r = B_{22} = B, \quad C_r = C_{02} = C, \quad D_r = D_0 = D.$$

Using these facts, it is easy to verify the above proposition. ∎

2.2. Preliminary analysis

We proceed now to do some preliminary analysis of recovery error $E(s)$. It turns out that the expression, $E(s) = L_t(s) - L(s)$, is not well suited for loop transfer recovery analysis. Realizing this, for the class of systems he considered, Goodman [22] related $E(s)$ to a matrix $M(s)$, here after called the recovery matrix. The following lemma generalizes Goodman's result for general nonstrictly proper systems and for both full and reduced order observer based controllers considered here.

Lemma 2.2.1. *Let Σ be stabilizable and detectable. Also, let $L_t(s) = F\Phi B$ be an admissible target loop, i.e. $L_t(s) \in \mathbf{T}(\Sigma)$. Then the error, $E(s)$, between the target loop transfer*

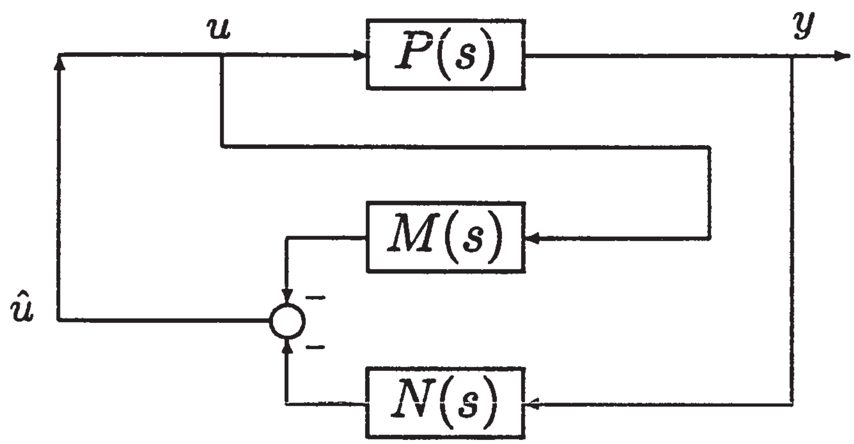


Figure 2.2.1: Plant and controller configuration.

function $L_t(s)$ and that realized by either of the controllers considered here, can be written in the form,

$$E(s) = M(s)[I_m + M(s)]^{-1}(I_m + F\Phi B). \tag{2.2.1}$$

Furthermore for all $\omega \in \Omega$,

$$E(j\omega) = 0 \text{ iff } M(j\omega) = 0$$

where Ω is the set of all $0 \leq |\omega| < \infty$ for which $L_t(j\omega)$ and $L(j\omega) = C(j\omega)P(j\omega)$ are well defined (i.e. all the required inverses exist). The expression for the recovery matrix $M(s)$ depends on the controller used. In particular, we have

$$M_f(s) = F(sI_n - A + K_fC)^{-1}(B - K_fD), \tag{2.2.2}$$

$$M_r(s) = F_2(sI - A_r + K_rC_r)^{-1}(B_r - K_rD_r). \tag{2.2.3}$$

Proof : See Appendix 2.A. ∎

A physical interpretation of the recovery matrix $M(s)$ can be given. To do so, one can view the controller as a device having two inputs, (1) the plant input u and (2) the plant output y as shown in 2.2.1. Then, $-M(s)$ is the transfer function from the plant input point to the controller output point while $\tilde{M}(s)$ is the transfer function from the plant input point to the estimated state $\hat{x}$. That is, one can write

$$\hat{U}(s) = -F\hat{X}(s) = -M(s)U(s) - N(s)Y(s),$$

and

$$\hat{X}(s) = \tilde{M}(s)U(s) + \tilde{N}(s)Y(s).$$

Here, depending on the type of controller used, the expressions for $M(s)$ are as given in (2.2.2) and (2.2.3). Also, $\tilde{M}(s)$ is such that $M(s) = F\tilde{M}(s)$. Moreover, for each type of controller, the expressions for $N(s)$ and $\tilde{N}(s)$ are as given below.

$$N_f(s) = F(sI - A + K_f C)^{-1} K_f \ , \ N_f(s) = F\tilde{N}_f(s) \, ,$$

$$N_r(s) = F_2(sI - A_r + K_r C_r)^{-1} G_r + [0, \, F_1 + F_2 K_{r1}] \ , \ N_r(s) = F\tilde{N}_r(s) \, .$$

In view of the above expressions, lemma 2.2.1 implies that whenever LTR is achieved, the controller output does not entail any feedback from the plant input point. On the other hand, the state estimate $\hat{x}$ in general depends on the plant input. The significance of lemma 2.2.1 can be seen in two ways. It converts the LTR analysis problem into a study of conditions under which $M(s)$ can be rendered zero. Also, it unifies the study of $M(s)$ for both types of controllers into a single mathematical framework. To see this explicitly, let us define an auxiliary system Σ_r characterized by the matrix quadruple (A_r, B_r, C_r, D_r). Then we have the following observation.

Observation 2.2.1. *The LTR mechanism for the given system Σ using a reduced order observer based controller can be studied using the auxiliary system Σ_r using a full order observer based controller constructed for it where in F_2 takes the place of F.*

In view of lemma 2.2.1 and observation 2.2.1, our study of LTR for both types of controllers is unified and reduces to the study of an appropriate recovery matrix $M(s)$. In order to further cement such a unification, we need to investigate the relationship between the structural properties of Σ_r and Σ. The following proposition delineates such a relationship.

Proposition 2.2.1.

1. *Σ_r is of (non-) minimum phase iff Σ is of (non-) minimum phase.*
2. *Σ_r is detectable iff Σ is detectable.*
3. *Invariant zeros of Σ_r are the same as those of Σ.*
4. *Orders of infinite zeros of Σ_r are reduced by one from those of Σ.*
5. *Σ_r is left invertible iff Σ is left invertible.*
6. $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{V}^+(\Sigma_r) = \mathcal{V}^+(\Sigma)$.
7. $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r) = \mathcal{S}^-(\Sigma) \cap \mho$, *where* $\mho := \{x \mid Cx \in \mathrm{Im}(D)\}$.
8. $\mathcal{S}^-(\Sigma_r) = \emptyset$ *iff Σ is left invertible and of minimum phase with no infinite zeros of order higher than one.*

Proof : See Appendix 2.B. ∎

Remark 2.2.1. *For a left invertible and minimum phase system Σ, it is simple to see that $\mathcal{S}^-(\Sigma_r) = \mathcal{S}^-(\Sigma) \cap \mho = \emptyset$ iff Σ has no infinite zeros of order higher than one. Also, for a nonstrictly proper single-input-single-output system Σ, $\mathcal{S}^-(\Sigma) = \mathcal{S}^-(\Sigma_r) = \mathcal{S}^-(\Sigma) \cap \mho = \emptyset$ iff it is of minimum phase.*

2.A. Appendix 2.A — Proof of Lemma 2.2.1

To simplify and to unify the proof of lemma 2.2.1 for both full and reduced order observer based controllers, fisrt we examine the following Luenberger observer based controller:

$$\begin{cases} \dot{v} &= Lv + Gu + Hy, \\ -u &= F\hat{x} = Pv + Vy, \end{cases} \tag{2.A.1}$$

where $v \in \Re^r$ with r being the order of the controller. It is well known (see e.g., [32]) that $\hat{x}$ is an asymptotic estimate of the state x provided that (a) L is an asymptotically stable matrix and (b) there exists a matrix $T \in \Re^{r \times n}$ satisfying the following conditions:

1. $TA - LT = HC$,

2. $G = TB - HD$,
3. $F = PT + VC$ and
4. $VD = 0$.

We have the following proposition given earlier in Niemann *et al* [32] although our proof of it is slightly different from that of [32].

Proposition 2.A.1. *Consider any admissible target loop transfer function $L_t(s) = F\Phi B$. Then the error $E(s)$ realized by Luenberger observer based controller of (2.A.1) is given by*

$$E(s) = M(s)[I_m + M(s)]^{-1}(I_m + F\Phi B), \tag{2.A.2}$$

where

$$M(s) = P(sI - L)^{-1}G. \tag{2.A.3}$$

Proof of Proposition 2.A.1 : It is straightforward to see that the transfer function of a Luenberger observer based controller is given by

$$\begin{aligned} \mathbf{C}(s) &= V + P(sI - L + GP)^{-1}(H - GV) \\ &= V + [I + P\Phi_\ell G]^{-1}P\Phi_\ell(H - GV) \\ &= [I + M(s)]^{-1}[V + P\Phi_\ell H], \end{aligned} \tag{2.A.4}$$

where $\Phi_\ell := (sI - L)^{-1}$. Also, using the fact that $TA - LT = HC$, it is trivial to verify that

$$P\Phi_\ell TB + P\Phi_\ell HC\Phi B - PT\Phi B = 0. \tag{2.A.5}$$

Then we have

$$\mathbf{C}(s)P(s) = [I + M(s)]^{-1}[V + P\Phi_\ell H](C\Phi B + D) \tag{2.A.6}$$

$$= [I + M(s)]^{-1}[VC\Phi B + P\Phi_\ell HC\Phi B + P\Phi_\ell HD] \tag{2.A.7}$$

$$= [I + M(s)]^{-1}[F\Phi B - PT\Phi B + P\Phi_\ell HC\Phi B + P\Phi_\ell HD] \tag{2.A.8}$$

$$= [I + M(s)]^{-1}[L_t(s) - PT\Phi B + P\Phi_\ell HC\Phi B + P\Phi_\ell TB - P\Phi_\ell G] \tag{2.A.9}$$

$$= [I + M(s)]^{-1}[L_t(s) - M(s)] \tag{2.A.10}$$

Note that we used the facts, $VD = 0$ to get (2.A.7) from (2.A.6); $F = PT + VC$ to get (2.A.8) from (2.A.7); $G = TB - HD$ to get (2.A.9) from (2.A.8), and finally (2.A.5) to get (2.A.10) from (2.A.9). Now it simple to show that

$$\begin{aligned} E(s) &= L_t(s) - \mathbf{C}(s)P(s) \\ &= M(s)[I + M(s)]^{-1}(I + F\Phi B). \end{aligned}$$

This completes the proof of proposition 2.A.1. □

Now, it is straightforward to verify that the full order observer based controller is a special case of Luenberger observer based controller in (2.A.1) with

$$\begin{cases} L = A - K_f C, & G = B - K_f D, \quad H = K_f, \\ P = F, & V = 0, \qquad T = I. \end{cases}$$

Similarly, the reduced order observer based controller is also a special case of (2.A.1) with

$$\begin{cases} L = A_{22} - K_r C_r, & G = B_{22} - K_r D_r, \qquad H = G_r, \\ P = F_2, & V = [0,\ F_1 + F_2 K_{r1}], \quad T = [-K_{r1},\ I]. \end{cases}$$

Hence, equations (2.2.2) and (2.2.3) of lemma 2.2.1 follow trivially from (2.A.2) and (2.A.3). ∎

2.B. Appendix 2.B — Proof of Proposition 2.2.1

Consider a linear time-invariant system characterized by

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du. \end{cases} \tag{2.B.1}$$

Without loss of generality, we can assume that the matrices C and D are of the form

$$C = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, we can partition the system (2.B.1) as,

$$\Sigma : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{0,1} & B_{1,1} \\ B_{0,2} & B_{1,2} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \\ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}. \end{cases} \tag{2.B.2}$$

Next, in this proof, we rewrite the special coordinate basis of section 2.2 slightly in an expanded format. That is, we can choose the transformations Γ_s, Γ_i and Γ_o such that

$$\Gamma_s^{-1}\begin{bmatrix} A_{11} & A_{12}-B_{0,1}C_{02} \\ A_{21} & A_{22}-B_{0,2}C_{02}\end{bmatrix}\Gamma_s$$

$$=\begin{bmatrix} A_a & L_{ab} & 0 & 0 & L_{af1} & L_{af0} & 0 \\ 0 & A_{b11} & C_{b2} & 0 & L_{bf11} & L_{bf10} & 0 \\ 0 & A_{b21} & A_{b22} & 0 & L_{bf21} & L_{bf20} & 0 \\ B_cE_{ca} & L_{cb} & 0. & A_c & L_{cf1} & L_{cf0} & 0 \\ E_{fa1} & E_{fb11} & E_{fb12} & E_{fc1} & A_{f11} & A_{f10} & A_{f12} \\ 0 & 0 & 0 & 0 & A_{f01} & A_{f00} & C_{f2} \\ B_{f2}E_{fa2} & B_{f2}E_{fb21} & B_{f2}E_{fb22} & B_{f2}E_{fc2} & A_{f21} & A_{f20} & A_{f22}\end{bmatrix},$$

$$\Gamma_s^{-1}\begin{bmatrix} B_{0,1} & B_{1,1} \\ B_{0,2} & B_{1,2}\end{bmatrix}\Gamma_i=\begin{bmatrix} B_{a0} & 0 & 0 & 0 \\ B_{b01} & 0 & 0 & 0 \\ B_{b02} & 0 & 0 & 0 \\ B_{c0} & 0 & 0 & B_c \\ B_{f01} & I & 0 & 0 \\ B_{f00} & 0 & 0 & 0 \\ B_{f02} & 0 & B_{f2} & 0\end{bmatrix}, \tag{2.B.3}$$

$$\Gamma_o^{-1}\begin{bmatrix} 0 & C_{02} \\ I & 0\end{bmatrix}\Gamma_s=\begin{bmatrix} C_{0a} & 0 & C_{0b2} & C_{0c} & 0 & 0 & C_{0f2} \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0\end{bmatrix}, \tag{2.B.4}$$

$$\Gamma_o^{-1}\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}\Gamma_s=\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}. \tag{2.B.5}$$

Here we have decomposed our state space into seven subspaces, $x = x_a \oplus x_{b1} \oplus x_{b2} \oplus x_c \oplus x_{f1} \oplus x_{fh} \oplus x_{fr}$. Here $x_a = x_a^- \oplus x_a^+$, i.e., x_a is related to all finite invariant zeros of the system; $x_b = x_{b1} \oplus x_{b2}$ where $x_{b2} = x_b \cap \operatorname{Ker}(C_b)$; x_c is the same space as in the standard s.c.b; finally $x_f = x_{f1} \oplus x_{fh} \oplus x_{fr}$. Note that x_f is related to the infinite zero structure of the given system Σ. In particular, x_{f1} is a part of the output which is related to infinite zeros of order one, x_{fh} is the rest of the output and it is related to infinite zeros of order higher than one, and x_{fr} is $x_f \cap \operatorname{Ker}(C_f)$.

We note that

$$\left(A_{22}, [B_{0,2}, B_{1,2}], \Gamma\begin{bmatrix} C_{02} \\ A_{12}\end{bmatrix}, \Gamma\begin{bmatrix} I & 0 \\ B_{0,1} & B_{1,1}\end{bmatrix}\right)$$

$$= \left(A_{22}, [B_{0,2},\ B_{1,2}], \begin{bmatrix} C_{02} \\ A_{12} - B_{0,1}C_{02} \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & B_{1,1} \end{bmatrix}\right),$$

where Γ is nonsingular and is given by

$$\Gamma = \begin{bmatrix} I & 0 \\ -B_{0,1} & I \end{bmatrix}.$$

Hence, it is sufficient to prove proposition 2.2.1 for the new reduced order system characterized by

$$\left(A_{22},\ [B_{0,2},\ B_{1,2}],\ \begin{bmatrix} C_{02} \\ A_{12} - B_{0,1}C_{02} \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & B_{1,1} \end{bmatrix}\right).$$

From (2.B.3) to (2.B.5), we obtain

$$A_{22} - B_{0,2}C_{02} = \begin{bmatrix} A_a & 0 & 0 & 0 \\ 0 & A_{b22} & 0 & 0 \\ B_cE_{ca} & 0 & A_{cc} & 0 \\ B_{f2}E_{fa2} & B_{f2}E_{b22} & B_{f2}E_{c2} & A_{f22} \end{bmatrix},$$

$$[\,B_{0,2} \quad B_{1,2}\,] = \begin{bmatrix} B_{a0} & 0 & 0 & 0 \\ B_{b02} & 0 & 0 & 0 \\ B_{c0} & 0 & 0 & B_c \\ B_{f02} & 0 & B_{f2} & 0 \end{bmatrix},$$

$$\begin{bmatrix} C_{02} \\ A_{12} - B_{0,1}C_{02} \end{bmatrix} = \begin{bmatrix} C_{0a} & C_{0b2} & C_{0c} & C_{0f2} \\ E_{a1} & E_{b12} & E_{c1} & A_{f12} \\ 0 & 0 & 0 & C_{f2} \\ 0 & C_{b2} & 0 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} I & 0 \\ 0 & B_{1,1} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is interesting to observe from the above equations that

$$\left(A_{22},\ [B_{0,2},\ B_{1,2}],\ \begin{bmatrix} C_{02} \\ A_{12} - B_{0,1}C_{02} \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & B_{1,1} \end{bmatrix}\right)$$

is already in the form of a special coordinate basis. Then all the properties listed in proposition 2.2.1 follows trivially from the properties of the special coordinate basis discussed earlier in section 2.2. ∎

Chapter 3

GENERAL LTR ANALYSIS

3.1. Introduction

This chapter deals with the general analysis of LTR mechanism using either of the controllers discussed in the previous chapter. Notationally, in all our general discussion here, we deal with the given system Σ characterized by the quadruple (A, B, C, D) and the full order observer based controller in which K_f is the observer gain. In view of observation 2.2.1, all the general discussion presented here can be particularized to the reduced order observer based controller with appropriate notational changes. In all our main theorems, we will however explicitly point out the capabilities of each type of controller as they might be different.

As evident from lemma 2.2.1, the nucleus of LTR analysis is the study of $M_f(s)$ to ascertain how and when it can or cannot be rendered zero. The required study of $M_f(s)$ can be undertaken in two ways, with or without the prior knowledge of F that prescribes the target loop transfer function $L_t(s)$. Note that the study of $M_f(s)$ without the prior knowledge of F imitates the traditional LQG design philosophy in which the two tasks of obtaining F and K_f are separated. Keeping this in mind, our goal in the next section is to study $M_f(s)$ without taking into account any specific characteristics of F. The following section is devoted to LTR analysis while taking into account appropriate characteristics of F. It complements the analysis of the earlier section. Decomposing $M_f(s)$ as $F\tilde{M}_f(s)$, the study of $M_f(s)$ without knowing F is the same one as the study of $\tilde{M}_f(s)$. A detailed study of $\tilde{M}_f(s)$ leads to two fundamental lemmas, one dealing with finite and another dealing with asymptotically infinite eigenstructure assignment to the observer dynamic matrix $A - K_f C$ by an appropriate design of K_f. These two lemmas reveal the limitations of the given system as a consequence of its structural properties in recovering an arbitrary target loop transfer function via either a full or reduced order observer based controller. Furthermore, they enable us to decompose $\tilde{M}_f(s)$ into three essential parts, $\tilde{M}_f^0(s)$, $\tilde{M}_f^\infty(s)$ and $\tilde{M}_f^e(s)$. The first part $\tilde{M}_f^0(s)$ can be rendered either exactly or asymptotically zero by an appropriate finite eigenstructure assignment to $A - K_f C$, while the second part $\tilde{M}_f^\infty(s)$ can be rendered asymptotically zero by an appropriate infinite eigenstructure assignment to $A - K_f C$. The third part $\tilde{M}_f^e(s)$ in general cannot be rendered zero, either exactly or asymptotically, by any means, although our analysis of $\tilde{M}_f^e(s)$ reveals a multitude of ways by which it can be shaped. All in all, the decomposition of $\tilde{M}_f(s)$ into several parts and the subsequent analysis of each part forms the gist of the LTR analysis discussed throughout this book. In particular, it leads to several important results given in this chapter. For example, theorem 3.2.2 characterizes the asymptotic behavior of loop transfer function as well as sensitivity and complementary sensitivity functions achievable by either full or reduced order observer based controllers. On the other hand, theorem 3.2.3 shows the subspace $\mathcal{S}^e \in \Re^m$ in which $\tilde{M}_f^e(s)$ can be rendered zero asymptotically, i.e, the projections of the target and achievable sensitivity and complementary sensitivity functions onto $\mathcal{S}^e$ can match each other asymptotically. Next, in section 3.3, theorems 3.3.1 and 3.3.3 develop the necessary and sufficient conditions a target loop transfer function $L_t(s)$ must satisfy so that it can be either exactly or asymptotically recoverable for the given system Σ. On the other hand, theorems 3.3.2 and 3.3.4 develop the necessary and sufficient conditions on Σ so that it has at least one either exactly or asymptotically recoverable target loop transfer function. Section 3.4 generalizes the results of sections 3.2 and 3.3 when recovery is important over a prescribed subspace of the control space. Furthermore, our analysis in this chapter reveals the mechanism of pole zero cancellation between the controller eigenvalues and the input or output decoupling zeros of Σ.

3.2. Recovery analysis for an arbitrary target loop

In this section, we consider that the target loop transfer function $L_t(s) = F\Phi B$ is arbitrary. That is, we do not take into account any specific characteristics of $L_t(s)$ in analyzing the LTR mechanism. Then, as implied by lemma 2.2.1, $\tilde{M}_f(s)$ as given below forms the basis of our study,

$$\tilde{M}_f(s) = (sI_n - A + K_f C)^{-1}(B - K_f D). \tag{3.2.1}$$

It is evident that the gain K_f is the only free design parameter in $\tilde{M}_f(s)$. First of all, in order to guarantee the closed-loop stability, K_f must be such that $A - K_f C$ is an asymptotically stable matrix. The remaining freedom in choosing K_f can then be used for the purpose of achieving LTR. We note that exact loop transfer recovery (ELTR) is possible for an arbitrary F iff

$$\tilde{M}_f(j\omega) = (j\omega I_n - A + K_f C)^{-1}(B - K_f D) \equiv 0.$$

However, due to the nonsingularity of $(j\omega I_n - A + K_f C)^{-1}$, the fact that $\tilde{M}_f(j\omega) \equiv 0$ implies that $B - K_f D \equiv 0$. The class of systems in which $B - K_f D$ can be rendered exactly zero is very restrictive, and hence one normally attempts to achieve asymptotic loop transfer recovery (ALTR), i.e. to render $\tilde{M}_f(j\omega)$ approximately zero in some sense. In order to analyze whether ALTR is possible, as mentioned in the introduction, we parameterize the gain K_f with a tuning parameter σ and there-by creating a family of controllers,

$$\mathbf{C}_f(s,\sigma) = F[sI_n - A + BF + K_f(\sigma)C - K_f(\sigma)DF]^{-1}K_f(\sigma). \tag{3.2.2}$$

In this case, $M_f(s)$ and $\tilde{M}_f(s)$ are functions of σ and are denoted respectively by $M_f(s,\sigma)$ and $\tilde{M}_f(s,\sigma)$. To proceed with our analysis, for clarity of presentation we will temporarily assume that $A - K_f C$ is nondefective. This allows us to expand $\tilde{M}_f(s,\sigma)$ and hence $M_f(s,\sigma)$ in a dyadic form,

$$\tilde{M}_f(s,\sigma) = \sum_{i=1}^{n} \frac{\tilde{R}_i}{s - \lambda_i} \tag{3.2.3}$$

where the residue $\tilde{R}_i$ is given by

$$\tilde{R}_i = W_i V_i^H [B - K_f(\sigma) D]. \tag{3.2.4}$$

Here W_i and V_i are respectively the right and left eigenvectors associated with the eigenvalue λ_i of $A - K_f(\sigma)C$ and they are scaled so that $WV^H = V^H W = I_n$ where

$$W = [W_1, W_2, \cdots, W_n] \quad \text{and} \quad V = [V_1, V_2, \cdots, V_n]. \tag{3.2.5}$$

In general, all λ_i, V_i and W_i are functions of σ. However, for economy of notation we will not show the dependence on σ explicitly unless it is needed for clarity.

Remark 3.2.1. *The assumption that $K_f(\sigma)$ is selected so that $A - K_f(\sigma)C$ is nondefective is not essential. However, it simplifies our presentation. A removal of this condition necessitates the use of generalized right and left eigenvectors of $A - K_f(\sigma)C$ instead of the right and left eigenvectors W_i and V_i and consequently the expansion of $\tilde{M}_f(s,\sigma)$ requires a double summation in place of the single summation used in (3.2.3).*

We are looking for conditions under which the i-th term of $\tilde{M}_f(s,\sigma)$ in (3.2.3) can be made zero for each $i = 1$ to n. There are only two possibilities to do so.

1. The first possibility is by assigning λ_i to any finite value in C^- while simultaneously rendering the corresponding residue $\tilde{R}_i$ zero either exactly or asymptotically, i.e. $\tilde{R}_i = 0$ or $\tilde{R}_i \to 0$ as $\sigma \to \infty$. Thus this possibility deals with finite eigenstructure assignment to $A - K_f(\sigma)C$.

2. The second possibility is to make $\frac{\tilde{R}_i}{s-\lambda_i} \to 0$ pointwise in s as $\sigma \to \infty$. This can be done by placing the eigenvalue $\lambda_i(\sigma)$ asymptotically at infinity while making sure

that the corresponding residue $\tilde{R}_i$ is uniformly bounded as $\sigma \to \infty$. It is important to recognize that placing λ_i asymptotically at infinity alone is not beneficial unless the corresponding residue $\tilde{R}_i$ is bounded. This amounts to assigning $W_i(\sigma)$ and $V_i(\sigma)$ such that $\tilde{R}_i = W_i(\sigma)V_i^H(\sigma)[B - K_f(\sigma)D]$ remains bounded while $\lambda_i \to \infty$ as $\sigma \to \infty$. Thus this possibility deals with infinite eigenstructure assignment to $A - K_f(\sigma)C$.

The above two possibilities of making terms of $\tilde{M}_f(s,\sigma)$ zero leads to the following two fundamental questions: (1) How many left eigenvectors of $A - K_f(\sigma)C$ can be assigned to the null space of $[B - K_f(\sigma)D]'$? and (2) How many eigenvalues of $A - K_f(\sigma)C$ can be placed at asymptotically infinite locations in $\mathcal{C}^-$ so that the corresponding residues are finite? The following two lemmas respectively answer these two questions.

Lemma 3.2.1. *Let λ_i and V_i be an eigenvalue and the corresponding left eigenvector of $A - K_f(\sigma)C$ for any gain $K_f(\sigma)$ such that it is asymptotically stable. Then the maximum possible number of $\lambda_i \in \mathcal{C}^-$ which satisfy the condition $\tilde{R}_i = 0$ is $n_a^- + n_b$. A total of n_a^- of these λ_i coincide with the system invariant zeros which are in $\mathcal{C}^-$ (the so called minimum phase zeros) and the remaining n_b eigenvalues can be assigned arbitrarily to any locations in $\mathcal{C}^-$. All the eigenvectors V_i that correspond to these $n_a^- + n_b$ eigenvalues span the subspace $\Re^n / \mathcal{S}^-(\Sigma)$. Moreover, the n_a^- eigenvectors V_i which correspond to the eigenvalues which coincide with the system invariant zeros in $\mathcal{C}^-$ coincide with the corresponding left state zero directions and span the subspace $\mathcal{V}^*(\Sigma)/\mathcal{V}^+(\Sigma)$.*

Proof : See Appendix 3.A. ∎

Remark 3.2.2. *Instead of rendering the $n_a^- + n_b$ residues $\tilde{R}_i$ mentioned in lemma 3.2.1 exactly zero, if one prefers, they can be rendered asymptotically zero as $\sigma \to \infty$. In that case n_a^- eigenvalues coincide asymptotically with the n_a^- minimum phase invariant zeros while the corresponding eigenvectors in the limit as $\sigma \to \infty$ coincide with the corresponding left state zero directions and span the subspace $\mathcal{V}^*(\Sigma)/\mathcal{V}^+(\Sigma)$.*

Lemma 3.2.2. *Let λ_i, W_i and V_i be an eigenvalue and the corresponding right and left eigenvectors of $A - K_f(\sigma)C$ for any gain $K_f(\sigma)$ such that it is asymptotically stable. The maximum number of eigenvalues of $A - K_f(\sigma)C$ that can be assigned arbitrarily to asymptotically infinite locations in C^- so that the corresponding $\tilde{R}_i$ are bounded as $|\lambda_i| \to \infty$ is $n_b + n_f$. Furthermore, all the corresponding left eigenvectors V_i of such eigenvalues asymptotically span the subspace $\Re^n/\mathcal{V}^*(\Sigma)$.*

Proof : It follows along the same lines as lemma 3.3 of Saberi et al [39]. ∎

As implied by lemma 3.2.1, in addition to n_a^- eigenvalues which coincide with the system minimum phase invariant zeros, there are n_b other eigenvalues which can be assigned arbitrarily to any locations in C^- such that $\tilde{R}_i \equiv 0$. This implies that $\tilde{R}_i$ corresponding to these n_b eigenvalues are identically zero and hence are bounded. Thus these n_b eigenvalues are included among the $n_b + n_f$ eigenvalues indicated in lemma 3.2.2. That is, there is a set of n_b eigenvalues which can be placed arbitrarily at either asymptotically finite locations in C^- as indicated by lemma 3.2.1 or at asymptotically infinite locations in C^- as indicated by lemma 3.2.2. Here after in order to conserve the controller band-width, we will assume that these n_b eigenvalues are always assigned to asymptotically finite locations.

Remark 3.2.3. *Consider the case when Σ is right invertible and has no infinite zeros. Note that this case includes the special case when Σ is a non-strictly proper single-input and single-output system. For this case, $n_b + n_f = 0$ and hence there is no eigenvalue, λ_i of $A - K_f(\sigma)C$ that can be assigned to an infinite location such that the corresponding $\tilde{R}_i$ is bounded.*

Lemmas 3.2.1 and 3.2.2 together tell us all the possibilities of rendering various terms of $\tilde{M}_f(s,\sigma)$ zero either exactly or asymptotically. There are altogether $n_a^- + n_b + n_f$ eigenvalues which can be assigned, some at finite and others at asymptotically infinite locations, so that the corresponding terms of $\tilde{M}_f(s,\sigma)$ in its dyadic expansion (3.2.3) are either exactly or asymptotically zero. Thus a question arises as to under what conditions on the given system, $n_a^- + n_b + n_f$ equals the dimension n so that any arbitrary admissible target loop can be recovered. This is explored in the following theorem.

Theorem 3.2.1. *Consider a stabilizable and detectable system Σ. Then any arbitrary admissible target loop is recoverable by either full or reduced order observer based controller, i.e., $\mathbf{T}_f^{\mathrm{R}}(\Sigma) = \mathbf{T}_r^{\mathrm{R}}(\Sigma) = \mathbf{T}(\Sigma)$, iff Σ is left invertible and of minimum phase.*

Proof : Let us take the case of full order observer based controller. The fact that Σ is left invertible and of minimum phase implies that $n_a^+ = n_c = 0$. Thus $n_a^- + n_b + n_f = n$. Hence the result follows from (2.2.1) and lemmas 3.2.1 and 3.2.2. Conversely, it is simple to see that the recoverability of all the admissible target loops implies that $V_i^H(B - K_f D) = 0$, $i = 1, \cdots, n$. Then by lemmas 3.2.1 and 3.2.2, we know that this is possible only when $n_a^- + n_b + n_f = n$. Hence, $n_a^+ = n_c = 0$, and thus Σ is left invertible and of minimum phase. In the case of reduced order observer based controller, in view of proposition 2.2.1 (i.e. item 6), we note that $n_a^+ + n_c$ corresponding to Σ_r is equal to zero iff Σ is left invertible and of minimum phase. Hence the result. ∎

For strictly proper systems and when full order observer based controller is used, the above results were first obtained by Doyle and Stein in their seminal paper [18]. For the case of reduced order observer based controller, the above results were given in [51], [29] and [43]. For general non-strictly proper systems, the above results follow from [4] and [10].

The required structural conditions for recovery of any arbitrary admissible target loop, as given in theorem 3.2.1 are not always met in practice. To see what is and what is not feasible when the given Σ is not left invertible and is of nonminimum phase, let us partition the dyadic expansion (3.2.3) of $\tilde{M}_f(s,\sigma)$ into four parts, each part having a particular type of characteristics,

$$\tilde{M}_f(s,\sigma) = \tilde{M}_f^-(s,\sigma) + \tilde{M}_f^b(s,\sigma) + \tilde{M}_f^\infty(s,\sigma) + \tilde{M}_f^e(s,\sigma), \tag{3.2.6}$$

where

$$\tilde{M}_f^-(s,\sigma) = \sum_{i=1}^{n_a^-} \frac{\tilde{R}_i^-}{s-\lambda_i^-}, \quad \tilde{M}_f^b(s,\sigma) = \sum_{i=1}^{n_b} \frac{\tilde{R}_i^b}{s-\lambda_i^b},$$

and

$$\tilde{M}_f^\infty(s,\sigma) = \sum_{i=1}^{n_f} \frac{\tilde{R}_i^\infty}{s-\lambda_i^\infty}, \quad \tilde{M}_f^e(s,\sigma) = \sum_{i=1}^{n_a^+ + n_c} \frac{\tilde{R}_i^e}{s-\lambda_i^e}.$$

Define the following sets where $n_e = n_a^+ + n_c$:

$$\Lambda^-(\sigma) := \{\lambda_i^-(\sigma) \mid i = 1, \cdots, n_a^-\}, \quad \Lambda^b(\sigma) := \{\lambda_i^b(\sigma) \mid i = 1, \cdots, n_b\},$$

$$\Lambda^\infty(\sigma) := \{\lambda_i^\infty(\sigma) \mid i = 1, \cdots, n_f\}, \quad \Lambda^e(\sigma) := \{\lambda_i^e(\sigma) \mid i = 1, \cdots, n_e\},$$

$$V^-(\sigma) := \{V_i^-(\sigma) \mid i = 1, \cdots, n_a^-\}, \quad V^b(\sigma) := \{V_i^b(\sigma) \mid i = 1, \cdots, n_b\},$$

$$V^\infty(\sigma) := \{V_i^\infty(\sigma) \mid i = 1, \cdots, n_f\}, \quad V^e(\sigma) := \{V_i^e(\sigma) \mid i = 1, \cdots, n_e\},$$

$$W^-(\sigma) := \{W_i^-(\sigma) \mid i = 1, \cdots, n_a^-\}, \quad W^b(\sigma) := \{W_i^b(\sigma) \mid i = 1, \cdots, n_b\},$$

$$W^\infty(\sigma) := \{W_i^\infty(\sigma) \mid i = 1, \cdots, n_f\}, \quad W^e(\sigma) := \{W_i^e(\sigma) \mid i = 1, \cdots, n_e\}.$$

Hereafter we will be using an over bar on a certain variable to denote its limit whenever it exists as $\sigma \to \infty$. For example, $\overline{\tilde{M}_f^e}(s)$ and $\overline{W}^e$ denote respectively the limits of $\tilde{M}_f^e(s,\sigma)$ and $W^e(\sigma)$ as $\sigma \to \infty$.

We now note that various parts of $\tilde{M}_f(s,\sigma)$ have the following interpretation:

1. $\tilde{M}_f^-(s,\sigma)$ contains n_a^- terms. The n_a^- eigenvalues of $A - K_f(\sigma)C$ represented in it form a set $\Lambda^-(\sigma)$. In accordance with lemma 3.2.1, there exists a gain $K_f(\sigma)$ such that $\tilde{M}_f^-(s,\sigma)$ can be rendered identically zero by assigning the elements of $\Lambda^-(\sigma)$ to coincide with the system minimum phase invariant zeros while the corresponding set of left eigenvectors $V^-(\sigma)$ coincides with the corresponding set of left state zero directions. In fact, $K_f(\sigma)$ can also be designed such that $\Lambda^-(\sigma)$ and $V^-(\sigma)$ approach asymptotically the set of system minimum phase invariant zeros and the corresponding set of left state zero directions as $\sigma \to \infty$. In this case, $\tilde{M}_f^-(s,\sigma) \to 0$ as $\sigma \to \infty$.

2. $\tilde{M}_f^b(s,\sigma)$ contains n_b terms. The n_b eigenvalues of $A - K_f(\sigma)C$ represented in it form a set $\Lambda^b(\sigma)$. In accordance with the lemmas 3.2.1 and 3.2.2, there exists a gain $K_f(\sigma)$ such that $\tilde{M}_f^b(s,\sigma)$ can be rendered identically zero by assigning the elements of $\Lambda^b(\sigma)$ arbitrarily to either asymptotically finite or infinite locations in $\mathcal{C}^-$ as $\sigma \to \infty$. As discussed earlier, in order to conserve the controller band-width, we will assume hereafter that these eigenvalues are assigned to asymptotically finite locations. Also, $K_f(\sigma)$ can be designed so that $\tilde{M}_f^b(s,\sigma) \to 0$ as $\sigma \to \infty$.

3. $\tilde{M}_f^\infty(s,\sigma)$ contains n_f terms. The n_f eigenvalues of $A - K_f(\sigma)C$ represented in it form a set $\Lambda^\infty(\sigma)$. In accordance with the lemma 3.2.2, there exists a gain $K_f(\sigma)$ such that $\tilde{M}_f^\infty(s,\sigma) \to 0$ as $\sigma \to \infty$ by assigning the elements of $\Lambda^\infty(\sigma)$ arbitrarily to asymptotically infinite locations in $\mathcal{C}^-$.

4. $\tilde{M}_f^e(s,\sigma)$ contains the remaining $n_e \equiv n_a^+ + n_c$ terms. It is nonexistent, i.e. $n_e = 0$, iff Σ is left invertible and of minimum phase. The n_e eigenvalues of $A - K_f(\sigma)C$ represented in $\tilde{M}_f^e(s,\sigma)$ form a set $\Lambda^e(\sigma)$. In view of lemmas 3.2.1 and 3.2.2, $\tilde{M}_f^e(s,\sigma)$ cannot in general be rendered zero either asymptotically or otherwise by any assignment of $\Lambda^e(\sigma)$ and the associated sets of right and left eigenvectors, $W^e(\sigma)$ and $V^e(\sigma)$. However, as will be discussed later on, $\tilde{M}_f^e(s,\sigma)$ can be shaped to have some desirable properties. Since (A, C) is assumed to be a detectable pair, except for the stable but unobservable eigenvalues of A, others among the remaining eigenvalues of $A - K_f(\sigma)C$ which are in $\overline{\Lambda}^e$ can be assigned to arbitrary locations in $\mathcal{C}^-$. These arbitrary locations can either be asymptotically finite or infinite. Moreover, assigning elements of $\overline{\Lambda}^e(\sigma)$ to asymptotically infinite locations increases unnecessarily controller band-width. Because of this, we assume that Λ^e is confined to finite locations in $\mathcal{C}^-$.

Since $\tilde{M}_f^-(s,\sigma)$ and $\tilde{M}_f^b(s,\sigma)$ can be rendered identically zero, for future use we can combine them into one term,

$$\tilde{M}_f^0(s,\sigma) = \tilde{M}_f^-(s,\sigma) + \tilde{M}_f^b(s,\sigma),$$

and rewrite $\tilde{M}_f(s,\sigma)$ as

$$\tilde{M}_f(s,\sigma) = \tilde{M}_f^0(s,\sigma) + \tilde{M}_f^\infty(s,\sigma) + \tilde{M}_f^e(s,\sigma). \tag{3.2.7}$$

We define likewise,

$$\Lambda^0(\sigma) = \Lambda^-(\sigma) \cup \Lambda^b(\sigma),$$

$$W^0(\sigma) = W^-(\sigma) \cup W^b(\sigma),$$

and

$$V^0(\sigma) = V^-(\sigma) \cup V^b(\sigma).$$

As the above discussion indicates, lemmas 3.2.1 and 3.2.2 form the heart of the underlying mechanism of LTR as they enable us to decompose $\tilde{M}_f(s,\sigma)$ and hence $M_f(s,\sigma)$ into several parts. They show clearly what is and what is not feasible under what conditions. Although they do not directly provide methods of obtaining the gain $K_f(\sigma)$, they do provide structural guide lines as to how certain eigenvalues and eigenvectors are to be assigned while indicating a multitude of ways in which freedom exists in assigning the other eigenvalues and eigenvectors of $A - K_f(\sigma)C$. These guidelines, in turn, can appropriately be channeled to come up with a design method to obtain an appropriate gain $K_f(\sigma)$. As will be discussed systematically in the next chapter, there exist essentially three methods of design to obtain appropriate $K_f(\sigma)$. These are (1) Kalman filter formalism which minimizes the H_2-norm of $\tilde{M}_f(s,\sigma)$, (2) Methods of minimizing H_∞-norm of $M_f(s,\sigma)$ and (3) Asymptotic time-scale and eigenstructure assignment (ATEA) method of [43], [40] by which $\tilde{M}_f(s,\sigma)$ can be shaped as desired in a number of ways. Leaving aside now the methods of design, let us at this stage simply define a set of gains $\mathcal{K}_f^*(\Sigma,\sigma)$ as follows:

Definition 3.2.1. *Consider the system Σ. Let $\mathcal{K}_f^*(\Sigma,\sigma)$ be a set of gains $K_f(\sigma) \in \Re^{n\times p}$ such that*

1. $A - K_f(\sigma)C$ *is stable for all* $\sigma > \sigma^*$ *where* $0 \le \sigma^* < \infty$,
2. *the limits as* $\sigma \to \infty$ *of the finite eigenvalues of* $A - K_f(\sigma)C$ *remain in* C^-,

$\dot{3}$. *if* $n_f = 0$, $\tilde{M}_f^0(s,\sigma)$ *is identically zero for all* σ,

$\bar{3}$. *if* $n_f \neq 0$, *as* $\sigma \to \infty$, $\tilde{M}_f^0(s,\sigma)$ *is either identically zero or asymptotically zero while the eigenvalues represented in* $\tilde{M}_f^0(s,\sigma)$ *tend to finite locations in* C^-, *and*

4. $\tilde{M}_f^\infty(s,\sigma) \to 0$ *as* $\sigma \to \infty$.

In a similar manner, $\mathcal{K}_r^*(\Sigma,\sigma)$ *is defined for the reduced order system* Σ_r.

Remark 3.2.4. *For the case when Σ does not have any infinite zeros, i.e. $n_f = 0$, any element $K_f(\sigma)$ of $\mathcal{K}_f^*(\Sigma,\sigma)$ is independent of σ and hence is bounded. On the other hand, if Σ has at least one infinite zero, i.e. $n_f \neq 0$, any element $K_f(\sigma)$ of $\mathcal{K}_f^*(\Sigma,\sigma)$ is dependent on σ. Moreover, $\|K_f(\sigma)\| \to \infty$ as $\sigma \to \infty$.*

It is obvious that $\mathcal{K}_f^*(\Sigma,\sigma)$ and $\mathcal{K}_r^*(\Sigma,\sigma)$, as defined above, are nonempty sets. We also note that whenever $K_f(\sigma)$ is chosen as an element of $\mathcal{K}_f^*(\Sigma,\sigma)$, the asymptotic limit as $\sigma \to \infty$ of $M_f^e(s,\sigma) \equiv F\tilde{M}_f^e(s,\sigma)$, namely $\overline{M}_f^e(s) \equiv F\overline{\tilde{M}}_f^e(s)$, is the ultimate error in recovery matrix $M_f(s,\sigma)$. As such, hereafter $\overline{M}_f^e(s)$ is called as the *recovery error matrix.* Theorem 3.2.2 given below characterizes the asymptotic behavior of the achieved loop transfer function as well as sensitivity and complementary sensitivity functions in terms of $\overline{M}_f^e(s)$.

Theorem 3.2.2. *Let the given system Σ be stabilizable and detectable. Also, let $L_t(s) = F\Phi B$ be an admissible target loop, i.e. $L_t(s) \in \mathbf{T}(\Sigma)$. Then for a full order observer based controller with gain $K_f(\sigma) \in \mathcal{K}_f^*(\Sigma,\sigma)$, we have pointwise in s, as $\sigma \to \infty$,*

$$E(s,\sigma) \to \overline{M}_f^e(s)[I_m + \overline{M}_f^e(s)]^{-1}(I_m + F\Phi B), \tag{3.2.8}$$

$$S_f(s,\sigma) \to S_t(s)[I_m + \overline{M}_f^e(s)], \tag{3.2.9}$$

$$T_f(s,\sigma) \to T_t(s) - S_t(s)\overline{M}^e_f(s), \tag{3.2.10}$$

and

$$\frac{|\,\sigma_i[S_f(j\omega,\sigma)] - \sigma_i[S_t(j\omega)]\,|}{\sigma_{max}[S_t(j\omega)]} \le \sigma_{max}[\overline{M}^e_f(j\omega)], \tag{3.2.11}$$

$$\frac{|\,\sigma_i[T_f(j\omega,\sigma)] - \sigma_i[T_t(j\omega)]\,|}{\sigma_{max}[S_t(j\omega)]} \le \sigma_{max}[\overline{M}^e_f(j\omega)]. \tag{3.2.12}$$

The above results are true for reduced order observer based controller as well, provided that the subscript f is changed to r and the quadruple (A, B, C, D) is changed to (A_r, B_r, C_r, D_r). Also, F in (3.2.8) to (3.2.12) is replaced by F_2.

Proof : Expressions (3.2.8) follows directly from the definition of of $\mathcal{K}^*_f(\Sigma,\sigma)$ in definition 3.2.1.

To prove (3.2.9) and (3.2.10), let us consider the following. From (2.2.1), we have

$$\begin{aligned} E_f(s,\sigma) &= F\Phi B - \mathbf{C}_f(s,\sigma)P(s) \\ &= M_f(s,\sigma)[I + M_f(s,\sigma)]^{-1}(I + F\Phi B), \end{aligned}$$

and hence

$$\begin{aligned} I + \mathbf{C}_f(s,\sigma)P(s) &= I + F\Phi B - E_f(s,\sigma) \\ &= I + F\Phi B - M_f(s,\sigma)[I + M_f(s,\sigma)]^{-1}(I + F\Phi B) \\ &= [I + M_f(s,\sigma)]^{-1}(I + F\Phi B). \end{aligned}$$

Thus we obtain

$$S_f(s,\sigma) = S_t(s)[I + M_f(s,\sigma)] \tag{3.2.13}$$

and

$$T_f(s,\sigma) = T_t(s) - S_t(s)M_f(s,\sigma). \tag{3.2.14}$$

It is simple to see that (3.2.9) and (3.2.10) follows from the definition of of $\mathcal{K}^*_f(\Sigma,\sigma)$ in definition 3.2.1.

We now proceed to show (3.2.11) and (3.2.12). Applying singular value inequalities to (3.2.13), we have for each $i = 1$ to m,

$$\sigma_i[S_f(j\omega,\sigma)] \leq \sigma_i[S_t(j\omega)] + \sigma_{max}[S_t(j\omega)M_f(j\omega,\sigma)],$$

and thus

$$\sigma_i[S_f(j\omega,\sigma)] - \sigma_i[S_t(j\omega)] \leq \sigma_{max}[S_t(j\omega)]\,\sigma_{max}[M_f(j\omega,\sigma)]. \tag{3.2.15}$$

Now rewriting (3.2.13) as,

$$S_t(s) = S_f(s,\sigma) - S_t(s)M_f(s,\sigma),$$

we have for each $i = 1$ to m,

$$\sigma_i[S_t(j\omega)] - \sigma_i[S_f(j\omega,\sigma)] \leq \sigma_{max}[S_t(j\omega)]\,\sigma_{max}[M_f(j\omega,\sigma)]. \tag{3.2.16}$$

Then in view of (3.2.15) and (3.2.16), we get

$$\frac{|\,\sigma_i[S_f(j\omega,\sigma)] - \sigma_i[S_t(j\omega)]\,|}{\sigma_{max}[S_t(j\omega)]} \leq \sigma_{max}[M_f(j\omega,\sigma)].$$

Next using singular value inequalities and proceeding as above, we get

$$\frac{|\,\sigma_i[T_f(j\omega,\sigma)] - \sigma_i[T_t(j\omega)]\,|}{\sigma_{max}[S_t(j\omega)]} \leq \sigma_{max}[M_f(j\omega,\sigma)].$$

This completes the proof of theorem 3.2.2. ∎

Remark 3.2.5. *Theorem 3.2.1 is a special case of the above theorem. In fact, if the given system Σ is left invertible and of minimum phase, then both $\tilde{M}_f^e(s,\sigma)$ and $\tilde{M}_r^e(s,\sigma)$ are nonexistent and hence $E_f(s,\sigma)$ as well as $E_r(s,\sigma)$ tend to zero pointwise in s as $\sigma \to \infty$ for all $s \in C$.*

As implied by theorem 3.2.2, the recovery error matrix $\overline{\tilde{M}_f^e}(s)$ plays a dominant role in the recovery process and hence it should be shaped to yield as best as possible the desired results. Shaping $\overline{\tilde{M}_f^e}(s)$ involves selecting the set of eigenvalues $\overline{\Lambda^e}$ represented in $\overline{\tilde{M}_f^e}(s)$

and the associated set of right and left eigenvectors $\overline{W}^c$ and $\overline{V}^c$. Such a selection can be done in a number of ways subject to the constraints imposed in selecting the eigenvectors [30]. However, note that though, no shaping may be necessary if $\overline{\tilde{M}}_f^e(s)$ turns out to be small. For certain class of systems $\overline{\tilde{M}}_f^e(s)$ is, in fact, small in some sense or other. Following a similar result of [39], one can prove easily that for a left invertible nonminimum phase system which is not necessarily strictly proper but which has all its nonminimum phase zeros far away from the band-width of the target loop transfer function, the norm of the recovery error matrix $\overline{\tilde{M}}_f^e(s)$ is indeed always small.

In multivariable systems, one interesting aspect of theorem 3.2.2 is that there could exist a subspace of the control space in which $\overline{\tilde{M}}_f^e(s)$ can be rendered zero. To pinpoint this, let

$$e_i = [B - K_f(\sigma)D]'\overline{V}_i \;\; , \;\; \overline{V}_i \in \overline{V}^e, \tag{3.2.17}$$

and let $\mathcal{E}^e$ be the subspace of $\Re^m$,

$$\mathcal{E}^e = \text{Span}\{e_i \mid \overline{V}_i \in \overline{V}^e\}. \tag{3.2.18}$$

Let the dimension of $\mathcal{E}^e$ be m^e. Now let

$$\mathcal{S}^e = \text{orthogonal complement of } \mathcal{E}^e \text{ in } \Re^m. \tag{3.2.19}$$

Let P^s be the orthogonal projection matrix onto $\mathcal{S}^e$. Then the following theorem pinpoints the directional behavior of $\tilde{M}_f(s,\sigma)$ and consequently the behavior of $S_f(s,\sigma)$ and $T_f(s,\sigma)$ as $\sigma \to \infty$.

Theorem 3.2.3. *Let Σ be stabilizable and detectable, and $L_t(s)$ be a member of the set admissible target loops $\mathbf{T}(\Sigma)$. Then for any $K_f(\sigma) \in \mathcal{K}_f^*(\Sigma,\sigma)$, the corresponding full order observer based controller satisfies, as $\sigma \to \infty$, pointwise in s,*

$$\tilde{M}_f(s,\sigma)P^s \to 0,$$

$$S_f(s,\sigma)P^s \to S_t(s)P^s$$

$$T_f(s,\sigma)P^s \to T_t(s)P^s,$$

where P^s is the orthogonal projection onto $\mathcal{S}^e \in \Re^m$ as given in (3.2.19). The above results are equally true for reduced order observer based controller, provided that the subscript f is changed to r, and quadruple (A, B, C, D) is changed to (A_r, B_r, C_r, D_r).

Proof : In view of the definitions of the matrix P^s and the subspaces $\mathcal{E}^e$ and $\mathcal{S}^e$, theorem 3.2.2 implies the results of theorem 3.2.3. ∎

In general, although $\tilde{M}_f(s,\sigma)$ and hence $S_t(s)$ and $T_t(s)$ are recoverable in a subspace such as $\mathcal{S}^e$, the loop transfer function $L_t(s)$ is not necessarily recoverable in that subspace $\mathcal{S}^e$ as can be seen from an example given below. However, this may not be as important as it seems since in most of the design schemes recovery of $L_t(s)$ is only a means to recover $S_t(s)$ and $T_t(s)$.

Example 3.1 : Consider a non-strictly proper system characterized by

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = C = D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is invertible with two nonminimum phase invariant zeros at $s = 1$ and $s = 2$. Let the target loop $L_t(s)$ and target sensitivity function $S_t(s)$ be specified by

$$F = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}.$$

Let

$$K_f(\sigma) = \begin{bmatrix} 10 & -9 \\ 4 & -3 \end{bmatrix}.$$

Then, it is easy to calculate that

$$L_t(s) = F(sI - A)^{-1}B = \begin{bmatrix} \frac{5}{s-3} & 0 \\ 0 & \frac{4}{s-2} \end{bmatrix},$$

$$S_t(s) = [I + L_t(s)]^{-1} = \begin{bmatrix} \frac{s-3}{s+2} & 0 \\ 0 & \frac{s-2}{s+2} \end{bmatrix},$$

$$L_f(s,\sigma) = C_f(s,\sigma)P(s) = \frac{\begin{bmatrix} 50s^3 - 190s^2 + 160s + 40 & -45s^3 + 90s^2 + 225s - 270 \\ 16s^3 - 32s^2 - 64s + 128 & -12s^3 - 72s^2 + 444s - 360 \end{bmatrix}}{s^4 - 32s^3 + 155s^2 - 232s + 84}$$

and

$$S_f(s,\sigma) = \frac{\begin{bmatrix} s^4 - 44s^3 + 83s^2 + 212s - 276 & 45s^3 - 90s^2 - 225s + 270 \\ -16s^3 + 32s^2 + 64s - 128 & s^4 + 18s^3 - 35s^2 - 72s + 124 \end{bmatrix}}{s^4 + 6s^3 + 13s^2 + 12s + 4}.$$

Now consider a subspace $\mathcal{S}^e$ having the orthogonal projection matrix P^s as

$$P^s = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

It is now straightforward to verify that

$$S_t(s)P^s = S_f(s,\sigma)P^s = \begin{bmatrix} \frac{S-3}{2(S+2)} & \frac{S-3}{2(S+2)} \\ \frac{S-2}{2(S+2)} & \frac{S-2}{2(S+2)} \end{bmatrix}.$$

This implies that the projections of target and achieved sensitivity functions onto the subspace $\mathcal{S}^e$ are equal to one another. On the other hand, we have

$$L_t(s)P^s = \frac{\begin{bmatrix} 5(s-2) & 5(s-2) \\ 4(s-3) & 4(s-3) \end{bmatrix}}{2(s-2)(s-3)}$$

and

$$L_f(s,\sigma)P^s = \frac{\begin{bmatrix} 5(s^3 - 20s^2 + 77s - 46) & 5(s^3 - 20s^2 + 77s - 46) \\ 4(s^3 - 26s^2 + 95s - 58) & 4(s^3 - 26s^2 + 95s - 58) \end{bmatrix}}{2(s^4 - 32s^3 + 155s^2 - 232s + 84)}.$$

Obviously, this implies that the projections of target and achieved loop transfer functions onto the subspace $\mathcal{S}^e$ are not equal, i.e. $L_t(s)P^s \neq L_f(s,\sigma)P^s$. □

In view of the directional behavior of $\overline{\tilde{M}_f^e}(s)$ as given by theorem 3.2.3, one could try to shape it in a particular way so as to obtain the recovery of sensitivity and complementary sensitivity functions in certain desired directions or one could try to shape $\overline{\tilde{M}_f^e}(s)$ so that the subspace $\mathcal{S}^e$ has as large a dimension as possible, i.e. the subspace $\mathcal{E}^e$ has as small a dimension as possible. In this regard, we note that we have already selected Λ^0 and Λ^∞ and the corresponding sets of eigenvectors $\overline{V}^0$ and $\overline{V}^\infty$ so that $\tilde{M}_f^0(s,\sigma)$ and $\tilde{M}_f^\infty(s,\sigma)$ tend to zero as $\sigma \to \infty$. We also note that although all the $n_a^+ + n_c$ vectors $\overline{V}_i \in \overline{V}^e$ can be selected to be linearly independent, the corresponding $e_i \equiv [B - K_f(\sigma)D]'\overline{V}_i$ need not be linearly independent. In fact for a given $e \neq 0$, the equation

$$e = [B - K_f(\sigma)D]'V,$$

has $n - m + 1$ linearly independent solutions for V. Of course, not all such $n - m + 1$ vectors could be admissible eigenvectors of $A - K_fC$ for different eigenvalues of $A - K_fC$ in C^-, and moreover some or all of these $n - m + 1$ vectors could also be linearly dependent on already selected eigenvectors in the sets $\overline{V}^0$ and $\overline{V}^\infty$. Thus the problem of shaping $\mathcal{E}^e$ is to find an admissible set of eigenvalues λ_i and vectors e_i, $i = 1$ to $n_a^+ + n_c$, which are not necessarily linearly independent, but the associated eigenvectors V_i of $A - K_fC$ satisfying $e_i = [B - K_f(\sigma)D]'\overline{V}_i$, $i = 1$ to $n_a^+ + n_c$, together with the vectors in the sets $\overline{V}^0$ and $\overline{V}^\infty$ form n linearly independent vectors. This problem of selecting an admissible set (λ_i, e_i) is very much related to the traditional problem of distributing the modes of a closed-loop system to various output components by an appropriate selection of the closed-loop eigenstructure. This traditional problem of 'shaping the output response characteristics' of a closed-loop system has been studied first by Moore [30] and Shaked [47] and more recently by Sogaard-Andersen [53] although to this date there exists no systematic design procedure.

The above discussion focuses how to shape the subspace $\mathcal{S}^e$ in which $\tilde{M}_f(s,\sigma)$, $S_t(s)$ and $T_t(s)$ are recovered. A practical problem of interest could be to achieve recovery of $\tilde{M}_f(s,\sigma)$ (or $M_f(s,\sigma)$), $S_t(s)$ and $T_t(s)$ in a prescribed subspace $\mathcal{S}^e$. We will discuss this aspect of the problem in section 3.4.

We will next examine the asymptotic behavior of open-loop eigenvalues of the full order observer based controller $\mathbf{C}_f(s,\sigma)$ and the mechanism of pole zero cancellation between the controller eigenvalues and the input or output decoupling zeros [38] of the system. It is important to know the eigenvalues of $\mathbf{C}_f(s,\sigma)$ as they are included among the invariant zeros of the closed-loop system [41] and hence affect the performance of it, e.g., command following. The controller transfer function is given by (3.2.2) while the eigenvalues of it are

$$\lambda[A - K_f(\sigma)C - BF + K_f(\sigma)DF].$$

To study the nature of these eigenvalues, let

$$\det[sI_n - A + K_f(\sigma)C] = \phi^0(s)\phi^\infty(s)\phi^e(s)$$

where $\phi^0(s)$, $\phi^\infty(s)$ and $\phi^e(s)$ are polynomials in s whose zeros are the eigenvalues of $A - K_f(\sigma)C$ that belong to the sets $\Lambda^0(\sigma)$, $\Lambda^\infty(\sigma)$ and $\Lambda^e(\sigma)$ respectively. Also, let

$$F\overline{\tilde{M}}^e_f(s) = \frac{R^e(s)}{\phi^e(s)} \tag{3.2.20}$$

where $R^e(s)$ is a polynomial matrix in s. Now consider the following:

$$\begin{aligned}
\det[sI_n &- A + K_f(\sigma)C + BF - K_f(\sigma)DF] \\
&= \det[sI_n - A + K_f(\sigma)C]\det[I_n + (sI_n - A + K_f(\sigma)C)^{-1}(B - K_f(\sigma)D)F] \\
&= \phi^0(s)\phi^\infty(s)\phi^e(s)\det[I_m + F(sI_n - A + K_f(\sigma)C)^{-1}(B - K_f(\sigma)D)] \\
&= \phi^0(s)\phi^\infty(s)\phi^e(s)\det[I_m + F\tilde{M}_f(s,\sigma)] \\
&\to \phi^0(s)\phi^\infty(s)\phi^e(s)\det[I_m + F\overline{\tilde{M}}^e_f(s)] \text{ as } \sigma \to \infty \\
&= \phi^0(s)\phi^\infty(s)\phi^e(s)\det[I_m + \frac{R^e(s)}{\phi^e(s)}] \\
&= \phi^0(s)\phi^\infty(s)\det[I_m\phi^e(s) + R^e(s)]/[\phi^e(s)]^{m-1}.
\end{aligned} \tag{3.2.21}$$

We note that the controller can be designed such that $\phi^0(s)$, $\phi^\infty(s)$ and $\phi^e(s)$ are coprime. Thus the open-loop eigenvalues of the controller of (3.2.2) are the zeros of $\phi^0(s)$, $\phi^\infty(s)$ and

$$\det[I_m\phi^e(s) + R^e(s)]/[\phi^e(s)]^{m-1}.$$

Thus Λ^0 and Λ^∞ are contained among the eigenvalues of the controller. Although Λ^0 and Λ^∞ are in C^-, there is no guarantee that the zeros of

$$\det[I_m\phi^e(s) + R^e(s)]/[\phi^e(s)]^{m-1}$$

are in C^-. Hence the controller may or may not be open-loop stable. In general, the loop transfer function $\mathbf{C}_f(s,\sigma)P(s)$ has $2n$ eigenvalues, n of them coming from the system Σ and the other n from the controller. However, there are several cancellations among the input or output decoupling zeros [38] of $\mathbf{C}_f(s,\sigma)P(s)$ and the controller eigenvalues. The following lemma 3.2.3 which is a slight generalization of a similar one in Goodman [22], explores such a cancellation.

Lemma 3.2.3. *Let λ be an eigenvalue of $A - K_f(\sigma)C$ and the corresponding left eigenvector V be such that $V^H[B - K_f(\sigma)D] = 0$. Then λ is an eigenvalue of $A - K_f(\sigma)C - BF + K_f(\sigma)DF$ with corresponding left eigenvector as V. Moreover, λ cancels an input decoupling zero of $\mathbf{C}_f(s,\sigma)P(s)$.*

Proof : See Appendix 3.B. ∎

Thus, in view of lemma 3.2.1, the above lemma implies that whatever may be the matrix F, if the controller is appropriately designed, there are $n_a^- + n_b$ cancellations among the eigenvalues of the controller and the input decoupling zeros of $\mathbf{C}_f(s,\sigma)P(s)$. As will be seen in the next section, there may be additional cancellations if F satisfies certain properties.

Remark 3.2.6. *Equation (3.2.21) and lemma 3.2.3 are equally true for reduced order observer based controller. In this case, notationally the quadruple (A,B,C,D), F and $\mathbf{C}_f(s,\sigma)$ are to be replaced respectively by (A_r,B_r,C_r,D_r), F_2 and $\mathbf{C}_r(s,\sigma)$.*

3.3. Analysis for recoverable target loops

In section 3.2, loop transfer recovery analysis is conducted without taking into account any knowledge of F. It involves essentially the study of the matrix $\tilde{M}_f(s)$ or $\tilde{M}_f(s,\sigma)$ as to when it can or cannot be rendered zero. This section complements the analysis of section 3.2 by taking into account the knowledge of F. Obviously then, the analysis of this section is a study of $M(s) = F\tilde{M}_f(s)$ or $M(s,\sigma) = F\tilde{M}_f(s,\sigma)$. One of the important questions that needs to be answered here is as follows. What class of target loops can be recovered exactly (or asymptotically) for a given system? Or equivalently, what are the necessary and sufficient conditions a target loop transfer function $L_t(s)$ has to satisfy so that it can exactly (or asymptotically) be recoverable for the given system? As it forms a coupling between analysis and design, characterization of $L_t(s)$ to determine whether it can be recovered either exactly or asymptotically for the given system, plays an extremely important role. Although the physical tasks of designing F and K are separable, one can benefit enormously by knowing ahead what kind of target loops are recoverable. The necessary and sufficient conditions developed here on $L_t(s)$ for its recoverability, turn out to be constraints on the finite and infinite zero structure of $L_t(s)$ as related to the corresponding structure of Σ. An interpretation of these conditions reveals that either exact or asymptotic recovery of $L_t(s)$ for general nonminimum phase systems is possible under a variety of conditions.

Another important question that arises before one undertakes formulating any target loop transfer function $L_t(s)$ for a given system Σ is as follows. What are the necessary and sufficient conditions on Σ so that it has at least one recoverable target loop? An answer to this question obviously helps a designer to remodel the given plant if necessary by appropriately modifying the number or type of plant inputs or outputs. To answer the question posed, we develop here an auxiliary system Σ^{ER} of Σ and show that the set of exactly recoverable target loops $\mathbf{T}^{ER}(\Sigma)$ is nonempty iff Σ^{ER} is stabilizable by a static output feedback controller. Similarly, another auxiliary system Σ^{R} of Σ is developed to show that the set of recoverable target loops $\mathbf{T}^{R}(\Sigma)$ is nonempty iff Σ^{R} is stabilizable by a static output feedback controller. A close look at these conditions reveals a surprising necessary condition; namely, strong stabilizability of Σ is necessary for it to have at least one, either exactly or asymptotically, recoverable target loop.

Finally, another aspect of analysis given here shows the mechanism of pole zero cancellation between the controller eigenvalues and the input or output decoupling zeros of Σ for the case when the target loop $L_t(s)$ is known.

We proceed now to give the following results regarding the exact recoverability of a target loop transfer function $L_t(s) = F\Phi B$ for a given system Σ.

Theorem 3.3.1. *Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily left invertible and not necessarily of minimum phase. Depending upon the controller used, the following condition is necessary and sufficient so that an admissible target loop transfer function $L_t(s) \in \mathbf{T}(\Sigma)$ is exactly recoverable:*

1. *For a full order observer based controller, the condition is that $\mathcal{S}^-(\Sigma) \subseteq \mathrm{Ker}\,(F)$.*

2. *For a reduced order observer based controller, the condition is that $\mathcal{S}^-(\Sigma) \cap \mho \subseteq \mathrm{Ker}\,(F)$.*

Thus the set of exactly recoverable target loops under each controller is characterized as follows:

(1) Full order observer based controller

$$\mathbf{T}_f^{\mathrm{ER}}(\Sigma) = \{\, L_t(s) \in \mathbf{T}(\Sigma) \mid \mathcal{S}^-(\Sigma) \subseteq \mathrm{Ker}\,(F)\,\}.$$

(2) Reduced order observer based controller :

$$\mathbf{T}_r^{\mathrm{ER}}(\Sigma) = \{L_t(s) \in \mathbf{T}(\Sigma) \mid \mathcal{S}^-(\Sigma) \cap \mho \subseteq \mathrm{Ker}\,(F)\}.$$

Proof : For the case of a full order observer based controller, we consider an auxiliary system characterized by

$$\Sigma_{au} : \begin{cases} \dot{x} = A'x + C'u + F'w, \\ z = B'x + D'u. \end{cases} \tag{3.3.1}$$

Also, with a state feedback law

$$u = -K_f' x,$$

the closed-loop transfer function from w to z, denoted here by $T_{zw}^{au}(s)$, is simply

$$T_{zw}^{au}(s) = M_f'(s).$$

Hence, the problem of finding an observer gain matrix such that $A-K_fC$ is asymptotically stable and that $M_f(s)=0$ is equivalent to the well-known disturbance decoupling problem with internal stability when the plant considered is Σ_{au} as given in (3.3.1). Then it follows from Stoorvogel [54] that the above disturbance decoupling problem with internal stability is solvable if and only if $\mathcal{S}^-(\Sigma) \subseteq \operatorname{Ker}(F)$.

Results for reduced order observer based controller are derived from similar arguments and the properties of Σ_r as given in proposition 2.2.1 are taken into account. ∎

Corollary 3.3.1. *Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A,B,C,D). Then*

1. $\mathbf{T}_f^{\mathrm{ER}}(\Sigma) = \mathbf{T}(\Sigma)$, *i.e. any admissible target loop is exactly recoverable by a full order observer based controller, iff Σ is left invertible and of minimum phase with no infinite zeros.*
2. $\mathbf{T}_r^{\mathrm{ER}}(\Sigma) = \mathbf{T}(\Sigma)$, *i.e. any admissible target loop is exactly recoverable by a reduced order observer based controller, iff Σ is left invertible and of minimum phase with no infinite zeros of order higher than one.*

Proof : For the case of a full order observer based controllers, it follows from the properties of s.c.b that $\mathcal{S}^-(\Sigma) = \emptyset$ iff Σ is of left invertible and of minimum phase and has no infinite zeros. Hence, the result follows from theorem 3.3.1. Results for a reduced order observer based controller follow from similar arguments when the properties of Σ_r given in proposition 2.2.1 are taken into account. ∎

Several interpretations emerge from the recoverability conditions on the target loops given in theorem 3.3.1. In fact the constraints given in theorem 3.3.1 are nothing more than constraints on the finite and infinite zero structure and invertibility properties of $L_t(s)$. Some interesting interpretations in this regard are exemplified below.

1. If Σ is not left invertible, any exactly recoverable $L_t(s)$ is not left invertible. On the other hand, left invertibility of Σ does not necessarily imply that an exactly

recoverable $L_t(s)$ is left invertible. That is, whenever Σ is left invertible, an exactly recoverable $L_t(s)$ could be either left invertible or not left invertible.

2. Any left invertible and exactly recoverable $L_t(s)$ must contain the nonminimum phase zero structure of Σ. An exactly recoverable but not left invertible $L_t(s)$ does not necessarily contain the nonminimum phase zero structure of Σ. This is illustrated in example 3.2.

3. For simplicity of presentation, let us assume that Σ is strictly proper, invertible and of uniform rank with relative degree q (i.e., all the infinite zeros of Σ are of the same order q). Then

   (a) the smallest order of infinite zero of $L_t(s) \in \mathbf{T}_f^{\mathrm{ER}}(\Sigma)$, i.e. $L_t(s)$ is exactly recoverable by a full order observer based controller, is greater than q (See also, corollary 3.3.2).

   (b) the smallest order of infinite zero of $L_t(s) \in \mathbf{T}_r^{\mathrm{ER}}(\Sigma)$, i.e. $L_t(s)$ is exactly recoverable by a reduced order observer based controller, is equal to or greater than q (See also, corollary 3.3.2).

We have the following corollary to theorem 3.3.1.

Corollary 3.3.2. *Consider a strictly proper, invertible and nonminimum phase system Σ. Also, let Σ be of uniform rank with relative degree q. Then*

1. *any target loop transfer function $L_t(s)$ which is invertible with the smallest order of infinite zeros greater than q and which contains the nonminimum phase zero structure of Σ is exactly recoverable by a full order observer based controller.*
2. *any target loop transfer function $L_t(s)$ which is invertible with the smallest order of infinite zeros equal to or greater than q and which contains the nonminimum phase zero structure of Σ is exactly recoverable by a reduced order observer based controller.*

Proof : See Appendix 3.C. ∎

Remark 3.3.1. *A special case of corollary 3.3.2 when Σ is strictly proper, invertible and of minimum phase with relative degree $q = 1$ was given earlier by Goodman (1984). Thus corollary 3.3.2 generalizes Goodman's result for both nonminimum phase systems and for systems with relative degree greater than unity.*

Example 3.2 : Consider an invertible system Σ characterized by

$$A = \begin{bmatrix} 1 & 0 & 0 & 4 & 1 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -5 & 1 & 1 \\ -5 & 1 & 1 & -10 & 0 \\ -5 & 1 & 1 & 0 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This system has three invariant zeros at $s = 1$, $s = -1$ and at $s = -5$. Let the target loop be defined by the triple (F, A, B) where

$$F = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then, it is straightforward to show that $A - BF$ is asymptotically stable and $\mathcal{S}^-(\Sigma)$ is a subset of $\mathrm{Ker}\,(F)$. Thus ELTR can be achieved. In fact, the controller defined below having the eigenvalues at -1, -2, -3, -4 and -5 achieves ELTR:

$$\hat{u} = -F\hat{x}$$

where

$$\dot{\hat{x}} = \begin{bmatrix} 1 & 0 & 0 & 1.5091 & 1.4909 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ -5 & 0 & 0 & -5.0110 & -1.9894 \\ -5 & 0 & 0 & -2.0106 & -4.9890 \end{bmatrix} \hat{x} + \begin{bmatrix} 2.4909 & -0.4909 \\ 1.0000 & 1.0000 \\ 1.0000 & 1.0000 \\ -4.9890 & 1.9894 \\ 2.0106 & -5.0110 \end{bmatrix} y.$$

However, it is simple to verify that the given $L_t(s) = F\Phi B$ is right invertible and is of minimum phase with one invariant zero at $s = -3.5$. Thus we can conclude that an exactly recoverable $L_t(s)$ need not contain the nonminimum phase zero structure of Σ. □

We have the following interesting corollary of theorem 3.3.1.

Corollary 3.3.3. *Consider a single-input single-output (SISO) non-strictly proper system Σ. Then a target loop transfer function $L_t(s) = F\Phi B$ is exactly recoverable by the full order observer based controller iff it contains the nonminimum phase zero structure of Σ.*

Proof : A single-input single-output non-strictly proper system is always invertible. Hence, the result follows from interpretation 2 given above. ∎

Our aim next is to develop the needed conditions on Σ so that $\mathbf{T}^{\mathrm{ER}}(\Sigma)$ is nonempty. We have the following theorem.

Theorem 3.3.2. *Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let $\overline{C}_f^{\mathrm{ER}}$ and $\overline{C}_r^{\mathrm{ER}}$ be any full rank matrices of dimensions $(n_a^- + n_b) \times n$ and $(n_a^- + n_b + m_f) \times n$, respectively, such that* $\mathrm{Ker}\,(\overline{C}_f^{\mathrm{ER}}) = \mathcal{S}^-(\Sigma)$, *and* $\mathrm{Ker}\,(\overline{C}_r^{\mathrm{ER}}) = \mathcal{S}^-(\Sigma) \cap \mathcal{U}$.

Also, define the auxiliary systems Σ_f^{ER} and Σ_r^{ER} which are respectively characterized by the matrix triples $(A, B, \overline{C}_f^{\mathrm{ER}})$, and $(A, B, \overline{C}_r^{\mathrm{ER}})$. Then we have the following results depending upon the controller used :

(1) Full order observer based controller :

$\mathbf{T}_f^{\mathrm{ER}}(\Sigma)$ *is nonempty iff Σ_f^{ER} is stabilizable by a static output feedback controller.*

(2) Reduced order observer based controller :

$\mathbf{T}_r^{\mathrm{ER}}(\Sigma)$ *is nonempty iff Σ_r^{ER} is stabilizable by a static output feedback controller.*

Proof : Let us first consider the case of full order observer based controller. It follows from theorem 3.3.1 that any admissible target loop $L_t(s) = F\Phi B$ is exactly recoverable iff

$\mathcal{S}^-(\Sigma) \subseteq \mathrm{Ker}\,(F)$. Hence, $\mathrm{Ker}\,(\overline{C}_f^{\mathrm{ER}}) = \mathcal{S}^-(\Sigma)$ implies $\mathrm{Ker}\,(\overline{C}_f^{\mathrm{ER}}) \subseteq \mathrm{Ker}\,(F)$ and $\mathrm{Im}\,(F') \subseteq \mathrm{Im}\,([\overline{C}_f^{\mathrm{ER}}]')$. Then we have $F = G\overline{C}_f^{\mathrm{ER}}$ for some constant matrix G. It is trivial to verify that the existence of an exactly recoverable target loop $L_t(s) = F\Phi B$ such that $A - BF$ is asymptotically stable is equivalent to the existence of a matrix G such that $A - BG\overline{C}_f^{\mathrm{ER}}$ is asymptotically stable.

Result for reduced order observer based controllers follow from similar arguments. This completes the proof of theorem 3.3.2. ∎

Theorems 3.3.1 and 3.3.2 deal with ELTR. Since the required conditions for ELTR in general are very severe, most often in practice one is interested only in ALTR. From its definition, it is easy to see that ALTR occurs, i.e., the recovery error matrix $\overline{M}_f^e(s)$ is zero, iff $F\overline{W}^e = 0$. We have the following results regarding ALTR.

Theorem 3.3.3. *Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily left invertible and not necessarily of minimum phase. Then the condition $\mathcal{V}^+(\Sigma) \subseteq \mathrm{Ker}\,(F)$ is necessary and sufficient so that an admissible target loop transfer function $L_t(s) \in \mathbf{T}(\Sigma)$, is recoverable by either full or reduced order observer based controller. Thus the set of recoverable target loops under either full or reduced order observer based controller is characterized by,*

$$\mathbf{T}_f^{\mathrm{R}}(\Sigma) = \mathbf{T}_r^{\mathrm{R}}(\Sigma) = \left\{\, L_t(s) \in \mathbf{T}(\Sigma) \mid \mathcal{V}^+(\Sigma) \subseteq \mathrm{Ker}\,(F) \,\right\}.$$

Proof : Let us consider the case of full order observer based controller first. Following the proof of theorem 3.3.1, it is simple to see that our problem is equivalent to the well-known almost disturbance decoupling problem with internal stability (ADDPS) for the auxiliary system Σ_{au} in (3.3.1). It is shown in Scherer [46] that the above ADDPS is solvable if and only if $\mathcal{V}^+(\Sigma) \subseteq \mathrm{Ker}\,(F)$. Here we adhere to the notion of closed-loop stability by excluding those cases where, in the limits as $\sigma \to \infty$, the finite eigenvalues of the closed-loop system are on the $j\omega$ axis.

The result for reduced order observer based controller follows from similar arguments when the properties of Σ_r given in proposition 2.2.1 are utilized. ∎

As in the case of ELTR, we can interpret the constraints imposed by theorem 3.3.3 in terms of the invertibility and the finite zero structures of $L_t(s)$ and Σ as follows.

1. If Σ is not left invertible, any recoverable $L_t(s)$ is not left invertible. On the other hand, left invertibility of Σ does not necessarily imply that a recoverable $L_t(s)$ is left invertible. That is, whenever Σ is left invertible, an recoverable $L_t(s)$ could be either left invertible or not left invertible.

2. Any left invertible and recoverable $L_t(s)$ must contain the nonminimum phase zero structure of Σ. A recoverable but not left invertible $L_t(s)$ does not necessarily contain the nonminimum phase zero structure of Σ.

We have the following corollary to theorem 3.3.3.

Corollary 3.3.4. *Consider a left invertible and nonminimum phase system Σ, which is not necessarily strictly proper. Then a target loop transfer function $L_t(s) = F\Phi B$ is recoverable by the full order observer based controller if it contains the nonminimum phase zero structure of Σ.*

Proof : Proposition 3.C.1 (see Appendix 3.C) and left invertiblity of Σ together imply that $\mathcal{V}^+(\Sigma) \subseteq \mathrm{Ker}\,(F)$. Hence the result follows. ∎

Example 3.3 : Consider an invertible system Σ characterized by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ -10 & 0 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

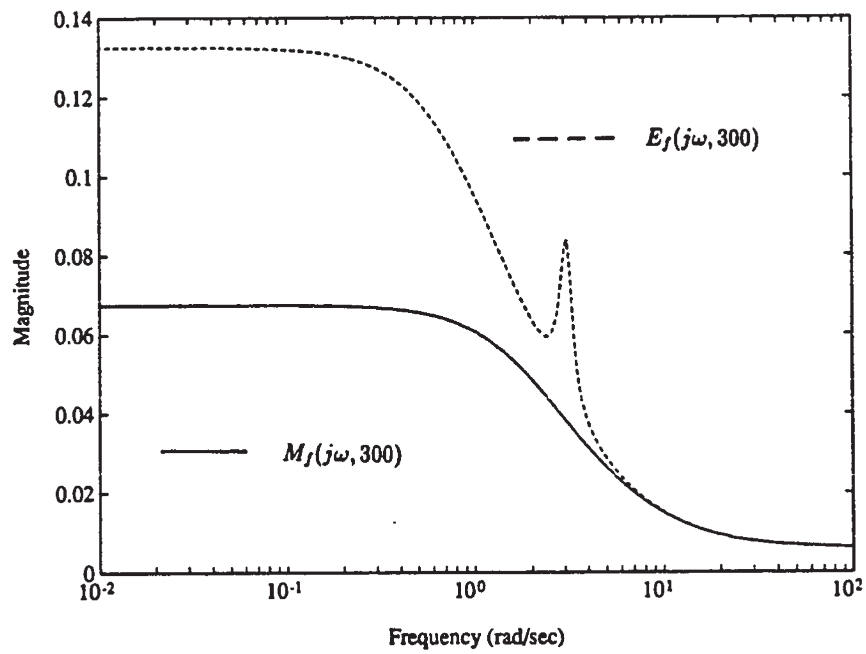


Figure 3.3.1: $\sigma_{max}[M_f(j\omega, 300)]$ and $\sigma_{max}[E_f(j\omega, 300)]$.

This plant has a nonminimum phase invariant zero at $s = 1$. Let the target loop be defined by the triple (F, A, B) where

$$F = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The triple (F, A, B) forms a minimum phase right invertible system and it does not contain the nonminimum phase zero structure of Σ. However, for this example it can be easily seen that $\mathcal{V}^+(\Sigma)$ is the span of $[1, \ 0, \ 0]'$ and hence it is contained in $\mathrm{Ker}\,(F)$. Thus in accordance with theorem 3.3.3, there exists a controller which achieves ALTR. In fact, a full order observer based controller having the eigenvalues at -2, $-\sigma$ and $-\sigma$ and $K(\sigma)$ as given below achieves ALTR,

$$K(\sigma) = \begin{bmatrix} 3\sigma & 0 \\ \sigma - 1 & 0 \\ 0 & \sigma - 0.5 \end{bmatrix}.$$

Figures 3.3.1 and 3.3.2 pertaining to the case of $\sigma = 300$ illustrate that ALTR is achieved.

□

Analogous to theorem 3.3.2, we have the following theorem 3.3.4 regarding the nonempti-

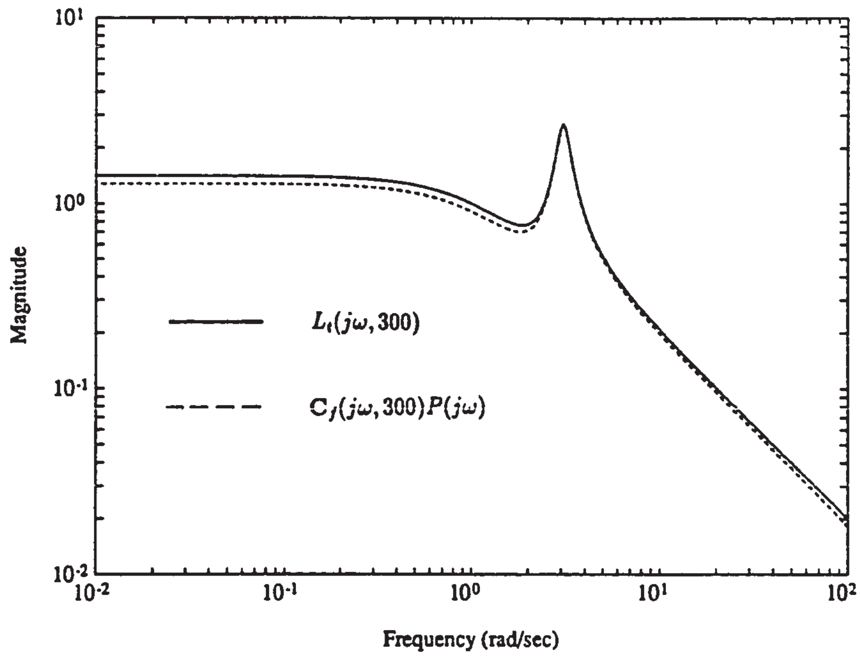


Figure 3.3.2: $\sigma_{max}[L_t(j\omega, 300)]$ and $\sigma_{max}[C_f(j\omega, 300)P(j\omega)]$.

ness of $\mathbf{T}^{R}(\Sigma)$.

Theorem 3.3.4. *Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let $\overline{C}^{R}$ be any full rank matrix with dimensions $(n_a^- + n_b + n_f) \times n$ such that $\mathrm{Ker}\,(\overline{C}^{R}) = \mathcal{V}^{+}(\Sigma)$. Also, define an auxiliary system Σ^{R} characterized by the matrix triple $(A, B, \overline{C}^{R})$. Then, under either a full or reduced order observer based controller, the given system Σ has at least one recoverable target loop, i.e. $\mathbf{T}^{R}(\Sigma)$ is nonempty, iff the auxiliary system Σ^{R} is stabilizable by a static output feedback controller.*

Proof : The proof follows along the same lines as that of theorem 3.3.2. ∎

Theorems 3.3.2 and 3.3.4 respectively give the necessary and sufficient conditions under which the set of exactly recoverable target loops $\mathbf{T}^{ER}(\Sigma)$, and the set of recoverable target loops $\mathbf{T}^{R}(\Sigma)$, are nonempty. However, the conditions given there are not conducive to any intuitive feelings. The following corollary gives a necessary condition which is surprising as well as intuitively appealing.

Corollary 3.3.5. *Strong stabilizability of a given system Σ is a necessary condition for it to have at least one, exactly or asymptotically, recoverable target loop by using either a full or reduced order observer based controller.*

Proof : For the case of full order observer based controller, it follows from (3.2.21) that whenever a target loop is exactly or asymptotically recoverable, the eigenvalues of the corresponding observer based controller are either exactly or asymptotically given by $\Lambda^0 \in \mathcal{C}^-$, $\Lambda^\infty \in \mathcal{C}^-$ and $\Lambda^e \in \mathcal{C}^-$. Hence, such a controller is asymptotically stable. By definition the given system Σ is strongly stabilizable.

Results for reduced order observers follow using similar arguments. ∎

We now proceed to discuss the possible cancellations between the eigenvalues of the controller and the input or output decoupling zeros of $\mathbf{C}_f(s,\sigma)$ or $\mathbf{C}_f(s,\sigma)P(s)$. Lemma 3.2.3 already discussed one such result which is a slight generalization of a similar one in Goodman [22]. The following lemma is also a slight generalization of a similar one in [22].

Lemma 3.3.1. *Let λ be an eigenvalue of $A - K_f(\sigma)C$ and the corresponding right eigenvector W be such that $FW = 0$. Then λ is an eigenvalue of $A - K_f(\sigma)C - BF + K_f(\sigma)DF$ with corresponding right eigenvector as W. Moreover, λ cancels an output decoupling zero of $\mathbf{C}_f(s,\sigma)$.*

Proof : It follows from some simple algebra. ∎

We have the following theorems.

Theorem 3.3.5. *If ELTR is achieved, i.e. if $E(j\omega,\sigma) = 0$ for all $0 \leq |\omega| < \infty$, then every eigenvalue of $A - K_f(\sigma)C - BF + K_f(\sigma)DF$ cancels either an output decoupling zero of $\mathbf{C}_f(s,\sigma)$ or an input decoupling zero of $\mathbf{C}_f(s,\sigma)P(s)$.*

Proof : ELTR is achieved iff either $FW_i = 0$ or $V_i^H[B - K_f(\sigma)D] = 0$ or both. Hence the result follows from lemmas 3.2.3 and 3.3.1. ∎

Theorem 3.3.6. *If ALTR is achieved, i.e. if $E(j\omega,\sigma) \to 0$ as $\sigma \to \infty$ for all $0 \leq |\omega| < \infty$, then every asymptotically finite eigenvalue of $A - K_f(\sigma)C - BF + K_f(\sigma)DF$ cancels either an output decoupling zero of $\mathbf{C}_f(s,\sigma)$ or an input decoupling zero of $\mathbf{C}_f(s,\sigma)P(s)$.*

Proof : If ALTR is achieved, then every asymptotically finite eigenvalue of $A - K_f(\sigma)C$ with corresponding right and left eigenvectors W_i and V_i must be such that either $FW_i = 0$ or $V_i^H[B - K_f(\sigma)D] = 0$ or both. Hence this result also follows from lemmas 3.2.3 and 3.3.1. ∎

In view of lemmas 3.2.3 and 3.3.1, and theorem 3.3.5, whenever ELTR occurs, there are n exact cancellations among the eigenvalues of the controller and the output decoupling zeros of $\mathbf{C}_f(s)$ or the input decoupling zeros of $\mathbf{C}_f(s)P(s)$.

Remark 3.3.2. *Lemma 3.3.1 and theorems 3.3.5 and 3.3.6 are equally true for reduced order observer based controller. In this case, notationally the quadruple (A,B,C,D), F and $\mathbf{C}_f(s)$ or $\mathbf{C}_f(s,\sigma)$ are to be replaced respectively by (A_r,B_r,C_r,D_r), F_2 and $\mathbf{C}_r(s)$ or $\mathbf{C}_r(s,\sigma)$.*

3.4. Recovery analysis in a given subspace

In the last two sections, we discussed recovery of a target loop transfer function $L_t(s) = F\Phi B$ when the recovery is required over the entire control space $\Re^m$ and when the knowledge of state feedback gain F is either unknown or known. The traditional LTR problem as treated in there, concentrates on recovering an open-loop transfer function $L_t(s)$ which has been formed to take into account the given design specifications. Actually, design specifications are normally formulated in terms of certain required closed-loop sensitivity and complementary sensitivity functions, $S_t(s) = [I_m + F\Phi B]^{-1}$ and $T_t(s) = I_m - S_t(s)$. In LQG/LTR design philosophy, these given specifications are reflected in formulating an open-loop transfer function called target loop transfer function. As discussed earlier, this aspect of determining a target loop transfer function is a first step in LQG/LTR design and falls in the category of loop shaping. Generating a target loop transfer function $L_t(s)$ at the present time is an engineering art and often involves the use of linear quadratic design in which the cost matrices are used as free design parameters to obtain the state feedback gain F and thus to obtain $L_t(s) = F\Phi B$ and $S_t(s) = [I_m + F\Phi B]^{-1}$. In the second step of design, the so called loop transfer recovery (LTR) design, $L_t(s)$ is recovered using a measurement feedback controller. Obviously, in the traditional LTR design where recovery is required over the entire control space $\Re^m$, the recovery of $L_t(s)$ implies the recovery of the corresponding sensitivity function $S_t(s)$ and hence the recovery of the complementary sensitivity function $T_t(s)$. Conversely, in a similar manner, the recovery of $S_t(s)$ or equivalently that of $T_t(s)$, implies the recovery of $L_t(s)$. In other words, when recovery is required over the entire control space $\Re^m$, recovering a certain target loop transfer function is equivalent to recovering a certain target sensitivity function. Thus, without loss of any freedom, historically, recovery of a target loop transfer function has been sought.

As seen in earlier sections, loop transfer recovery in the entire control space $\Re^m$ is not possible in general. This may force a designer to seek recovery only in a chosen subspace $\mathcal{S}$ of the control space $\Re^m$. In that case, it is natural to think of recovering the projections of both the target loop $L_t(s)$ and the sensitivity function $S_t(s)$ onto $\mathcal{S}$. However, as seen in example 3.1, one may obtain the projections of achieved and target sensitivity functions onto $\mathcal{S}$ matching each other, but the projection of the correspondingly achieved loop transfer function may or may not match that of the target loop. This implies that the designer may have to choose between matching the projections onto $\mathcal{S}$ of (1) achieved and target sensitivity functions, and (2) achieved and target loop transfer functions. Since, most often design specifications are given in terms of sensitivity functions, it is natural to choose matching the projections onto $\mathcal{S}$ of achieved and target sensitivity functions. In view of this, in this section, we focus on recovery of sensitivity functions over a subspace. For the case when $\mathcal{S}$ equals $\Re^m$, obviously the sensitivity recovery formulation of this section coincides with the conventional LTR formulation. Thus this section can indeed be viewed as a generalization of the notion of traditional LTR to cover recovery over either the entire or any specified subspace $\mathcal{S}$ of the control space $\Re^m$.

A brief outline of this section is as follows. At first, precise definitions dealing with the sensitivity recovery problem are given. Then, lemma 3.4.1 is developed generalizing lemma 2.2.1. It formulates the condition for the recoverability of a sensitivity function in $\mathcal{S}$ in terms of a matrix $M^s(s)$. Then, theorem 3.4.1 specifies the required conditions on Σ so that asymptotic sensitivity recovery in $\mathcal{S}$ is possible for any arbitrarily specified sensitivity function $S_t(s)$. Similarly, theorems 3.4.2 and 3.4.4 specify the necessary and sufficient conditions respectively for exact and asymptotic recoverability of a sensitivity function when the knowledge of $S_t(s)$ is known. In an analogous manner, theorems 3.4.3 and 3.4.5 respectively establish the necessary and sufficient conditions so that sets of exactly or asymptotically recoverable sensitivity functions of the given system Σ for a specified subspace $\mathcal{S}$, are nonempty. An important aspect of recovery analysis in a subspace is to determine the maximum possible dimension of a recoverable subspace $\mathcal{S}$. Our results here in this regard show that for a left invertible nonminimum phase system, whatever may be the given target sensitivity and complementary sensitivity functions and whatever may be the number of nonminimum phase invariant zeros, there exists at least one $m-1$ dimensional subspace $\mathcal{S}$ of $\Re^m$ in which recovery of sensitivity and complementary sensitivity functions is possible by using either a full or reduced order observer based controllers.

We have the following formal definitions.

Definition 3.4.1. *The set of admissible target sensitivity function* $\mathbf{S}(\Sigma)$ *for a given system* Σ *is defined as follows:*

$$\mathbf{S}(\Sigma) := \left\{ S_t(s) \in \mathcal{M}^{m\times m}(\mathcal{R}_p) \mid S_t(s) = [I_m + L_t(s)]^{-1},\, L_t(s) \in \mathbf{T}(\Sigma) \right\}.$$

Definition 3.4.2. *Given* $S_t(s) \in \mathbf{S}(\Sigma)$ *and a subspace* $\mathcal{S} \in \Re^m$*, we say* $S_t(s)$ *is exactly recoverable in the subspace* $\mathcal{S}$ *if there exists a* $\mathbf{C}(s) \in \mathcal{M}^{m\times p}(\mathcal{R}_p)$ *such that*

(i) *the closed-loop system comprising of the controller* $\mathbf{C}(s)$ *and* $P(s)$ *as in the configuration of figure 1.2.1 is asymptotically stable, and*

(ii) $S(s)P^{s} = S_t(s)P^{s}$, *where* $S(s)$ *is the achieved sensitivity function and* P^{s} *is the orthogonal projection matrix onto* $\mathcal{S}$.

Definition 3.4.3. *Given* $S_t(s) \in \mathbf{S}(\Sigma)$ *and a subspace* $\mathcal{S} \in \Re^m$, *we say* $S_t(s)$ *is asymptotically recoverable in the subspace* $\mathcal{S}$ *if there exists a parameterized family of controllers* $\mathbf{C}(s,\sigma) \in \mathcal{M}^{m \times p}(\mathcal{R}_p)$, *where* σ *is a scalar parameter taking positive values, such that*

(i) *the closed-loop system comprising of* $\mathbf{C}(s,\sigma)$ *and* $P(s)$ *as in the configuration of figure 1.2.1 is asymptotically stable for all* $\sigma > \sigma^*$ *where* $0 \leq \sigma^* < \infty$, *and*

(ii) $S(s,\sigma)P^{s} = S_t(s)P^{s}$ *as* $\sigma \to \infty$. *Moreover, the limits, as* $\sigma \to \infty$, *of the finite eigenvalues of the closed-loop system should remain in* $\mathcal{C}^-$. *Here,* $S(s,\sigma)$ *is the achieved sensitivity function and* P^{s} *is the orthogonal projection matrix onto* $\mathcal{S}$.

Definition 3.4.4. *Given* $S_t(s) \in \mathbf{S}(\Sigma)$ *and a subspace* $\mathcal{S} \in \Re^m$, *we say that* $S_t(s)$ *is recoverable in the subspace* $\mathcal{S}$ *if* $S_t(s)$ *is either exactly or asymptotically recoverable in* $\mathcal{S}$.

Definition 3.4.5.

1. *The set of exactly recoverable* $S_t(s) \in \mathbf{S}(\Sigma)$ *in the given subspace* $\mathcal{S}$ *is denoted by* $\mathbf{S}^{\mathrm{ER}}(\Sigma,\mathcal{S})$.
2. *The set of recoverable* $S_t(s) \in \mathbf{S}(\Sigma)$ *in the given subspace* $\mathcal{S}$ *is denoted by* $\mathbf{S}^{\mathrm{R}}(\Sigma,\mathcal{S})$.
3. *The set of admissible* $S_t(s) \in \mathbf{S}(\Sigma)$ *which are asymptotically recoverable but not exactly recoverable in the given subspace* $\mathcal{S}$ *is denoted by* $\mathbf{S}^{\mathrm{AR}}(\Sigma,\mathcal{S})$.

Obviously, $\mathbf{S}^{\mathrm{R}}(\Sigma,\mathcal{S}) = \mathbf{S}^{\mathrm{ER}}(\Sigma,\mathcal{S}) \cup \mathbf{S}^{\mathrm{AR}}(\Sigma,\mathcal{S})$.

As usual, subscripts f and r are used respectively to distinguish the above sets for full and reduced order observer based controllers. Also, we note that the above definitions 3.4.1

to 3.4.5 are natural extensions of the corresponding definitions 1.2.1 to 1.2.5 given earlier in Chapter 1. In fact, the definitions 3.4.1 to 3.4.5 generalize the concept of recovery to a subspace and enable us to reanalyze all the results of the previous sections to cover recovery in a given subspace $\mathcal{S}$.

The following lemma is analogous to lemma 2.2.1.

Lemma 3.4.1. *Let the given system Σ be stabilizable and detectable. Also, let $L_t(s) = F\Phi B$ be an admissible target loop, i.e. $L_t(s) \in \mathbf{T}(\Sigma)$. Then $E^s(s)$, the projection onto a given subspace $\mathcal{S} \in \Re^m$ of the error between the achieved sensitivity function $S(s)$ and the target sensitivity function $S_t(s)$, is given by*

$$E^s(s) = [I_m + F\Phi B]^{-1} M^s(s) \tag{3.4.1}$$

where

$$M^s(s) = M(s)P^s. \tag{3.4.2}$$

Furthermore for all $\omega \in \Omega$,

$$E^s(j\omega) = 0 \quad \textit{iff} \quad M^s(j\omega) = 0,$$

where Ω is the set of all $0 \leq |\omega| < \infty$ for which $S_t(j\omega)$ and $S(j\omega)$ are well defined (i.e., all required inverses exist). Here the expression for $M(s)$ depends on the controller used. In particular, for full and reduced order observer based controllers considered in this chapter, the needed expressions are as in (2.2.2) and (2.2.3).

Proof : It is obvious. ∎

The following observation pertains to the case when $\mathcal{S} = \Re^m$.

Observation 3.4.1. *If $\mathcal{S} = \Re^m$, then $S_t(s) = [I_m + L_t(s)]^{-1}$ is exactly recoverable in $\mathcal{S}$ iff the corresponding target loop transfer function $L_t(s)$ is exactly recoverable in $\mathcal{S}$. Similarly, if $\mathcal{S} = \Re^m$, then $S_t(s)$ is asymptotically recoverable in $\mathcal{S}$ iff $L_t(s)$ is asymptotically recoverable in $\mathcal{S}$.*

In view of the results of observation 3.4.1, for the case when $\mathcal{S} = \Re^m$, the recoverability of any sensitivity function in $\Re^m$ does indeed imply the recoverability of the corresponding target loop transfer function in $\Re^m$. This implies that when $\mathcal{S} = \Re^m$, definitions 3.4.1 to 3.4.5 are equivalent to the definitions 1.2.1 to 1.2.5 given earlier in chapter 1. On the other hand, definitions 3.4.1 to 3.4.5 generalize the concept of recovery to a subspace and thus enable us to reanalyze all the results of the previous two sections to cover recovery in a given subspace $\mathcal{S}$.

To proceed with the recovery analysis, let V^s be a matrix whose columns form an orthogonal basis of $\mathcal{S} \in \Re^m$. Assume that the columns of V^s are scaled so that the norm of each column is unity. Let $P^s = V^s(V^s)'$ be the unique orthogonal projection matrix onto $\mathcal{S}$. Then, define two auxiliary systems Σ_f^s and Σ_r^s characterized, respectively, by the quadruples (A, BV^s, C, DV^s) and $(A_r, B_rV^s, C_r, D_rV^s)$. Now treating each auxiliary system as the given system, one can rediscuss here mutatis mutandis all the results of sections 3.2 and 3.3. In particular, we have the following theorems.

Theorem 3.4.1. *Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $\mathcal{S} \in \Re^m$. Then*

(1) any admissible sensitivity function $S_t(s)$ of Σ, i.e., $S_t(s) \in \mathbf{S}(\Sigma)$, is asymptotically recoverable in $\mathcal{S}$ by full order observer based controller if the auxiliary system Σ_f^s is left invertible and of minimum phase;

(2) any admissible sensitivity function $S_t(s)$ of Σ, i.e., $S_t(s) \in \mathbf{S}(\Sigma)$, is asymptotically recoverable in $\mathcal{S}$ by reduced order observer based controller if the auxiliary system Σ_r^s is left invertible and of minimum phase.

Proof : It is obvious in view of theorem 3.2.1. ∎

Theorem 3.4.1 is concerned with the recovery analysis when F is arbitrary or unknown. As in section 3.3, one can formulate the recovery conditions for a known F as follows.

Theorem 3.4.2. *Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $\mathcal{S} \in \Re^m$. Then depending on the controller used, an admissible sensitivity function $S_t(s)$ of Σ, i.e. $S_t(s) \in \mathbf{S}(\Sigma)$, is exactly recoverable in $\mathcal{S}$ iff the following condition is satisfied:*

1. *For a full order observer based controller, the condition is that $\mathcal{S}^-(\Sigma_f^s) \subseteq \operatorname{Ker}(F)$.*
2. *For a reduced order observer based controller, the condition is that $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r^s) \subseteq \operatorname{Ker}(F)$.*

Thus, the set of exactly recoverable sensitivity functions in the given subspace $\mathcal{S}$ is characterized as follows:

(1) Full order observer based controller :

$$\mathbf{S}_f^{ER}(\Sigma, \mathcal{S}) = \left\{ S_t(s) \in \mathbf{S}(\Sigma) \mid \mathcal{S}^-(\Sigma_f^s) \subseteq \operatorname{Ker}(F) \right\}.$$

(2) Reduced order observer based controller :

$$\mathbf{S}_r^{ER}(\Sigma, \mathcal{S}) = \left\{ S_t(s) \in \mathbf{S}(\Sigma) \mid \begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r^s) \subseteq \operatorname{Ker}(F) \right\}.$$

Proof : The proof is a consequence of theorem 3.3.1. ∎

Remark 3.4.1. *If the given system Σ is strictly proper, i.e. $D = 0$, then it is simple to verify that*

$$\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r^s) = \mathcal{S}^-(\Sigma^s) \cap \{x \mid Cx \in \operatorname{Im}(DV^s)\}$$

and

$$\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{V}^+(\Sigma_r^s) = \mathcal{V}^+(\Sigma^s) \cap \{x \mid Cx \in \operatorname{Im}(DV^s)\}$$

which are not true in general for non-strictly proper systems.

In what follows, we give a necessary and sufficient condition under which $\mathbf{S}^{\mathrm{ER}}(\Sigma,\mathcal{S})$ is non-empty for the given subspace $\mathcal{S} \in \Re^m$. We have the following theorem.

Theorem 3.4.3. *Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A,B,C,D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $\mathcal{S} \in \Re^m$. Let $\overline{C}_f^s$ and $\overline{C}_r^s$ be any full rank matrices such that*

1. $\mathrm{Ker}\,(\overline{C}_f^s) = \mathcal{S}^-(\Sigma_f^s)$,
2. $\mathrm{Ker}\,(\overline{C}_r^s) = \begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r^s)$.

Next, define two auxiliary systems Σ_f^{aus} and Σ_r^{aus}, which are respectively characterized by the matrix triples $(A,B,\overline{C}_f^s)$ and $(A,B,\overline{C}_r^s)$. Then we have the following results depending upon the controller used :

(1) Full order observer based controller :

$\mathbf{S}_f^{\mathrm{ER}}(\Sigma,\mathcal{S})$ *is nonempty iff* Σ_f^{aus} *is stabilizable by a static output feedback controller.*

(2) Reduced order estimator based controller :

$\mathbf{S}_r^{\mathrm{ER}}(\Sigma,\mathcal{S})$ *is nonempty iff* Σ_r^{aus} *is stabilizable by a static output feedback controller.*

Proof : The proof is a consequence of theorem 3.3.2. ∎

The following theorem deals with asymptotic recoverability of $S_t(s)$.

Theorem 3.4.4. *Consider a stabilizable and detectable system Σ characterized by a matrix quadruple (A,B,C,D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $\mathcal{S} \in \Re^m$. Then depending on the controller used, an admissible sensitivity function $S_t(s)$ of Σ, i.e. $S_t(s) \in \mathbf{S}(\Sigma)$, is recoverable in $\mathcal{S}$ iff the following condition is satisfied:*

1. *For a full order observer based controller, the condition is that* $\mathcal{V}^+(\Sigma_f^s) \subseteq \mathrm{Ker}\,(F)$.

2. *For a reduced order observer based controller, the condition is that* $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{V}^+(\Sigma_r^s) \subseteq \operatorname{Ker}(F)$.

Thus the set of asymptotically recoverable sensitivity functions in the given subspace $\mathcal{S}$ *is characterized as follows:*

(1) Full order observer based controller :

$$\mathbf{S}_f^{\mathrm{R}}(\Sigma, \mathcal{S}) = \left\{ S_t(s) \in \mathbf{S}(\Sigma) \mid \mathcal{V}^+(\Sigma_f^s) \subseteq \operatorname{Ker}(F) \right\}.$$

(2) Reduced order observer based controller :

$$\mathbf{S}_r^{\mathrm{R}}(\Sigma, \mathcal{S}) = \left\{ S_t(s) \in \mathbf{S}(\Sigma) \mid \begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{V}^+(\Sigma_r^s) \subseteq \operatorname{Ker}(F) \right\}.$$

Proof : The proof is a consequence of theorem 3.3.3. ∎

Again, as in theorem 3.4.3, we have the following theorem regarding non-emptiness of the set $\mathbf{S}^{\mathrm{AR}}(\Sigma, \mathcal{S})$.

Theorem 3.4.5. *Consider a stabilizable and detectable system* Σ *characterized by a matrix quadruple* (A, B, C, D)*, which is not necessarily of minimum phase and which is not necessarily left invertible. Let* V^s *be a matrix whose columns form an orthogonal basis of a given subspace* $\mathcal{S} \in \Re^m$*. Let* $\overline{C}_f^{sa}$ *and* $\overline{C}_r^{sa}$ *be any full rank matrices such that* $\operatorname{Ker}(\overline{C}_f^{sa}) = \mathcal{V}^+(\Sigma_f^s)$ *and* $\operatorname{Ker}(\overline{C}_r^{sa}) = \begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{V}^+(\Sigma_r^s)$*. Then depending on the controller used, we have the following results:*

(1) Full order observer based controller :

$\mathbf{S}_f^{\mathrm{R}}(\Sigma, \mathcal{S})$ *is nonempty iff an auxiliary system* Σ_f^{sa} *characterized by the matrix triple* $(A, B, \overline{C}_f^{sa})$ *is stabilizable by a static output feedback controller.*

(2) Reduced order observer based controller :

$\mathbf{S}_r^{\mathrm{R}}(\Sigma, \mathcal{S})$ *is nonempty iff an auxiliary system* Σ_r^{sa} *characterized by the matrix triple* $(A, B, \overline{C}_r^{sa})$ *is stabilizable by a static output feedback controller.*

Proof : The proof is a consequence of theorem 3.3.4. ∎

An important aspect that arises when one is interested in recovery analysis in a subspace is to determine the maximum possible dimension of a recoverable subspace $\mathcal{S}$. In this regard, our goal in what follows, as in [39], is to prove that whatever may be the given target loop transfer function and whatever may be the number of nonminimum phase zeros, there exists at least one $m-1$ dimensional subspace $\mathcal{S}$ of $\Re^m$ in which the given target sensitivity function is always recoverable either by a full or a reduced order observer based controller. To prove this result, for simplicity of presentation, we will make a technical assumption that all the nonminimum phase invariant zeros of Σ have geometric multiplicity equal to one.

We next state two lemmas which lead to the intended result.

Lemma 3.4.2. *Let the given system Σ be left invertible and let z, x and w be respectively an invariant zero, the associated right state and input zero directions of Σ. Then we have the following properties.*

1. *The auxiliary system Σ_f^s is left invertible.*
2. *Every invariant zero and the associated right state zero direction of Σ_f^s are also the invariant zero and the associated right state zero direction of Σ.*
3. *z and x are respectively an invariant zero and the associated right state zero direction of Σ_f^s iff $w \in \mathcal{S}$.*

Proof : See Appendix 3.D. ∎

Remark 3.4.2. *Lemma 3.4.2 is equally true if we replace Σ and Σ_f^s by Σ_r and Σ_r^s.*

Now let z_i, x_i and w_i, $i = 1$ to n_a^+, be respectively a nonminimum phase invariant zero and the associated right state and input zero directions of the given system Σ. Since Σ is assumed to be stabilizable and detectable, we have $w_i \neq 0$ for all $i = 1$ to n_a^+. Because if $w_i = 0$, then by definition,

$$(z_i I_n - A)x_i = Bw_i = 0 \;\; , \;\; Cx_i + Dw_i = Cx_i = 0.$$

This implies that z_i is an output decoupling zero of Σ. But this contradicts the detectability of Σ as $z_i \in \mathcal{C}^+$. Next let us define for each $i = 1$ to n_a^+,

$$\mathcal{N}_i = \mathrm{Ker}\,(w_i').$$

Since $w_i \neq 0$, each $\mathcal{N}_i$ is an $m-1$ dimensional subspace. We have the following lemma.

Lemma 3.4.3. *There exists at least one nonzero vector* $e \in \Re^m$ *such that*

$$e \notin \bigcup_{i=1}^{n_a^+} \mathcal{N}_i.$$

Proof : The proof is by induction. Lemma is trivially true when $n_a^+ = 1$. Assume that the given lemma is true for $n_a^+ = k$. Then there exists a vector $0 \neq v \in \Re^m$ such that

$$v \notin \bigcup_{i=1}^{k} \mathcal{N}_i.$$

To proceed with the proof, let us first assume that $v \notin \mathcal{N}_{k+1}$. Then $e = v \notin \bigcup_{i=1}^{k+1} \mathcal{N}_i$ and hence the result.

On the other hand, assume that $v \in \mathcal{N}_{k+1}$. First select nonzero scalar numbers α_i, $i = 1$ to $k+1$, such that $\alpha_i \neq \alpha_j$ if $i \neq j$. Then we have for all $i = 1$ to $k+1$, $\alpha_i v \in \mathcal{N}_{k+1}$. Since $\mathcal{N}_{k+1}$ has only a dimension of $m-1$, there exists a vector $0 \neq w \in \Re^m$ such that $w \notin \mathcal{N}_{k+1}$. Now define for each $i = 1$ to $k+1$,

$$x_i = \alpha_i v + w \neq 0.$$

Because of the fact that $\alpha_i \neq \alpha_j$ if $i \neq j$, we note that $x_i \neq x_j$ if $i \neq j$. Moreover $x_i \notin \mathcal{N}_{k+1}$ for all $i = 1$ to $k+1$ since $w \notin \mathcal{N}_{k+1}$. Now if $x_i \in \bigcup_{j=1}^{k} \mathcal{N}_j$ for all $i = 1$ to $k+1$, then there exists two distinct vectors among x_i, $i = 1$ to $k+1$, say x_s and x_t for some integers s and t, such that both are contained in some $\mathcal{N}_\beta$ for some $\beta \leq k$. Thus $0 \neq (\alpha_s - \alpha_t)v = (x_s - x_t) \in \mathcal{N}_\beta$. This implies that $v \in \mathcal{N}_\beta$ and thus contradicts the inductive hypothesis. Hence there exists at least one x_i for some $i \leq k+1$, such that

$$x_i \notin \bigcup_{j=1}^{k} \mathcal{N}_j \text{ and that } e = x_i \notin \bigcup_{j=1}^{k+1} \mathcal{N}_j.$$

Hence the result. ∎

Thus in view of lemma 3.4.3, there exists at least one e such that

$$e'w_i \neq 0 \text{ for all } i = 1 \text{ to } n_a^+. \tag{3.4.3}$$

We have the following theorem.

Theorem 3.4.6. *Let the given system Σ be left invertible with nonminimum phase invariant zeros having geometric multiplicity equal to unity. Then there exists at least one $m-1$ dimensional subspace $\mathcal{S}$ of $\Re^m$ such that any admissible target sensitivity functions $S_t(s)$ of Σ, i.e., $S_t(s) \in \mathbf{S}(\Sigma)$, is recoverable, either by a full or a reduced order observer based controller, in $\mathcal{S}$.*

Proof : For the case of full order observer based controller, we select e as in (3.4.3). Define $\mathcal{S}$ as

$$\mathcal{S} = \text{ The orthogonal complement of the subspace spanned by } e \text{ in } \Re^m.$$

Then it is trivial to see $\mathcal{S}$ has a dimension of $m-1$ and that $w_i \notin \mathcal{S}$ for all $i = 1$ to n_a^+. Because if $w_i \in \mathcal{S}$, say $w_i = V^s v_i \in \mathcal{S}$, then $e'w_i = 0$ which is a contradiction. In view of lemma 3.4.2, this implies that Σ_f^s is left invertible and of minimum phase. This in turn implies the results of theorem 3.4.6.

The result for reduced order observer based controller follows from similar arguments and the properties of Σ_r as given in proposition 2.2.1. ∎

3.A. Appendix 3.A — Proof of Lemma 3.2.1

Let λ_i and V_i be an eigenvalue and the corresponding left eigenvector of $A - KC$ for any gain K. To show that there are at most $n_a^- + n_b$ left eigenvectors of $A - KC$ for any gain K such that the corresponding $\lambda_i \in C^-$ and that $V_i^H(B - KD) = 0$, consider the dual system Σ_t characterized by (A_t, B_t, C_t, D_t) where

$$A_t = A', \quad B_t = C', \quad C_t = B', \quad D_t = D'.$$

Let $\mathcal{V}_t$ be the subspace of all right eigenvectors V_t of $(A_t - B_tK_t)$ for some K_t such that $(C_t - D_tK_t)V_t = 0$. Observe that $\mathcal{V}_t$ is a stable $(A_t,\ B_t)$–invariant subspace. Furthermore, $\mathcal{V}_t$ is in the kernel of $(C_t - D_tK_t)$. Hence $\mathcal{V}_t$ is a subset of $\mathcal{V}^-(A_t,\ B_t,\ C_t,\ D_t)$. The largest possible dimension of $\mathcal{V}^-(A_t,\ B_t,\ C_t,\ D_t)$ is $n_a^- + n_b$. Hence, there are at most $n_a^- + n_b$ left eigenvectors of $A - KC$ for any gain K such that the corresponding $\lambda_i \in \mathcal{C}^-$ and that $V_i^H(B - KD) = 0$.

We now proceed to determine the necessary gain K to assign such eigenvalues. Without loss of generality we can assume that the given system is represented by the s.c.b as given in Chapter 1. Then consider a gain K of the form,

$$K = \begin{bmatrix} B_{0a}^- & L_{af}^- & L_{ab}^- \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & L_{bf} & K_{bb} \\ B_{0c} & 0 & 0 \\ B_{0f} & 0 & 0 \end{bmatrix}$$

where K_{bb} is selected such that $\lambda(A_{bb} - K_{bb}C_b)$ are in $\mathcal{C}^-$. Let V_{a-} and V_b respectively be any left eigenvectors of A_{aa}^- and $A_{bb} - K_{bb}C_b$. It can easily be verified that $\lambda(A_{aa}^-)$ and $\lambda(A_{bb} - K_{bb}C_b)$ are among the eigenvalues of $A - KC$ and that $[V_{a-}^H,\ 0,\ 0,\ 0,\ 0]^H$ and $[0,\ 0,\ V_b^H,\ 0,\ 0]^H$ are the associated left eigenvectors of $A - KC$. Furthermore, it is easy to verify that

$$[V_{a-}^H,\ 0,\ 0,\ 0,\ 0](B - KD) = [V_{a-}^H,\ 0,\ 0,\ 0,\ 0] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_c \\ 0 & B_f & 0 \end{bmatrix} = 0$$

and similarly

$$[0,\ 0,\ V_b^H,\ 0,\ 0](B - KD) = 0.$$

Finally, in view of the properties of s.c.b, it is straightforward to see that such vectors $[V_{a-}^H,\ 0,\ 0,\ 0,\ 0]^H$ and $[0,\ 0,\ V_b^H,\ 0,\ 0]^H$ respectively span the subspaces x_a^- and x_b. Moreover, x_a^- spans $\mathcal{V}^*(A,B,C,D)/\mathcal{V}^+(A,B,C,D)$ and hence the result. ∎

3.B. Appendix 3.B — Proof of Lemma 3.2.3

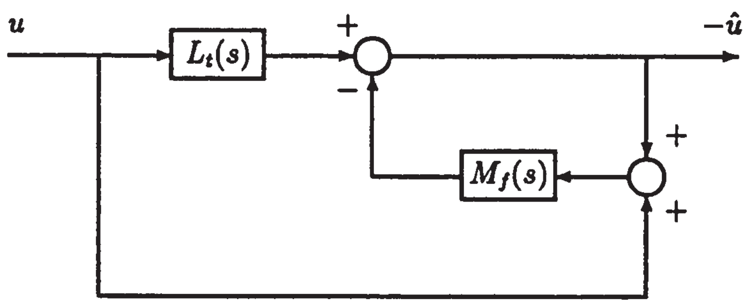
For economy of notations, we drop the dependency on σ throughout this proof. Noting from Lemma 2.2.1 that

$$E_f(s) := L_t(s) - \mathbf{C}_f(s)P(s) = M_f(s)[I + M_f(s)]^{-1}[I + L_t(s)],$$

we obtain,

$$\begin{aligned}\mathbf{C}_f(s)P(s) &= L_t(s) - M_f(s)[I + M_f(s)]^{-1}[I + L_t(s)] \\ &= [I + M_f(s)]^{-1}[L_t(s) - M_f(s)],\end{aligned}$$

from which $\mathbf{C}_f(s)P(s)$ can be interpreted in terms of a block diagram given below.



In view of the block diagram, it is straightforward to write a state-space realization of $\mathbf{C}_f(s)P(s)$ as

$$\begin{cases} \dot{\tilde{x}} = \begin{bmatrix} A & 0 \\ (B - K_pD)F & A - K_pC - BF + K_pDF \end{bmatrix} \tilde{x} + \begin{bmatrix} B \\ B - K_pD \end{bmatrix} u, \\ -\hat{u} = [\,F, \quad -F\,]\,\tilde{x}. \end{cases}$$

Let λ be an eigenvalue of $A - K_pC$ and the corresponding left eigenvector V be such that $V^H(B - K_pD) = 0$. It is simple then to verify that

$$[0, \quad V^H] \begin{bmatrix} \lambda I - A & 0 \\ -(B - K_pD)F & \lambda I - A + K_pC + (B - K_pD)F \end{bmatrix} = 0$$

and

$$[0, \quad V^H] \begin{bmatrix} B \\ B - K_pD \end{bmatrix} = 0.$$

This shows that λ is an input decoupling zero of $\mathbf{C}_f(s)P(s)$ and thus the result follows. ∎

3.C. Appendix 3.C — Proof of Corollary 3.2.2

Again, without loss of generality, let us assume that the given system is represented by the s.c.b. First, we have the following propositions.

Proposition 3.C.1. *The fact that the target loop transfer function $F\Phi B$ contains the nonminimum phase zero structure of Σ, which is left invertible but not necessarily strictly proper, implies that $\mathcal{V}^+(\Sigma) \subseteq \text{Ker}(F)$, i.e., the span of $x_a^+ \subseteq \text{Ker}(F)$.*

Proof : Let x_{ri} and w_{ri}, $i = 1$ to n_a^+, be the right state and input zero directions associated with the invariant zeros z_i, $i = 1$ to n_a^+, which are in C^+. It is easy to show (for example, see [39]) that

$$\text{Span}\{x_{ri},\ i = 1, 2, \cdots, n_a^+\} = \text{Span of } x_a^+.$$

We also have for all $i = 1$ to n_a^+,

$$(z_i I_n - A)x_{ri} = Bw_{ri} \quad \text{and} \quad Cx_{ri} + Dw_{ri} = 0.$$

Since the target loop transfer function $F\Phi B$ contains the same nonminimum phase zero structure as Σ, the above implies

$$(z_i I_n - A)x_{ri} = Bw_{ri} \quad \text{and} \quad Fx_{ri} + 0 \cdot w_{ri} = 0.$$

for all $i = 1$ to n_a^+. Hence the span of $x_a^+ \subseteq \text{Ker}(F)$. This proves proposition 3.C.1. □

Proposition 3.C.2. *Let Σ be strictly proper, invertible and of uniform rank with relative degree q. Then*

1. *the smallest order of infinite zero of $L_t(s) = F\Phi B$ is greater than q implies that $\mathcal{S}^-(\Sigma)/\mathcal{V}^+(\Sigma) \subseteq \text{Ker}(F)$, i.e., the span of $x_f \subseteq \text{Ker}(F)$.*
2. *the smallest order of infinite zero of $L_t(s) = F\Phi B$ is equal to or greater than q implies that $\{\mathcal{S}^-(\Sigma) \cap \mho\}/\mathcal{V}^+(\Sigma) \subseteq \text{Ker}(F)$, i.e., the span of $x_f \cap \text{Ker}(C) \subseteq \text{Ker}(F)$.*

Proof : Let us first prove item 1. Given that Σ is strictly proper, invertible and of uniform rank with relative degree q implies that the matrices A, B, C and D are of the form,

$$A=\begin{bmatrix} A_{aa} & L_a & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ E_a & E_1 & E_2 & E_3 & \cdots & E_q \end{bmatrix}, \quad B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix},$$

$$C=\begin{bmatrix} 0 & I & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad D=0.$$

It is then straightforward to verify that

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{q-1}B \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & I \\ 0 & 0 & 0 & \cdots & I & X \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & I & \cdots & X & X \\ 0 & I & X & \cdots & X & X \\ I & X & X & \cdots & X & X \end{bmatrix}, \tag{3.C.1}$$

where X denotes the submatrices, which are not necessarily zero. The fact that the smallest order of infinite zero of $L_t(s)$ is greater than q implies that

$$FB=FAB=FA^2B=\cdots=FA^{q-1}B=0.$$

Then in view of (3.C.1), we can conclude that F is of the form,

$$F=[X,0,0,\cdots,0].$$

This implies that the span of $x_f \subseteq \mathrm{Ker}\,(F)$.

On the other hand, the fact that the smallest order of infinite zero of $L_t(s)$ is equal to or greater than q implies that

$$FB=FAB=FA^2B=\cdots=FA^{q-2}B=0.$$

Then in view of (3.C.1), we see that F is of the form,

$$F=[X,X,0,\cdots,0].$$

This implies that the span of $x_f \cap \operatorname{Ker}(C) \subseteq \operatorname{Ker}(F)$. This completes the proof of proposition 3.C.2. □

The results of corollary 3.3.2 follow from the properties of s.c.b, theorem 3.3.1, propositions 3.C.1 and 3.C.2. ∎

3.D. Appendix 3.D — Proof of Lemma 3.4.2

Assume that Σ_f^s is not left invertible. Then it is well known that for any complex number z_1, there exist $0 \neq x_1 \in \Re^n$ and $v_1 \in \Re^m$ such that

$$\begin{bmatrix} z_1 I_n - A & -BV^s \\ C & DV^s \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \end{bmatrix} = 0.$$

This implies that

$$\begin{bmatrix} z_1 I_n - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 \\ V^s v_1 \end{bmatrix} = 0.$$

Since Σ is left invertible, this then in turn implies that z_1 is an invariant zero of Σ. This is a contradiction and hence Σ_f^s is left invertible. To prove the second property of the lemma, let z_s, x_s and w_s be respectively an invariant zero, the associated right state and input zero directions of Σ_f^s. Then by definition, we have

$$\begin{bmatrix} z_s I_n - A & -BV^s \\ C & DV^s \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix} = 0.$$

Thus we note that

$$\begin{bmatrix} z_s I_n - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_s \\ V^s w_s \end{bmatrix} = 0.$$

This proves the second property of the lemma. Let us next prove the sufficiency part of property 3. Let $w = V^s v$, then

$$\begin{bmatrix} zI_n - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0$$

implies that

$$\begin{bmatrix} zI_n - A & -BV^s \\ C & DV^s \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = 0.$$

As Σ_f^s is left invertible, the above implies that z and x are an invariant zero and the associated right state zero direction of Σ_f^s. To prove necessity, assume that z and x are an invariant zero and the associated right state zero direction of Σ_f^s. Then there exists a w_s such that

$$(zI_n - A)x = BV^s w_s, \;\; Cx + DV^s w_s = 0.$$

In view of this and by the definition of z, x and w, we have

$$BV^s w_s = Bw, \quad DV^s w_s = Dw.$$

Since $[B', \; D']'$ is assumed to be of full rank, it implies then that $w \in \mathcal{S}$. ∎

Chapter 4

GENERAL LTR DESIGN

4.1. Introduction

Chapter 3 considers the analysis of loop transfer recovery problem for general continuous systems when either full or reduced order observer based controllers are used. Also, the analysis given there corresponding to full and reduced order observer based controllers has been unified into a single mathematical frame work. As such, for general discussion, only full order observer based controllers are considered. The analysis of chapter 3 focuses on four fundamental issues, (1) recoverability of a target loop when it is arbitrarily given, (2) recoverability of a target loop taking into account its specific characteristics, (3) establishment of necessary and sufficient conditions on the given system so that it has at least one recoverable target loop transfer function or sensitivity function, and (4) recoverability of a sensitivity function in a specified subspace of the control space. All this analysis shows some fundamental limitations of the given system as a consequence of its structural properties. In particular, the analysis of chapter 3 decomposes the so called recovery matrix $M_f(s,\sigma)$ where σ is a tuning parameter of the observer gain, into three parts, namely $M_f^0(s,\sigma)$, $M_f^\infty(s,\sigma)$ and $M_f^e(s,\sigma)$. For any arbitrary target loop transfer function $L_t(s)$, the first part $M_f^0(s,\sigma)$ can identically be rendered zero by appropriate finite eigenstructure assignment to the observer dynamic matrix $A-K_f(\sigma)C$, while the second part $M_f^\infty(s,\sigma)$ can be rendered zero asymptotically as $\sigma \to \infty$ by appropriate infinite eigenstructure assignment to $A - K_f(\sigma)C$. The third part called ***recovery error matrix*** $M_f^e(s,\sigma)$ cannot in general be rendered zero either exactly or asymptotically by any means for an arbitrary target loop transfer function. However, it turns out that there exists a considerable amount of freedom to shape the recovery error matrix in a desired way. Thus the analysis of chapter 3 while shows explicitly what is feasible and what is not and helps to set meaningful design goals at the onset of design.

The task of this chapter is to develop different design methods for both full and reduced order observer based controllers in order to recover whatever is recoverable while shaping the recovery error appropriately. Three different design methods are developed. The first one is an asymptotic time-scale structure and eigenstructure assignment scheme (ATEA), and the other two are optimization based designs. ATEA design method yields a controller design which achieves ***any chosen*** recovery error matrix among a set of admissible recovery error matrices. On the other hand, one of the optimization based design methods leads to a controller that achieves a recovery error matrix having the infimum H_∞ norm, while the other does the same except it achieves a recovery error matrix having the infimum H_2 norm. The traditional design method developed first by Doyle and Stein [18] for left invertible minimum phase systems using full order observer based controllers, turns out to be an optimization based design in which H_2 norm of the recovery error matrix is minimized. In optimization methods, one normally generates a sequence of observer gains by solving parameterized algebraic Riccati equations. As the parameter tends to a certain value, the corresponding sequence of H_2 norms (or H_∞ norms depending on the method) of the resulting recovery matrices tends to a limit which is the infimum of the H_2 norm (or H_∞ norm) of the recovery matrix over the set of all possible observer gains. A suboptimal solution is obtained when one selects an observer gain corresponding to a particular value of the parameter. In the process of generating a sequence of suboptimal solutions, the mathematical optimization procedure follows a particular path and shapes the recovery matrix accordingly. That is, there is no freedom to shape the recovery matrix directly, and one has to be content with whatever the mathematical optimization procedure yields. In contrast with this, since parameterization of ATEA design procedure is explicit rather than being implicit via algebraic Riccati equations of optimization based methods, ATEA design procedure allows all the available design freedom to shape the recovery matrix as desired within the structural constraints imposed by the given system. Thus, an important difference between ATEA and optimization based designs that needs to be emphasized is this. ATEA is capable of achieving any admissible recovery error matrix where as optimization based methods always lead to a particular recovery error matrix having the infimum H_∞ or H_2 norm depending on the method used. For the case when the given target loop transfer function is recoverable, all the three design methods are capable of recovering it. We note that the observer gain is in general not unique, even for the case when the given target loop transfer function is recoverable.

Let us next clear some notational considerations in presenting our design methods. In our general discussion that is to follow, as in chapter 3, we always deal with the full order observer based controller and the given system Σ along with the given target loop transfer function $L_t(s)$. As is clear from chapter 2, in a full order observer based controller, the observer gain K_f is the *free* design parameter while the *fixed* parameters of design are (a) matrices (A, B, C, D) which characterize the given system Σ, and (b) the state feed back gain F which specifies the given target loop transfer function $L_t(s)$. On the other hand, in a reduced order observer based controller, the observer gain K_r is the *free* design parameter while the *fixed* parameters of design are (a) matrices (A_r, B_r, C_r, D_r) which characterize a reduced order subsystem Σ_r of the given system Σ, and (b) a part F_2 of the state feed back gain F that corresponds to the states which are being estimated by the observer. Thus, if one is interested in a reduced order observer based controller design, for all the design methods discussed here, the free as well as fixed parameters K_f, (A, B, C, D) and F should respectively be replaced by K_r, (A_r, B_r, C_r, D_r) and F_2. This replacement is only for the purpose of design. Once the observer gain K_r is obtained, the implementation of the controller is as in (2.1.12).

This chapter is organized as follows. Section 4.2 reviews the design constraints as well as the available freedom in terms of finite and asymptotically infinite eigenstructure assignment to the observer dynamic matrix $A - K_f(\sigma)C$. Section 4.3 develops the general ATEA method of design. Also, in section 4.3, a simplification of ATEA is given to arrive at a design for exact loop transfer recovery whenever it is feasible. Section 4.4 develops both optimization methods, one dealing with the H_2 norm and the other with the H_∞ norm minimization of the recovery error matrix. Section 4.5 considers the generalized design task of recovering the target sensitivity and complementary sensitivity functions over a subspace of the control space. For this purpose, following chapter 3, an auxiliary system of the given system is constructed so that all the three designs developed earlier can be readily applied for the new design task. Section 4.6 discusses the relative advantages of the ATEA and optimization based design methods.

4.2. Design constraints and available freedom

It is clear from chapters 2 and 3, the recovery matrix $M_f(s,\sigma)$ where σ is a tunable parameter of observer gain $K_f(\sigma)$, plays a central role in both LTR analysis and design since loop transfer recovery error is zero if and only if $M_f(s,\sigma)$ is zero. In chapter 3, we have decomposed $M_f(s,\sigma)$ into three parts,

$$M_f(s,\sigma) = M_f^0(s,\sigma) + M_f^\infty(s,\sigma) + M_f^e(s,\sigma), \tag{4.2.1}$$

and studied each part in detail. To facilitate our discussion here, let us review the essential aspects of the above decomposition by considering one part of it at a time.

<u>Discussion on $M_f^0(s,\sigma)$</u> : This term depends on a set of $n_a^- + n_b$ eigenvalues $\Lambda^0(\sigma) = \Lambda^-(\sigma) \cup \Lambda^b(\sigma)$ of $A - K_f(\sigma)C$, and the corresponding set of right and left eigenvectors, $W^0(\sigma) = W^-(\sigma) \cup W^b(\sigma)$, and $V^0(\sigma) = V^-(\sigma) \cup V^b(\sigma)$. For any arbitrary target loop transfer function, $M_f^0(s,\sigma)$ can identically (irrespective of the value of σ) be rendered

zero. To accomplish this, the set of n_a^- eigenvalues $\Lambda^-(\sigma)$ and the corresponding set of left eigenvectors $V^-(\sigma)$ must be selected to coincide respectively with the set of plant minimum phase invariant zeros and the corresponding left state zero directions of Σ. However, the set of n_b eigenvalues $\Lambda^b(\sigma)$ can be assigned arbitrarily either at asymptotically finite or infinite locations in $\mathcal{C}^-$, while the corresponding set of left eigenvectors $V^b(\sigma)$ is in the null space of matrix $[B - K_f(\sigma)D]'$. If one prefers, $M_f^0(s,\sigma)$ can be rendered zero asymptotically as $\sigma \to \infty$. This can be done by selecting $\Lambda^-(\sigma)$ and the corresponding set of left eigenvectors $V^-(\sigma)$ so that their asymptotic limits $\overline{\Lambda}^-$ and $\overline{V}^-$ coincide respectively with the set of plant minimum phase invariant zeros and the corresponding left state zero directions of Σ. Also, although $\Lambda^b(\sigma)$ can be selected arbitrarily in $\mathcal{C}^-$, the corresponding set of left eigenvectors $V^b(\sigma)$ of $A - K_f(\sigma)C$ must be such that its asymptotic limit $\overline{V}^b$ is in the null space of matrix $[B - K_f(\sigma)D]'$.

Discussion on $M_f^\infty(s,\sigma)$: This term depends on a set of n_f eigenvalues $\Lambda^\infty(\sigma)$ of $A - K_f(\sigma)C$ and the corresponding set of right and left eigenvectors, $W^\infty(\sigma)$ and $V^\infty(\sigma)$. For any arbitrary target loop transfer function, $M_f^\infty(s,\sigma)$ can be rendered zero asymptotically as $\sigma \to \infty$. For this purpose, the set of n_f eigenvalues $\Lambda^\infty(\sigma)$ can be assigned arbitrarily at asymptotically infinite locations in $\mathcal{C}^-$. However, for every $\lambda_i^\infty(\sigma) \in \Lambda^\infty(\sigma)$, the corresponding right and left eigenvectors $W_i^\infty(\sigma)$ and $V_i^\infty(\sigma)$ must be such that $W_i^\infty(\sigma)[V_i^\infty(\sigma)]^H[B - K_f(\sigma)D]$ is uniformly bounded as $\sigma \to \infty$. We note that there exists complete freedom in the way $\lambda_i^\infty(\sigma) \in \Lambda_\infty(\sigma)$ tends to infinity as $\sigma \to \infty$, i.e., the asymptotic direction and the rate at which each $\lambda_i^\infty(\sigma)$ goes to infinity can be dictated as desired by the designer.

Discussion on $M_f^e(s,\sigma)$: This term depends on a set of $n_a^+ + n_c$ eigenvalues $\Lambda^e(\sigma)$ of $A - K_f(\sigma)C$ and the corresponding set of right and left eigenvectors $W^e(\sigma)$ and $V^e(\sigma)$. It is nonexistent if the given system is of minimum phase and left invertible. For general systems and for an arbitrary target loop transfer function, $M_f^e(s,\sigma)$ can never be rendered zero in any way by any means.

To summarize the above development, the first two terms of $M_f(s,\sigma)$, namely $M_f^0(s,\sigma)$ and $M_f^\infty(s,\sigma)$, can be rendered zero either exactly or asymptotically by appropriate eigenstructure assignment to the observer dynamic matrix $A - K_f(\sigma)C$, but the third term $M_f^e(s,\sigma)$ cannot in general be rendered zero for an arbitrary target loop transfer function. Thus, $M_f^e(s,\sigma)$ or its asymptotic limit as $\sigma \to \infty$, namely $\overline{M}_f^e(s)$, can be termed as *error recovery matrix*. Although $M_f^e(s,\sigma)$ cannot in general be rendered zero, clearly, there exists ample freedom in assigning the parameters $\Lambda^e(\sigma)$, $W^e(\sigma)$ and $V^e(\sigma)$ on which $M_f^e(s,\sigma)$ depends. $\Lambda^e(\sigma)$ can be assigned at any (either asymptotically finite or infinite) locations in C^- subject to the condition that any unobservable but stable eigenvalues of the given system Σ must be included among $\Lambda^e(\sigma)$. Also, there exists complete freedom consistent with the results of [30] in assigning the right and left eigenvector sets $W^e(\sigma)$ and $V^e(\sigma)$. Thus, one can shape the recovery error matrix $M_f^e(s,\sigma)$ by selecting appropriately $\Lambda^e(\sigma)$, $W^e(\sigma)$ and $V^e(\sigma)$. In other words, for any given system, there exists a set of admissible recovery error matrices, and such a set can be denoted as $\mathcal{M}_f^e(\Sigma,\sigma)$. Then, one naturally seeks a design method which leads to a chosen recovery error matrix $\mathcal{M}_f^e(\Sigma,\sigma)$ among the set of admissible recovery error matrices $\mathcal{M}_f^e(\Sigma,\sigma)$. In the following section, we will give an asymptotic time-scale and eigenstructure assignment (ATEA) design method capable of achieving any chosen $M_f^e(s,\sigma) \in \mathcal{M}_f^e(\Sigma,\sigma)$. In section 4.4, we will describe two optimization based design methods, one method leads to a design that yields the infimum H_∞ norm of the recovery error matrix as the tuning parameter $\sigma \to \infty$, while the other yields the infimum H_2 norm. We emphasize that the (ATEA) design method can lead to any chosen recovery error matrix, where as the optimization based design methods yield a particular recovery error matrix having either the infimum H_∞ norm or H_2 norm depending on the method used.

The above discussion pertains to the case where the target loop transfer function $L_t(s) = F\Phi B$ is arbitrarily specified. Apparently, one acquires additional freedom when specific properties of F are taken into account. For example, as stated in theorem 3.3.3, any admissible target loop transfer function $L_t(s)$ of Σ, i.e., $L_t(s) \in \mathbf{T}(\Sigma)$, is recoverable by either full or reduced order observer based controller iff $\mathcal{V}^+(\Sigma) \subseteq \mathrm{Ker}\,(F)$. Thus, whenever $L_t(s)$ is an element of $\mathbf{T}^{\mathrm{R}}(\Sigma)$, $\overline{M}_f^e(s)$ is zero irrespective of the way the set of $n_a^+ + n_c$ eigenvalues belonging to $\Lambda^e(\sigma)$ and the associated right and left eigenvector sets $W^e(\sigma)$ and $V^e(\sigma)$ are selected. Similarly, as stated in theorem 3.3.1, any admissible target loop transfer function is exactly recoverable, i.e., an element of $\mathbf{T}_f^{\mathrm{ER}}(\Sigma)$, iff $\mathcal{S}^-(\Sigma) \subseteq \mathrm{Ker}\,(F)$ for the case of full order observer based controller. Now whenever $L_t(s)$ is an element of $\mathbf{T}_f^{\mathrm{ER}}(\Sigma)$, both $\overline{M}_f^e(s)$ and $\overline{M}_f^\infty(s)$ are zero irrespective of the way the sets of eigenvalues $\Lambda^e(\sigma)$ and $\Lambda^\infty(\sigma)$, and the associated eigenvector sets $W^e(\sigma)$, $V^e(\sigma)$, $W^\infty(\sigma)$ and $V^\infty(\sigma)$, are selected. In fact, in this case, all eigenvalues of $A - K_f(\sigma)C$ can be assigned to finite locations in $\mathcal{C}^-$. Moreover, since there is no necessity of assigning asymptotically infinite eigenvalues, there is no need either to parameterize the observer gain K_f in terms of the tuning parameter σ.

4.3. Observer design by 'ATEA'

The previous section summarizes the available design freedom as well as constraints in assigning the eigenstructure of observer dynamic matrix for loop transfer recovery. We describe here a design procedure which follows the asymptotic time-scale and eigenstructure assignment (ATEA) concepts proposed originally in [43], [44] and developed fully in [10]. At first in subsection 4.3.1, we give a design procedure for an arbitrarily specified target loop transfer function, i.e. without taking into account any specific characteristics of F. This is the most general design procedure. Also, when the given $L_t(s)$ is asymptotically recoverable, it entails additional freedom in selecting some eigenvalues and eigenvectors. However, the general ATEA procedure of subsection 4.3.1 still yields a design which asymptotically recovers $L_t(s)$. On the other hand, for exactly recoverable target loop transfer functions, F satisfies $\mathcal{S}^-(\Sigma) \subseteq \mathrm{Ker}\,(F)$. Because of this, one needs to assign only a finite eigenstructure to $A - K_f(\sigma)C$. For this case, the general ATEA design procedure of subsection 4.3.1 can be simplified greatly and such a simplified design is presented in subsection 4.3.3.

4.3.1. General 'ATEA' design

The ATEA design method is decentralized in nature. It uses the special coordinate basis (s.c.b) of the given system Σ (See theorem 1.3.1). The specified finite eigenstructure of $A-K_f(\sigma)C$ is assigned by working with subsystems which represent the finite zero structure of the given system (See equations (1.3.3) to (1.3.6)) . Similarly the specified asymptotically infinite eigenstructure of $A-K_f(\sigma)C$ is assigned appropriately by working with subsystems which represent the infinite zero structure of the given system (See equation (1.3.8) for each $i = 1$ to m_f).

There are two issues in formulating the observer dynamic matrix $A - K_f(\sigma)C$ by an appropriate selection of $K_f(\sigma)$. The first issue is eigenvalue assignment and the second one is corresponding eigenvector assignment. We will focus on one issue at a time. Let us first consider the eigenvalue assignment. As discussed earlier, some eigenvalues of $A - K_f(\sigma)C$ are constrained while some others are free to be assigned to either asymptotically finite or infinite locations in C^-. To be specific,

1. $\Lambda^-(\sigma)$ must coincide either exactly or asymptotically with the set of plant minimum phase invariant zeros,
2. $\Lambda^b(\sigma)$ and $\Lambda^e(\sigma)$ can be assigned to either asymptotically finite or infinite locations, and
3. $\Lambda^\infty(\sigma)$ have to be assigned to asymptotically infinite locations.

In this section, in order to conserve controller band-width, both $\Lambda^b(\sigma)$ and $\Lambda^e(\sigma)$ are assigned to asymptotically finite locations. Let us next examine carefully the freedom available in assigning $\Lambda^\infty(\sigma)$ to asymptotically infinite locations. As is clear from earlier discussion, there exists complete freedom in the way each $\lambda_i^\infty(\sigma) \in \Lambda^\infty(\sigma)$ tends to infinity as $\sigma \to \infty$, i.e., both the asymptotic direction and the rate at which $\lambda_i^\infty(\sigma)$ goes to infinity can be dictated as desired by the designer. In other words, the freedom available in assigning every asymptotically infinite eigenvalue $\lambda_i^{\infty}(\sigma)$ manifests itself in two ways:

1. in choosing the asymptotic directions along which the eigenvalues tend to infinity, and

2. in choosing the rates at which the eigenvalues tend to infinity.

To reflect both these types of freedom, we subdivide $\Lambda^{\infty}(\sigma)$ for asymptotically large values of σ into $r \le n_f$ sets,

$$\frac{\Lambda_1}{\mu_1}, \frac{\Lambda_2}{\mu_2}, \cdots, \frac{\Lambda_r}{\mu_r}. \tag{4.3.1}$$

Here Λ_ℓ is a set of n_ℓ numbers all in C^- and Λ_ℓ is closed under complex conjugation. Also $\sum_{\ell=1}^{r} n_\ell = n_f$. Apparently, the elements of Λ_ℓ, $\ell = 1$ to r, define the asymptotic directions of asymptotically infinite or fast eigenvalues while the small parameters μ_ℓ, $\ell = 1$ to r, which are some functions of σ, define the rates at which these eigenvalues go to infinity.

In summary, regarding the eigenvalues, a designer has the freedom to specify

i. the asymptotic limits $\overline{\Lambda}^b$ and $\overline{\Lambda}^e$ of $\Lambda^b(\sigma)$ and $\Lambda^e(\sigma)$, and

ii. Λ_ℓ and μ_ℓ, $\ell = 1$ to r.

We note that $\overline{\Lambda}^b$ and $\overline{\Lambda}^e$ in addition to $\overline{\Lambda}^-$ define the asymptotically finite eigenvalues of $A - K_f(\sigma)C$, while Λ_ℓ and μ_ℓ, $\ell = 1$ to r, define the asymptotically infinite eigenvalues.

Let us now look at the constraints and design freedom available in assigning the eigenvectors of $A - K_f(\sigma)C$. The set of right eigenvectors $\overline{V}^-$ is constrained to coincide with the corresponding set of state zero directions of the plant. Moreover, $\mathrm{Im}(\overline{V}^-)$ coincides with the subspace $\mathcal{V}^*(\Sigma)/\mathcal{V}^+(\Sigma)$. On the other hand, the set of eigenvectors $\overline{V}^b$ is constrained to be in the null space of $[B - K_f(\sigma)D]'$. In view of the particular structure of s.c.b, it can be seen then that every element $\overline{V}_i^b$ of $\overline{V}^b$ is constrained to be of the form $[0,\ 0,\ (V_i^b)^H,\ 0,\ 0]^H$. In other words, the set $\overline{V}^b$ can be represented in a matrix notation as $[0,\ 0,\ (V^{bb})^H,\ 0,\ 0]^H$ where V^{bb} is a $n_b \times n_b$ matrix. Thus the selection of $\overline{V}^b$ to be in the null space of $[B - K_f(\sigma)D]'$ is equivalent to any arbitrary selection of V^{bb} consistent with the freedom available in assigning it [30]. Again in view of the properties of s.c.b, we note that the columns of $\overline{V}^b$ span the subspace $\Re^n/\{\mathcal{S}^+(\Sigma) \cup \mathcal{S}^-(\Sigma)\}$. Next, there is also freedom available in specifying $\overline{W}^e$. It can easily shown (see for example [39]) that $\mathrm{Im}(\overline{W}^e)$ coincides with the subspace $\mathcal{V}^+(\Sigma)$. Again owing to the special structure of s.c.b, $\overline{W}^e$ has the special matrix form $[0,\ (W^{e+})^H,\ 0,\ (W^{ec})^H,\ 0]^H$ where $W^{ee} \equiv [(W^{e+})^H,\ (W^{ec})^H]^H$ is a $n_e \times n_e$ matrix. Thus an appropriate selection of $\overline{W}^e$ is equivalent to any arbitrary selection of W^{ee} consistent with the freedom available in assigning it [30].

Now an assignment of both asymptotically finite and infinite eigenvalues and the corresponding eigenvectors to $A - K_f(\sigma)C$ can be viewed as a problem for asymptotic time-scale and eigenstructure assignment (ATEA). In order to have a well defined separation of time-scales, we will assume throughout the paper that

$$\mu_\ell/\mu_{\ell+1} \rightarrow 0 \text{ as } \mu_{\ell+1} \rightarrow 0. \tag{4.3.2}$$

We emphasize that the freedom that exists in specifying the asymptotically infinite eigenstructure of $A - K_f(\sigma)C$ reflects itself in specifying the fast time-scale structure. The asymptotic directions of asymptotically infinite eigenvalues can be specified by the sets Λ_ℓ, $\ell = 1$ to r, where r is an integer less than or equal to n_f. The relative fastness of time-scales is specified by the small positive parameters μ_ℓ, $\ell = 1$ to r, which are appropriate functions of the tuning parameter σ so that (4.3.2) is true as $\sigma \rightarrow \infty$. We note that there is also a constraint on the infinite eigenstructure; namely, for every asymptotically infinite eigenvalue $\lambda_i^\infty(\sigma)$, the corresponding right and left eigenvectors $W_i^\infty(\sigma)$ and $V_i^\infty(\sigma)$ of $A - K_f(\sigma)C$ must be such that $W_i^\infty(\sigma)[V_i^\infty(\sigma)]^H[B - K_f(\sigma)D]$ is uniformly bounded as $\sigma \rightarrow \infty$. This constraint, however, is automatically taken into account by the ATEA design procedure given in this section.

In what follows, we give a step by step ATEA design algorithm. In view of the above discussion, the input parameters of the algorithm are $\overline{\Lambda}^b$, V^{bb}, $\overline{\Lambda}^e$, W^{ee}, Λ_ℓ and μ_ℓ, $\ell = 1$

to r, as well as the integer r. In fact, the primary inputs are (1) $\overline{\Lambda}^e$ and W^{ee} which shape the resulting recovery error matrix $\overline{M}_f^e(s)$ and (2) Λ_ℓ and μ_ℓ, $\ell = 1$ to r, which control the time-scale structure of the observer and thus have a strong impact on the resulting gain of the controller. The rest of the input parameters, namely $\overline{\Lambda}^b$ and V^{bb}, are secondary inputs. Our algorithm can be divided into three steps. Steps 1 and 2 deal respectively with subsystem designs to assign the asymptotically finite and infinite eigenstructures. In step 3, subsystem designs of steps 1 and 2 are put together to form a composite design for the given system.

Step 1 : This step deals with the assignment of asymptotically finite eigenstructure (i.e., slow time-scale structure) and makes use of subsystems (1.3.3) to (1.3.6) of Σ. The minimum phase invariant zeros $\lambda(A_{aa}^-)$ of Σ are left alone to form some of the eigenvalues of $A - K_f(\sigma)C$, namely the set $\overline{\Lambda}^-$, while the corresponding left eigenvectors of $A - K_f(\sigma)C$ coincide with the the corresponding left state zero directions of Σ. To place the set of eigenvalues $\overline{\Lambda}^b$ and left eigenvectors $\overline{V}^b$, choose a gain K^b such that $\lambda(A_{bb} - K^bC_b)$ coincides with $\overline{\Lambda}^b$ while V^{bb} coincides with the set of left eigenvectors of $A^{bbc} = A_{bb} - K^bC_b$. Note that the existence of such a K^b is guaranteed by property 1.3.2 as long as the eigenvector set $\overline{V}^b$ is consistent with the freedom available in assigning it [30]. Next, in order to place the set of eigenvalues $\overline{\Lambda}^e$ and right eigenvectors W^{ee}, let us first form matrices A^{ee} and C^e as follows:

$$A^{ee} = \begin{bmatrix} A_{aa}^+ & 0 \\ B_cE_{ca}^+ & A_{cc} \end{bmatrix}, \quad C^e = \begin{bmatrix} C^{e0} \\ C^{e1} \end{bmatrix} = \begin{bmatrix} C_{0a}^+ & C_{0c} \\ E_a^+ & E_c \end{bmatrix}, \tag{4.3.3}$$

where

$$E_a^+ = [(E_{1a}^+)', (E_{2a}^+)', \ldots, (E_{m_fa}^+)']', \quad E_{ia} = [E_{ia}^+, E_{ia}^-], \quad E_c = [E_{1c}', E_{2c}', \ldots, E_{m_fc}']'.$$

Now select a gain

$$K^e = \begin{bmatrix} K^{a+} \\ K^c \end{bmatrix} = [K^{e0}, K^{e1}]$$

such that the set of eigenvalues and right eigenvectors of $A^{eec} = A^{ee} - K^eC^e$ coincide with $\overline{\Lambda}^e$ and W^{ee} respectively. Again note that the existence of such a K^e is guaranteed by property 1.3.2 as long as the eigenvector set W^{ee} is consistent with the freedom available in assigning it [30]. For future use, let us define

$$A^{ee1} = A^{ee} - K^{e0}C^{e0} \ , \ K^{e0} = \begin{bmatrix} K^{a0+} \\ K^{c0} \end{bmatrix},$$

and partition K^{e1} as

$$K^{e1} = \left[\, K^{e11}, \ K^{e12}, \ \cdots, \ K^{e1m_f} \, \right] \tag{4.3.4}$$

where K^{e1i} is a $n_e \times 1$ dimensional vector.

Step 2 : This step deals with the assignment of asymptotically infinite eigenstructure (i.e., the fast time-scale structure) and makes use of subsystems, $i = 1$ to m_f, represented by (1.3.8). This step exists only when $n_f > 0$ since otherwise there is no need to assign any asymptotically infinite eigenstructure, and hence we assume here that $n_f > 0$. As discussed earlier, there is complete freedom to specify any $r \leq n_f$ fast time-scales. In particular, one can always choose $r = 1$. For generality, we will keep r as arbitrarily given. The freedom in assigning the fast time-scales is reflected in specifying the sets Λ_ℓ, and the small positive parameters μ_ℓ, $\ell = 1$ to r. Our design to assign an appropriate fast time-scale structure is again decentralized. We deal with one single input single output system at a time as represented by (1.3.8) for a particular value of i, $i = 1$ to m_f. Thus to proceed with our design, we need to distribute the designer specified elements of the sets Λ_ℓ, and the parameters μ_ℓ, $\ell = 1$ to r, among m_f subsystems. There exists a complete freedom in such a distribution and hence it can be done in a number of ways. Let the subsystem i be assigned r_i time-scales for some $r_i \leq q_i$. Let

$$\frac{\Lambda_{ij}}{\mu_{ij}}, \ j = 1 \text{ to } r_i,$$

be the asymptotically infinite eigenvalues that need to be assigned to subsystem i. Let n_{ij} be the number of eigenvalues corresponding to the time-scale t/μ_{ij}. That is, let Λ_{ij} contain n_{ij} elements. As usual, the set Λ_{ij} is assumed to be closed under complex conjugation. Also, in order to have a well defined separation of time-scales in subsystem i, we will assume that

$$\mu_{ij}/\mu_{ij+1} \to 0 \text{ as } \mu_{ij+1} \to 0 \text{ for all } j = 1 \text{ to } r_i - 1. \tag{4.3.5}$$

We note that when $r = 1$, all μ_{ij} are equal to a single parameter μ and all r_i are equal to unity. That is, there is only one time-scale to be assigned to all subsystems. In this case, σ can be taken as $1/\mu$. With these preliminaries, we are now ready to design the i-th subsystem. At first, we will design a gain matrix K_{ij} for each time-scale t/μ_{ij}, $j = 1$ to r_i. Define a $n_{ij} \times n_{ij}$ dimensional matrix G_{ij} and a $1 \times n_{ij}$ dimensional matrix C_{ij} having the following structure:

$$G_{ij} = \begin{bmatrix} 0 & I_{n_{ij}-1} \\ 0 & 0 \end{bmatrix} \text{ and } C_{ij} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Choose a $n_{ij} \times 1$ dimensional gain vector K_{ij} such that $\lambda(G^c_{ij})$ coincides with Λ_{ij} where $G^c_{ij} = G_{ij} - K_{ij}C_{ij}$. Owing to the special structure of G_{ij} and C_{ij} , such a K_{ij} always exists. Let K_{ij} be partitioned as

$$K_{ij} = \begin{bmatrix} K_{ijc} \\ K_{ijd} \end{bmatrix},$$

where K_{ijd} is a scalar. Moreover, the nonsingularity of G^c_{ij} implies that K_{ijd} is nonzero. Next, the gains K_{ij}, $j = 1$ to r_i, obtained above are put together to form a composite gain vector which will induce the required fast time-scales in the i-th subsystem. Define the scalar numbers J_{ij} as

$$J_{i1} = 1 \ , \ J_{ij} = \prod_{\ell=1}^{j-1} K_{i\ell d} \ , \ j = 2 \text{ to } r_i.$$

Let

$$\alpha_{i0} = 0$$

and

$$\alpha_{ij} = \sum_{k=1}^{j} n_{ik} \ , \ j = 1 \text{ to } r_i.$$

Note that $\alpha_{ir_i} = q_i$. Also, let for each $j = 1$ to r_i,

$$\epsilon_{i\alpha_{ij-1}+1} = \epsilon_{i\alpha_{ij-1}+2} = \cdots = \epsilon_{i\alpha_{ij}} = \mu_{ij}$$

and

$$\eta_i = \prod_{k=1}^{q_i} \epsilon_{ik}. \tag{4.3.6}$$

Also, define a scaling matrix S_{ij} as

$$S_{ij} = \text{Diag}\left[\prod_{\ell=\alpha_{ij-1}+2}^{q_i} \epsilon_{i\ell}\,, \prod_{\ell=\alpha_{ij-1}+3}^{q_i} \epsilon_{i\ell}\,, \cdots, \prod_{\ell=\alpha_{ij}+1}^{q_i} \epsilon_{i\ell} \right]. \tag{4.3.7}$$

In (4.3.7), for $j = r_i$, the product $\prod_{\ell=q_i+1}^{q_i} \epsilon_{i\ell}$ is taken as unity. Now let,

$$\tilde{K}_{ij}(\sigma) = \frac{1}{\eta_i} J_{ij} S_{ij} K_{ij},$$

and

$$\tilde{K}_i(\sigma) = \left[\tilde{K}'_{i1}(\sigma),\ \tilde{K}'_{i2}(\sigma),\ \cdots,\ \tilde{K}'_{ir_i}(\sigma)\right]'. \tag{4.3.8}$$

The above design is rather simple when $r_i = 1$. For this case, let $\overline{\mu}_i$ denote the small parameter. Then

$$\tilde{K}_i(\sigma) = \frac{1}{(\overline{\mu}_i)^{q_i}} \left[(\overline{\mu}_i)^{q_i-1}\hat{K}_{i1},\ (\overline{\mu}_i)^{q_i-2}\hat{K}_{i2},\ \cdots,\ \hat{K}_{iq_i}\right]' \tag{4.3.9}$$

where $\hat{K}_{ij}, j = 1$ to q_i, are selected such that $\lambda(G_i^c)$ are as desired where,

$$G_i^c = -\begin{bmatrix} \hat{K}_{i1},\ \hat{K}_{i2}, \cdots,\ \hat{K}_{iq_i-1} & \hat{K}_{iq_i} \\ -I_{q_i-1} & 0 \end{bmatrix}'.$$

Here we did not discuss any eigenvector assignment. However, it turns out that our eventual design is such that the eigenvectors corresponding to the asymptotically infinite eigenvalues are naturally assigned to appropriate locations so that $M_f^{\infty}(j\omega, \sigma) \to 0$ as $\sigma \to \infty$.

Step 3 : In this step, various gains calculated in steps 1 and 2 are put together to form a composite observer gain for the given system Σ. Define $\tilde{K}^{e1}$ as

$$\tilde{K}^{e1}(\sigma) = \begin{bmatrix} \tilde{K}^{a1+}(\sigma) \\ \tilde{K}^{c1}(\sigma) \end{bmatrix} = \left[\tilde{K}^{e11},\ \tilde{K}^{e12},\ \cdots,\ \tilde{K}^{e1m_f}\right],\quad \tilde{K}^{e1i} = \frac{1}{\eta_i} J_{ir_i} K_{ir_i d} K^{e1i}. \tag{4.3.10}$$

For the case when $r_i = 1$, K_{i1d} is the same as $\hat{K}_{iq_i}$ and η_i is the same as $(\overline{\mu}_i)^{q_i}$. Assume $n_f > 0$ and define the observer gain $K_f(\sigma)$ as

$$K_f(\sigma) = \Gamma_1 \tilde{K}(\sigma) \Gamma_2^{-1} \tag{4.3.11}$$

where

$$\tilde{K}(\sigma) = \begin{bmatrix} B_{0a}^- & L_{af}^- + \tilde{H}_{af}^- & L_{ab}^- + \tilde{H}_{ab}^- \\ B_{0a}^+ + K^{a0+} & L_{af}^+ + \tilde{H}_{af}^+ + \tilde{K}^{a1+}(\sigma) & L_{ab}^+ + \tilde{H}_{ab}^+ \\ B_{0b} & L_{bf} + \tilde{H}_{bf} & K^b \\ B_{0c} + K^{c0} & L_{cf} + \tilde{H}_{cf} + \tilde{K}^{c1}(\sigma) & L_{cb} + \tilde{H}_{cb} \\ B_{0f} & L_f + \tilde{K}_f(\sigma) & 0 \end{bmatrix} \tag{4.3.12}$$

and where

$$\tilde{K}_f(\sigma) = \text{Diag}\left[\tilde{K}_1(\sigma),\ \tilde{K}_2(\sigma),\ \cdots,\tilde{K}_{m_f}(\sigma)\right],$$

$$L_f = \left[L_1',\ L_2',\ \cdots,L_{m_f}'\right]'$$

while the gains $\tilde{H}_{af}^+$, $\tilde{H}_{ab}^+$, $\tilde{H}_{af}^-$, $\tilde{H}_{ab}^-$, $\tilde{H}_{bf}$, $\tilde{H}_{cf}$ and $\tilde{H}_{cb}$ are arbitrary but finite. We have the following theorem.

Theorem 4.3.1. *Consider a full order observer based controller with its gain given by (4.3.11) where n_f is assumed to be greater than zero. Then we have the following properties:*

1. *There exists a σ^* such that for all $\sigma > \sigma^*$, the designed observer is asymptotically stable. Furthermore, it has the time-scale structure t, t/μ_{ij}, $j = 1$ to r_i, $i = 1$ to m_f. That is, the eigenvalues of the observer as $\mu_r \to 0$ are given by*

$$\overline{\Lambda}^- + 0(\mu_r)\,,\ \overline{\Lambda}^b + 0(\mu_r)\,,\ \overline{\Lambda}^c + 0(\mu_r)\,,$$

$$\frac{\Lambda_{ij}}{\mu_{ij}} + 0(1) \text{ for } j = 1 \text{ to } r_i \text{ and } i = 1 \text{ to } m_f.$$

   Moreover, if $\tilde{H}_{af}^- = 0$ and $\tilde{H}_{bf} = 0$, some finite eigenvalues of $A - K_f(\sigma)C$ are exactly equal to $\overline{\Lambda}^-$ and $\overline{\Lambda}^b$ for all σ rather than asymptotically tending to $\overline{\Lambda}^-$ and $\overline{\Lambda}^b$.

2. *LTR is achieved as intended in the sense that as $\sigma \to \infty$,*

$$M_f(s,\sigma) \to \overline{M}_f^c(s) \text{ pointwise in } s.$$

Proof : See Appendix 4.A. ∎

Remark 4.3.1. *For the case when $n_f = 0$, the observer gain obtained in the above ATEA procedure is independent of σ and is simply given by*

$$K_f(\sigma) = \Gamma_1 \begin{bmatrix} B_{0a}^- & L_{ab}^- \\ B_{0a}^+ + K^{a0+} & L_{ab}^+ \\ B_{0b} & K^b \\ B_{0c} + K^{c0} & L_{cb} \end{bmatrix} \Gamma_2^{-1}. \tag{4.3.13}$$

Moreover, such an observer gain places the eigenvalues of $A - K_f(\sigma)C$ precisely at $\overline{\Lambda}^- \cup \overline{\Lambda}^b \cup \overline{\Lambda}^e$ and $\overline{M}_f^e(s)$ is exactly rather than asymptotically attained, i.e. $M_f(s,\sigma) = \overline{M}_f^e(s)$.

Remark 4.3.2. *We emphasize that whenever $L_t(s)$ is an element of $\mathbf{T}^R(\Sigma)$, $\overline{M}_f^e(s)$ is zero irrespective of the way the set of $n_a^+ + n_c$ eigenvalues belonging to $\Lambda^e(\sigma)$ and the associated right and left eigenvector sets $W^e(\sigma)$ and $V^e(\sigma)$ are selected.*

As it can be easily seen, the ATEA design is decentralized. Required time-scale structure and eigenstructure are assigned to the subsystems of the given system Σ. The calculations involved in subsystem designs do not explicitly require the value of the tuning parameter σ. σ enters only in (4.3.8) or (4.3.9) where subsystem designs are put together to form a composite gain which assigns the required time-scale structure. Thus σ truly and directly acts as a tuning parameter and controls the degree of fastness of fast time-scales.

4.3.2. A helicopter control system design

We present here a helicopter control system design to illustrate the ATEA design algorithm. The following is a mathematical model for a typical modern attack helicopter operating near hover. The dynamics of the helicopter were originally modeled by a twelfth-order model that consists of an eighth-order model representing the rigid-body dynamics and two second-order models representing the advancing and regressing rotor tip-path plane modes. Main rotor collective pitch, lateral cyclic pitch, longitudinal cyclic pitch and tail rotor collective pitch are the control inputs.

An eighth-order design model that did not contain the tip-path plane modes were developed from the 12th-order model and it is given in [21],

$$\dot{x} = Ax + Bu,$$

where

$$x = \begin{bmatrix} \text{forward velocity, ft/s} \\ \text{lateral velocity, ft/s} \\ \text{heave velocity, ft/s} \\ \text{roll rate, rad/s} \\ \text{pitch rate, rad/s} \\ \text{yaw rate, rad/s} \\ \text{roll angle, rad/s} \\ \text{pitch angle, rad/s} \end{bmatrix}, \quad u = \begin{bmatrix} \text{main rotor collective pitch} \\ \text{lateral cyclic pitch} \\ \text{longitudinal cyclic pitch} \\ \text{tail rotor collective pitch} \end{bmatrix},$$

$$A = \begin{bmatrix} -0.0199 & -0.0058 & -0.0058 & -0.0151 & 0.0232 & 0.0006 & 0 & -0.6652 \\ -0.0452 & -0.0526 & -0.0061 & -0.0260 & -0.0155 & 0.0148 & 0.6648 & -0.0003 \\ -0.0788 & -0.0747 & -0.3803 & 0.0008 & -0.0048 & 0.0420 & 0.0228 & 0.0102 \\ 0.4557 & -2.5943 & -0.1787 & -2.9979 & -0.5308 & 0.4155 & 0 & 0 \\ 0.3688 & 0.1931 & -0.1753 & 0.0710 & -0.5943 & 0.0013 & 0 & 0 \\ 1.0939 & 0.7310 & -0.0358 & 0.4058 & 0.4069 & -0.4940 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0.0005 & -0.0154 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9994 & 0.0343 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -0.0456 & -0.0083 & 0.4735 & -0.0016 \\ -0.0369 & 0.2785 & 0.0086 & 0.3600 \\ -3.1126 & -0.0032 & 0.0076 & 0.0002 \\ -2.4241 & 20.8327 & 1.0196 & 9.1903 \\ -0.3205 & 0.2538 & -6.3329 & -0.0648 \\ 5.7889 & -2.6208 & 2.3832 & -11.0904 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The measured outputs are the heave velocity, roll, pitch and yaw rates, i.e.,

$$y = Cx = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x.$$

The target loop is specified by the following state feedback gain matrix,

$$F = \begin{bmatrix} 0.0243 & 0.0242 & -0.5524 & -0.0002 & 0.0008 & -0.0136 & -0.0073 & -0.0046 \\ 0.0686 & -0.1072 & 0.4990 & 0.0193 & 0.0483 & 0.0901 & 0.2389 & 0.1105 \\ -0.0459 & -0.0245 & 0.0597 & -0.0085 & -0.3616 & -0.0040 & 0.0154 & -0.6622 \\ -0.1120 & -0.0332 & -0.2841 & -0.0431 & -0.1254 & -0.1650 & -0.0569 & -0.1708 \end{bmatrix}.$$

It is simple to verify that the given system $(A, B, C, 0)$ is invertible with invariant zeros at $\{0,\ 0,\ -0.00075649 \pm j0.013932\}$. Hence, it is of nonminimum phase.

In what follows, we proceed with the ATEA design.

Step 0 : Using the *Linear System Toolbox* developed by Lin *et al.* [26], we obtain the s.c.b decomposition of the given system $(A, B, C, 0)$ as follows,

$$\tilde{A}=\begin{bmatrix} 0.007833 & -0.011183 & 0 & 0 & -0.009834 & -38.224759 & 31.881866 & 1.683855 \\ 0.023953 & -0.009346 & 0 & 0 & 0.021831 & -26.734750 & 81.818861 & 3.219285 \\ 0 & 0 & 0 & 0 & 0 & 46.641361 & -73.009653 & -3.224812 \\ 0 & 0 & 0 & 0 & 0 & 0 & -48.807083 & -1.675088 \\ -0.078800 & 0.074700 & -0.021285 & 0.105341 & -0.379227 & -0.000034 & 0.001694 & 0.043684 \\ 0.455700 & 2.594300 & 1.859568 & 1.864994 & -0.065245 & -3.024800 & -0.542191 & 0.477555 \\ 0.368800 & -0.193100 & 0.191564 & -0.369178 & -0.174135 & 0.073264 & -0.623340 & -0.002869 \\ 1.093900 & -0.731000 & 0.477545 & -1.224680 & -0.038601 & 0.414174 & 0.319397 & -0.510120 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Gamma_1=\begin{bmatrix} 1 & 0 & 0.819370 & -0.572017 & 0.023863 & 0.000650 & -0.074215 & 0.001117 \\ 0 & -1 & -0.572864 & -0.819358 & -0.039541 & 0.010483 & -0.008645 & -0.023724 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.021440 & -0.032082 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.020477 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Gamma_3 = \begin{bmatrix} -0.321277 & -0.000050 & -0.000410 & -0.000045 \\ 0.037720 & 0.053232 & 0.025161 & 0.043965 \\ 0.019535 & 0.002260 & -0.156469 & 0.002787 \\ -0.172414 & -0.012120 & -0.039783 & -0.099982 \end{bmatrix}$$

and

$$n_a^- = 2, \quad n_a^+ = 2, \quad n_b = 0, \quad n_c = 0, \quad n_f = 4.$$

Step 1 : This step deals with the assignment of asymptotically finite eigenstructure. For this example, $n_b = 0$. Hence, there is no need to assign the eigenstructure for $\overline{\Lambda}^b$ and $\overline{V}^b$. Also, in this example,

$$A^{ee} = A_{aa}^+ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C^e = E_a^+ = \begin{bmatrix} -0.021285 & 0.105341 \\ 1.859568 & 1.864994 \\ 0.191564 & -0.369178 \\ 0.477545 & -1.224680 \end{bmatrix}.$$

Let us specify,

$$\overline{\Lambda}^e = \{-0.001, \ -0.001\} \quad \text{and} \quad W^{ee} = [W^{ee1}, W^{ee2}] = \begin{bmatrix} 1 & -0.851846 \\ 0 & 0.523793 \end{bmatrix},$$

so that $\overline{M}_f^e(s)$ is prescribed as

$$\overline{M}_f^e(s) = \frac{0.001}{s + 0.001} \begin{bmatrix} -0.140254 & 0.117916 & -0.052177 & 0.281234 \\ -0.157749 & -0.980041 & -0.172435 & 0.060003 \\ 0.280960 & -0.126562 & 0.115732 & -0.538118 \\ 0.546280 & 0.029670 & 0.253212 & -0.982759 \end{bmatrix}.$$

Then we obtain

$$K^e = 10^{-3} \times \begin{bmatrix} 0 & 0.386579 & 0 & 0.588698 \\ 0 & 0.150740 & 0 & -0.586986 \end{bmatrix}.$$

Step 2 : This step deals with the assignment of asymptotically infinite eigenstructure. Let us specify $r = 1$ and

$$\Lambda_{11} = \Lambda_{21} = \Lambda_{31} = \Lambda_{41} = \{-1\}.$$

We obtain

$$\tilde{K}_f(\sigma) = \begin{bmatrix} \sigma & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix}$$

and

$$\tilde{K}^e(\sigma) = 10^{-3} \times \begin{bmatrix} 0 & 0.386579\sigma & 0 & 0.588698\sigma \\ 0 & 0.150740\sigma & 0 & -0.586986\sigma \end{bmatrix}.$$

Step 3 : The observer gain matrix is then given by $K_f(\sigma) =$

$$\begin{bmatrix} -0.006046+0.023863\sigma & -0.015174+0.000880\sigma & 0.024727-0.074215\sigma & 0.000715+0.001935\sigma \\ -0.005099-0.039541\sigma & -0.026581+0.010138\sigma & -0.011691-0.008645\sigma & 0.015997-0.023580\sigma \\ -0.379227+\sigma & -0.000034 & 0.001694 & 0.043684 \\ -0.065245 & -3.024800+\sigma & -0.542191 & 0.477555 \\ -0.174135 & 0.073264 & -0.623340+\sigma & -0.002869 \\ -0.038601 & 0.414174 & 0.319397 & -0.510120+\sigma \\ 0 & 1+3.4522\times10^{-6}\sigma & 0.000500 & -0.015400+3.1454\times10^{-5}\sigma \\ 0 & -3.0866\times10^{-6}\sigma & 0.999400 & 0.034300+1.2019\times10^{-5}\sigma \end{bmatrix}.$$

This $K_f(\sigma)$ places two observer eigenvalues exactly at $\{-0.00075649 \pm j0.013932\}$ and the remaining eigenvalues asymptotically at $\{-0.001, -0.001, -\sigma, -\sigma, -\sigma, \}$. Plots of maximum singular values of $\overline{M}_f^e(j\omega)$ and $M_f(j\omega,\sigma)$ for several values of σ are shown in Figure 4.3.1. The figure shows that $M_f(j\omega,\sigma)$ tends to $\overline{M}_f^e(j\omega)$ as $\sigma \to \infty$. □

4.3.3. Design for exactly recoverable target loops

As discussed in the previous subsection, in general in ATEA design, some eigenvalues are assigned to finite locations and some others are assigned to asymptotically infinite locations. Obviously, ATEA design discussed there yields a family of parameterized controllers $\mathbf{C}_f(s,\sigma)$. Depending upon the design requirements, one then chooses a particular member of this family that corresponds to a particular value of the tuning parameter σ. However, for the case when the given target loop is exactly recoverable (i.e., $L_t(s) \in \mathbf{T}^{\mathrm{ER}}(\Sigma)$), there is no need for generating a sequence of controllers. In fact, all the eigenvalues of $A - K_f(\sigma)C$ can be assigned to finite locations and hence ATEA design procedure can drastically be

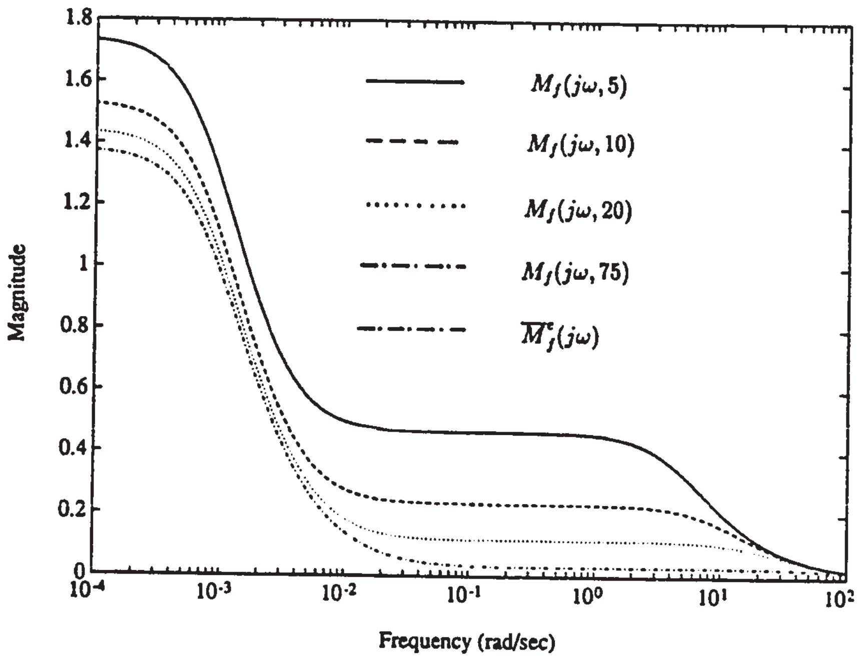


Figure 4.3.1: Maximum singular values of $\overline{M}_f^e(j\omega)$ and $M_f(j\omega,\sigma)$.

simplified. In this case, design involves only finite eigenstructure assignment, and no fast time-scale structure assignment is required. The intent of this section is to describe the available design freedom and a step by step design for an appropriate finite eigenstructure assignment to $A - K_f(\sigma)C$ for exact loop transfer function recovery (ELTR) whenever it is feasible.

Note that for exact recoverable case, the observer gain K_f is not parameterized as a function of σ and thus the presence of σ is dropped in all our notations. Following the interpretations of different partitions of $M_f(s)$ as in section 4.2, the available design freedom whenever $L_t(s) \in \mathbf{T}^{\text{ER}}(\Sigma)$ can be described as follows:

1. A set of n_a^- eigenvalues of $A - K_f(\sigma)C$, namely Λ^-, must be chosen to coincide exactly with the set of plant minimum phase invariant zeros while the corresponding left eigenvectors of $A - K_f(\sigma)C$ must coincide exactly with the the corresponding left state zero directions of Σ so that $M_f^-(s)$ is rendered zero.

2. A set of n_b eigenvalues of $A - K_f(\sigma)C$, namely Λ^b, can be assigned arbitrarily at finite

locations in $\mathcal{C}^-$. Moreover, the eigenvector set V^b corresponding to these eigenvalues can be selected freely within the constraints defined in [30]. However, V^b must be selected to be in the null space of $(B - K_f D)'$ so that $M_f^b(s)$ is rendered zero.

3. A set of $n_a^+ + n_c$ eigenvalues of $A - K_f(\sigma)C$, namely Λ^e, can be assigned arbitrarily at finite locations in $\mathcal{C}^-$ subject to the condition that any unobservable but stable eigenvalues of the given system must be included among Λ^e. Moreover, the eigenvector set W^{ee} corresponding to these eigenvalues can be selected freely within the constraints defined in [30]. We note that under condition $\mathcal{S}^-(\Sigma) \subseteq \text{Ker}\,(F)$, $M_f^e(s)$ is zero irrespective of how Λ^e and W^{ee} are selected. Also, we note that $n_a^+ + n_c = 0$ if the given system is of minimum phase and left invertible.

4. A set of n_f eigenvalues of $A - K_f(\sigma)C$, namely Λ^d, can be assigned arbitrarily at any finite locations in $\mathcal{C}^-$. (The sets Λ^∞ and V^∞ are renamed here as Λ^d and V^d because of the finiteness of the eigenvalues.) Moreover, the eigenvector set V^d corresponding to these eigenvalues can be selected freely within the constraints defined in [30]. Again, under the condition $\mathcal{S}^-(\Sigma) \subseteq \text{Ker}\,(F)$, $M_f^\infty(s)$ is zero irrespective of how Λ^d and V^d are selected.

We now move on to give the design steps to obtain K_f which assigns an appropriate finite eigenstructure to $A - K_f(\sigma)C$ so that the observer based controller achieves ELTR.

Step 1a : This step deals with the assignment of finite eigenstructure to the subsystem (1.3.5) of s.c.b. Choose a gain K^b such that $\lambda(A_{bb} - K^b C_b)$ coincides with Λ^b, a set of n_b designer specified eigenvalues all in $\mathcal{C}^-$. Note that the existence of such a K^b is guaranteed by property 1.3.2. Also, in our design, the eigenvectors of $A_{bb} - K^b C_b$ can be assigned in any chosen way consistent with the freedom available in assigning them [30]. Owing to the properties of s.c.b, our design always results in an eigenvector set V^b corresponding to the eigenvalues Λ^b of $A - K_f(\sigma)C$, in the null space of $(B - K_f D)'$ so that $M_f^b(s) = 0$.

Step 1b : This step deals with the assignment of finite eigenstructure to the subsystems (1.3.4), (1.3.6), and (1.3.8) of s.c.b. Let A^g and C^g be defined as

$$A^g = \begin{bmatrix} A^+_{aa} & 0 & L^+_{af}C_f \\ B_cE^+_{ca} & A_{cc} & L_{cf}C_f \\ B_fE^+_a & B_fE_c & A_f \end{bmatrix}, \quad C^g = \begin{bmatrix} C^+_{0a} & C_{0c} & C_{0f} \\ 0 & 0 & C_f \end{bmatrix}. \tag{4.3.14}$$

Also, let $\Lambda^g \equiv \Lambda^e \cup \Lambda^d$ be a set of $n^+_a + n_c + n_f$ designer specified eigenvalues all in $\mathcal{C}^-$ subject to the condition that any unobservable but stable eigenvalues of the given system must be included among Λ^g. Now select a gain K^g such that $\lambda(A^g - K^gC^g)$ coincides with Λ^g. Again note that the existence of such a K_g is guaranteed by property 1.3.2. Also, the eigenvectors of $A^g - K^gC^g$ can be assigned in any chosen way consistent with the freedom available in assigning them [30]. Let us next partition K^g as

$$K^g = \begin{bmatrix} K^{a0+} & K^{a1+} \\ K^{c0} & K^{c1} \\ K^{f0} & K^{f1} \end{bmatrix}.$$

Step 2 : In this step, K^b and K^g calculated in step 1 are put together into a composite matrix. Let

$$\tilde{K} = \begin{bmatrix} B^-_{0a} & L^-_{af} & L^-_{ab} \\ B^+_{0a} + K^{a0+} & K^{a1+} & L^+_{ab} \\ B_{0b} & L_{bf} & K^b \\ B_{0c} + K^{c0} & K^{c1} & L_{cb} \\ B_{0f} + K^{f0} & K^{f1} & 0 \end{bmatrix}. \tag{4.3.15}$$

Finally define the observer gain K_f as

$$K_f = \Gamma_1 \tilde{K} \Gamma_2^{-1}. \tag{4.3.16}$$

We have the following theorem.

Theorem 4.3.2. *Consider a full order observer based controller with its gain given by (4.3.16). Then the eigenvalues of the observer are given by Λ^-, Λ^b and Λ^g. Moreover, the observer based controller using the gain given in (4.3.16) achieves ELTR.*

Proof : It follows from the properties of s.c.b and some simple algebra. ∎

Remark 4.3.3. *We note that in general the observer gain for ELTR is not unique.*

4.3.4. A helicopter control system design (cont.)

We continue here with the helicopter control system design discussed earlier in subsection 4.3.2. In some practical problems, the conditional stability of the closed-loop system is allowed. In fact, the closed-loop system in the original design of Garrard *et al* [21] is unstable since there are two closed-loop eigenvalues at $\{0.004877 \pm j0.017982\}$ in their design. It is simple to verify that for the helicopter system described in subsection 4.3.2, ELTR is achievable via a reduced order observer based controller if we are allowed to place two closed-loop eigenvalues at the origin. The following is such a reduced order observer based controller obtained using the ATEA algorithm discussed in the previous subsection.

$$\begin{cases} \dot{v} = A_{comp}v + B_{comp}y, \\ -u = C_{comp}v + D_{comp}y, \end{cases},$$

where

$$A_{comp} = \begin{bmatrix} 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ -0.000544 & -0.665443 & 0.007833 & 0.011183 \\ 0.665702 & 0.000103 & -0.023953 & -0.009346 \end{bmatrix},$$

$$B_{comp} = \begin{bmatrix} 0.000000 & 1.000000 & 0.000500 & -0.015400 \\ 0.000000 & 0.000000 & 0.999400 & 0.034300 \\ -0.009834 & -0.008232 & -0.021579 & -0.000281 \\ -0.021831 & 0.015586 & -0.003752 & 0.000592 \end{bmatrix},$$

$$C_{comp} = \begin{bmatrix} -0.007300 & -0.004600 & 0.024300 & 0.024200 \\ 0.238900 & 0.110500 & 0.068600 & -0.107200 \\ 0.015400 & -0.662200 & -0.045900 & -0.024500 \\ -0.056900 & -0.170800 & -0.112000 & -0.033200 \end{bmatrix},$$

and

$$D_{comp} = \begin{bmatrix} -0.552777 & 0.000069 & -0.001213 & -0.014147 \\ 0.504876 & 0.018221 & 0.044136 & 0.092720 \\ 0.059573 & -0.008787 & -0.357982 & -0.003470 \\ -0.285460 & -0.043521 & -0.116801 & -0.164337 \end{bmatrix}.$$

□

4.4. Optimization based design methods

As is clear from chapter 2, the whole notion of LTR is to render the recovery matrix $M_f(s) = F(sI_n - A + K_fC)^{-1}(B - K_fD)$ small in some sense or other. The ATEA design method views this task from the perspective of asymptotic time-scale and eigenstructure assignment to the observer dynamic matrix. An alternative method is to view it as finding a gain K_f which minimizes some (say, either H_2 or H_∞) norm of $M_f(s)$. That is, one can cast the LTR design as a straightforward mathematical optimization problem. A suboptimal or optimal solution to such an optimization problem provides the needed observer gain. There is some historical basis to casting the LTR problem as such. In their seminal work, considering only left invertible and minimum phase systems, Doyle and Stein [18] propose a design method based on Kalman filter formalism in which the intensity of a ficticious input process noise is used as the tuning parameter σ. As $\sigma \to \infty$, their method yields an observer gain which renders $M_f(s,\sigma)$ asymptotically zero and thus achieves ALTR. It looks, however, mysterious why and how such a gain achieves ALTR for the class of problems considered by [18]. It turns out, as proved later on by Goodman [22], that the procedure of [18] minimizes the H_2 norm of the recovery matrix $M_f(s)$ as $\sigma \to \infty$. That is, the procedure of [18] yields a sequence of suboptimal solutions to H_2 norm minimization of $M_f(s)$. These suboptimal solutions are parameterized in terms of σ; and in the limit as $\sigma \to \infty$ of the sequence of corresponding $\|M_f(s,\sigma)\|_{H_2}$ is the infimum of $\|M_f(s)\|_{H_2}$ over the set of all possible gains. The infimum of $\|M_f(s)\|_{H_2}$ happens to be zero for left invertible and minimum phase systems. In view of this historic perspective, in this section, we also cast the loop transfer recovery problem for general not necessarily left invertible and not necessarily minimum phase systems, as a standard H_2 or H_∞ optimization problem. To facilitate this, we consider the following auxiliary system,

$$\Sigma_a : \begin{cases} \dot{x} = A'x + C'u + F'w, \\ y = x, \\ z = B'x + D'u. \end{cases} \tag{4.4.1}$$

Here w is treated as an exogenous disturbance input to Σ_a while u is the control input. The variables y and z are respectively considered as the measured and desired outputs.

Suppose that one uses a state feedback law to generate the control u,

$$u = -K_f' x. \tag{4.4.2}$$

It is then simple to verify that the closed-loop transfer function from w to z, denoted by $T_{zw}(s)$, is indeed equal to $M'(s)$. Now the LTR design problem can be cast as the task of obtaining a K_f such that (1) the auxiliary system Σ_a under the control law (4.4.2) is asymptotically stable, and (2) the norm (H_2 or H_∞) of $M_f(s)$ is minimized. There exists a vast literature on H_2 or H_∞ minimization methods. Borrowing from such a literature, subsection 4.4.1 discusses algorithms for H_2 minimization of $M_f(s)$ while subsection 4.4.3 does the same for H_∞ minimization. We want to emphasize that the optimization problem is cast here in terms of minimizing an appropriate norm of recovery matrix $M_f(s)$ rather than the recovery error $E(s) = L_t(s) - L_f(s)$.

It is well known that an optimal solution for either H_2 or H_∞ minimization of $M_f(s)$ does not necessarily exist, and the infimum of $\|M_f(s)\|_{H_2}$ or $\|M_f(s)\|_{H_\infty}$ is in general nonzero. For a class of target loops, however, the infimum of $\|M_f(s)\|_{H_2}$ or $\|M_f(s)\|_{H_\infty}$ is in fact zero, and it can be attained by a finite gain K_f. This is the class of exactly recoverable target loops $\mathbf{T}^{\text{ER}}(\Sigma)$. Also, for recoverable but not exactly recoverable target loops $\mathbf{T}^{\text{R}}(\Sigma)$, the infimum of $\|M_f(s)\|_{H_2}$ or $\|M_f(s)\|_{H_\infty}$ is zero, and it can be attained only asymptotically by using larger and larger gain K_f. Whether the infimum of $\|M_f(s)\|_{H_2}$ or $\|M_f(s)\|_{H_\infty}$ is zero or not, for general target loops, one needs to generate a sequence of gains having the property that the limit of H_2 or H_∞ norms of the correspondingly generated recovery matrices is the infimum of $\|M_f(s)\|_{H_2}$ or $\|M_f(s)\|_{H_\infty}$ over the set of all possible gains. A suboptimal solution results when one uses a gain corresponding to a particular member of the sequence. In H_2 optimization, an observer gain is generated via the solution of an algebraic Riccati equation (called hereafter H_2-ARE) parameterized in terms of a tuning parameter σ. A sequence of suboptimal gains is generated by tending σ to ∞. Let γ^* be the infimum of $\|M_f(s)\|_{H_\infty}$ over the set of all possible gains. Then given a parameter γ greater than γ^*, in H_∞ optimization, one generates a gain by solving an algebraic Riccati equation (called here after H_∞-ARE) parameterized in terms of γ so that the resulting $\|M_f(s,\gamma)\|_{H_\infty}$ is strictly less than γ. By gradually reducing γ, one obtains a sequence of suboptimal gains.

For simplicity but without loss of generality, we assume throughout this section that the matrix D is of the form,

$$D = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}.$$

Also, we partition the matrices B and C as

$$B = [B_0, \quad B_1] \quad \text{and} \quad C = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix},$$

and let $A_1 = A - B_0C_0$.

4.4.1. H_2-optimization design algorithm

In this subsection, we consider H_2 norm minimization of $M_f(s)$ or equivalently $T_{zw}(s)$. At first, let us look at calculating the infimum value of $\|M_f(s)\|_{H_2}$ as is done very elegantly in a recent work by Stoorvogel [55]. We first recall the following lemma.

Lemma 4.4.1. *Assume that (A,C) is detectable. Then the infimum of $\|M_f(s)\|_{H_2}$ over all the stabilizing observer gains is given by* Trace$\{F\overline{P}F'\}$, *where $\overline{P} \in \Re^{n\times n}$ is the unique positive semi-definite matrix satisfying:*

i. $\tilde{F}(\overline{P}) = \begin{bmatrix} A\overline{P} + \overline{P}A' + BB' & \overline{P}C' + BD' \\ C\overline{P} + DB' & DD' \end{bmatrix} \geq 0,$

ii. rank $\tilde{F}(\overline{P}) =$ normrank$\{C(sI_n - A)^{-1}B + D\}$ $\forall s \in C^+/C^o$,

iii. rank $\begin{bmatrix} [sI - A', & -C'] \\ & \tilde{F}(\overline{P}) \end{bmatrix} = n +$ normrank$\{C(sI_n - A)^{-1}B + D\}$ $\forall s \in C^+/C^o$.

Here normrank$\{\cdot\}$ denotes the rank of matrix $\{\cdot\}$ over the field of rational functions.

Proof : See Stoorvogel [54]. ∎

In general, as discussed earlier, the infimum of $\|M_f(s)\|_{H_2}$ can only be obtained asymptotically. In what follows, we proceed to introduce a basic algorithm of obtaining a sequence of parameterized observer gains $K_f(\sigma)$ for a system Σ such that the H_2 norm of the corresponding recovery matrix, which is also parameterized by σ and is denoted by $M_f(s,\sigma)$, tends to the infimum of $\|M_f(s)\|_{H_2}$ as $\sigma \to \infty$. The algorithm consists of the following two steps:

Step 1 : Solve the following parameterized algebraic Riccati equation (H_2-ARE) for a chosen fixed value of the parameter σ,

$$A_1P + PA_1' - PC_0'C_0P - \sigma PC_1'C_1P + B_1B_1' + \frac{1}{\sigma}I_n = 0, \tag{4.4.3}$$

for its positive definite solution P. We note that a unique positive definite solution P of (4.4.3) always exists for all $\sigma > 0$. Obviously, P is a function of σ and is denoted by $P(\sigma)$.

Step 2 : Let

$$K_f(\sigma) = [B_0 + P(\sigma)C_0',\ \sigma P(\sigma)C_1']. \tag{4.4.4}$$

We have the following theorem.

Theorem 4.4.1. *Consider a full order observer based controller with its gain given by (4.4.4). Let $M_f(s,\sigma)$ be the resulting recovery matrix. Then, we have*

$$\lim_{\sigma\to\infty} P(\sigma) = \overline{P}$$

Moreover, $\|M(s,\sigma)\|_{H_2}$ tends to the infimum of $\|M_f(s)\|_{H_2}$ as $\sigma \to \infty$, i.e.,

$$\lim_{\sigma\to\infty} \|M(s,\sigma)\|_{H_2} = \text{Trace}\,\{F\overline{P}F'\}.$$

Proof : See Appendix 4.B. ∎

In view of theorem 4.4.1, it is apparent that as σ takes on larger and larger values, the design algorithm given above generates a sequence of observer gains having the property that the limit of the correspondingly generated $\|M_f(s,\sigma)\|_{H_2}$ is the infimum of $\|M_f(s)\|_{H_2}$ over the set of all possible gains. A suboptimal solution results when one uses a particular value of the parameter σ. However, for some particular class of systems, e.g. the well-known *regular problems*†, the infimum value of $\|M_f(s)\|_{H_2}$ can be achieved with the following observer gain [17],

$$K_f = B_0 + PC_0', \tag{4.4.5}$$

where P is the positive semi-definite solution of

$$A_1P + PA_1' - PC_0'C_0P + B_1B_1' = 0.$$

The resulting infimum value of $\|M_f(s)\|_{H_2}$ is given by,

$$\|M_f(s)\|_{H_2} = \text{Trace}\{FPF'\}.$$

Note that in this discussion, the observer gain K_f and thus the resulting recovery matrix is not parameterized as a function of σ. We note that for a regular problem when $\|M_f(s)\|_{H_2} = 0$, the observer gain K_f as given in (4.4.5) achieves exact loop transfer recovery (ELTR). There is a larger class of systems than the class of regular systems, for which $\|M_f(s)\|_{H_2} = 0$. However, no optimization based method exists yet in the literature to generate the needed gain to achieve $\|M_f(s)\|_{H_2} = 0$, whenever it is possible, for systems other than the class of regular systems. On the other hand, a direct design procedure based on ATEA, which achieves ELTR *whenever it can be done*, was presented earlier in subsection 4.3.3.

Another special case of design that is of interest is as follows. Consider a left invertible minimum phase system Σ which is non-strictly proper. Let the observer gain $K_f(\sigma)$ be given by

$$K_f(\sigma) = [B_0,\ \sigma P(\sigma)C_1'],$$

where $P(\sigma) := P$ is the positive definite solution of

$$A_1P + PA_1' - \sigma PC_1'C_1P + B_1B_1' = 0.$$

†Regular problems are the class of problems where D is surjective impling that Σ is right invertible and has no infinite zeros, and where Σ has no invariant zeros on the $j\omega$ axis.

It is simple to show then that the observer gain $K_f(\sigma)$ chosen as above achieves asymptotic loop transfer recovery (ALTR), i.e. the resulting $\|M_f(s,\sigma)\|_{H_2}$ tends to zero asymptotically as $\sigma \to \infty$. This is a generalization, for non-strictly proper left invertible minimum phase systems, of the result given by Doyle and Stein [18] who treat only strictly proper left invertible minimum phase systems. The above result has been given earlier by Chen, Saberi, Bingulac and Sannuti [4].

It is of interest to investigate what type of time-scale structure and eigenstructure is assigned to the observer dynamic matrix $A - K_f(\sigma)C$ by the gain $K_f(\sigma)$ obtained via the basic algorithm of equations (4.4.3) and (4.4.4). Obviously, the basic algorithm renders $M_f^0(s,\sigma)$ and $M_f^\infty(s,\sigma)$ zero as $\sigma \to \infty$, while shaping $M_f^e(s,\sigma)$ in a particular way so that the infimum of $\|M_f(s)\|_{H_2}$ is attained as $\sigma \to \infty$. In so doing, among all the possible choices for the time-scale structure and eigenstructure of $A - K_f(\sigma)C$, it selects a particular choice which can be deduced from the results of cheap and singular control problems as analyzed in [45] (see also, [58] and [40]). We have the following results.

1. As $\sigma \to \infty$, the asymptotic limits of the set of n_a^- eigenvalues $\Lambda^-(\sigma)$ and the associated set of left eigenvectors $V^-(\sigma)$ of $A - K_f(\sigma)C$ coincide respectively with the set of plant minimum phase invariant zeros and the corresponding left state zero directions of Σ. Also. as $\sigma \to \infty$, some of the n_b eigenvalues in $\Lambda^b(\sigma)$ coincide with the stable but uncontrollable eigenvalues of Σ while the rest of them coincide with what are called 'compromise' zeros of Σ [45]. Moreover, the asymptotic limits of the associated left eigenvectors, namely $V^b(\sigma)$, fall in the null space of matrix $[B - K_f(\sigma)D]'$. This renders $M_f^0(s,\sigma)$ zero asymptotically as $\sigma \to \infty$.

2. As $\sigma \to \infty$, the set of n_f eigenvalues $\Lambda^\infty(\sigma)$ of $A - K_f(\sigma)C$ tend to asymptotically infinite locations in such a way that $M_f^\infty(s,\sigma) \to 0$. The time-scale structure assigned to these eigenvalues depends on the infinite zero structure of Σ (see for details in [45]). Also, the eigenvalues assigned to each fast time-scale follow asymptotically a Butter-

worth pattern.

3. As $\sigma \to \infty$, the asymptotic limits of n_a^+ eigenvalues in $\Lambda^e(\sigma)$ coincide with the mirror images of nonminimum phase invariant zeros of Σ, while the associated set of left eigenvectors of $A - K_f(\sigma)C$ coincide with the corresponding right input zero directions of Σ. The rest of n_c eigenvalues of $\Lambda^e(\sigma)$, as $\sigma \to \infty$, tend to some unnamed finite locations, while the associated left eigenvectors follow some unnamed directions. This shapes the limit of recovery matrix, $M_f^e(s)$ in a particular way so that the infimum of $\|M_f(s)\|_{H_2}$ is attained as $\sigma \to \infty$.

To conclude, as ATEA design procedure does in general, the basic algorithm of equations (4.4.3) and (4.4.4) renders $M_f^0(s,\sigma)$ and $M_f^\infty(s,\sigma)$ zero asymptotically as $\sigma \to \infty$. Moreover, it shapes $M_f^e(s)$ in a particular way so that the infimum of $\|M_f(s)\|_{H_2}$ is attained as $\sigma \to \infty$. In contrast to this, ATEA design procedure of section 4.3 allows complete available freedom to shape the limit of recovery matrix $\overline{M}_f^e(s)$ in a chosen manner within the design constraints imposed by the structural properties of the given system.

4.4.2. A helicopter control system design (cont.)

We continue here the helicopter control system design discussed earlier in subsections 4.3.2 and 4.3.4. It is simple to verify that for the given helicopter system of subsection 4.3.2, $\overline{P} = 0$ satisfies all the three conditions of Lemma 4.4.1. Hence, by theorem 4.4.1, there exists a sequence of observer gain matrices $K_f(\sigma)$ such that the corresponding recovery matrices,

$$\lim_{\sigma \to \infty} \|M_f(s,\sigma)\|_{H_2} = \mathrm{Trace}\,(F\overline{P}F') = 0.$$

This is clearly demonstrated by the plots of maximum singular values of $M_f(j\omega,\sigma)$ for several values of σ as shown in Figure 4.4.1. The observer gain matrices for these plots are generated using the H_2-optimization based algorithm given in the previous subsection. □

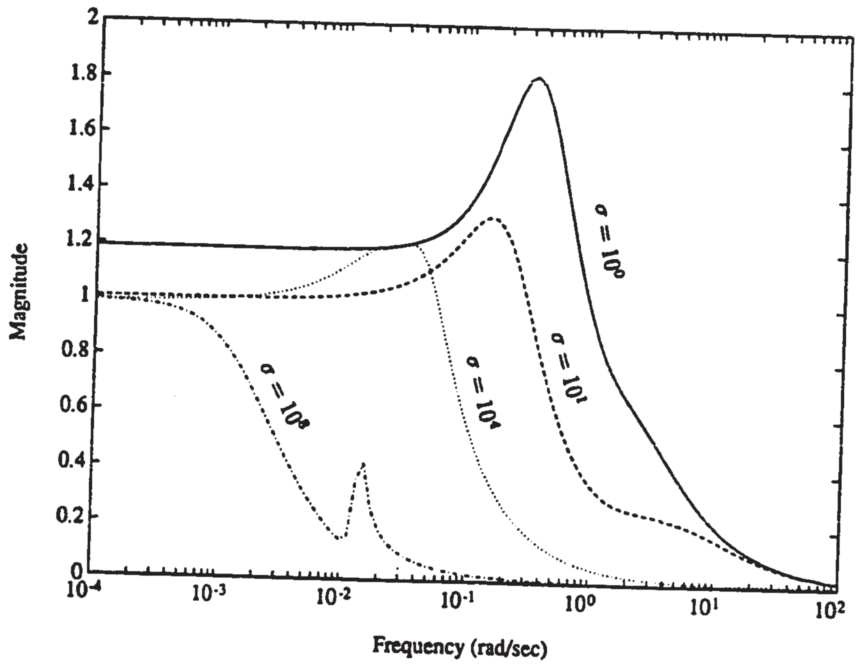


Figure 4.4.1: Maximum singular values of $M_f(j\omega, \sigma)$.

4.4.3. H_∞-optimization design algorithm

In this subsection, we consider H_∞ norm minimization of $M_f(s)$ or equivalently $T_{zw}(s)$. Unlike in H_2 norm minimization case of previous subsection, for general systems, there are no direct methods available to compute exactly the infimum of $\|M_f(s)\|_{H_\infty}$ which is denoted here by γ^*. However, there are iterative algorithms that can be used to approximate γ^*, at least in principle, to any arbitrary degree of accuracy (See for example [36]). Recently though, for a particular class of problems, i.e. when Σ is left invertible and has no invariant zeros on the $j\omega$ axis, such an infimum γ^* has explicitly been calculated in [5] and [6].

We now proceed to present a basic algorithm of computing the observer gain matrix K_f such that the resulting H_∞-norm of the recovery matrix $M_f(s, \gamma)$, is less than a priori given scalar $\gamma > \gamma^*$. The algorithm is as follows:

Step 0 : Choose a value $\epsilon = 1$.

Step 1 : Solve the following algebraic Riccati equation (H_∞-ARE),

$$A_1P + PA_1' - PC_0'C_0P - \frac{1}{\epsilon}PC_1'C_1P + B_1B_1' + \frac{1}{\gamma^2}PF'FP + \epsilon I_n = 0, \tag{4.4.6}$$

for P. Evidently, the solution P of the above H_∞-ARE is a function of γ and is denoted by $P(\gamma)$.

Step 2 : If $P(\gamma) > 0$ go to **Step 3**. Otherwise, decrease ϵ and go to **Step 1**. Note that for $\gamma > \gamma^*$, it is shown in [60] that there always exists a sufficiently small scalar $\epsilon^* > 0$ such that the H_∞-ARE (4.4.6) has a unique positive definite solution $P(\gamma)$ for each $\epsilon \in (0, \epsilon^*)$.

Step 3 : Let

$$K_f(\gamma) = [B_0 + P(\gamma)C_0', \ \frac{1}{2\epsilon}P(\gamma)C_1']. \tag{4.4.7}$$

We have the following theorem.

Theorem 4.4.2. *Consider a full order observer based controller with its gain taken as in (4.4.7). Let $M_f(s,\gamma)$ be the resulting recovery matrix. Then, $\|M_f(s,\gamma)\|_{H_\infty}$ is strictly less than γ, and tends to γ^* as $\gamma \to \gamma^*$.*

Proof : It follows simply from the results of [60]. ∎

Remark 4.4.1. *We note that γ acts here as a tuning parameter. Since to start with, one does not know γ^*, a particular prescribed value for γ may turn out to be less than γ^*. In that case, the H_∞-ARE (4.4.6) does not have any positive definite solution even for a sufficiently small ϵ. Then, one has to increase the value of γ and try to solve the H_∞-ARE once again for $P(\gamma) > 0$. One has to repeat this procedure as many times as necessary.*

For the special case of *regular problems*, there exists a method of generating the gain without the need to introduce another parameter ϵ, and is given by [17],

$$K_f(\gamma) = B_0 + P(\gamma)C_0', \tag{4.4.8}$$

where $P(\gamma) := P$ is the positive semi-definite solution of

$$A_1P + PA_1' - PC_0'C_0P + B_1B_1' + \frac{1}{\gamma^2}PF'FP = 0,$$

such that $\lambda(A_1' - C_0'C_0P + \gamma^{-2}F'FP) \subseteq C^-$. A full order observer based controller with its gain taken as in (4.4.8) results in $\|M_f(s,\gamma)\|_{H_\infty}$ being strictly less than γ.

Apparently, the gain $K_f(\gamma)$ obtained via the basic H_∞-optimization algorithm of equations (4.4.6) and (4.4.7) assigns a particular time-scale structure and eigenstructure to the observer dynamic matrix $A - K_f(\gamma)C$. An investigation into the exact nature of time-scale structure and the eigenstructure of $A - K_f(\gamma)C$ as $\gamma \rightarrow \gamma^*$ is still an open research problem. But we like to point out that, as ATEA design procedure does in general, the basic H_∞-optimization algorithm renders the corresponding $M_f^0(s,\gamma)$ and $M_f^\infty(s,\gamma)$ zero asymptotically as $\gamma \rightarrow \gamma^*$. Also, the corresponding $M_f^e(s)$ is shaped in a particular way so that the infimum of $\|M_f(s)\|_{H_\infty}$ is attained as $\gamma \rightarrow \gamma^*$. In so doing, in addition to $\Lambda^\infty(\gamma)$, some elements of $\Lambda^e(\gamma)$ may be pushed to infinite locations in C^- as $\gamma \rightarrow \gamma^*$.

4.4.4. A helicopter control system design (cont.)

We continue here the helicopter control system design discussed earlier in subsections 4.3.2, 4.3.4 and 4.4.2. For the given helicopter system as in subsection 4.3.2, it can be verified that γ^*, the infimum of $\|M_f(s)\|_{h_\infty}$, is approximately equal to 1. Let the observer gain $K_f(\gamma)$ be calculated using H_∞-optimization based algorithm discussed in the previous subsection. The plots of maximum singular values of $M_f(j\omega,\gamma)$ for several values of γ as given in Figure 4.4.2, clearly demonstrate that $\|M_f(j\omega,\gamma)\|_{H_\infty} \rightarrow \gamma^*$ as $\gamma \rightarrow \gamma^*$. □

4.5. Design for recovery over a specified subspace

Sections 4.3 and 4.4 consider the conventional LTR design problem which seeks the recovery over the entire control space. Here, given a subspace $\mathcal{S}$ of $\Re^m$, the interest is to design an observer so that the achieved and target sensitivity and complementary sensitivity functions projected onto the subspace $\mathcal{S}$ match each other either exactly or asymptotically. The conditions under which such a design is possible are given in chapter 3. To recapitulate these conditions, let V^s be a matrix whose columns form an orthogonal basis of the given

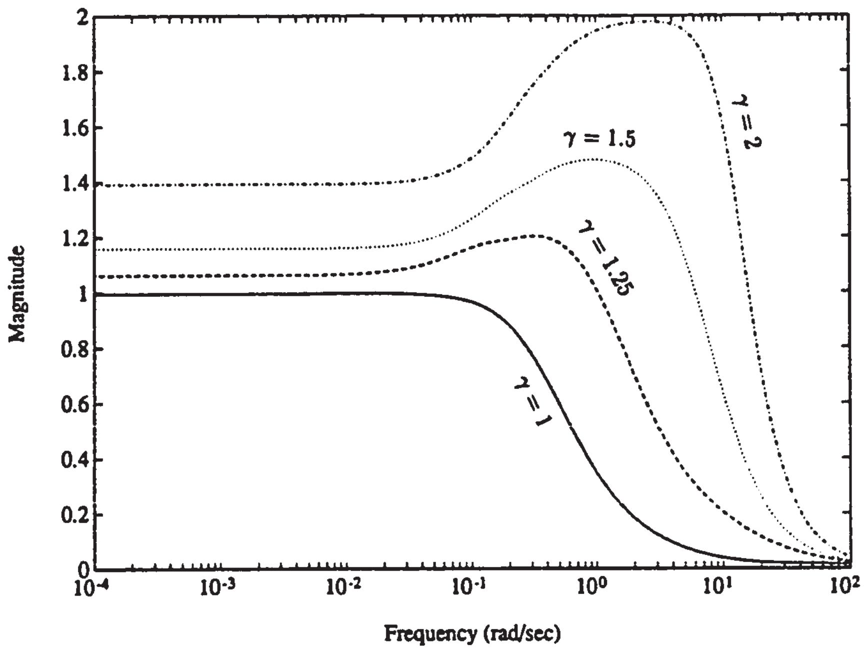


Figure 4.4.2: Maximum singular values of $M_f(j\omega, \gamma)$.

subspace $\mathcal{S}$ of $\Re^m$. Also, given the system Σ characterized by $(A,\ B,\ C,\ D)$, let us define an auxiliary system Σ^s characterized by the matrix triple $(A,\ BV^s,\ C,\ DV^s)$. Also, let $L_t(s) = F\Phi B$ be the specified target loop transfer function. Then the analysis given in Part 1 implies the following:

1. The projections of achievable and target sensitivity and complementary sensitivity functions onto the subspace $\mathcal{S}$ match each other exactly iff $\mathcal{S}^-(\Sigma^s) \subseteq \mathrm{Ker}\,(F)$.

2. The projections of achievable and target sensitivity and complementary sensitivity functions onto the subspace $\mathcal{S}$ match each other asymptotically iff $\mathcal{V}^+(\Sigma^s) \subseteq \mathrm{Ker}\,(F)$.

Thus the task of designing observers for either exact or asymptotic recovery over a subspace collapses to the task discussed in earlier sections except that one needs to use Σ^s instead of Σ. The following example illustrates this.

Example 4.5 : Consider a system Σ characterized by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This system is left invertible and of nonminimum phase with invariant zeros at $s = 1$, $s = 2$, $s = 3$ and at $s = 4$. Now consider a specified subspace $\mathcal{S}$ which is a span of

$$V^s = \begin{bmatrix} 0.4433 & -0.4553 & -0.0027 \\ 0.3802 & 0.5771 & -0.7128 \\ 0.6006 & -0.4719 & -0.1664 \\ 0.5462 & 0.4867 & 0.6813 \end{bmatrix}.$$

It is simple to verify that the auxiliary system Σ^s characterized by $(A,\ BV^s,\ C,\ DV^s)$ is left invertible and of minimum phase. Hence the projections of target and achievable sensitivity and complementary sensitivity functions onto V^s can match each other asymptotically. To exemplify this, let the target loop be specified by,

$$F = \begin{bmatrix} 0 & 0 & 0 & 200 & 100 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 50 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 & 0 & 30 & 0 \\ 50 & 0 & 0 & 0 & 0 & 0 & 0 & 25 \end{bmatrix}.$$

Let us choose $K_f(\sigma)$ with $\sigma = 1000$ as

$$K_f(\sigma) = \begin{bmatrix} 0 & 10707293 & 240.6328 & -717.12942 & -18.09924 \\ 0 & -1324480 & -35.68825 & 106.64317 & 2.70284 \\ 0 & -6883343 & -215.567 & 643.91894 & 16.36376 \\ 0 & 258551 & 9.26071 & -27.598585 & -0.701356 \\ 1 & 1.76203 & 2.185\times10^{-7} & -6.51\times10^{-7} & -1.655\times10^{-8} \\ -0.4570667 & 3978.336 & 144.08447 & 68.78912 & 1.74812 \\ 3.57747 & -9798.08346 & 40.0091 & 23.90837 & -3.03008 \\ 0.185196 & -354.2448 & 1.115087 & -3.32316 & 125.04055 \end{bmatrix}$$

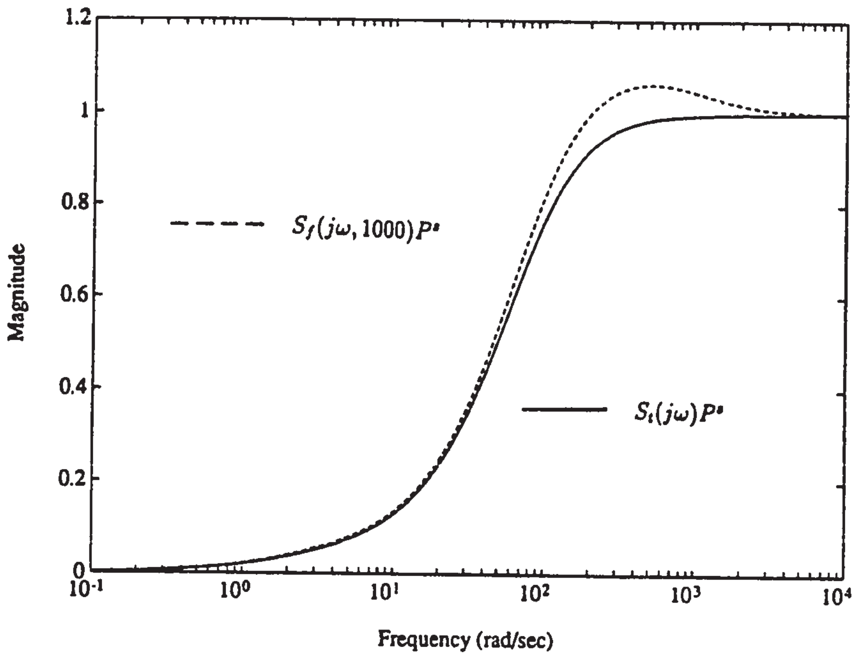


Figure 4.5.1: Maximum singular values of $S_t(j\omega)P^s$ and $S_f(j\omega, 1000)P^s$.

so that the observer eigenvalues are placed at -1000, -1000, -1, -2, -3, -4, -5 and -6. Let the orthogonal projection matrix onto the subspace $\mathcal{S}$ be $P^s = V^s(V^s)'$. Then the resulting $M_f(j\omega,\sigma)P^s$, $E_f(j\omega,\sigma)P^s$, $S_f(j\omega,\sigma)P^s$ and $S_t(j\omega)P^s$ are plotted with respect to ω over a given range of ω in figures 4.5.1 and 4.5.2.

It is easy to note that $M_f(j\omega,\sigma)P^s$ is approximately zero while $S_f(j\omega,\sigma)P^s$ is close to $S_t(j\omega)P^s$. Also, note that the minimum singular values of $S_f(j\omega,\sigma)P^s$ and $S_t(j\omega)P^s$ are identically zero due to the singularity of P^s.

4.6. Comparison of 'ATEA' and 'ARE' based algorithms

A comparison of optimal or suboptimal design schemes based on solving Algebraic Riccati equations (ARE's) as described in section 4.4 and the asymptotic time-scale and eigenstructure assignment (ATEA) design schemes of section 4.3, is in order. In this regard, Saberi, Chen and Sannuti [40] discuss several relative advantages and disadvantages of ATEA and ARE based designs. Here we look at ATEA design and optimization based designs from two different perspectives, (1) numerical simplicity and (2) flexibility to use all the available

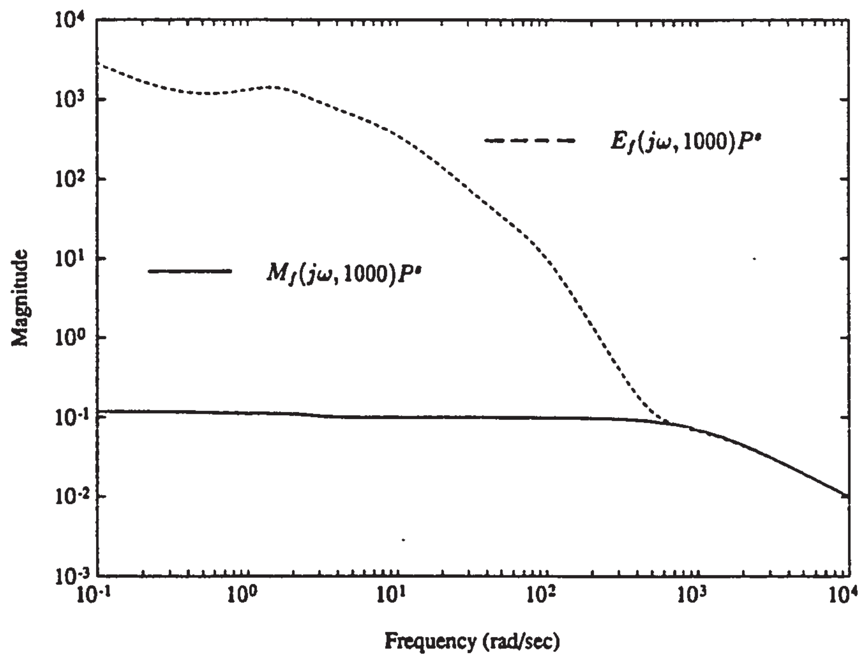


Figure 4.5.2: Maximum singular values of $M_f(j\omega, 1000)P^s$ and $E_f(j\omega, 1000)P^s$.

freedom.

Let us first consider the numerical aspects of those design algorithms. It is clear that the central part of either optimization based design of section 4.4 lies in obtaining the positive definite solutions of parameter dependent ARE's repeatedly for different values of the parameter. As is well known, these ARE's become numerically 'stiff' when the concerned parameter takes values close to a critical value. To be specific, the H_2-ARE becomes stiff as the parameter σ becomes large, where as the H_∞-ARE becomes stiff when γ approaches γ^*. This is due to the interaction of fast and slow dynamics inherent in such equations. Thus, the numerical difficulties accrue not only due to the required repeated solutions of ARE's but also due to the 'stiffness' of such equations. On the other hand, as is clear from section 4.3, ATEA adopts a decentralized design procedure and in so doing removes both the obstacles of repetitive solution of algebraic equations and their stiffness. That is, in ATEA, in order not to allow the interaction between the slow and various fast time-scales, the needed design to assign asymptotically finite and infinite eigenstructure to the observer dynamic matrix is done separately. The tuning parameter merely adjusts the relative fastness of fast time-scales and it is introduced only in composing the two separately designed gains together into a composite gain. This procedure presents no numerical difficulties whatsoever as the parameter takes larger and larger values.

Another factor that is of great importance in selecting a design procedure is the flexibility it offers to utilize all the available design freedom. As summarized in section 4.2, there exists considerable amount of freedom to shape the recovery matrix by an appropriate eigenstructure assignment to the observer dynamic matrix. Such a freedom can be utilized to shape $\overline{M}_f^e(j\omega)$, the limit of the recovery matrix, with respect to ω. Any optimization based method adopts a particular way of shaping $\overline{M}_f^e(j\omega)$ as dictated by the mathematical minimization procedure. For example, as discussed earlier, in H_2 optimization $\overline{M}_f^e(j\omega)$ is shaped by assigning some of the eigenvalues of $A - K_fC$ at the mirror images of the nonminimum phase invariant zeros of Σ, while the associated set of left eigenvectors of $A - K_fC$ coincide with the corresponding right input zero directions of Σ. Such a shaping obviously limits the available design freedom, and may or may not be desirable from an engineering point of view. Next, available design freedom can also be utilized to characterize appropriately the behavior of asymptotically infinite or otherwise called fast eigenvalues of $A - K_fC$. What we mean by the behavior of fast eigenvalues is (a) their asymptotic directions and (b) the rate at which they go to infinity, i.e., the fast time-scale structure of $A - K_fC$. As demonstrated in [40], the behavior of fast eigenvalues has a dramatic effect on the resulting controller band-width. Again, optimization based design methods fix the behavior of fast eigenvalues in a particular way that may or may not be favorable to the designer's goals. We believe that the ability to utilize all the available design freedom is a valuable asset; in particular, exploring such a freedom in the space in which complete recovery is not feasible is of dire importance. ATEA design methods of section 4.3 put all the available design freedom in the hands of designer and hence are preferable to optimization based designs of section 4.4. However, a clear advantage of the optimization based schemes is that at the onset of design, they do not require much systematic planning and hence are straightforward to apply. In fact, one simply (!) solves the relevant ARE's repeatedly for several values of tuning parameter until a gain obtained for one particular value of the parameter is appropriate for a suboptimal design. Admittedly, ATEA design does not have such a simplicity. One needs in ATEA design to come up with an appropriate utilization of the available design freedom and thus the selection of available design parameters in order to meet the practical design specifications. But this perhaps can be done by a simple iterative adjustment. Such a procedure is still computationally inexpensive as the required calculations for ATEA design are straightforward and do not involve any 'stiff' equations.

4.A. Appendix 4.A — Proof of Theorem 4.3.1

We shall prove theorem 4.3.1 in the following two cases:

Case 1 : The given system Σ is strictly proper, i.e. $D = 0$.

Without loss of generality, we will assume that the given system is in the form of s.c.b. Here we note that for strictly proper system, u_0, y_0, C_0 and B_0 are nonexistent. Hence, we have

$$C^e = [\,E_a^+ \quad E_c\,], \quad K^e = K^{e1}, \quad \tilde{K}^{a+}(\sigma) = \tilde{K}^{a1+}(\sigma), \quad \tilde{K}^c(\sigma) = \tilde{K}^{c1}(\sigma)$$

and

$$\tilde{K}(\sigma) = \begin{bmatrix} L_{af}^- + \tilde{H}_{af}^- & L_{ab}^- + \tilde{H}_{ab}^- \\ L_{af}^+ + \tilde{H}_{af}^+ + \tilde{K}^{a+}(\sigma) & L_{ab}^+ + \tilde{H}_{ab}^+ \\ L_{bf} + \tilde{H}_{bf} & K^b \\ L_{cf} + \tilde{H}_{cf} + \tilde{K}^c(\sigma) & L_{cb} + \tilde{H}_{cb} \\ L_f + \tilde{K}_f(\sigma) & 0 \end{bmatrix}.$$

Then by renaming the variables $x_0 = [(x_a^-)', x_b']'$ and $x_e = [(x_a^+)', x_c']'$, we can rewrite the observer dynamic matrix $A - K_f(\sigma)C$ as,

$$A - K_f(\sigma)C = \begin{bmatrix} A^{00} & 0 & -\tilde{H}_{0f}C_f \\ A^{e0} & A^{ee} & -[\tilde{H}_{ef} + \tilde{K}^e(\sigma)]C_f \\ B_f E^0 & B_f C^e & A_f - \tilde{K}_f(\sigma)C_f - L_f C_f \end{bmatrix}, \tag{4.A.1}$$

where

$$A^{00} = \begin{bmatrix} A^-_{aa} & -\tilde{H}^-_{as}C_s \\ 0 & A^{bbc} \end{bmatrix}, \; A^{e0} = \begin{bmatrix} 0 & -\tilde{H}^+_{as}C_s \\ B_cE^-_{ca} & -\tilde{H}_{cs}C_s \end{bmatrix}, \; A^{ee} = \begin{bmatrix} A^+_{aa} & 0 \\ B_cE^+_{ca} & A_{cc} \end{bmatrix},$$

$$\tilde{H}_{0f} = \begin{bmatrix} \tilde{H}^-_{af} \\ \tilde{H}_{bf} \end{bmatrix}, \; \tilde{H}_{ef} = \begin{bmatrix} \tilde{H}^+_{af} \\ \tilde{H}_{cf} \end{bmatrix}.$$

We prove the TSS properties of the observer on a transposed system Σ_t whose closed-loop dynamic matrix is a transpose of A_o. Consider Σ_t,

$$\dot{x}_0 = (A^{00})'x_0 + (A^{e0})'x_e + \sum_{\ell=1}^{m_u} E'_{\ell 0}x_{\ell q_\ell} \tag{4.A.2}$$

$$\dot{x}_e = (A^{ee})'x_e + \sum_{\ell=1}^{m_u} E'_{\ell e}x_{\ell q_\ell} \tag{4.A.3}$$

and for each $i = 1$ to m_u,

$$\dot{x}_i = A'_{q_i}x_i - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [\tilde{H}'_{0i}x_0 + (\tilde{H}'_{ei} + [\tilde{K}^{ei}(\sigma)]')x_e + \tilde{K}'_i(\sigma)x_i] + \sum_{\ell=1}^{m_u} E'_{\ell i}x_{\ell q_\ell} \tag{4.A.4}$$

where

$$E_{i0} = [E^-_{ia}, E_{ib}], \; E_{ie} = [E^+_{ia}, E_{ic}],$$

$$\tilde{H}_{0f} = [\tilde{H}_{01}, \tilde{H}_{02}, \cdots, \tilde{H}_{0m_u}], \; \tilde{H}_{ef} = [\tilde{H}_{e1}, \tilde{H}_{e2}, \cdots, \tilde{H}_{em_u}],$$

$$x = [x'_0, x'_e, x'_1, \cdots, x'_{m_u}]', \; x_i = [x_{i1}, x_{i2}, \cdots, x_{iq_i}]'.$$

Let us adopt the following scaling and transformation of variables,

$$\boldsymbol{x}_0 = x_0, \; \boldsymbol{x}_e = x_e, \; \boldsymbol{x}_{iq_i} = x_{iq_i} + (K^{ei})'x_e, \; \boldsymbol{x}_{ik} = \prod_{\ell=k+1}^{q_i} \epsilon_{i\ell}\, x_{ik}, \; i = 1 \text{ to } q_i - 1.$$

We next define,

$$\tilde{X}_{ij} = [x_{i\alpha_{ij-1}+1}, x_{i\alpha_{ij-1}+2}, \cdots, x_{i\alpha_{ij}}]'$$

and

$$\boldsymbol{X}_{ij} = [\boldsymbol{x}_{i\alpha_{ij-1}+1}, \boldsymbol{x}_{i\alpha_{ij-1}+2}, \cdots, \boldsymbol{x}_{i\alpha_{ij}}]'$$

so that

$$\boldsymbol{X}_{ij} = S_{ij}\tilde{X}_{ij} \text{ for } j = 1 \text{ to } r_i - 1,$$

and

$$\boldsymbol{X}_{ir_i} = S_{ir_i}\tilde{X}_{ir_i} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} K'_{ei}\boldsymbol{x}_e$$

where S_{ij} is as defined in (4.3.7). Then (4.A.2) to (4.A.4) can be rewritten as

$$\dot{\boldsymbol{x}}_0 = (A^{00})'\boldsymbol{x}_0 + D_{0e}\boldsymbol{x}_e + \sum_{\ell=1}^{m_u} D_{0\ell}[0\,,\,1]\boldsymbol{X}_{\ell r_\ell} \tag{4.A.5}$$

$$\dot{\boldsymbol{x}}_e = (A^{eec})'\boldsymbol{x}_e + \sum_{\ell=1}^{m_u} D_{e\ell}[0\,,\,1]\boldsymbol{X}_{\ell r_\ell} \tag{4.A.6}$$

and for each $i = 1$ to m_u,

$$\mu_{i1}\dot{\boldsymbol{X}}_{i1} = G'_{i1}\boldsymbol{X}_{i1} - H_{i1}\sum_{j=1}^{r_i} J_{ij}K'_{ij}\boldsymbol{X}_{ij} + \sum_{\ell=1}^{m_u} D_{i1\ell}[0\,,\,1]\boldsymbol{X}_{\ell r_\ell} + D_{i10}\boldsymbol{x}_0 + D_{i1e}\boldsymbol{x}_e \tag{4.A.7}$$

$$\mu_{ij}\dot{\boldsymbol{X}}_{ij} = G'_{ij}\boldsymbol{X}_{ij} + H_{ij}\boldsymbol{X}_{ij-1} + \sum_{\ell=1}^{m_u} D_{ij\ell}[0\,,\,1]\boldsymbol{X}_{\ell r_\ell} + D_{ije}\boldsymbol{x}_e \text{ for } j = 2 \text{ to } r_i \tag{4.A.8}$$

where

$$H_{ij} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The zero elements in $n_{ij} \times n_{ij-1}$ dimensional matrix H_{ij} are of appropriate dimension and may or may not exist depending upon the values of n_{ij} and n_{ij-1}. Also, various coefficient matrices in the above equations are as follows:

$$D_{0e} = (A^{e0})' - \sum_{\ell=1}^{m_u} E'_{\ell 0}K'_{e\ell}\,,\ D_{0\ell} = E'_{\ell 0}\,,\ D_{e\ell} = E'_{\ell e}$$

$$D_{i10} = -\eta_i H_{i1}\tilde{H}'_{0i}\,,\ D_{i1e} = -\eta_i H_{i1}\tilde{H}'_{ei} - \sum_{\ell=1}^{m_u} \mathcal{E}'_{\ell i1}(K^{e\ell})'$$

$$[\mathcal{E}_{\ell i1}\,, \mathcal{E}_{\ell i2}\,, \cdots, \mathcal{E}_{\ell ir_i}] = E_{\ell i}\text{Diag}[\prod_{k=1}^{q_i} \epsilon_{ik},\ \prod_{k=2}^{q_i} \epsilon_{ik},\ \cdots, \epsilon_{iq_i}]$$

$$D_{ije} = -\sum_{\ell=1}^{m_u} \mathcal{E}'_{\ell ij}(K^{e\ell})' \text{ for } j = 2 \text{ to } r_i - 1$$

$$D_{ij\ell} = \mathcal{E}'_{\ell ij} \text{ for } j = 1 \text{ to } r_i - 1$$

$$D_{ir_ie} = -\sum_{\ell=1}^{m_u} \mathcal{E}'_{\ell ir_i}(K^{e\ell})' + \mu_{ir_i}\begin{bmatrix} 0 \\ 1 \end{bmatrix}(K^{ei})'(A^c_{ee})'$$

and

$$D_{ir_i\ell} = \mathcal{E}'_{\ell ir_i} + \mu_{ir_i} \begin{bmatrix} 0 \\ 1 \end{bmatrix} K'_{ei} D_{e\ell}. \tag{4.A.9}$$

Although (4.A.5) to (4.A.8) are in singularly perturbed form, their time-scale structure properties are not transparent. In order to bring various time-scales into focus, we adopt another transformation of variables. Let for each $i = 1$ to m_u,

$$X_{ir_i} = \boldsymbol{X}_{ir_i} \text{ and } X_{ij} = \boldsymbol{X}_{ij} + \mathcal{K}_{ij+1} X_{ij+1} \text{ for } j = 1 \text{ to } r_i - 1 \tag{4.A.10}$$

where $n_{ij-1} \times n_{ij}$ dimensional matrix

$$\mathcal{K}_{ij} = \begin{bmatrix} 0 & 0 \\ K'_{ijc} & K_{ijd} \end{bmatrix}.$$

Then it is straightforward to verify that (4.A.7) and (4.A.8) can be rewritten as

$$\mu_{i1}\dot{X}_{i1} = (G^c_{i1})' X_{i1} + \sum_{\ell=1}^{m_u} D_{i1\ell}[0\,,\,1] X_{\ell r_\ell} + D_{i10}\boldsymbol{x}_0 + D_{i1e}\boldsymbol{x}_e$$
$$+ \sum_{k=2}^{r_i} \frac{\mu_{i1}}{\mu_{ik}} \mathcal{K}_{i2}\mathcal{K}_{i3} \cdots \mathcal{K}_{ik} \left[(G^c_{ik})' X_{ik} + H_{ik} X_{ik-1} + \sum_{\ell=1}^{m_u} D_{ik\ell}[0\,,\,1] X_{\ell r_\ell} + D_{ike}\boldsymbol{x}_e \right],$$
$$\mu_{ij}\dot{X}_{ij} = (G^c_{ij})' X_{ij} + H_{ij} X_{ij-1} + \sum_{\ell=1}^{m_u} D_{ij\ell}[0\,,\,1] X_{\ell r_\ell} + D_{ije}\boldsymbol{x}_e$$
$$+ \sum_{k=j+1}^{r_i} \frac{\mu_{ij}}{\mu_{ik}} \mathcal{K}_{ij+1}\mathcal{K}_{ij+2} \cdots \mathcal{K}_{ik} \left[(G^c_{ik})' X_{ik} + H_{ik} X_{ik-1} + \sum_{\ell=1}^{m_u} D_{ik\ell}[0\,,\,1] X_{\ell r_\ell} + D_{ike}\boldsymbol{x}_e \right]$$
$$\text{for } j = 2 \text{ to } r_i - 1\,,$$
$$\mu_{ir_i}\dot{X}_{ir_i} = (G^c_{ir_i})' X_{ir_i} + H_{ir_i} X_{ir_i-1} + \sum_{\ell=1}^{m_u} D_{ir_i\ell}[0\,,\,1] X_{\ell r_\ell} + D_{ir_ie}\boldsymbol{x}_e\,. \tag{4.A.11}$$

Since the interconnection matrices in the coupled equations (4.A.11) tend to null matrices as $\sigma \to \infty$, the time-scale structure property of the observer follows directly from singular perturbation theory. To show this more explicitly, we next do Lyapunov analysis of the above dynamic system. For this purpose all the small parameters are redefined as

$$\mu_{ij} = \epsilon^{a_{ij}} \tag{4.A.12}$$

for some some positive scalars a_{ij} where

$$\epsilon = \frac{1}{\sigma}.$$

Then in view of the property (4.3.5), we note that

$$a_{ij} > a_{ij+1} \text{ for all } j = 1 \text{ to } r_i - 1.$$

Also, we can rewrite (4.A.11) as

$$\begin{aligned}
\epsilon^{a_{i1}} \dot{X}_{i1} &= (G^c_{i1})' X_{i1} + \epsilon^{d^*_{i1}} \left[\mathcal{D}^*_{i10} \boldsymbol{x}_0 + \mathcal{D}^*_{i1e} \boldsymbol{x}_e + \sum_{\ell=1}^{m_u} \mathcal{D}^*_{i1\ell} X_{\ell r_\ell} + \sum_{k=1}^{r_i} K^*_{1ik} X_{ik} \right], \\
\epsilon^{a_{ij}} \dot{X}_{ij} &= (G^c_{ij})' X_{ij} + H_{ij} X_{ij-1} + \epsilon^{d^*_{ij}} \left[\mathcal{D}^*_{ije} \boldsymbol{x}_e + \sum_{\ell=1}^{m_u} \mathcal{D}^*_{ij\ell} X_{\ell r_\ell} + \sum_{k=j}^{r_i} K^*_{jik} X_{ik} \right] \\
&\qquad \text{for } j = 2 \text{ to } r_i - 1, \\
\epsilon^{a_{ir_i}} \dot{X}_{ir_i} &= (G^c_{ir_i})' X_{ir_i} + H_{ir_i} X_{ir_i - 1} + \epsilon^{d^*_{ir_i}} \left[\mathcal{D}^*_{ir_i e} \boldsymbol{x}_e + \sum_{\ell=1}^{m_u} \mathcal{D}^*_{ir_i\ell} X_{\ell r_\ell} \right]
\end{aligned} \tag{4.A.13}$$

for some positive scalars d^*_{ij} and for some appropriately defined interconnection coefficient matrices. It is important to note that all the interconnection matrices are bounded as $\epsilon \to 0$. Let

$$d = \frac{1}{2} \min\{d^*_{ij} \; ; \; i = 1 \text{ to } m_u \text{ and } j = 1 \text{ to } r_i\}.$$

Then

$$d_{ij} \equiv d^*_{ij} - d > 0 \text{ for all } i \text{ and } j.$$

Also, let us define

$$X_0 = \epsilon^d \boldsymbol{x}_0 \text{ and } X_e = \epsilon^d \boldsymbol{x}_e.$$

Then we can rewrite (4.A.5), (4.A.6) and (4.A.13) as

$$\dot{X}_0 = (A^{00})' X_0 + D_{0e} X_e + \epsilon^d \sum_{\ell=1}^{m_u} D_{0\ell}[0\,,\,1] X_{\ell r_\ell}\,,$$

$$\dot{\boldsymbol{x}}_e = (A^{eec})' \boldsymbol{x}_e + \epsilon^d \sum_{\ell=1}^{m_u} D_{e\ell}[0\,,\,1] X_{\ell r_\ell}\,,$$

and for each $i = 1$ to m_u,

$$\epsilon^{a_{i1}} \dot{X}_{i1} = (G^c_{i1})' X_{i1} + \epsilon^{d_{i1}} \left[\mathcal{D}_{i10} X_0 + \mathcal{D}_{i1e} X_e + \sum_{\ell=1}^{m_u} \mathcal{D}_{i1\ell} X_{\ell r_\ell} + \sum_{k=1}^{r_i} K_{1ik} X_{ik} \right],$$

$$\epsilon^{a_{ij}}\dot{X}_{ij} = (G^c_{ij})'X_{ij} + H_{ij}X_{ij-1} + \epsilon^{d_{ij}}\left[\mathcal{D}_{ije}X_e + \sum_{\ell=1}^{m_u}\mathcal{D}_{ij\ell}X_{\ell r_\ell} + \sum_{k=j}^{r_i}K_{jik}X_{ik}\right]$$
$$\text{for } j = 2 \text{ to } r_i - 1\,,$$
$$\epsilon^{a_{ir_i}}\dot{X}_{ir_i} = (G^c_{ir_i})'X_{ir_i} + H_{ir_i}X_{ir_i-1} + \epsilon^{d_{ir_i}}\left[\mathcal{D}_{ir_ie}X_e + \sum_{\ell=1}^{m_u}\mathcal{D}_{ir_i\ell}X_{\ell r_\ell}\right]. \tag{4.A.14}$$

To proceed with a Lyapunov analysis of the above system, let us select the positive definite matrices P_0, P_e and P_{ij}, $i = 1$ to m_u and $j = 1$ to r_i, satisfying the following Lyapunov equations:

$$P_0A'_{00} + A_{00}P_0 = -I,$$

$$P_eA'_{ee} + A_{ee}P_e = -I,$$

$$P_{ij}(G^c_{ij})' + G^c_{ij}P_{ij} = -I.$$

We next define a Lyapunov function,

$$V(X) = X'_0P_0X_0 + c_eX'_eP_eX_e + \sum_{i=1}^{m_u}\sum_{j=1}^{r_i}c_{ij}X'_{ij}P_{ij}X_{ij} \tag{4.A.15}$$

where c_e and c_{ij} are some positive scalars that are yet to be selected. It is then easy to show that dV/dt calculated along the trajectory of (4.A.14) satisfies the following:

$$\frac{dV}{dt} \le -\|X_0\|^2 + 2\|P_0\|\,\|D_{0e}\|\,\|X_0\|\,\|X_e\| + 2\epsilon^d\sum_{k=1}^{m_u}\|P_0\|\,\|D_{0k}\|\,\|X_0\|\,\|X_{kr_k}\|$$
$$-c_e\|X_e\|^2 + 2c_e\epsilon^d\sum_{k=1}^{m_u}\|P_e\|\,\|D_{ek}\|\,\|X_e\|\,\|X_{kr_k}\|$$
$$+\sum_{i=1}^{m_u}\left\{-\frac{c_{i1}}{\epsilon^{a_{i1}}}\|X_{i1}\|^2 + 2\frac{c_{i1}}{\epsilon^{a_{i1}}}\epsilon^{d_{i1}}\|X_{i1}\|\,\|P_{i1}\|\left[\|\mathcal{D}_{i10}\|\,\|X_0\|\right.\right.$$
$$\left.+\|\mathcal{D}_{i1e}\|\,\|X_e\| + \sum_{k=1}^{m_u}\|\mathcal{D}_{i1k}\|\,\|X_{kr_k}\| + \sum_{k=1}^{r_i}\|K_{1ik}\|\,\|X_{ik}\|\right]$$
$$+\sum_{j=2}^{r_i-1}\left[-\frac{c_{ij}}{\epsilon^{a_{ij}}}\|X_{ij}\|^2 + 2\frac{c_{ij}}{\epsilon^{a_{ij}}}\|P_{ij}\|\,\|X_{ij}\|\,\|X_{ij-1}\| + 2\frac{c_{ij}}{\epsilon^{a_{ij}}}\epsilon^{d_{ij}}\|X_{ij}\|\|P_{ij}\|\right.$$
$$\left.\left(\|\mathcal{D}_{ije}\|\,\|X_e\| + \sum_{k=1}^{m_u}\|\mathcal{D}_{ijk}\|\,\|X_{kr_k}\| + \sum_{k=j}^{r_i}\|K_{jik}\|\,\|X_{ik}\|\right)\right]$$

$$
-\frac{c_{ir_i}}{\epsilon^{a_{ir_i}}}\|X_{ir_i}\|^2 + 2\frac{c_{ir_i}}{\epsilon^{a_{ir_i}}}\|P_{ir_i}\|\|X_{ir_i}\|\|X_{ir_i-1}\| + 2\frac{c_{ir_i}}{\epsilon^{a_{ir_i}}}\epsilon^{d_{ir_i}}\|X_{ir_i}\|\|P_{ir_i}\|
\left[\|\mathcal{D}_{ir_ie}\|\,\|X_e\| + \sum_{k=1}^{m_u}\|\mathcal{D}_{ir_ik}\|\,\|X_{kr_k}\|\right]\Bigg\}
$$

$$
= -\left[\|X_0\|, \|X_e\|, \|X_{11}\|, \cdots, \|X_{1r_1}\|, \cdots, \|X_{m_u1}\|, \cdots, \|X_{m_ur_{m_u}}\|\right] R(\epsilon)
\left[\|X_0\|, \|X_e\|, \|X_{11}\|, \cdots, \|X_{1r_1}\|, \cdots, \|X_{m_u1}\|, \cdots, \|X_{m_ur_{m_u}}\|\right]'. \quad (4.A.16)
$$

Let us next choose

$$
c_e > \|P_0\|^2 \|D_{0e}\|^2. \quad (4.A.17)
$$

In order to facilitate the selection of coefficients c_{ij}, let

$$
d_i = \frac{1}{r_i + 1} \min\{d_{ij};\ j = 1 \text{ to } r_i\}
$$

and define

$$
b_{ij} = (r_i + 1 - j)d_i < d_{ij},\ j = 1 \text{ to } r_i. \quad (4.A.18)
$$

Then for each $i = 1$ to m_u and $j = 1$ to r_i, select

$$
c_{ij} = \epsilon^{a_{ij} - b_{ij}}.
$$

Here we note that for $j = 1$ to $r_i - 1$,

$$
b_{ij} > b_{ij+1}.
$$

Then the matrix $R(\epsilon)$ is given as,

$$
R(\epsilon) = \begin{bmatrix} R_{0e} & \star & \star & \cdots & \star \\ \star & R_1 & \star & \cdots & \star \\ \star & \star & R_2 & \cdots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & R_{m_u} \end{bmatrix} \quad (4.A.19)
$$

where $\star$'s represent appropriate dimensional submatrices which tend to null matrices as $\epsilon \to 0$. Also,

$$
R_{0e} = \begin{bmatrix} 1 & -\|P_0\|\|D_{0e}\| \\ -\|P_0\|\|D_{0e}\| & c_e \end{bmatrix} > 0 \quad (4.A.20)
$$

whenever c_e is as in (4.A.17). Furthermore, for each $i = 1$ to m_u,

$$R_i = \begin{bmatrix} \frac{1}{\epsilon^{b_{i1}}} - \star & \frac{\|P_{i2}\|}{\epsilon^{b_{i2}}} & \cdots & \star & \star \\ -\frac{\|P_{i2}\|}{\epsilon^{b_{i2}}} & \frac{1}{\epsilon^{b_{i2}}} - \star & \ddots & \star & \star \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \star & \star & \ddots & \frac{1}{\epsilon^{b_{ir_i-1}}} - \star & -\frac{\|P_{ir_i}\|}{\epsilon^{b_{ir_i}}} \\ \star & \star & \cdots & -\frac{\|P_{ir_i}\|}{\epsilon^{b_{ir_i}}} & \frac{1}{\epsilon^{b_{ir_i}}} - \star \end{bmatrix}. \tag{4.A.21}$$

Now in view of (4.A.18), it is straightforward to verify that R_i, for each $i = 1$ to m_u, is positive definite for ϵ sufficiently small. Then in view of the special structure of $R(\epsilon)$ as in (4.A.19), there exists an ϵ^* such that for any $\epsilon < \epsilon^*$, $R(\epsilon)$ is indeed a positive definite matrix and thus the stability of the observer dynamics is guaranteed. This completes our Lyapunov analysis for **Case 1.**

So far, we proved that the ATEA algorithm yields an admissible observer gain $K(\sigma)$ in the sense that A_o is a stable matrix for sufficiently large σ and that it has the required time-scale structure. In what follows, we will show that $K(\sigma)$ achieves LTR in the sense that

$$M_f(s,\sigma) = F[sI_n - A + K_f(\sigma)C]^{-1}B \rightarrow \overline{M}_e(s) \text{ pointwise in } s \tag{4.A.22}$$

as $\sigma \rightarrow \infty$. In view of (4.3.12), it can be seen easily that $K_f(\sigma)$ has the following form,

$$K_f(\sigma) = T(\sigma)\Gamma(\sigma)N + Q \tag{4.A.23}$$

where

$$\Gamma(\sigma) = \text{Diag}[\frac{1}{\eta_1}, \frac{1}{\eta_2}, \cdots, \frac{1}{\eta_{m_u}}] \ , \ N = [I_{m_u}, 0] ,$$

$$Q = \begin{bmatrix} L_{af}^+ + \tilde{H}_{af}^+ & L_{as}^+ + \tilde{H}_{as}^+ \\ L_{af}^- + \tilde{H}_{af}^- & L_{as}^- + \tilde{H}_{as}^- \\ L_{bf} + \tilde{H}_{bf} & K_b \\ L_{cf} + \tilde{H}_{cf} & L_{cs} + \tilde{H}_{cs} \\ L_f & 0 \end{bmatrix}, \tag{4.A.24}$$

while $T(\sigma)$ satisfies

$$T(\sigma) \rightarrow B_m T \tag{4.A.25}$$

as $\sigma \to \infty$ where

$$B_m = \begin{bmatrix} K^{a+} \\ 0 \\ 0 \\ K^c \\ B_f \end{bmatrix}, \quad T = \text{Diag}[J_{1r_1} K_{1r_1 d},\, J_{2r_2} K_{2r_2 d},\, \cdots,\, J_{m_u r_{m_u}} K_{m_u r_{m_u} d}]. \tag{4.A.26}$$

It is shown in [7] that the triple $(C,\, A,\, B_m)$ forms a left invertible and a minimum phase system. Thus it follows from the results of [43] that

$$[sI_n - A + K_f(\sigma)C]^{-1} B_m \to 0 \text{ pointwise in } s \tag{4.A.27}$$

as $\sigma \to \infty$. Next let

$$B = [B_m\,, 0] + B^e \tag{4.A.28}$$

where

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_f & 0 \end{bmatrix} \quad \text{and} \quad B^e = \begin{bmatrix} -K^{a+} & 0 \\ 0 & 0 \\ 0 & 0 \\ -K^c & B_c \\ 0 & 0 \end{bmatrix}. \tag{4.A.29}$$

Then we have

$$\begin{aligned} M_f(s,\sigma) &= F[sI_n - A + K_f(\sigma)C]^{-1} B \\ &= F[sI_n - A + K_f(\sigma)C]^{-1}([B_m\,, 0] + B^e) \\ &\to F[sI_n - A + K_f(\sigma)C]^{-1} B^e \end{aligned} \tag{4.A.30}$$

as $\sigma \to \infty$. We will next show that $M_f(s,\sigma) \to \overline{M}_f^e(s)$ as $\sigma \to \infty$. To simplify the notation, we reorder some variables and rewrite $A - K_f(\sigma)C$ as in (4.A.1). We note that

$$\tilde{K}^e(\sigma)\Gamma^{-1}(\sigma) = K^e T \tag{4.A.31}$$

and

$$\tilde{K}_f(\sigma)\Gamma^{-1}(\sigma) \to B_f T \tag{4.A.32}$$

as $\sigma \to \infty$. Let $\lambda^{ei}(\sigma)$ and $W^{ei}(\sigma)$ respectively be an eigenvalue and eigenvector of $A - K_f(\sigma)C$ represented in $M^e(s,\sigma)$. Let us partition $W^{ei}(\sigma)$ as

$$W^{ei}(\sigma) = [(W^{e0i})^H(\sigma)\,, (W^{eei})^H(\sigma)\,, (W^{e\infty i})^H(\sigma)]^H. \tag{4.A.33}$$

It is then easy to show that as $\sigma \to \infty$,

$$\lambda^{ei}(\sigma) \to \overline{\lambda}^{ei} \in \overline{\Lambda}^{e},$$

$$W^{e0i}(\sigma) \to 0 \ , \ W^{eei}(\sigma) \to \overline{W}^{ei} \ , \ W^{e\infty i}(\sigma) \to C_f'\Gamma^{-1}(\sigma)T^{-1}C^e\overline{W}^{eei} \to 0 \qquad (4.A.34)$$

where $\overline{\lambda}^{ei}$ and $\overline{W}^{eei}$ are respectively an eigenvalue and eigenvector of A^{eec}. Now in view of the fact

$$[\overline{V}^0, \overline{V}^e, \overline{V}^\infty]^H[\overline{W}^0, \overline{W}^e, \overline{W}^\infty] = I_n,$$

we note that $\overline{V}^0$ and $\overline{V}^\infty$ are of the form,

$$\overline{V}^0 = [*, 0, *]' \text{ and } \overline{V}^\infty = [*, 0, *]',$$

where $*$ denotes some finite value not necessarily zero. Hence,

$$\overline{V}^0 B^e = 0 \quad \text{and} \quad \overline{V}^\infty B^e = 0. \qquad (4.A.35)$$

Thus we can rewrite (4.A.30) as

$$\begin{aligned} M_f(s,\sigma) &\to F[sI_n - A + K_f(\sigma)C)^{-1}B^e \\ &= \sum_{i=1}^{n} \frac{FW_i(\sigma)V_i^H(\sigma)B^e}{s-\lambda_i} \\ &\to \sum_{i=1}^{n_e} \frac{F\overline{W}^{ei}(\overline{V}^{ei})^H B^e}{s-\overline{\lambda}^{ei}} \\ &= \overline{M}_f^e(s). \end{aligned} \qquad (4.A.36)$$

Next by partitioning F as

$$F = [F_0, F_e, F_\infty],$$

and letting

$$B^{ee} = \begin{bmatrix} -K^{a+} & 0 \\ -K^c & B_c \end{bmatrix},$$

we note that

$$M_f(s,\sigma) \to \overline{M}_f^e(s) = \sum_{i=1}^{n_e} \frac{F_e\overline{W}^{eei}(\overline{V}^{eei})^H B^{ee}}{s-\overline{\lambda}^{ei}} = F_e(sI_{n_e} - A^{eec})^{-1}B^{ee} \qquad (4.A.37)$$

where

$$[\overline{V}^{ee1}, \overline{V}^{ee2}, \cdots, \overline{V}^{een_e}] = [\overline{W}^{ee1}, \overline{W}^{ee2}, \cdots, \overline{W}^{een_e}]^{-H}.$$

This completes the LTR analysis of ATEA algorithm for **Case 1.**

Case 2 : The given system Σ is non-strictly proper.

Without loss of generality, we will assume that the given system is in the form of s.c.b. Then by renaming the variables $x_0 = [(x_a^-)', x_b']'$ and $x_e = [(x_a^+)', x_c']'$, we can rewrite the observer dynamic matrix $A - K_f(\sigma)C$ as,

$$A - K_f(\sigma)C = \begin{bmatrix} A^{00} & 0 & -\tilde{H}_{0f}C_f \\ A^{e0} & A^{ee1} & -[\tilde{H}_{ef} + \tilde{K}^{e1}(\sigma)]C_f \\ B_fE^0 & B_fC^{e1} & A_f - \tilde{K}_f(\sigma)C_f - L_fC_f \end{bmatrix}, \quad (4.A.38)$$

where

$$A^{00} = \begin{bmatrix} A_{aa}^- & -\tilde{H}_{ab}^-C_b \\ 0 & A_{bb} - K^bC_b \end{bmatrix}, \quad \tilde{H}_{0f} = \begin{bmatrix} \tilde{H}_{af}^- \\ \tilde{H}_{bf} \end{bmatrix}, \quad E^0 = \begin{bmatrix} E_a^- & E_b \end{bmatrix},$$

$$A^{e0} = \begin{bmatrix} -K^{a0+}C_{0a}^- & -\tilde{H}_{ab}^+C_b - K^{a0+}C_{0b} \\ B_cE_{ca}^- - K_{a0}^-C_{0a}^- & -\tilde{H}_{cb}C_b - K^{c0}C_{0b} \end{bmatrix},$$

$$A^{ee1} = A^{ee} - K^{e0}C^{e0} = \begin{bmatrix} A_{aa}^+ - K^{a0+}C_{0a}^+ & -K^{a0+}C_{0c} \\ B_cE_{ca}^+ - K^{c0}C_{0a}^+ & A_{cc} - K^{c0}C_{0c} \end{bmatrix},$$

$$\tilde{H}_{ef} = \begin{bmatrix} K^{a0+}C_{0f} + \tilde{H}_{af}^+C_f \\ K^{c0}C_{0f} + \tilde{H}_{cf}C_f \end{bmatrix}.$$

The fact that the ATEA algorithm yields an admissible observer gain $K_f(\sigma)$ in the sense that $A - K_f(\sigma)C$ is a stable matrix for sufficiently large σ and that it has the required time-scale structure follows along the lines as in **Case 1.** In what follows, we will show that $K_f(\sigma)$ achieves LTR in the sense that

$$M_f(s,\sigma) = F[sI_n - A + K_f(\sigma)C]^{-1}[B - K(\sigma)D] \to \overline{M}_f^e(s) \text{ pointwise in } s \quad (4.A.39)$$

as $\sigma \to \infty$. In view of (4.3.12), let us partition $K_f(\sigma)$ as,

$$K_f(\sigma) = \tilde{K}_0 + [0 \quad \overline{K}(\sigma)]$$

$$= \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^+ + K^{a0+} & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} + K^{c0} & 0 & 0 \\ B_{0f} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & L_{af}^- + \tilde{H}_{af}^- & L_{ab}^- + \tilde{H}_{ab}^- \\ 0 & L_{af}^+ + \tilde{H}_{af}^+ + \tilde{K}^{a1+}(\sigma) & L_{ab}^+ + \tilde{H}_{ab}^+ \\ 0 & L_{bf} + \tilde{H}_{bf} & K^b \\ 0 & L_{cf} + \tilde{H}_{cf} + \tilde{K}^{c1}(\sigma) & L_{cb} + \tilde{H}_{cb} \\ 0 & L_f + \tilde{K}_f(\sigma) & 0 \end{bmatrix} \quad (4.A.40)$$

Then we have,

$$\overline{B} = B - K(\sigma)D = B - \tilde{K}_0 D = \begin{bmatrix} 0 & 0 & 0 \\ -K^{a0+} & 0 & 0 \\ 0 & 0 & 0 \\ -K^{c0} & 0 & B_c \\ 0 & B_f & 0 \end{bmatrix},$$

and

$$\overline{A} = A - \tilde{K}_0 C$$

$$= \begin{bmatrix} A_{aa}^- & 0 & L_{ab}^- C_b & 0 & L_{af}^- C_f \\ -K^{a0+} C_{0a}^- & A_{aa}^+ - K^{a0+} C_{0a}^+ & L_{ab}^+ C_b - K^{a0+} C_{0b} & -K^{a0+} C_{0c} & L_{af}^+ C_{0f} \\ 0 & 0 & A_{bb} & 0 & L_{bf} C_f \\ B_c E_{ca}^- - K^{c0} C_{0a}^- & B_c E_{ca}^+ - K^{c0} C_{0a}^+ & L_{cb} C_b - K^{c0} C_{0b} & A_{cc} - K^{c0} C_{0c} & L_{cf} C_f - K^{c0} C_{0f} \\ B_f E_a^- & B_f E_a^+ & B_f E_b & B_f E_c & A_f \end{bmatrix}$$

$$\overline{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix}.$$

With these definitions, we can write $M_f(s,\sigma)$ as

$$M_f(s,\sigma) = F\left[sI_n - \overline{A} + \overline{K}(\sigma)\overline{C}\right]^{-1}\overline{B}.$$

Then in view of (4.A.40), it can be seen easily that $\overline{K}(\sigma)$ has the form,

$$\overline{K}(\sigma) = T(\sigma)\Gamma(\sigma)N + Q,$$

where

$$\Gamma(\sigma) = \mathrm{Diag}\left[\frac{1}{\eta_1}, \frac{1}{\eta_2}, \cdots, \frac{1}{\eta_{m_f}}\right], \quad N = [I_{m_f}, 0],$$

and

$$Q = \begin{bmatrix} L_{af}^- + \tilde{H}_{af}^- & L_{ab}^- + \tilde{H}_{ab}^- \\ L_{af}^+ + \tilde{H}_{af}^+ & L_{ab}^+ + \tilde{H}_{ab}^+ \\ L_{bf} + \tilde{H}_{bf} & K^b \\ L_{cf} + \tilde{H}_{cf} & L_{cb} + \tilde{H}_{cb} \\ L_f & 0 \end{bmatrix},$$

while $T(\sigma)$ satisfies

$$T(\sigma) \to B_m T$$

as $\sigma \to \infty$ where

$$B_m = \begin{bmatrix} 0 \\ K^{a1+} \\ 0 \\ K^{c1} \\ B_f \end{bmatrix}, \quad T = \mathrm{Diag}\left[J_{1r_1}K_{1r_1d},\, J_{2r_2}K_{2r_2d},\, \cdots,\, J_{m_f r_{m_f}}K_{m_f r_{m_f} d}\right].$$

It is shown in [7] tha. the triple $(\overline{C}, \overline{A}, B_m)$ forms a left invertible and a minimum phase system. Thus, it follows from the results of [43] that

$$[sI_n - A + K_f(\sigma)C]^{-1}B_m \to 0 \text{ pointwise in } s$$

as $\sigma \to \infty$. Next let

$$\overline{B} = [0\,, B_m\,, 0] + B^e$$

where

$$B^e = \begin{bmatrix} 0 & 0 & 0 \\ -K^{a0+} & -K^{a1+} & 0 \\ 0 & 0 & 0 \\ -K^{c0} & -K^{c1} & B_c \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we have

$$M_f(s,\sigma) \to F[sI_n - \overline{A} + \overline{K}(\sigma)\overline{C}]^{-1}B^e$$

as $\sigma \to \infty$. Let us now partition F as

$$F = [\,F_0 \quad F_e \quad F_\infty\,]$$

and define

$$B^{ee} = \begin{bmatrix} -K^{a0+} & -K^{a1+} & 0 \\ -K^{c0} & -K^{c1} & B_c \end{bmatrix}.$$

It follows from the results of **Case 1** that

$$M_f(s,\sigma) \to \overline{M}^e_f(s) = \sum_{i=1}^{n_e} \frac{F_e \overline{W}^{eei}(\overline{V}^{eei})^H B^{ee}}{s - \overline{\lambda}^{ei}} = F_e(sI_{n_e} - A^{ee} + K^e C^e)^{-1}B^{ee}.$$

where

$$\left[\overline{V}^{ee1}, \overline{V}^{ee2}, \cdots, \overline{V}^{een_e}\right] = \left[\overline{W}^{ee1}, \overline{W}^{ee2}, \cdots, \overline{W}^{een_e}\right]^{-H}.$$

This completes the proof of theorem 4.3.1. ∎

4.B. Appendix 4.B — Proof of Theorem 4.4.1

Let $\epsilon = 1/\sqrt{\sigma}$, and define the following perturbed system,

$$\Sigma_{a\epsilon} : \begin{cases} \dot{x} = A'x + C'u + F'w, \\ z = B_\epsilon' x + D_\epsilon' u, \end{cases} \tag{4.B.1}$$

where

$$B_\epsilon = [B_0,\ B_1,\ \epsilon I_n,\ 0], \quad \text{and} \quad D_\epsilon = \begin{bmatrix} I_{m_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon I_{m-m_0} \end{bmatrix}.$$

Consider the state feedback law,

$$u = -K_f'(\sigma)x \tag{4.B.2}$$

with gain $K_f'(\sigma)$ defined by,

$$K_f'(\sigma) = (D_\epsilon D_\epsilon')^{-1}(PC + D_\epsilon B_\epsilon'), \tag{4.B.3}$$

where P is the positive definite solution of

$$AP + PA' + B_\epsilon B_\epsilon' - (PC' + B_\epsilon D_\epsilon')(D_\epsilon D_\epsilon')^{-1}(CP + D_\epsilon B_\epsilon') = 0. \tag{4.B.4}$$

We note that D_ϵ' is injective. Then, it is shown in Stoorvogel [55] that the state feedback law (4.B.2), minimizes the H_2 norm of the transfer function from w to z, namely $T_{zw}(s,\sigma)$, as $\sigma \to \infty$ (or $\epsilon \to 0$). The proof of the first part of theorem 4.4.1 follows now by recognizing that (4.B.3) and (4.B.4) are respectively equivalent to (4.4.4) and (4.4.3). The rest of theorem 4.4.1 follows trivially from Stoorvogel [55]. ∎

Chapter 5

A STABLE COMPENSATOR DESIGN FOR ALTR

5.1. Introduction

In this chapter, a new compensator structure is proposed for ALTR for general nonminimum phase non-strictly proper systems. The proposed compensator is (a) open-loop stable, (b) guarantees closed-loop stability and above all (c) requires much smaller values of gain than the conventional observer based controller for the same degree of loop-transfer recovery. The fact that the new compensator requires much smaller values of gain than the conventional controller results in several practical advantages, the most important among them being the reduction in controller band-width and freedom from the owes of control saturation. Trade-off between the value of gain and the degree of loop transfer recovery as well as the bounds on singular values of sensitivity and complementary sensitivity functions is presented in this chapter.

Our central observation in this chapter is this. When one is restricted to the frame work of observer theory, the link from the control signal u to the observer via the control distribution matrix B is always present in the design configuration such as the one depicted in figure 2.2.1. In these observers, when $K_f(\sigma)$ and $K_r(\sigma)$ are designed to achieve ALTR, the effect of the above control-link on the output of observer based controller (namely $\hat{u}$) vanishes asymptotically as $\sigma \to \infty$ for the case when ALTR is achievable. However,

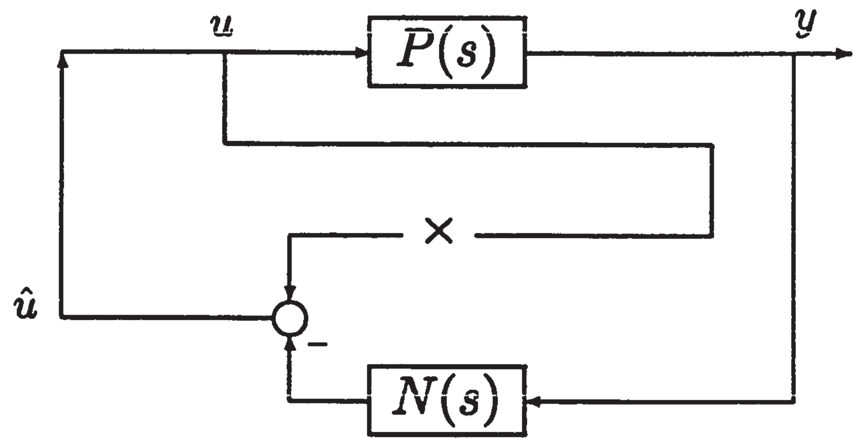


Figure 5.1.1: Plant and compensator configuration.

effect of the above link on $\hat{x}$ is in general nonzero and hence the need for the above link in the conventional observers. Based on this observation, we are inspired to remove the above mentioned link structurally right from the beginning of the design. In other words, to develop the appropriate full and reduced order compensators, we consider the configuration illustrated in figure 5.1.1, where $N(s)$ is exactly the same as the one in configuration 2.2.1. Once the link from the control input to the controller or what is now called a compensator, is removed we embark on a new design philosophy which is outside the realm of observer theory and hence the separation principle is no longer valid. Without the backing or blessing of the separation principle, one has to prove that the design objectives of closed-loop stability and recovering the target loop shape can both be simultaneously achieved. We intend to do exactly this.

Our design philosophy is deceptively very simple. Except for structurally omitting the link mentioned earlier, our compensators are exactly the same as the conventional observer based controllers. For example, in full order compensator design for LTR, we plan to obtain a $K_f(\sigma)$ such that (a) $A - K_f(\sigma)C$ has all its eigenvalues in the left half s plane (i.e., the compensator is open-loop stable), and (b) achieves asymptotic loop transfer recovery (ALTR). For this purpose, we can use any of the existing methods of obtaining such a $K_f(\sigma)$. Thus, our compensator design is parallel in all respects to the conventional observer design except for omitting the link mentioned earlier. Although our compensator structurally differs from the observer in a very simple way, it has a profound effect on the gain required for closed-loop stability and for ALTR. We show theoretically that for the same gain, the difference between the target loop transfer function and the one achieved by our compensator is always much smaller than that that can be achieved by the observer based controller. But since our design method is also an asymptotic method, the above theoretical result does not reveal the whole story. The proof that our method works is evident from our examples. We solved numerically many examples that appeared in the open literature, and noticed that the amount of gain required for the same degree of recovery by our compensator is orders of magnitude less than what is required by an observer based controller. This obviously has a profound impact on the practical implementation of LQG/LTR schemes. Some specific attributes of our compensator are as follows:

1. Low values of gain obviously results in low compensator band-width and hence much of the output noise that occurs at relatively high frequencies is filtered out. Furthermore, low values of gain relieves the design from ever present owes of actuator saturation. To emphasize this, we refer to Sogaard-Andersen and Niemann [50] who recently studied the design trade-offs between the level of loop transfer recovery and the necessary gain required by an observer based controller. A major conclusion of their study is that the target loop transfer recovery design cannot always be achieved even when modest and practically meaningful constraints are imposed on the size of the observer gain. Furthermore, contrary to what has been discussed in the literature (e.g., Friedland [20]; Baumgartner et al [3]), their study indicates that a high-gain from controller input to controller output affects the entire control-loop and in particular the control-noise signal ratio and the control-command signal ratio.

2. Since the given plant is of minimum phase, it is always possible to design an open-loop

stable compensator to guarantee the overall closed-loop stability (Vidyasagar [57]). Our design results in an open-loop stable compensator. The advantages of having such a compensator cannot be over emphasized. As it is known (Shaw [48]), open-loop unstable compensators result in poor overall system sensitivity to plant parameter variations. Furthermore, physical realizability of open-loop unstable compensators is rather difficult.

5.2. Compensator structure

Analogous to observer based controllers, there are two compensator structures, one full order type of dynamic order n and another reduced order type of dynamic order $n-p+m_0$. We now proceed to give the structural details of these compensators.

5.2.1. Full order stable compensator

The dynamic equations of the full order stable compensator are

$$\begin{cases} \dot{z} = [A - K_f(\sigma)C]z + K_f(\sigma)y, \\ u = \hat{u} = -Fz. \end{cases} \tag{5.2.1}$$

The transfer function of the compensator is,

$$\mathbf{C}_c(s,\sigma) = F[sI - A + K_f(\sigma)C]^{-1}K_f(\sigma). \tag{5.2.2}$$

We note that the above controller does not depend on the matrix D of the given plant. In the parameterized family of controllers given in (5.2.2), the only free design variable is the parameterized gain $K_f(\sigma)$. We need to parameterize $K_f(\sigma)$ in such a way that there exists a σ_1^* so that for all $\sigma > \sigma_1^*$, the controller $\mathbf{C}_c(s,\sigma)$ is open-loop stable while capable of achieving ALTR. That is, the design of $K_f(\sigma)$ is to be done to meet the following goals:

1. [**Stability of the closed-loop system**] The closed-loop system comprising the given system Σ and the full order compensator is asymptotically stable, i.e., there

exists a σ_2^* such that for all $\sigma > \sigma_2^*$, we have

$$\mathrm{Re}\,[\lambda(A_{cl}(\sigma))] < 0,$$

where

$$A_{cl}(\sigma) = \begin{bmatrix} A - K_f(\sigma)C - K_f(\sigma)DF & K_f(\sigma)C \\ -BF & A \end{bmatrix}. \tag{5.2.3}$$

Moreover, the limits of all finite eigenvalues of $A_{cl}(\sigma)$ remain in C^-.

2. [**Loop transfer recovery**] The achieved loop transfer function $L_c(j\omega, \sigma)$,

$$L_c(j\omega,\sigma) = \mathbf{C}_c(j\omega,\sigma)P(j\omega),$$

is asymptotically equal to the target loop $L_t(j\omega)$ as $\sigma \to \infty$, i.e., $\mathbf{C}(j\omega,\sigma)P(j\omega) \to L_t(j\omega)$ pointwise in ω as $\sigma \to \infty$.

3. [**Open-loop stability of the compensator**] The compensator is open-loop asymptotically stable, i.e., there exists a σ_1^* such that for all $\sigma > \sigma_1^*$, we have

$$\mathrm{Re}\,[\lambda(A - K_f(\sigma)C)] < 0.$$

5.2.2. Reduced order stable compensator

As in the case of reduced order observer based controller, without loss of generality, let us assume that matrices C and D are transformed into the form,

$$C = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} D_0 \\ 0 \end{bmatrix}.$$

and partition the dynamic equations of the given plant Σ as in (2.1.3). Then the dynamic equations of a reduced order compensator are given by

$$\begin{cases} \dot{v} = [A_r - K_r(\sigma)C_r]v + G_r(\sigma)y, \\ u = \hat{u} = -F_2 v - [0,\ F_1 + F_2 K_{r1}(\sigma)]y. \end{cases} \tag{5.2.4}$$

Here we note that all the submatrices in (5.2.4) are the same as those defined in (2.1.3) to (2.1.11). The transfer function of such a reduced order compensator is given by,

$$\mathbf{C}_{rc}(s,\sigma) = F_2[sI - A_r + K_r(\sigma)C_r]^{-1}G_r(\sigma) + [0,\ F_1 + F_2K_{r1}(\sigma)]. \tag{5.2.5}$$

In the parameterized family of controllers given in (5.2.5), the only free design variable is the parameterized gain $K_r(\sigma)$. We need to parameterize $K_r(\sigma)$ in such a way that there exists a σ_{1r}^* so that for all $\sigma > \sigma_{1r}^*$, the controller $\mathbf{C}_{rc}(s,\sigma)$ is open-loop stable while capable of achieving ALTR. That is, the design of $K_r(\sigma)$ is to be done to meet the following goals:

1. [**Stability of the closed-loop system**] The closed-loop system comprising the given system Σ and the full order compensator is asymptotically stable, i.e., there exists a σ_{2r}^* such that for all $\sigma > \sigma_{2r}^*$, we have

$$\mathrm{Re}\,[\lambda(A_{clr}(\sigma))] < 0,$$

where

$$A_{clr}(\sigma) = \begin{bmatrix} A_r - K_r(\sigma)C_r - K_r(\sigma)D_rF_2 & A_{21} - K_r(\sigma)D_rF_1 & K_r(\sigma)C_r \\ -B_{11}F_2 & A_{11} - B_{11}F_1 & A_{12} \\ -B_{22}F_2 & A_{21} - B_{22}F_1 & A_{22} \end{bmatrix}.$$

Moreover, the limits of all finite eigenvalues of $A_{clr}(\sigma)$ remain in $\mathcal{C}^-$.

2. [**Loop transfer recovery**] The achieved loop transfer function $L_{rc}(j\omega,\sigma)$,

$$L_{rc}(j\omega,\sigma) = \mathbf{C}_{rc}(j\omega,\sigma)P(j\omega),$$

is asymptotically equal to the target loop $L_t(j\omega)$ as $\sigma \to \infty$, i.e., $\mathbf{C}_{rc}(j\omega,\sigma)P(j\omega) \to L_t(j\omega)$ pointwise in ω as $\sigma \to \infty$.

3. [**Open-loop stability of the compensator**] The compensator is open-loop asymptotically stable, i.e., there exists a σ_{1r}^* such that for all $\sigma > \sigma_{1r}^*$, we have

$$\mathrm{Re}\,[\lambda(A_r - K_r(\sigma)C_r)] < 0.$$

5.3. Properties of compensators

In this section, we will show the advantages of new compensator structure over the conventional observer based controller. The following lemma characterizes the recovery error between the target and the achieved loop transfer functions.

Lemma 5.3.1. *The error $E_c(s,\sigma)$ between the target loop transfer function $L_t(s)$ and $L_c(s,\sigma)$, the one realized by a full order compensator, is given by*

$$E_c(s,\sigma) = M_f(s,\sigma),$$

where

$$M_f(s,\sigma) = F[\Phi^{-1} + K_f(\sigma)C]^{-1}[B - K_f(\sigma)D]. \tag{5.3.1}$$

Similarly, the error $E_{rc}(s,\sigma)$ between the target loop transfer function $L_t(s)$ and $L_{rc}(s,\sigma)$, the one realized by a reduced order compensator, is given by

$$E_{rc}(s,\sigma) = M_r(s,\sigma),$$

where

$$M_r(s,\sigma) = F_2[\Phi_r^{-1} + K_r(\sigma)C_r]^{-1}[B_r - K_r(\sigma)D_r], \tag{5.3.2}$$

and where $\Phi_r = (sI - A_r)^{-1}$.

Proof : See Appendix 5.A. ∎

Theorem 5.3.1. *Consider a stabilizable and detectable system Σ characterized by the matrix quadruple (A,B,C,D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let $L_t(s)$ be any recoverable target loop transfer function of Σ, i.e., $L_t(s) \in \mathbf{T}^{\mathrm{R}}(\Sigma)$, then $L_t(s)$ can be recovered via either a full or a reduced order type of compensator.*

Proof : See Appendix 5.B. ∎

We now pursue the advantages of the compensator structure over the conventional observer based structure. We have the following theorem.

Theorem 5.3.2. *Consider a stabilizable and detectable nonminimum phase plant. Assume that the same gain $K_f(\sigma)$ is used for both a full order observer based controller and a full order compensator. Let σ be such that $\sigma_{max}[M_f(j\omega,\sigma)]$ is small (say, $\ll 1$ but nonzero) for all ω. Furthermore, assume that*

$$\sigma_{min}[L_t(j\omega)] = \sigma_{min}[F(j\omega - A)^{-1}B] \gg 1 \text{ for all } \omega \in D_c, \tag{5.3.3}$$

for some frequency region of interest, D_c. Then for all $\omega \in D_c$, the mismatch between the target loop transfer function and the one achieved by the full order compensator is always less than the corresponding one achieved by the full order observer based controller. More specifically, we have

$$\sigma_{max}[E_f(j\omega,\sigma)] \gg \sigma_{max}[E_c(j\omega,\sigma)] \text{ for all } \omega \in D_c. \tag{5.3.4}$$

Similarly, assume that the same gain $K_r(\sigma)$ is used for both the reduced order observer based controller and the reduced order compensator. Let σ be such that $\sigma_{max}[M_r(j\omega,\sigma)]$ is small (say, $\ll 1$ but nonzero) for all ω. Furthermore, assume that (5.3.3) is true. Then for all $\omega \in D_c$, the mismatch between the target loop transfer function and the one achieved by the reduced order compensator is always less than the corresponding one achieved by the reduced order observer based controller. More specifically, we have

$$\sigma_{max}[E_r(j\omega,\sigma)] \gg \sigma_{max}[E_{rc}(j\omega,\sigma)] \text{ for all } \omega \in D_c. \tag{5.3.5}$$

Proof : Let us first consider the case of full order compensator and full order observer based controller. Recalling the expression for $E_f(j\omega,\sigma)$ from (2.2.1), we have

$$\begin{aligned}\sigma_{max}[E_f(j\omega,\sigma)] &= \sigma_{max}\{M_f(j\omega,\sigma)[I_m + M_f(j\omega,\sigma)]^{-1}[I_m + F\Phi(j\omega)B]\} \\ &\geq \sigma_{max}[M_f(j\omega,\sigma)]\sigma_{min}\{[I_m + M_f(j\omega,\sigma)]^{-1}\}\sigma_{min}[I_m + F\Phi(j\omega)B]\end{aligned}$$

$$= \frac{\sigma_{max}[M_f(j\omega,\sigma)]\sigma_{min}[I_m + F\Phi(j\omega)B]}{\sigma_{max}[I_m + M_f(j\omega,\sigma)]}$$

$$\geq \sigma_{max}[E_c(j\omega,\sigma)]\alpha(\omega,\sigma),$$

where

$$\alpha(\omega,\sigma) = \frac{\sigma_{min}[F\Phi(j\omega)B] - 1}{1 + \sigma_{max}[M_f(j\omega,\sigma)]}.$$

Now by our assumption, $\sigma_{max}[M_f(j\omega,\sigma)]$ is $\ll 1$ and $\sigma_{min}[F\Phi(j\omega)B]$ is $\gg 1$ for all $\omega \in D_c$ and hence $\alpha(\omega,\sigma)$ is $\gg 1$ for all $\omega \in D_c$. Thus

$$\sigma_{max}[E_f(j\omega,\sigma)] \gg \sigma_{max}[E_c(j\omega,\sigma)] \text{ for all } \omega \in D_c.$$

Results for the reduced order compensator and the reduced order observer based controller follow from similar arguments. ∎

Remark 5.3.1. *It is well known (Doyle and Stein [19]) that in order to have good properties in command following and disturbance rejection, the target loop transfer function $L_t(j\omega)$ has to be large and consequently, the minimum singular value $\sigma_{min}[L_t(j\omega)]$ should be large in the desired frequency region. Thus, the condition (5.3.3) is always satisfied in all practical situations.*

Remark 5.3.2. *Due to the sign $\gg$ in (5.3.4) and (5.3.5), theorem 5.3.2 clearly shows that the compensator structure requires much smaller value of gain and hence the controller bandwidth than that of the observer based structure for the same degree of recovery.*

Next we compare the sensitivity and complementary sensitivity functions achievable by a full and a reduced order compensators, with those achievable by a full and a reduced order observer based controllers. Let us define the sensitivity and complementary sensitivity functions achievable by a particular compensator as,

$$S_*(s,\sigma) = [I_m + L_*(s,\sigma)]^{-1} \quad \text{and} \quad T_*(s,\sigma) = I_m - S_*(s,\sigma)$$

where $L_*(s,\sigma)$ is the correspondingly achieved loop transfer function. Here, the subscript $*$ will be replaced respectively by c or rc, when the controller used is a full order compensator or a reduced order compensator.

We have the following result.

Theorem 5.3.3. *Consider a general stabilizable and detectable nonminimum phase plant. Assume that the same gain $K_f(\sigma)$ is used for both the full order observer based controller and the full order compensator. Let σ be such that $\sigma_{max}[M_f(j\omega,\sigma)]$ is small (say, $\ll 1$ but nonzero) for all ω. Furthermore, assume that (5.3.3) is true. Then for all $\omega \in D_c$, we have*

$$\sigma_{max}[S_f(j\omega,\sigma) - S_t(j\omega)] \gg \sigma_{max}[S_c(j\omega,\sigma) - S_t(j\omega)] \tag{5.3.6}$$

and

$$\sigma_{max}[T_f(j\omega,\sigma) - T_t(j\omega)] \gg \sigma_{max}[T_c(j\omega,\sigma) - T_t(j\omega)]. \tag{5.3.7}$$

Similarly, assume that the same gain $K_r(\sigma)$ is used for both the reduced order observer based controller and the reduced order compensator. Let σ be such that $\sigma_{max}[M_r(j\omega,\sigma)]$ is small (say, $\ll 1$ but nonzero) for all ω. Furthermore, assume that (5.3.3) is true. Then for all $\omega \in D_c$, we have

$$\sigma_{max}[S_r(j\omega,\sigma) - S_t(j\omega)] \gg \sigma_{max}[S_{rc}(j\omega,\sigma) - S_t(j\omega)] \tag{5.3.8}$$

and

$$\sigma_{max}[T_r(j\omega,\sigma) - T_t(j\omega)] \gg \sigma_{max}[T_{rc}(j\omega,\sigma) - T_t(j\omega)]. \tag{5.3.9}$$

Proof : Rewriting equation (3.2.13), we have

$$S_f(s,\sigma) - S_t(s) = S_t(s)M_f(s,\sigma). \tag{5.3.10}$$

We also note that

$$\begin{aligned} I_m + L_c(s,\sigma) &= I_m + L_t(s) - M_f(s,\sigma) \\ &= \{I_m - M_f(s,\sigma)[I_m + L_t(s)]^{-1}\}[I_m + L_t(s)] \end{aligned}$$

and hence

$$S_c(s,\sigma) - S_t(s) = S_t(s)M_f(s,\sigma)[I_m + L_t(s) - M_f(s,\sigma)]^{-1}. \qquad (5.3.11)$$

From (5.3.10) and (5.3.11), we obtain

$$S_f(s,\sigma) - S_t(s) = [S_c(s,\sigma) - S_t(s)][I_m + L_t(s) - M_f(s,\sigma)].$$

Now it is simple to see that under the assumptions of theorem 5.3.3,

$$\sigma_{max}[S_f(j\omega,\sigma) - S_t(j\omega)] \gg \sigma_{max}[S_c(j\omega,\sigma) - S_t(j\omega)], \quad \forall\omega \in D_c.$$

This proves (5.3.6). Also, (5.3.7) to (5.3.9) follow along similar arguments. ∎

The above theorem shows once again that the compensator structure is much better than the conventional observer based structure.

5.4. Examples

In what follows we consider four examples, including the helicopter control problem discussed in subsections 4.3.2, 4.3.4 and 4.4.2, to illustrate the theoretical results of section 5.3. We illustrate the advantages of the compensator structure in two different ways. At first, we select the same gain $K_f(\sigma)$ or $K_r(\sigma)$ for both the observer based structure and the compensator structure, and then for several values of σ, we compare the performance of these two controller structures by plotting with respect to frequency for a given frequency range, (i) the target and achieved loop transfer functions and (ii) the maximum singular value of the loop transfer recovery errors, $E_c(j\omega,\sigma)$ and $E_f(j\omega,\sigma)$. In another type of comparison, we fix a prior the required degree of recovery by specifying a highest tolerable value for the maximum singular value of the loop transfer recovery error. Then, we obtain for both the controller structures the norm of the gain which meets the given specification. We also obtain the resulting 0-db band-width as well as the eigenvalues of the controller. The comparisons by both the methods show explicitly that the compensator structure for the controller has much better recovery properties than the observer based structure.

Example 5.1 : Consider a system Σ characterized by

$$\dot{x} = \begin{bmatrix} -10 & 0 & -20 \\ 0 & 1 & -20 \\ 2.45 & 1.25 & -20 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = [0 \quad 0 \quad 1] x + 0 \cdot u,$$

which is a single-input and a single-output nonminimum phase system with invariant zeros at $s = -10$ and $s = 1$. The geometric subspace $\mathcal{V}^+(\Sigma)$ for this example is the span of $[0 \quad 1 \quad 0]'$. Let a target loop, $L_t(s)$, be specified by the gain matrix,

$$F = [12 \quad 0 \quad 8].$$

Then it is simple to verify that $L_t(s) \in \mathbf{T}^{\mathrm{R}}(\Sigma)$, i.e., $L_t(s)$ is recoverable. Now let us use H_2-optimization based algorithm of Chapter 4 to obtain $K_f(\sigma)$. Figure E.1 (A) and table E.1 (A) give the magnitude of the target loop transfer function as well as that of the two recovered loop transfer functions, one for the full order compensator and another for the full order observer based controller, for several values of the tuning parameter σ. On the other hand, for the same degree of recovery, figure E.1 (B) and table E.1 (B) show (1) the maximum singular value graphs of the two different controller transfer functions, and (2) the required values of gains and the eigenvalues of the controllers. These numerical results show clearly that the compensator structure has much better recovery properties than the observer based controller. □

Example 5.2 : Consider the example given in [58],

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.2 \end{bmatrix} x + \begin{bmatrix} -0.5 & 1.25 \\ -2.5 & -2.5 \\ 0.3 & -1.25 \\ 1.5 & 3.5 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u,$$

which is square, invertible and of nonminimum phase with two invariant zeros at $s = -8$ and $s = 1$. The geometric subspace $\mathcal{V}^+(\Sigma)$ for this example is the span of

$$[-0.138675 \quad -0.693375 \quad 0.138675 \quad 0.693375]'.$$

Now let a target loop, $L_t(s)$, be specified by the gain matrix,

$$F = \begin{bmatrix} -16.8910 & 0.5782 & -19.1586 & 1.0317 \\ -290.0338 & 7.0068 & -295.0560 & 8.0112 \end{bmatrix}.$$

It is straightforward to verify that $L_t(s) \in \mathbf{T}^{\mathrm{R}}(\Sigma)$, i.e., $L_t(s)$ is recoverable. Here, we used ATEA algorithm of Chapter 4 to obtain the following gain $K_f(\sigma)$,

$$K_f(\sigma) = \begin{bmatrix} 3.75(1-\sigma) & 0.75\sigma - 1.25 \\ 25(\sigma - 1) & 2.5(1-2\sigma) \\ 0.95(5\sigma - 1) & 0.25(1-3\sigma) \\ 5(1-5\sigma) & 6\sigma - 0.7 \end{bmatrix}.$$

This gain places the eigenvalues of $A - K_f(\sigma)C$, one precisely at -8 and others asymptotically at -1, $-\sigma$ and $-\sigma$. Figure E.2 (A) and table E.2 (A) give the maximum and minimum singular values of the target loop transfer function as well as those of the two recovered loop transfer functions, one for the full order compensator and another for the full order observer based controller, for several values of the tuning parameter σ. On the other hand, for the same degree of recovery, figure E.2 (B) and table E.2 (B) show (1) the maximum singular value graphs of the two different controller transfer functions, and (2) the required values of gains and the eigenvalues of the controllers. Again, these numerical results show clearly that the compensator structure has much better recovery properties than the observer based controller. □

Example 5.3 : Consider the following system Σ characterized by

$$\dot{x} = \begin{bmatrix} -25 & -25 & 1 & -25 \\ 0 & 0 & 0 & 1 \\ -6 & 1 & 0.3 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u,$$

which is square and invertible with one nonminimum phase invariant zero at $s = 0.3$. The geometric subspace $\mathcal{V}^{+}(\Sigma)$ for this example is the span of $[0 \quad 0 \quad 1 \quad 0]'$. Now let a target loop, $L_t(s) = F\Phi B$, be specified by the following gain matrix,

$$F = \begin{bmatrix} -13 & 50 & 0 & 10 \\ 11 & 250 & 0 & 50 \end{bmatrix}.$$

It is trivial to see that $\mathcal{V}^+(\Sigma) \subseteq \text{Ker}\,(F)$ and hence $L_t(s) \in \mathbf{T}^{\text{R}}(\Sigma)$, i.e., $L_t(s)$ is recoverable. Here, we used ATEA algorithm of Chapter 4 to obtain the following gain matrix,

$$K_r(\sigma) = \begin{bmatrix} 2.3 & 0 \\ 0 & \sigma \end{bmatrix}$$

for both reduced order observer based controller and reduced order stable compensator. This gain matrix $K_r(\sigma)$ places the eigenvalues of $A_r - K_r(\sigma)C_r$ precisely at -2 and $-(\sigma+2)$. Figure E.3 (A) and table E.3 (A) give the maximum and minimum singular values of the target loop transfer function as well as those of the two recovered loop transfer functions, one for the reduced order compensator and another for the reduced order observer based controller, for several values of the tuning parameter σ. Once again, these numerical results show clearly that the compensator structure has much better recovery properties than the observer based controller. □

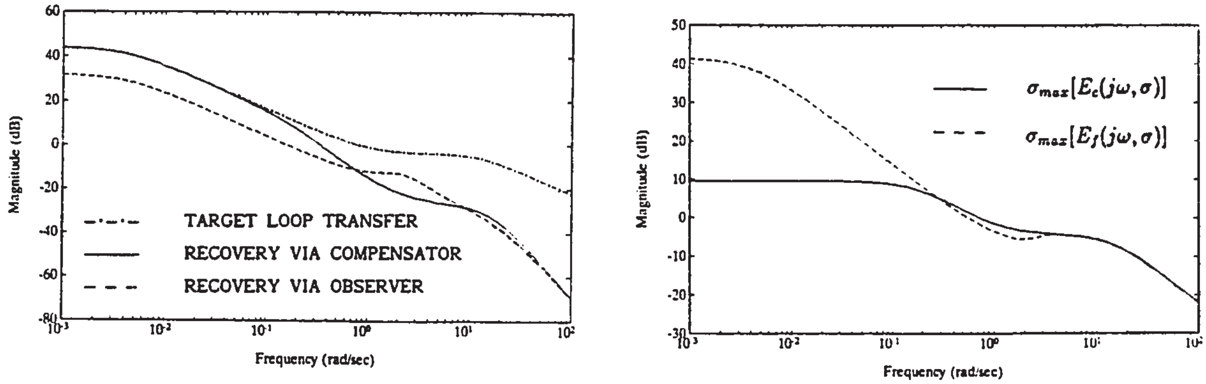
Example 5.4 : We continue here the helicopter control system design as discussed in subsections 4.3.2, 4.3.4 and 4.4.2 of Chapter 4. In what follows, we will apply our compensator structure to this problem and show that it also yields a much better recovery performance than the observer based controller, although the given target loop specified by F as given in subsection 4.3.2 is not recoverable. The justification of applying the compensator structure to LTR design for non-recoverable target loops remains a subject of our future investigation. We will pick one $K_f(\sigma)$ obtained from each design algorithm, namely, ATEA, H_2-optimization or H_∞-optimization based algorithm, and compare the performance of observer based and compensator structures by plotting the maximum singular values of $E_f(j\omega,\sigma)$ and $E_c(j\omega,\sigma)$. The results for ATEA algorithm are summarized in Tables E.5 (A1), E.5 (A2), and Figure E.5 (A), while the results for H_2-optimization based algorithm are given in Tables E.5 (B1), E.5 (B2), and Figure E.5 (B), and that for H_∞-optimization based algorithm are presented in Tables E.5 (C1), E.5 (C2), and Figure E.5 (C). □

TABLE E.1 (A)

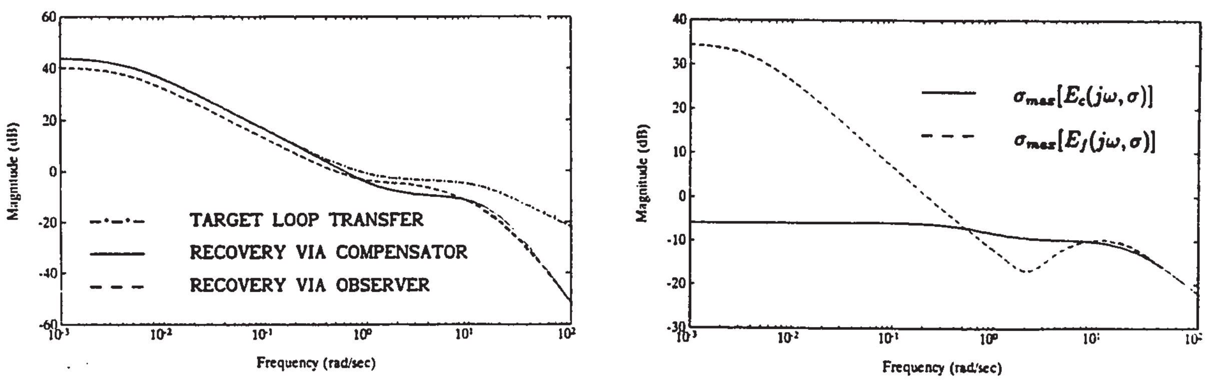
Supremum of Maximum Singular Values of Mismatch Functions

Frequncy Range: 0.001 to 100 rad/sec

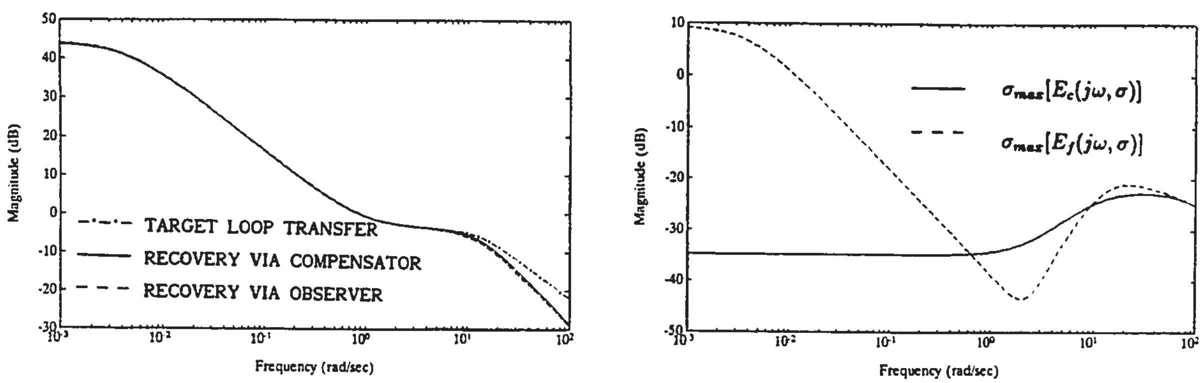
	TUNING PARAMETER	Sup $\{\sigma_{max}[E_f(j\omega,\sigma)]\}$	Sup $\{\sigma_{max}[E_c(j\omega,\sigma)]\}$
CASE 1	σ = 5	117.5629	3.0258
CASE 2	σ = 20	52.7333	0.5086
CASE 3	σ = 100	2.8949	0.0182



CASE 1



CASE 2



CASE 3

FIGURE E.1 (A)

TABLE E.1 (B)

Comparison of Observer Based Controller vs Stable Compensator

For the Same Degree of Recovery

DEGREE OF RECOVERY

$$\text{Sup}\{\sigma_{max}[E_f(j\omega)]\} \cong \text{Sup}\{\sigma_{max}[E_c(j\omega)]\} \cong 0.5086$$

for $0.001 \le \omega \le \infty$ rad/sec

	OBSERVER BASED CONTROLLER	STABLE COMPENSATOR
GAIN NORM-2:	4463.3	24.2299
EIGENVALUES:	-2386.9	-24.1102
	-9.9988	-13.0881
	-0.9935	-0.6338
(0 dB) BANDWIDTH:	18498 rad/sec	46.7 rad/sec

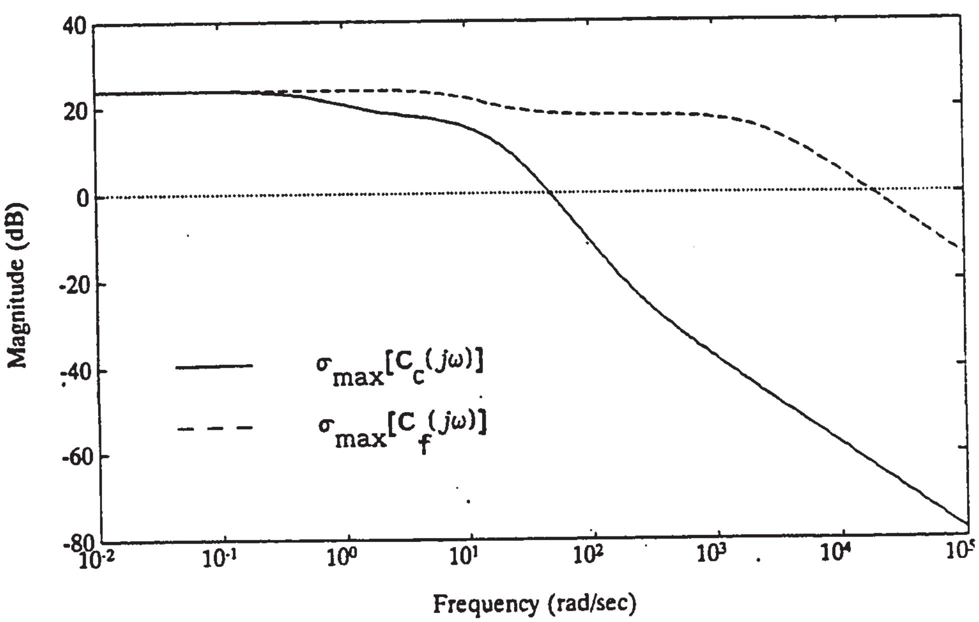


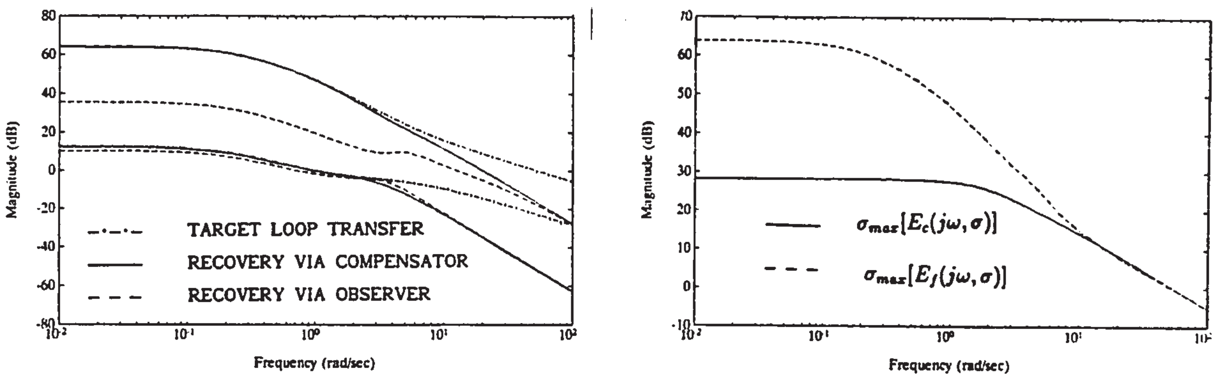
Figure E.1 (B) : Maximum singular values of observer based controller and stable compensator

TABLE E.2 (A)

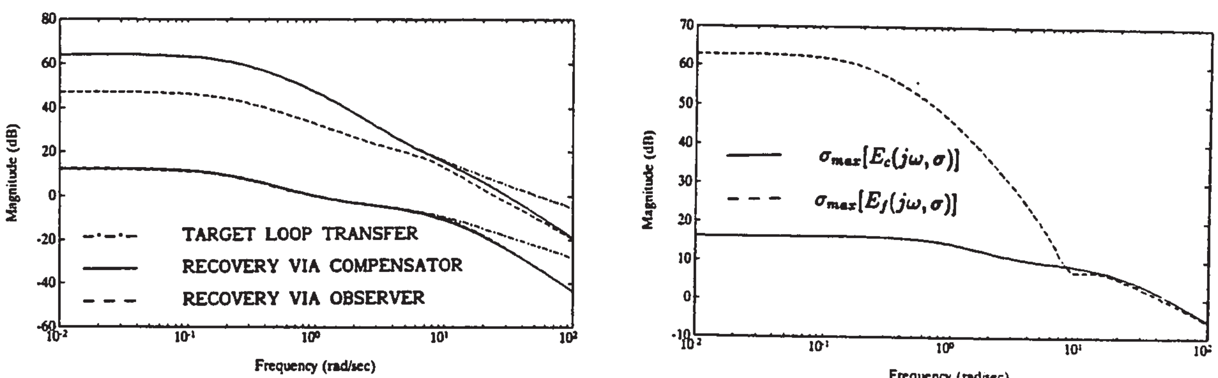
Supremum of Maximum Singular Values of Mismatch Functions

Frequncy Range: 0.01 to 100 rad/sec

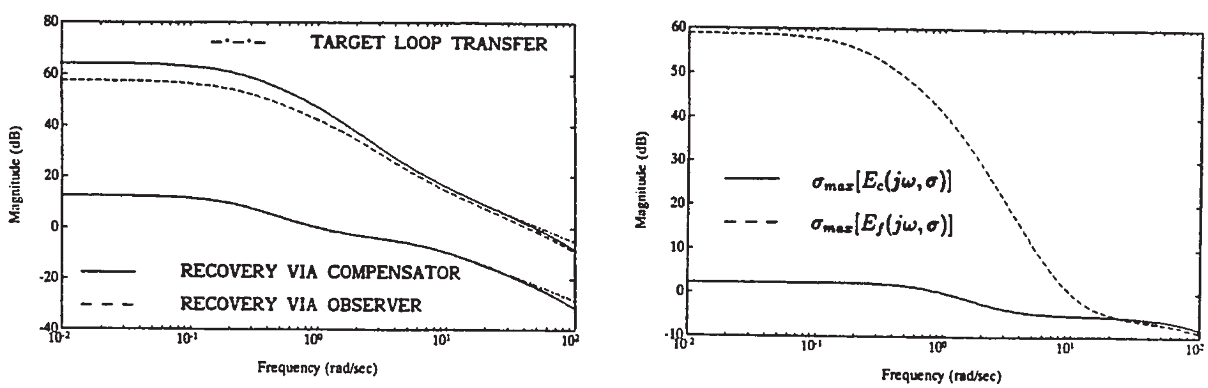
	TUNING PARAMETER	Sup $\{\sigma_{max}[E_f(j\omega,\sigma)]\}$	Sup $\{\sigma_{max}[E_c(j\omega,\sigma)]\}$
CASE 1	$\sigma = 5$	1570.7	25.9476
CASE 2	$\sigma = 20$	1397.7	6.4868
CASE 3	$\sigma = 100$	876.3	1.2975



CASE 1



CASE 2



CASE 3

FIGURE E.2 (A)

TABLE E.2 (B)

Comparison of Observer Based Controller vs Stable Compensator

For the Same Degree of Recovery

DEGREE OF RECOVERY

$\text{Sup}\{\sigma_{max}[E_f(j\omega)]\} \cong \text{Sup}\{\sigma_{max}[E_c(j\omega)]\} \cong 6.4868$

for $0.01 \le \omega \le \infty$ rad/sec

	OBSERVER BASED CONTROLLER	STABLE COMPENSATOR
GAIN NORM-2:	1.0844×10^6	713.11
EIGENVALUES:	-29563	-20
	-29522	-17.8815
	-8	-8
	-1.0034	-1.1185
(0 dB) BANDWIDTH:	1.0092×10^7 rad/sec	7794 rad/sec

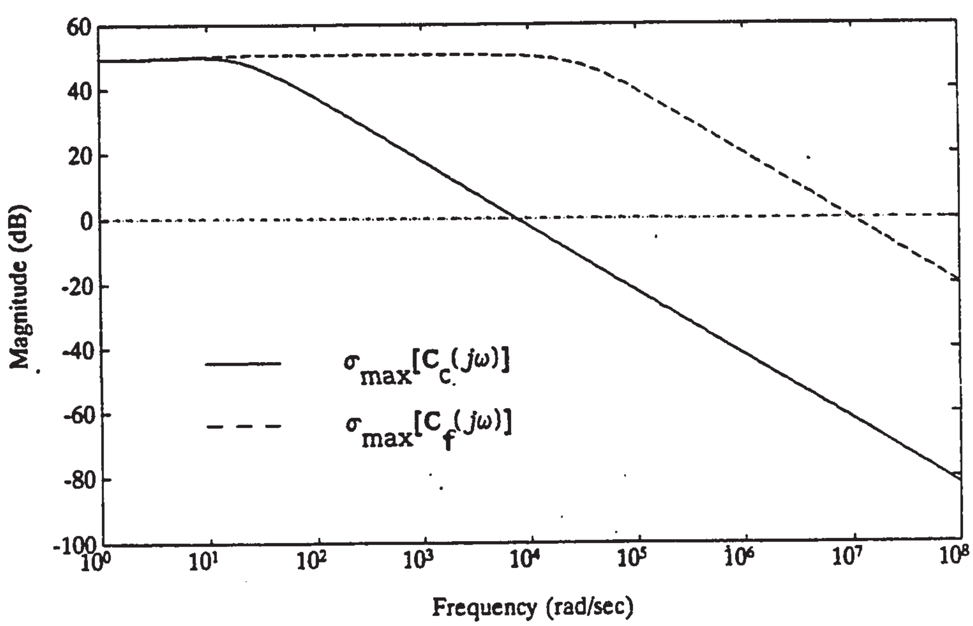


Figure E.2 (B) : Maximum singular values

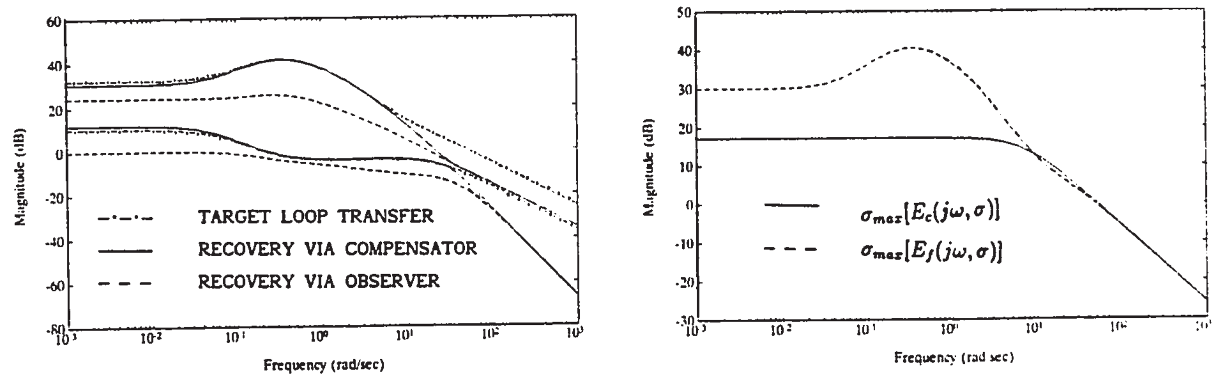
of observer based controller and stable compensator

TABLE E.3 (A)

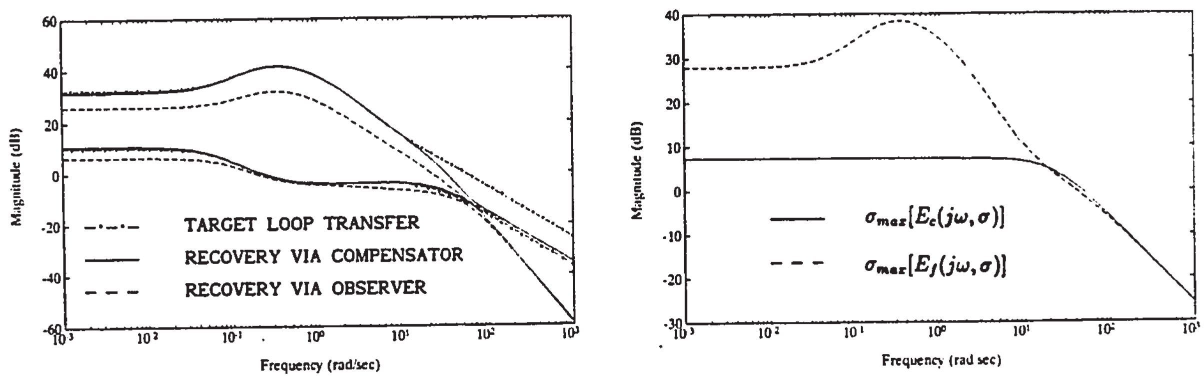
Supremum of Maximum Singular Values of Mismatch Functions

Frequncy Range: 0.001 to 1000 rad/sec

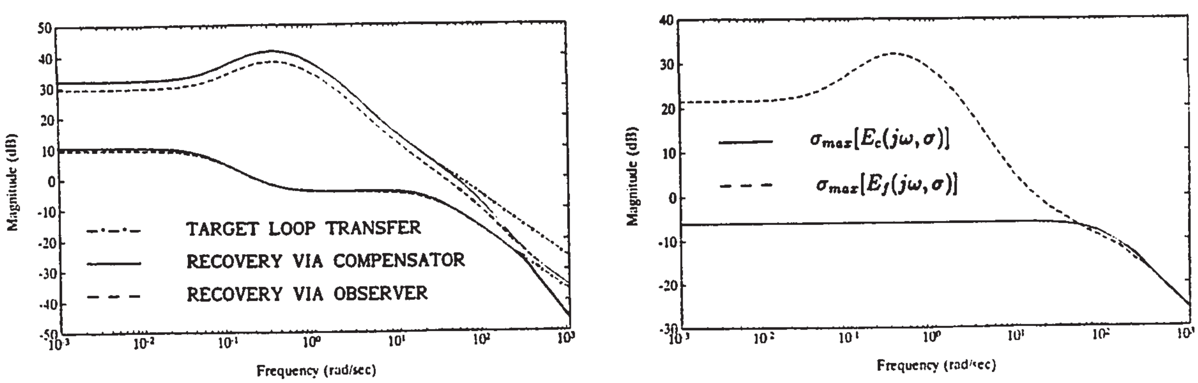
	TUNING PARAMETER	Sup $\{\sigma_{max}[E_f(j\omega,\sigma)]\}$	Sup $\{\sigma_{max}[E_c(j\omega,\sigma)]\}$
CASE 1	σ = 5	106.2351	7.2843
CASE 2	σ = 20	84.1034	2.3177
CASE 3	σ = 100	39.8388	0.4999



CASE 1



CASE 2



CASE 3

FIGURE E.3 (A)

Table E.5 (A1) : $K_f(\sigma)$ obtained from ATEA with $\sigma = 75$.

$$\begin{bmatrix} 1.78367774 & 0.05085816 & -5.54137522 & 0.14584105 \\ -2.97066524 & 0.73378480 & -0.66008964 & -1.75249421 \\ 74.62077330 & -0.00003431 & 0.00169392 & 0.04368414 \\ -0.06524472 & 71.97519966 & -0.54219109 & 0.47755509 \\ -0.17413468 & 0.07326398 & 74.37666021 & -0.00286912 \\ -0.03860068 & 0.41417413 & 0.31939682 & 74.48987981 \\ 0.00000000 & 1.00025892 & 0.00050000 & -0.01304098 \\ 0.00000000 & -0.00023150 & 0.99940000 & 0.03520146 \end{bmatrix}$$

Table E.5 (A2) : Suprema of $\sigma_{max}[E_f(j\omega,\sigma)]$ and $\sigma_{max}[E_c(j\omega,\sigma)]$ over $\omega \in (0.0001, 100)$ rad/sec.

Supremum of $\sigma_{max}[E_f(j\omega,\sigma)]$	Supremum of $\sigma_{max}[E_c(j\omega,\sigma)]$
114.0949	1.3725

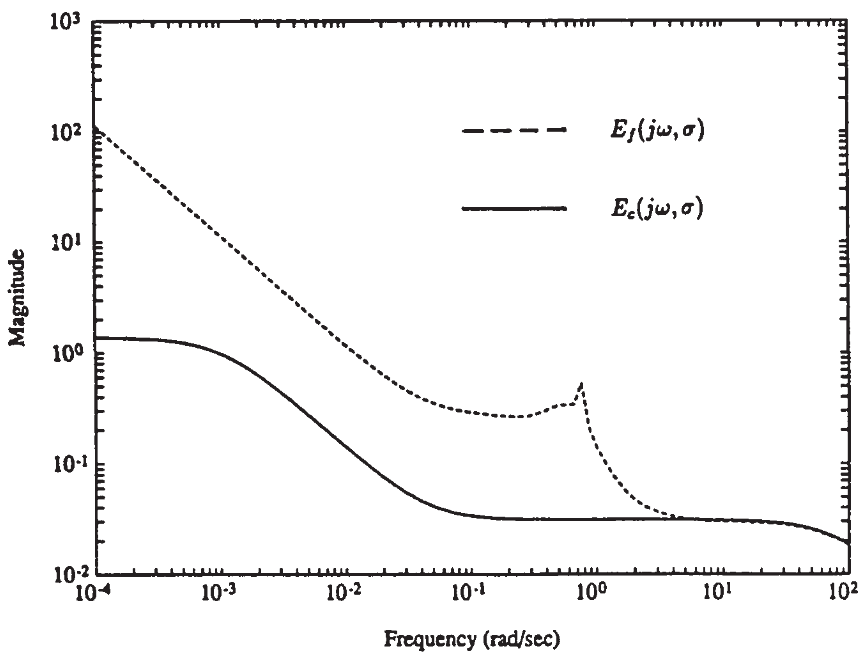


Figure E.5 (A) : Maximum singular values of $E_f(j\omega,\sigma)$ and $E_c(j\omega,\sigma)$.

Table E.5 (B1) : $\bar{K}_f(\sigma)$ obtained from H_2-optimization based algorithm with $\sigma = 10$.

$$\begin{bmatrix} 1.08575467 & 1.87579279 & 1.94020615 & 5.34047118 \\ -1.07142150 & -5.34940429 & 1.72470288 & 0.31681747 \\ 8.77244772 & 0.23431184 & -0.10022108 & -3.72267550 \\ 0.23431184 & 68.10414224 & -0.70094374 & -15.01264925 \\ -0.10022108 & -0.70094374 & 19.33832603 & -2.88740014 \\ -3.72267550 & -15.01264925 & -2.88740014 & 37.48117992 \\ -0.12793544 & -0.30905087 & 0.41767027 & 0.29480109 \\ -0.18626875 & -0.47613891 & 0.25980950 & -1.05729445 \end{bmatrix}$$

Table E.5 (B2) : Suprema of $\sigma_{max}[E_f(j\omega,\sigma)]$ and $\sigma_{max}[E_c(j\omega,\sigma)]$ over $\omega \in (0.0001, 100)$ rad/sec.

Supremum of $\sigma_{max}[E_f(j\omega,\sigma)]$	Supremum of $\sigma_{max}[E_c(j\omega,\sigma)]$
983.9667	1.3152

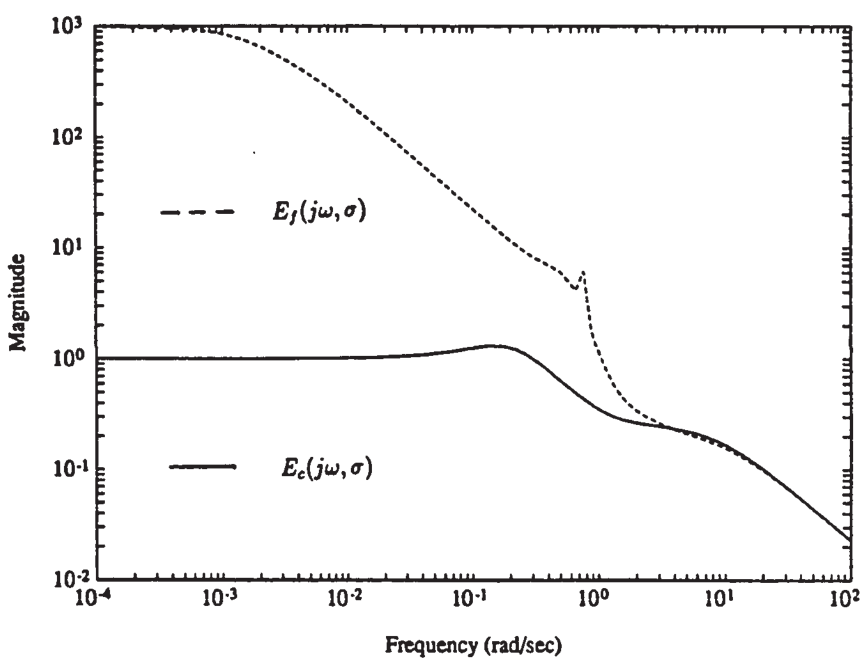


Figure E.5 (B) : Maximum singular values of $E_f(j\omega,\sigma)$ and $E_c(j\omega,\sigma)$.

Table E.5 (C1) : $K_f(\gamma)$ obtained from H_∞-optimization based algorithm with $\gamma = 1.5$.

$$\begin{bmatrix} 11.29282005 & 31.26778629 & 19.23846790 & 40.13781739 \\ -5.11144793 & -15.89047969 & -6.20137729 & -14.87306201 \\ 4.79116869 & 0.42589058 & 0.12665929 & -1.59218797 \\ 0.42589058 & 36.82490566 & 0.12580871 & -6.89988100 \\ 0.12665929 & 0.12580871 & 10.54345906 & -0.90272812 \\ -1.59218797 & -6.89988100 & -0.90272812 & 21.03886210 \\ -0.43799026 & -1.33778900 & -0.34831755 & -1.10520413 \\ -4.29490235 & -12.35299443 & -6.92834644 & -15.18565659 \end{bmatrix}$$

Table E.5 (C2) : Suprema of $\sigma_{max}[E_f(j\omega,\gamma)]$ and $\sigma_{max}[E_c(j\omega,\gamma)]$ over $\omega \in (0.0001, 100)$ rad/sec.

Supremum of $\sigma_{max}[E_f(j\omega,\gamma)]$	Supremum of $\sigma_{max}[E_c(j\omega,\gamma)]$
1932.5496	1.4785

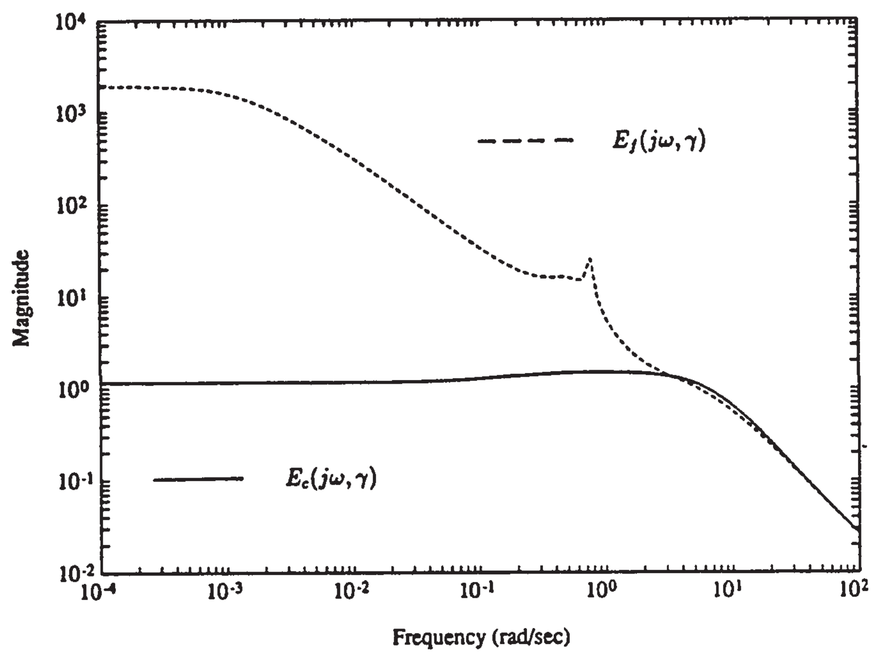


Figure E.5 (C) : Maximum singular values of $E_f(j\omega,\gamma)$ and $E_c(j\omega,\gamma)$.

5.A. Appendix 5.A — Proof of Lemma 5.2.1

For the sake of simplicity, we drop the σ-dependency on all of the variables. To unify the proof of lemma 5.3.1 for both full and reduced compensators, consider first the following compensator:

$$\begin{cases} \dot{v} &= Lv + Hy, \\ -u &= Pv + Vy, \end{cases} \tag{5.A.1}$$

where $v \in \Re^r$ with r being the order of the compensator. We assume that there exists a matrix $T \in \Re^{r \times n}$ such that the following conditions are satisfied:

1. $TA - LT = HC$,
2. $F = PT + VC$, and
3. $VD = 0$.

Also, let

4. $G = TB - HD$.

We have the following proposition.

Proposition 5.A.1. *Consider any admissible target loop transfer function $L_t(s) = F\Phi B$. Then the recovery error $E(s)$ realized by the compensator of (5.A.1) is given by*

$$E(s) = M(s) = P(sI - L)^{-1}G. \tag{5.A.2}$$

Proof of Proposition 5.A.1 : It is straightforward to see that the transfer function of the compensator (5.A.1) is given by

$$\mathbf{C}(s) = V + P\Phi_\ell H,$$

where $\Phi_\ell := (sI - L)^{-1}$. Also, using the fact that $TA - LT = HC$, it is trivial to verify that

$$P\Phi_\ell TB + P\Phi_\ell HC\Phi B - PT\Phi B = 0.$$

Then we have

$$\begin{aligned}\mathbf{C}(s)P(s) &= (V + P\Phi_\ell H)(C\Phi B + D)\\ &= P\Phi_\ell HC\Phi B + P\Phi_\ell HD + VC\Phi B\\ &= PT\Phi B - P\Phi_\ell TB + P\Phi_\ell HD + VC\Phi B\\ &= F\Phi B - P\Phi_\ell G.\end{aligned}$$

Hence,

$$E(s) = L_t(s) - \mathbf{C}(s)P(s) = M(s) = P\Phi_\ell G.$$

This completes the proof of proposition 5.A.1. □

Now, it is straightforward to verify that the full order compensator is a special case of (5.A.1) with

$$\begin{cases} L = A - K_f C, \quad G = B - K_f D, \quad H = K_f, \\ P = F, \qquad V = 0, \qquad T = I. \end{cases}$$

Similarly, the reduced order compensator is also a special case of (5.A.1) with

$$\begin{cases} L = A_{22} - K_r C_r, \quad G = B_{22} - K_r D_r, \quad H = G_r, \\ P = F_2, \quad V = [0,\ F_1 + F_2 K_{r1}], \quad T = [-K_{r1},\ I]. \end{cases}$$

Hence, equations (5.3.1) and (5.3.2) of lemma 5.3.1 follow trivially from (5.A.2). ■

5.B. Appendix 5.B — Proof of Theorem 5.2.1

It is shown in Chapters 3 and 4 that whenever a target loop is recoverable, there exist gain matrices $K_f(\sigma)$ and $K_r(\sigma)$ such that $A - K_f(\sigma)C$ and $A_r - K_r(\sigma)A_r$ are asymptotically stable matrices for all $\sigma > \sigma^*$ where $0 \le \sigma^* < \infty$ and in the limits the finite eigenvalues of these matrices belong to $\mathcal{C}^-$. Also, $K_f(\sigma)$ and $K_r(\sigma)$ guarantee that

$$E_c(s,\sigma) = M_f(s,\sigma) \to 0 \text{ pointwise in } s \text{ as } \sigma \to \infty, \tag{5.B.1}$$

and

$$E_{rc}(s,\sigma) = M_r(s,\sigma) \to 0 \text{ pointwise in } s \text{ as } \sigma \to \infty. \tag{5.B.2}$$

Hence, such $K_f(\sigma)$ and $K_r(\sigma)$ achieve loop transfer recovery and yield open-loop stability of either a full or a reduced order type of compensator. Next, we will show that the stable compensators designed as such can also achieve the asymptotic stability of the closed-loop system.

Full order compensator :

The dynamic matrix of the closed-loop system with a full order compensator is $A_{cl}(\sigma)$ as given in (5.2.3). Then consider the following reductions:

$$\begin{aligned}
\det[sI_{2n} - A_{cl}(\sigma)] &= \det\begin{bmatrix} sI_n - A + K_f(\sigma)C + K_f(\sigma)DF & -K_f(\sigma)C \\ BF & sI_n - A \end{bmatrix} \\
&= \det\begin{bmatrix} \Phi^{-1} + K_f(\sigma)DF & -K_f(\sigma)C \\ \Phi^{-1} + BF & \Phi^{-1} \end{bmatrix} \\
&= \det\begin{bmatrix} \Phi^{-1} + K_f(\sigma)DF & -K_f(\sigma)C \\ BF - K_f(\sigma)DF & \Phi^{-1} + K_f(\sigma)C \end{bmatrix} \\
&= \det\begin{bmatrix} \Phi^{-1} + BF & \Phi^{-1} \\ BF - K_f(\sigma)DF & \Phi^{-1} + K_f(\sigma)C \end{bmatrix}.
\end{aligned}$$

Now using Schur's formula for the determinant of a partitioned matrix, we have

$$\begin{aligned}
&\det[sI_{2n} - A_{cl}(\sigma)] \\
&\quad = \det[\Phi^{-1} + K_f(\sigma)C]\det[\Phi^{-1} + BF - \Phi^{-1}(\Phi^{-1} + K_f(\sigma)C)^{-1}(B - K_f(\sigma)D)F] \\
&\quad = \det[\Phi^{-1} + K_f(\sigma)C]\det[\Phi^{-1}]\det[I_n + \Phi BF - (\Phi^{-1} + K_f(\sigma)C)^{-1}(B - K_f(\sigma)D)F] \\
&\quad = \det[\Phi^{-1} + K_f(\sigma)C]\det[\Phi^{-1}]\det[I_m + F\Phi B - F(\Phi^{-1} + K_f(\sigma)C)^{-1}(B - K_f(\sigma)D)] \\
&\quad = \det[\Phi^{-1} + K_f(\sigma)C]\det[\Phi^{-1}]\det[I_m + F\Phi B - M_f(s,\sigma)] \\
&\quad \rightarrow \det[\Phi^{-1} + K_f(\sigma)C]\det[\Phi^{-1}]\det[I_m + F\Phi B] \text{ as } \sigma \rightarrow \infty \\
&\quad = \det[\Phi^{-1} + K_f(\sigma)C]\det[\Phi^{-1} + BF].
\end{aligned}$$

This shows that the closed-loop system with a full order compensator, is asymptotically stable for all $\sigma > \sigma_2^*$ for some nonnegative σ_2^*. It is also obvious that the limits of finite eigenvalues of $A_{cl}(\sigma)$ are in $\mathcal{C}^-$.

Reduced order compensator :

We first note the following:

$$sI_n - A \equiv \Phi^{-1} = \begin{bmatrix} \Phi_{11}^{-1} & -A_{12} \\ -A_{21} & \Phi_{22}^{-1} \end{bmatrix},$$

where $\Phi_{11} = (sI - A_{11})^{-1}$ and $\Phi_{22} = \Phi_r = (sI - A_{22})^{-1}$. Hence

$$\Phi^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix} = \Phi_a,$$

where

$$\Phi_a = \begin{bmatrix} -A_{12} \\ \Phi_{22}^{-1} \end{bmatrix}.$$

Thus

$$F\Phi\Phi_a = F_2. \tag{5.B.3}$$

We have the following series of reductions:

$$\det[sI_{2n-p+m_0} - A_{clr}(\sigma)]$$

$$= \det\begin{bmatrix} \Phi_{22}^{-1} + K_r(\sigma)C_r + K_r(\sigma)D_rF_2 & -A_{21} + K_r(\sigma)D_rF_1 & -K_r(\sigma)C_r \\ B_{11}F_2 & \Phi_{11}^{-1} + B_{11}F_1 & -A_{12} \\ B_{22}F_2 & -A_{21} + B_{22}F_1 & \Phi_{22}^{-1} \end{bmatrix}$$

$$= \det\begin{bmatrix} \Phi_{22}^{-1} + K_r(\sigma)D_rF_2 & -A_{21} + K_r(\sigma)D_rF_1 & -K_r(\sigma)C_r \\ -A_{12} + B_{11}F_2 & \Phi_{11}^{-1} + B_{11}F_1 & -A_{12} \\ \Phi_{22}^{-1} + B_{22}F_2 & -A_{21} + B_{22}F_1 & \Phi_{22}^{-1} \end{bmatrix}$$

$$= \det\begin{bmatrix} \Phi_{22}^{-1} + K_r(\sigma)D_rF_2 & -A_{21} + K_r(\sigma)D_rF_1 & -K_r(\sigma)C_r \\ -A_{12} + B_{11}F_2 & \Phi_{11}^{-1} + B_{11}F_1 & -A_{12} \\ (B_{22} - K_r(\sigma)D_r)F_2 & (B_{22} - K_r(\sigma)D_r)F_1 & \Phi_{22}^{-1} + K_r(\sigma)C_r \end{bmatrix}$$

$$= \det\begin{bmatrix} \Phi_{22}^{-1} + B_{22}F_2 & -A_{21} + B_{22}F_1 & \Phi_{22}^{-1} \\ -A_{12} + B_{11}F_2 & \Phi_{11}^{-1} + B_{11}F_1 & -A_{12} \\ (B_{22} - K_r(\sigma)D_r)F_2 & (B_{22} - K_r(\sigma)D_r)F_1 & \Phi_{22}^{-1} + K_r(\sigma)C_r \end{bmatrix}$$

$$= \det\begin{bmatrix} \Phi_{11}^{-1} + B_{11}F_1 & -A_{12} + B_{11}F_2 & -A_{12} \\ -A_{21} + B_{22}F_1 & \Phi_{22}^{-1} + B_{22}F_2 & \Phi_{22}^{-1} \\ (B_r - K_r(\sigma)D_r)F_1 & (B_r - K_r(\sigma)D_r)F_2 & \Phi_r^{-1} + K_r(\sigma)C_r \end{bmatrix}$$

$$= \det \begin{bmatrix} \Phi^{-1} + BF & \Phi_a \\ (B_r - K_r(\sigma)D_r)F & \Phi_r^{-1} + K_r(\sigma)C_r \end{bmatrix}$$

$$= \det[\Phi_r^{-1} + K_r(\sigma)C_r] \det[\Phi^{-1} + BF - \Phi_a(\Phi_r^{-1} + K_r(\sigma)C_r)^{-1}(B_r - K_r(\sigma)D_r)F]$$

$$= \det[\Phi_r^{-1} + K_r(\sigma)C_r] \det[\Phi^{-1}]$$

$$\cdot \det[I_n + [\Phi B - \Phi\Phi_a(\Phi_r^{-1} + K_r(\sigma)C_r)^{-1}(B_r - K_r(\sigma)D_r)]F]$$

$$= \det[\Phi_r^{-1} + K_r(\sigma)C_r] \det[\Phi^{-1}]$$

$$\cdot \det[I_m + F\Phi B - F\Phi\Phi_a(\Phi_r^{-1} + K_r(\sigma)C_r)^{-1}(B_r - K_r(\sigma)D_r)] \tag{5.B.4}$$

$$= \det[\Phi_r^{-1} + K_r(\sigma)C_r] \det[\Phi^{-1}]$$

$$\cdot \det[I_m + F\Phi B - F_2(\Phi_r^{-1} + K_r(\sigma)C_r)^{-1}(B_r - K_r(\sigma)D_r)] \tag{5.B.5}$$

$$= \det[\Phi_r^{-1} + K_r(\sigma)C_r] \det[\Phi^{-1}] \det[I_m + F\Phi B - M_r(s,\sigma)]$$

$$\rightarrow \det[\Phi_r^{-1} + K_r(\sigma)C_r] \det[\Phi^{-1}] \det[I_m + F\Phi B] \quad \text{as } \sigma \rightarrow \infty$$

$$= \det[\Phi_r^{-1} + K_r(\sigma)C_r] \det[\Phi^{-1}] \det[I_m + \Phi BF] \tag{5.B.6}$$

$$= \det[\Phi_r^{-1} + K_r(\sigma)C_r] \det[\Phi^{-1} + BF]. \tag{5.B.7}$$

Note that we used (5.B.3) in order to get (5.B.5) from (5.B.4). This shows that the closed-loop system with the reduced order compensator, is asymptotically stable for all $\sigma > \sigma_{2r}^*$ for some nonnegative σ_{2r}^*. It is also obvious that the limits of finite eigenvalues of $A_{clr}(\sigma)$ are in C^-. This completes the proof of theorem 5.3.1. ∎

Chapter 6

CONCLUDING REMARKS

6.1. Summary of the thesis

In this thesis, we present a fairly complete theory of loop transfer recovery, using both observer based controllers and open-loop stable compensators, for multivariable linear continuous time systems. In Chapters 2 and 3, we deal with issues concerning the analysis of loop transfer recovery problem using full and reduced order observer based controllers for general non-strictly proper systems. The results for the full and reduced order observer based controllers are unified in the same framework. There are several fundamental results given in these chapters. Based on the structural properties of the given system, we decompose the *recovery matrix*, which characterizes the loop transfer recovery error between the target loop transfer function and that that can be achieved by the observer based controllers, into three distinct parts for any arbitrarily specified target loop transfer function. The first part of the recovery matrix can be rendered exactly zero by finite eigenstructure assignment of the observer dynamic matrix, while the second part can be rendered arbitrarily close to zero by an appropriate asymptotically infinite eigenstructure assignment. The third part in general cannot be rendered zero, either exactly or asymptotically, by any means although there exists a multitude of ways to shape it. Such a decomposition of the recovery matrix helps us to discover the subspace of the control space in which target sensitivity and complementary sensitivity functions can be either exactly or asymptotically recovered. Moreover, it helps to formulate explicit singular value bounds on the *recovery error matrix*, which is the limit of the recovery matrix. All this analysis is given for an arbitrarily specified target loop transfer function. Thus it shows the limitations of the given system in recovering the target loop transfer functions as a consequence of its structural properties, namely finite and infinite zero structure and invertibility. On the other hand, the next issue of our analysis concentrates on characterizing the required necessary and sufficient conditions on the target loop transfer functions so that they are either exactly or asymptotically recoverable by means of observer based controllers for the given system. The conditions developed here on a target loop transfer function for its recoverability, turn out to be constraints on its finite and infinite zero structure as related to the corresponding structure of the given system. We next move on to find the necessary and sufficient conditions on the given system such that it has at least one recoverable target loop. In this regard, we show that strong stabilizability of the given system is necessary for it to have at least one recoverable target loop. Since recovery in all control loops in general is not feasible, we concentrate next in developing the necessary and sufficient conditions under which either exact or asymptotic recovery of target sensitivity and complementary sensitivity functions is possible in any specified subspace of the control space. This generalizes the traditional notion of LTR to cover recoverability in a subspace. We prove next that for left invertible non-strictly proper systems irrespective of the number of nonminimum phase zeros and irrespective of the nature of the target loop transfer function, there exists at least one $m-1$ dimensional subspace of m dimensional control space, in which the target sensitivity and complementary sensitivity functions can always be recovered by an appropriate design of the controller. Inherent in all the issues discussed here is the characterization of the resulting controller eigenvalues and possible pole zero cancellations. Such an investigation is important in view of the fact, controller eigenvalues become the invariant zeros of the closed-loop system and thus affect the performance with respect to command following and other design objectives.

Chapter 4 concerns with design issues of loop transfer recovery for general not necessarily left invertible, not necessarily of minimum phase, and non-strictly proper systems. After reviewing the necessary design constraints and the available design freedom, three different design methods are developed. The first method is an asymptotic time-scale structure and eigenstructure assignment (ATEA) scheme. The other two methods are optimization based; one minimizes the H_2 norm of a recovery matrix related to the loop transfer recovery error and the other minimizes the H_∞ norm of the same. All three methods of design give explicit methods of obtaining observer gain parameterized in terms of a tuning parameter. In optimization based designs, the gain is implicitly parameterized via the solution of parameterized nonlinear algebraic Riccati equations (ARE's). On the other hand, ATEA design does not require any solution of nonlinear algebraic equations; here the tuning parameter enters the design only in forming a composite gain from several subsystem designs, and thus it truly acts as a tuning parameter. All three methods of design yield a sequence of controllers as the tuning parameter takes on different values. In optimization based methods, as the tuning parameter tends to a certain critical value, the corresponding sequence of H_2 norms (or H_∞ norms) of the resulting recovery matrices tends to a limit which is the infimum of the H_2 norm (or H_∞ norm) of the recovery matrix over all possible observer gains. In so doing, the optimization based methods shape the recovery error in a particular way which may or may not be meaningful from an engineering point of view. On the other hand. ATEA method has the flexibility to utilize all the available design freedom to shape the recovery error to meet the designer's needs within the constraints imposed by the structural properties of the given system. Also, ATEA method can easily be modified and simplified to yield an observer design that achieves ELTR whenever it is feasible. In contrast with ATEA design, optimal or suboptimal design schemes do not require much prior planning but involve solving repeatedly the parameterized ARE's for different values of the parameter. However, these ARE's invariably become 'stiff' as the parameter takes values closer to certain critical value. Besides the conventional LTR design task which seeks the recovery over the entire control space, another generalized task which seeks recovery only over a specified subspace of the control space is also considered in this chapter. All the design methods developed here are implemented in a 'Matlab' software package. We also illustrate several aspects of ATEA as well as optimization based algorithms using helicopter attitude and rate command control system.

Chapter 5 deals with the design of practical controllers. Although observer based controllers can recover all recoverable target loops, in connection with ALTR which is the goal in practice, there are some inherent problems in using observer based controllers. More specifically, observer based controllers require high values of gain. The use of high gain brings with it the problems associated with high controller band-width and owes of signal saturation. To liberate the designer from these difficulties, we advocate the use of compensator structure for the controller, which is originally proposed in [8] and [9] for strictly proper systems. As in the case with observer based controllers, the compensator structure can also recover any recoverable target loop. Moreover, the compensator structure uses values of gains orders of magnitude less than what the observer based controller does for the same degree of recovery. This is shown both theoretically as well as by a bank of the numerical examples including helicopter control system. Also, theoretical bounds on sensitivity and complementary sensitivity functions obtained here confirm the advantages of using the compensator structure over the observer based controller structure. In short, we believe that the use of compensator structure for the controller brings the design procedure of LTR into practical domain.

6.2. Future research topics

Despite recent progresses, there are still some open problems in loop transfer recovery and the related areas.

As it is demonstrated in Example 5.4, in some cases, the open-loop stable compensator structure proposed in Chapter 5 can be applied to LTR design for non-recoverable target loops to achieve a good recovery performance. The further examinations on the theoretical aspects of such an approach could be considered as a future research subject. Another interesting problem is to search for a controller structure for general nonminimum phase systems and non-recoverable target loops. The new structure may not be open-loop stable but shall yield a better recovery performance than that achieved by the conventional observer based controller.

Historically, LTR arose as an attempt to recover the impressive robustness properties of the state feedback laws designed via linear quadratic optimal control theory. Such impressive properties are useless if the corresponding target loop transfer function is not recoverable, i.e. neither ELTR nor ALTR is feasible. The issue of developing of a step by step algorithm to generate the target loops (or full state feedback laws), which have certain useful properties and which are either exactly or asymptotically recoverable, could be a very important research subject in multivariable linear control system design.

Finally, we note that this thesis mainly concerns with the theory of loop transfer recovery for general non-strictly proper and nonminimum phase continuous-time multivariable linear systems. We refer the interested readers to [12] and [13] for a complete analysis and design of loop transfer recovery problem for general non-strictly proper discrete-time systems. In [12] and [13], all the results of loop transfer recovery problem for discrete time systems using prediction, current and reduced order estimator based controllers are unified in a single framework.

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