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Theory of LTR for non-minimum phase systems, recoverable target loops, and recovery in a subspace

Part 2. Design

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This Part focuses on the design of full order observer based controllers for the recovery of target loop transfer function or sensitivity and complimentary sensitivity functions when the given system is not necessarily left invertible and not necessarily of minimum phase. Four design tasks are considered. The first task concerns with arbitrarily specified target loop transfer functions and develops an observer design which has the capability to shape the inevitable recovery error according to the designer's needs whenever they are feasible. The second task considers observer design for exactly recoverable target loop transfer functions. The third task is similar to the second one in that it makes use of the specific properties of the target loop transfer function, but it considers observer design for asymptotically recoverable target loop transfer functions. The fourth task can be thought of as a generalization of the second and third tasks, and it considers observer design so that the achieved and target sensitivity and complimentary sensitivity functions match each other either exactly or asymptotically over a given subspace of the control space whenever it is possible. For all these tasks, observer design constraints and the available design freedom are reviewed. In view of the available freedom, possible specifications on the time-scale and/or eigenstructure of the observer dynamic matrix are formulated. In the case of first task, the conventional approach of designing observer based controllers by Kalman filter formalism which requires solving algebraic Riccati equations, is shown to have several fundamental limitations. A method of design based on asymptotic time-scale and eigenstructure assignment (ATEA) developed here overcomes these limitations. For the other tasks, no design methods other than the ones developed here are available in the literature. All the developed design methods are implemented in a 'Matlab' software package. A bank of examples illustrate that the proposed methods of design are capable of directly exploiting all the available freedom so as to achieve the desired results.

1. Introduction and problem statement

As is well known and as discussed earlier in Part 1 (Saberi *et al.* 1991) the basic loop transfer recovery problem is concerned with analysing and possibly designing an observer based controller which can achieve the same robustness properties as those of a state feedback controller. To be specific, consider a plant $\tilde{\Sigma}$

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\tilde{\mathbf{u}}, \quad \tilde{\mathbf{y}} = \tilde{\mathbf{C}}\tilde{\mathbf{x}} \tag{1.1}$$

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where the state vector $\tilde{\mathbf{x}} \in \mathfrak{R}^n$, output vector $\tilde{\mathbf{y}} \in \mathfrak{R}^p$ and input vector $\tilde{\mathbf{u}} \in \mathfrak{R}^m$. Without loss of generality, assume that $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ are of maximal rank. Let us also assume that $\tilde{\Sigma}$ is stabilizable and detectable. Let the state feedback control law

$$\tilde{\mathbf{u}} = -\tilde{\mathbf{F}}\tilde{\mathbf{x}} \tag{1.2}$$

be such that:

(*a*) the closed-loop system is asymptotically stable, i.e. eigenvalues of $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ lie in the left half s-plane; and

(*b*) the open-loop transfer function when the loop is broken at the input point of the plant meets the given frequency dependent specifications.

Then $L(s)$, $S(s)$ and $T(s)$, the target loop transfer function, sensitivity and complimentary sensitivity functions are

$$L(s) = \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$$

$$S(s) = [I_m + L(s)]^{-1}$$

and

$$T(s) = I_m - S(s) = [I_m + L(s)]^{-1}L(s) \tag{1.3}$$

where $\tilde{\mathbf{\Phi}} = (sI - \tilde{\mathbf{A}})^{-1}$ and I_m denotes an identity matrix of dimension $m \times m$. On the other hand, let

$$\hat{\mathbf{u}} = -\tilde{\mathbf{F}}\hat{\mathbf{x}},$$

$$\dot{\hat{\mathbf{x}}} = (\tilde{\mathbf{A}} - \tilde{\mathbf{K}}\tilde{\mathbf{C}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}})\hat{\mathbf{x}} + \tilde{\mathbf{K}}\tilde{\mathbf{y}} \tag{1.4}$$

be a full order observer based control law where $\tilde{\mathbf{K}}$ is an observer gain. Thus $L_0(s)$, $S_0(s)$ and $T_0(s)$, the obtainable loop transfer function and sensitivity and complimentary sensitivity functions are given by

$$L_0(s) = C(s)P(s),\ P(s) = \tilde{\mathbf{C}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$$

$$S_0(s) = [I_m + L_0(s)]^{-1}$$

and

$$T_0(s) = I_m - S_0(s) = [I_m + L_0(s)]^{-1}L_0(s) \tag{1.5}$$

where $C(s)$ is the observer based controller transfer function,

$$C(s) = \tilde{\mathbf{F}}[sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}\tilde{\mathbf{C}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}}]^{-1}\tilde{\mathbf{K}} \tag{1.6}$$

Thus the goal of loop transfer recovery problem is to design a $\tilde{\mathbf{K}}$ such that the mismatch function $E(j\omega)$ with $E(s)$ defined as

$$E(s) = L(s) - L_0(s) \tag{1.7}$$

is either exactly zero or in some sense approximately zero over the frequency range of interest. More precisely, we say exact LTR (ELTR) is achieved if

$$C(s)P(s) = L(s) \text{ for all } s.$$

Achieving ELTR is in general not possible. In an attempt to achieve 'approximate' LTR, one normally parameterizes $C(s)$ as a function of a tuning parameter σ. In observer based controllers, the gain $\tilde{\mathbf{K}}$ is the only free design variable and thus parameterizing it as a function of σ, a family of controllers $C(s, \sigma)$ are obtained

$$C(s, \sigma) = \tilde{\mathbf{F}}[sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}}]^{-1}\tilde{\mathbf{K}}(\sigma) \tag{1.8}$$

We say asymptotic LTR (ALTR) is achieved if

$$C(s, \sigma)P(s) \rightarrow L(s) \text{ pointwise in } s$$

as $\sigma \rightarrow \infty$, or equivalently $E(s, \sigma) \rightarrow 0$ pointwise in s as $\sigma \rightarrow \infty$. Achievability of ALTR enables the designer to choose a member of the family of controllers that corresponds to a particular value of σ which achieves a desired level of recovery.

In Part 1 (Saberi *et al.* 1991), we considered general not necessarily invertible and not necessarily of minimum phase plants and analysed the mechanism of ELTR and ALTR via a full order observer based controller. The analysis there, while showing that neither ELTR nor ALTR can in general be achieved, focuses on three fundamental issues. The first issue is concerned with what can and what cannot be achieved for a given system and for an arbitrarily specified target loop transfer function, while the second issue is concerned with the development of necessary or/and sufficient conditions a target loop has to satisfy so that it can either exactly or asymptotically be recovered for the given system. The third issue deals with the development of method(s) to test whether recovery is possible in a given subspace of the control space or not, i.e. to test whether projections of target and achievable sensitivity and complimentary sensitivity functions onto a given subspace match each other or not. Thus the third issue generalizes the traditional notion of LTR. All this analysis shows some fundamental limitations of the given system as a consequence of its structural properties, namely finite and infinite zero structure and invertibility. It also discovers a multitude of ways in which freedom exists to shape the recovery error in a desired way. Thus it helps to set meaningful design goals at the onset of design. In this part of the paper, we develop design methods for all the issues identified and analysed earlier in Part 1. These methods are capable of utilizing in a direct and user friendly way all the available freedom discovered in Part 1. Four different design tasks are considered. The first one concerns with arbitrarily specified target loop transfer functions and develops an observer design which has the capability to shape the inevitable recovery error according to the designer's needs whenever they are feasible. In this case, the design does not exploit any specific characteristics of the target loop except using it as a goal for the design. Why should one consider an observer design for an arbitrary target loop transfer function? Well, it is traditional to do so. Separation principle which lets us to separate the state feedback and observer designs into two distinct and decoupled tasks, is rooted deeply in modern control theory. This hidden philosophy of using separation principle has been the heart of LQG/LTR and as such development of an observer design method for recovery of an arbitrarily specified target loop transfer function is a consequence of this philosophy. Next we like to move away from this traditional design philosophy and consider design schemes where in properties of the given loop transfer function could be taken into account. The analysis carried out in Part 1 facilitates such a task. More specifically, the analysis of Part 1 clearly points out the necessary and sufficient conditions under which a loop transfer function is either exactly or asymptotically recoverable. This analysis helps the designer to set meaningful goals at the onset of design. That is, although the actual physical tasks of first designing a target loop and then designing an observer based controller are separable, one can bridge or link these two tasks philosophically at the onset of design. In view of this, it is natural then to seek design schemes for exact or asymptotic recovery of a given loop transfer function whenever it is feasible to do so. Thus our second and third design problems are concerned with the development of design procedures respectively for exact and asymptotic recovery. After developing design schemes for either arbitrarily specified or exactly recoverable or asymptotically recoverable target loops, we move on next to generalization of these schemes. In particular, as revealed by our analysis in Part 1, recovery in all control loops as desired by the designer in general is not feasible in MIMO systems which are not necessarily left invertible and are of non-minimum phase. Thus one may seek to recover sensitivity and complimentary sensitivity functions (or some generalized recovery as will be clear from the context) over only a subspace of the control space. Again the analysis carried out in Part 1 shows the conditions under which recovery in a given subspace is feasible or not. Thus our fourth design problem is concerned with the development of design procedures for either exact or asymptotic recovery in a given subspace whenever it is feasible.

Any time when one deals with asymptotic recovery, one first considers a family of controllers $C(s, \sigma)$ parameterized with a tuning parameter σ and then selects a particular controller in the family which meets a desired level of recovery. On the other hand, for exact recovery whenever it is feasible, no such parameterization is necessary. Also, as is well known and as is discussed in Part 1 (Saberi *et al.* 1991), when asymptotic recovery is considered, the observer gain $\tilde{\mathbf{K}}(\sigma)$ tends to infinity as $\sigma \to \infty$. This implies that some of the eigenvalues of the observer dynamic matrix, $\tilde{\mathbf{A}}_0 = \tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}}$, tend to finite values while the rest tend to infinity at different rates along some asymptotes as $\sigma \to \infty$. In other words, whenever asymptotic recovery is considered, design of $\tilde{\mathbf{K}}(\sigma)$ involves both multiple time-scale structure assignment and finite eigenstructure assignment to $\tilde{\mathbf{A}}_0$. Where as whenever exact recovery is considered, design of $\tilde{\mathbf{K}}$ involves only finite eigenstructure assignment to $\tilde{\mathbf{A}}_0$.

Observer design for LTR in the existing literature is mostly focused on left invertible and minimum phase plants and that too in connection with the first design task outlined earlier. There exists three methods of determining the required observer gain for left invertible and minimum phase plants. These methods are, (1) Kalman filter formalism (Doyle and Stein 1979), (2) direct eigenstructure placement method (Sogaard-Andersen 1989) and (3) asymptotic eigenstructure and time-scale structure assignment (ATEA) method (Saberi and Sannuti 1990). Kalman filter formalism has been well studied and well understood for left invertible and minimum phase plants. In it, the observer eigenstructure is controlled by varying the intensity of the input process noise, i.e. the tuning parameter σ is the intensity of the input process noise. Here an appropriate high gain is obtained by solving a parameter dependent algebraic Riccati equation (ARE) and hence such a design can be referred to as ARE based design. In direct eigenstructure placement method, some of the eigenvalues of the observer are placed at the plant finite (invariant) zeros while the rest of them are placed far away in the negative half s-plane. However, there is a fundamental difficulty in placing the far away eigenvalues. One has to make sure that the residues associated with the far away eigenvalues remain uniformly bounded as these eigenvalues are pushed to infinity. There is no direct way of assuring this. In ATEA method, observer gain is parameterized directly in terms of σ rather than being done indirectly via a parameter dependent ARE. The parameter σ comes into play only in changing the degree of fastness of various time-scales. In connection with general non-minimum phase plants, since no direct design method is yet available it has been suggested in the literature that ARE based design be used for non-minimum phase plants as well and accept the consequent recovery error as being imminent. Such an approach necessitates a careful study of what ARE based design does to the recovery error. Realizing this, Zhang and Freudenberg (1987, 1990) recently for the first time developed expressions for the resulting asymptotic behavior of loop transfer and sensitivity functions when ARE based design is used. More recently, Niemann and Jannerup (1990) have expanded further on the results of Zhang and Freudenberg (1987, 1990).

In this paper for general systems we present new design methods dealing with all the four problems outlined earlier. Before we present a new design method for asymptotic recovery of arbitrarily specified target loop transfer functions (i.e. the first design problem), we need to motivate and justify the rationale behind developing an alternative design procedure to that of ARE based, especially in view of the simplicity of use and general availability of software for ARE based designs. For this purpose, we provide an in depth analysis of ARE based design regarding the achievable asymptotic limit of loop transfer recovery error, the way it is shaped, the amount of gain required for a chosen level of recovery and finally the resulting asymptotic behaviour of observer eigenstructure. Our study compliments and extends the results of Zhang and Freudenberg (1987, 1990). It reveals several serious limitations as listed below of ARE based design.

(1) Our analysis in Part 1 discovers a multitude of ways in which freedom exists to shape the loops or equivalently the recovery error. ARE based design chooses to shape the loops in a particular way among an array of such available choices. For left invertible and minimum phase plants, since ALTR is always possible, the particular way ARE based approach accomplishes the design does not play a critical role although it results in an unnecessarily high controller gain and band-width. However, for general systems, ability to utilize all the available design freedom is of paramount importance. The path taken by ARE based design to shape the loop is not necessarily the best path and hence one needs to explore all the available design freedom; especially exploring such a freedom in the subspace in which complete recovery is not possible is a dire necessity.

(2) The time-scale structure assigned to the observer dynamics by ARE based design is fixed by the infinite zero structure of the given system. This lack of freedom to assign any chosen time-scale structure results in a high gain consequently in a higher than the minimum necessary controller band-width.

(3) In ARE based design, due to the implicit parameterization of gain, a non-linear algebraic Riccati equation has to be solved repeatedly. Such a solution is numerically cumbersome and becomes 'stiff', especially for large σ owing to the interaction of several slow and fast time-scales. This can be brushed off as being a numerical problem. Nevertheless, it is an important limitation in practice.

Besides the above enumerated limitations, ARE based design is basically an asymptotic recovery scheme. Contrary to this, the scope of various design schemes proposed in this paper is much broader. Our aim here is to develop design schemes for all the four design problems outlined earlier. For exact LTR design, except the scheme given by Chen *et al.* (1990 a) for left invertible and minimum phase systems, no methods other than the ones given in this paper are available in the literature for general systems. In connection with asymptotic recovery design either for arbitrarily specified or asymptotically recoverable target loops, the design presented here is an extension and generalization of our earlier ATEA design scheme (Saberi and Sannuti 1990). It overcomes all the limitations of ARE based design. Moreover, observer gain in ATEA method is parameterized directly in terms of σ. Also, the design equations can be solved without explicitly requiring a value for σ. When the observer is implemented either by soft or hard ware, the value for σ can be adjusted on line. Since the effect of such a 'knob' on the performance and robustness of a given plant is straightforward, it should be very appealing from a practical point of view. Also, the required feedback gain matrix can be calculated in a decentralized manner using subsystems of a given system. Such a decentralized method, obviously reduces the computational complexity of designing a large scale system. By adopting a standard method of design for each subsystem, the mechanics of performing the design are simplified. Since the necessary design in each time-scale is done separately, it alleviates the stiffness problems that arise due to interaction of different time-scales. Furthermore, the computations required in ATEA design do not involve arbitrarily small or large numbers. Since ATEA design algorithm is more sophisticated than ARE based design and as it offers more flexibility and freedom to shape the recovery error as well as the asymptotically finite and infinite eigenstructure of the observer dynamic matrix, it is somewhat complex than ARE based algorithm. However, it is straightforward and easy to implement it. In fact, we have already implemented it into a 'Matlab' software package.

This paper is organized as follows. Section 2 deals with the first design task, i.e. developing an appropriate design when specified target loops are arbitrary. In particular, at first § 2.1 discusses observer design constraints and specifications. Next, § 2.2 examines and illustrates systematically various limitations of ARE based design. To overcome all these existing limitations, § 2.3 develops an asymptotic time-scale structure and eigenstructure assignment (ATEA) method. Sections 3 and 4 respectively deal with the design for exactly and asymptotically recoverable target loop transfer functions. Section 5 deals with the design to recover sensitivity and complimentary sensitivity functions in a given subspace whenever such a recovery is possible. All the design methods presented here are implemented in a 'Matlab' software package. Section 6 draws conclusions of our work.

As in Part 1, throughout this paper, A' denotes the transpose of A, A^H denotes the complex conjugate transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. $\lambda(A)$ and $\mathrm{Re}\,[\lambda(A)]$ respectively denote the set of eigenvalues and real parts of eigenvalues of A. Similarly, $\sigma_{\max}[A]$ and $\sigma_{\min}[A]$ respectively denote the maximum and minimum singular values of A. Ker $[V]$ and Im $[V]$ denote respectively the kernel and the image of V. The open left and closed right half s-planes are respectively denoted by $\mathscr{C}^-$ and $\mathscr{C}^+$.

2. Design for arbitrary target loops

In this section, we consider observer design for arbitrarily specified target loop transfer functions. As explained earlier, in this case we deal with a family of parameterized controllers $C(s, \sigma)$. The parameterization of any controller must be done in such a way that $C(s, \sigma)$ has certain asymptotic properties. Both the asymptotically finite and infinite eigenstructures of the observer dynamic matrix $\tilde{\mathbf{A}}_0 = \tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}}$ must satisfy certain properties in order to have appropriate recovery. However, there exists also an abundant amount of freedom in assigning certain parts of either asymptotically finite or infinite eigenstructure to $\tilde{\mathbf{A}}_0$. Let us next briefly review the LTR mechanism as analysed in Part 1 so as to familiarize ourselves with the necessary design constraints and the available design freedom. We recall that $E(s, \sigma)$, the mismatch function between the target loop transfer function $L(s)$ and the achievable one $L_0(s)$, is given by

$$E(s, \sigma) = M(s, \sigma)[I_m + M(s, \sigma)]^{-1}(I_m + \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}) \tag{2.1}$$

where

$$M(s, \sigma) = \tilde{\mathbf{F}}[sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}}]^{-1}\tilde{\mathbf{B}} \tag{2.2}$$

As indicated in Lemma 3.1 of Part 1, $E(j\omega, \sigma) = 0$ iff $M(j\omega, 0) = 0$. Also, as indicated by (3.6)–(3.9) of Part 1, we can expand $M(s, \sigma)$ dyadically

$$M(s, \sigma) = \sum_{i=1}^{n} \frac{R_i(\sigma)}{s - \lambda_i(\sigma)} \tag{2.3}$$

where the residue $R_i(\sigma)$ is given by

$$R_i(\sigma) = \tilde{\mathbf{F}} W_i(\sigma) V_i^H(\sigma)\tilde{\mathbf{B}} \tag{2.4}$$

Here $W_i(\sigma)$ and $V_i(\sigma)$ are respectively the right and left eigenvectors associated with an eigenvalue $\lambda_i(\sigma)$ of $\tilde{\mathbf{A}}_0$ and they are scaled so that $W(\sigma)V^H(\sigma) = V^H(\sigma)W(\sigma) = I_n$ where

$$W(\sigma) = [W_1(\sigma) \quad W_2(\sigma) \quad \dots \quad W_n(\sigma)] \quad \text{and} \quad V(\sigma) = [V_1(\sigma) \quad V_2(\sigma) \quad \dots \quad V_n(\sigma)] \tag{2.5}$$

As is evident from (2.3) and (2.4), for an arbitrary $\tilde{\mathbf{F}}$, $M(s, \sigma)$ can be rendered zero asymptotically by rendering either $\tilde{\mathbf{B}}V_i(\sigma)$ zero or by pushing λ_i to infinity while keeping $W_i(\sigma)V_i^H(\sigma)\tilde{\mathbf{B}}$ uniformly bounded. To develop guide lines when and how this can be done, let us partition $M(s, \sigma)$ into four parts

$$M(s, \sigma) = M_-(s, \sigma) + M_b(s, \sigma) + M_\infty(s, \sigma) + M_e(s, \sigma) \tag{2.6}$$

where

$$M_-(s, \sigma) = \sum_{i=1}^{n_a^-} \frac{R_i(\sigma)}{s - \lambda_i(\sigma)}$$

$$M_b(s, \sigma) = \sum_{i=n_a^-+1}^{n_a^-+n_b} \frac{R_i(\sigma)}{s - \lambda_i(\sigma)}$$

$$M_\infty(s, \sigma) = \sum_{i=n_a^-+n_b+1}^{n_a^-+n_b+n_f} \frac{R_i(\sigma)}{s - \lambda_i(\sigma)}$$

and

$$M_e(s, \sigma) = \sum_{i=n-n_a^--n_b-n_f+1}^{n} \frac{R_i(\sigma)}{s - \lambda_i(\sigma)}$$

Let $\Lambda_-(\sigma)$, $\Lambda_b(\sigma)$, $\Lambda_\infty(\sigma)$ and $\Lambda_e(\sigma)$ be the sets of eigenvalues of $\tilde{\mathbf{A}}_0$ associated respectively with the parts $M_-(s,\sigma)$, $M_b(s,\sigma)$, $M_\infty(s,\sigma)$ and $M_e(s,\sigma)$. Similarly, to correspond with this partition of eigenvalues, partition the right and left eigenvectors of $\tilde{\mathbf{A}}_0$ into sets $W_-(\sigma)$, $W_b(\sigma)$, $W_\infty(\sigma)$, $W_e(\sigma)$, $V_-(\sigma)$, $V_b(\sigma)$, $V_\infty(\sigma)$ and $V_e(\sigma)$. Also, let us use an over bar on a certain variable to denote its limit whenever it exists as $\sigma \to \infty$. For example, $\bar{M}_e(s)$ and $\bar{W}_e$ denote respectively the limits of $M_e(s,\sigma)$ and $W_e(\sigma)$ as $\sigma \to \infty$.

2.1. *Design constraints and specifications*

As analysed in Part 1, various parts of $M(s,\sigma)$ as given in (2.6) reveal several design constraints and the available design freedom. Let us enumerate them one at a time.

(1) The set of n_a^- eigenvalues $\Lambda_-(\sigma)$ and the corresponding set of left eigenvectors $V_-(\sigma)$ of $\tilde{\mathbf{A}}_0$ must be selected so that their asymptotic limits $\bar{\Lambda}_-$ and $\bar{V}_-$ coincide respectively with the set of plant minimum phase invariant zeros and the corresponding left state zero directions of $\tilde{\Sigma}$. Such a choice of eigenvalues and eigenvectors renders $\bar{M}_-(s)$ zero. Also, if one prefers, $\Lambda_-(\sigma)$ and $V_-(\sigma)$ can be designed to be independent of σ, i.e. $\Lambda_-(\sigma) \equiv \bar{\Lambda}_-$ and $V_-(\sigma) \equiv \bar{V}_-$ for all σ. Such a choice is some times beneficial and renders $M_-(s,\sigma) \equiv 0$.

(2) The set of n_b eigenvalues $\Lambda_b(\sigma)$ can be assigned arbitrarily either at asymptotically finite or infinite locations in $\mathscr{C}^-$, whiler the corresponding set of left eigenvectors $V_b(\sigma)$ of $\tilde{\mathbf{A}}_0$ must be such that their asymptotic limit $\bar{V}_b$ is in the null space of matrix $\tilde{\mathbf{B}}'$ so as to render $\bar{M}_b(s)$ zero. In order to conserve the controller band-width, it will be assumed that elements of $\bar{\Lambda}_b$ are assigned to finite locations. Also, if one prefers, $\Lambda_b(\sigma)$ and $V_b(\sigma)$ can be designed to be independent of σ, i.e. $\Lambda_b(\sigma) \equiv \bar{\Lambda}_b$ and $V_b(\sigma) \equiv \bar{V}_b$ for all σ. In this case, $M_b(s,\sigma) \equiv 0$.

(3) The set of $n_a^+ + n_c$ eigenvalues $\Lambda_e(\sigma)$ can be assigned arbitrarily at any (either asymptotically finite or infinite) locations in $\mathscr{C}^-$ subject to the condition that any unobservable but stable eigenvalues of the given system must be included among $\Lambda_e(\sigma)$. Also, there exists a complete freedom consistent with the results of Moore (1976) in assigning the right and left eigenvector sets $W_e(\sigma)$ and $V_e(\sigma)$ and hence $\bar{W}^e$ and $\bar{V}_e$. But in general $\bar{\Lambda}_e$, $\bar{W}_e$ and $\bar{V}_e$ cannot be assigned such that $\bar{M}_e(s)$ is zero. However, there exists a multitude of ways to assign $\bar{\Lambda}_e$ and $\bar{W}_e$ (and hence $\bar{V}_e$) so that the recovery error $\bar{M}_e(s)$ can be shaped to have certain desired directional properties or it is as small as it could be. Although theoretically there exists complete freedom in assigning $\Lambda_e(\sigma)$ to either asymptotically finite or infinite locations, however as will be demonstrated by means of an example later on, $M_e(j\omega,\sigma)$ can be unbounded as $\sigma \to \infty$ whenever any elements of $\Lambda_e(\sigma)$ are assigned to asymptotically infinite locations in $\mathscr{C}^-$. Moreover, assigning $\Lambda_e(\sigma)$ to asymptotically infinite locations increases unnecessarily controller band-width and hence we assume that all $\bar{\Lambda}_e$ are finite. We note that $n_a^+ + n_c = 0$ and hence $M_e(s,\sigma)$ is non-existent if the given system is of minimum phase and left invertible.

(4) The set of n_f eigenvalues $\Lambda_\infty(\sigma)$ can be assigned arbitrarily at asymptotically infinite locations in $\mathscr{C}^-$. However for every $\lambda_i(\sigma) \in \Lambda_\infty(\sigma)$, the corresponding

right and left eigenvectors. $W_i(\sigma)$ and $V_i(\sigma)$ must be such that $W_i(\sigma)V_i^H(\sigma)\tilde{\mathbf{B}}$ is uniformly bounded as $\sigma \to \infty$. This renders $\bar{M}_\infty(s)$ zero. We note that there exists complete freedom in the way $\lambda_i(\sigma) \in \Lambda_\infty(\sigma)$ tends to infinity as $\sigma \to \infty$, i.e. the asymptotic direction and the rate at which each $\lambda_i(\sigma)$ goes to infinity can be dictated as desired by the designer.

Let us expand more on the freedom available in assigning every asymptotically infinite eigenvalue $\lambda_i(\sigma) \in \Lambda_\infty(\sigma)$. As mentioned above, this freedom manifests itself in two ways:

(1) in choosing the asymptotic directions along which the eigenvalues tend to infinity; and

(2) in choosing the rates at which the eigenvalues tend to infinity.

To reflect both these types of freedom, let $\Lambda_\infty(\sigma)$ for asymptotically large values of σ be subdivided into $r \leqslant n_f$ sets,

$$\frac{\Lambda_1}{\mu_1}, \quad \frac{\Lambda_2}{\mu_2}, \quad \ldots, \quad \frac{\Lambda_r}{\mu_r} \tag{2.7}$$

Here Λ_l is a set of n_l numbers all in $\mathscr{C}^-$ and Λ_l is closed under complex conjugation. Also $\sum_{l=1}^{r} n_l = n_f$. Apparently, the elements of Λ_l, $l = 1$ to r, define the asymptotic directions of asymptotically infinite or fast eigenvalues while the small parameters μ_l, $l = 1$ to r, which are some functions of σ, define the rates at which these eigenvalues go to infinity. Thus a designer has the freedom to specify:

(1) $\bar{\Lambda}_b$ and $\bar{\Lambda}_e$ which in addition to $\bar{\Lambda}_-$ define the asymptotically finite eigenvalues of $\tilde{\mathbf{A}}_0$; and

(2) Λ_l and μ_l, $l = 1$ to r, which define the asymptotically infinite eigenvalues of $\tilde{\mathbf{A}}_0$.

An assignment of both asymptotically finite and infinite eigenvalues and the corresponding eigenvectors to a system can be viewed as an asymptotic time-scale and eigenstructure assignment (ATEA) to it. More formally, we define a time-scale structure (TSS) of a system as follows:

Definition 2.1

A system defined by the dynamic equation

$$\dot{\hat{\mathbf{x}}} = (\tilde{\mathbf{A}} - \tilde{\mathbf{C}}\tilde{\mathbf{K}}(\sigma))\hat{\mathbf{x}} \equiv \tilde{\mathbf{A}}_0\hat{\mathbf{x}} \tag{2.8}$$

is said to have a time-scale structure (TSS)

$$t, t/\mu_1, t/\mu_2, \ldots, t/\mu_r$$

if $\lambda(\tilde{\mathbf{A}}_0)$ approach

$$\Lambda_0, \Lambda_1/\mu_1, \Lambda_2/\mu_2, \ldots, \Lambda_r/\mu_r$$

as $\sigma \to \infty$ where Λ_0 and Λ_l, $l = 1$ to r are a set of finite elements in $\mathscr{C}^-$ and μ_l, $l = 1$ to r, are some small positive parameters dependent on σ.

In order to have a well defined separation of time-scales, we will assume throughout the paper that

$$\mu_l/\mu_{l+1} \to 0 \text{ as } \mu_{l+1} \to 0 \tag{2.9}$$

The point of view of TSS expressed by the above definition is slightly different from other definitions used in the literature (see Saberi 1988 for more details).

It has been suggested (Niemann and Jannerup 1990) to push $\bar{\Lambda}_e$ to ∞ so that $M_e(s, \sigma)$ can be constant over all frequencies. We give below a counter example to show that if the number of non-minimum phase invariant zeros is greater than one, such an idea does not work in general. Also, as implied by Lemma 3.2 of Part 1, pushing $\bar{\Lambda}_e$ to ∞ in general results in an unbounded peaking of $M(j\omega, \sigma)$ since the residue $R_i(\sigma)$ corresponding to an eigenvalue in $\lambda_i \in \bar{\Lambda}_e$ can grow faster than λ_i itself as $\sigma \to \infty$. Such an unbounded peaking causes unmodelled dynamics and noise at high frequencies to pass through the entire loop.

Example 2.1

Let $\tilde{\Sigma}$ be characterized by

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and $\tilde{\mathbf{C}} = \tilde{\mathbf{B}}'$. This system has three invariant zeros at $s = 1$, 2 and 3. Let the target loop transfer function $\tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$ be specified by

$$\tilde{\mathbf{F}} = [40 \quad -250 \quad 320 \quad 15]$$

Since this is a SISO system, the observer gain is fully specified by the intended eigenvalues of the observer. Let

$$\tilde{\mathbf{K}}(\sigma) = \begin{bmatrix} 0{\cdot}5\sigma^5 + 1{\cdot}5\sigma^4 + 2\sigma^3 + 2\sigma^2 + 1{\cdot}5\sigma + 1{\cdot}5 \\ -\sigma^5 - 6\sigma^4 - 14\sigma^3 - 20\sigma^2 - 24\sigma - 15 \\ 0{\cdot}5\sigma^5 + 4{\cdot}5\sigma^4 + 15\sigma^3 + 27\sigma^2 + 40{\cdot}5\sigma + 41{\cdot}5 \\ \sigma^2 + 3\sigma + 6 \end{bmatrix}$$

Then it is straightforward to verify that the above $\tilde{\mathbf{K}}(\sigma)$ results in observer eigenvalues at $-\sigma^2$, $-\sigma$, $-\sigma$ and $-\sigma$ precisely. The resulting $M(s, \sigma)$ is

$$M(s, \sigma) = \frac{N(s, \sigma)}{D(s, \sigma)}$$

where

$$\begin{aligned} N(s, \sigma) = 15s^3 &- (430\sigma^5 + 3000\sigma^4 + 8380\sigma^3 + 13720\sigma^2 + 19020\sigma + 17070)s^2 \\ &+ (1580\sigma^5 + 10620\sigma^4 + 28800\sigma^3 + 46320\sigma^2 + 63180\sigma + 55145)s \\ &- (1190\sigma^5 + 7740\sigma^4 + 20580\sigma^3 + 32760\sigma^2 + 44280\sigma + 38130) \end{aligned}$$

and

$$D(s, \sigma) = (s, \sigma^2)(s, \sigma)^3$$

By examining the behaviour of $M(j\omega, \sigma)$ at frequencies close to σ, we find that $\|M(j\sigma, \sigma)\|$ is of the order of $152\sigma^2$ for $\sigma \gg 1$. This shows that $M(j\omega, \sigma)$ is unbounded as $\sigma \to \infty$.

In summary we note the following: $M_-(s, \sigma)$ can be rendered zero either exactly or asymptotically by letting $\Lambda_-(\sigma)$ and $V_-(\sigma)$ to coincide either exactly or asymptotically with the set of plant minimum phase invariant zeros and the corresponding plant left state zero directions. $M_b(s, \sigma)$ can be rendered zero either exactly or asymptotically by assigning $\Lambda_b(\sigma)$ arbitrarily in $\mathscr{C}^-$ while $V_b(\sigma)$ is assigned either exactly or asymptotically to be in the null space of matrix $\tilde{\mathbf{B}}'$. $M_e(s, \sigma)$ can never be rendered zero either exactly or asymptotically by any means. However, it is important to recognize that the asymptotic recovery error matrix $\bar{M}_e(s)$ which depends on $\bar{\Lambda}_e$, $\bar{V}_e$ and $\bar{W}_e$ can be shaped in an infinite number of ways. As discussed in Part 1, $\bar{\Lambda}_e$ and $\bar{W}_e$ (and hence $\bar{V}_e$) can be assigned appropriately so that $\bar{M}_e(s)$ has certain chosen directional properties or it is as small (in some norm sense) as it could be. Needless to say such a freedom is of paramount importance to the designer. Next $M_\infty(s, \sigma)$ can be rendered asymptotically zero by selecting every element of $\Lambda_\infty(\sigma)$ arbitrarily and by assigning the associated right and left eigenvectors $W_i(\sigma)$ and $V_i(\sigma)$ such that $W_i(\sigma)V_i^H(\sigma)\tilde{\mathbf{B}}$ is uniformly bounded as $\sigma \to \infty$. The freedom in assigning $\Lambda_\infty(\sigma)$, $W_i(\sigma)$ and $V_i(\sigma)$ can be viewed as freedom to assign the asymptotic time-scale structure (TSS) and hence the infinite eigenstructure of the observer dynamic matrix. This freedom of assigning appropriate TSS, as will be seen later on, has an overwhelming impact on the value of controller gain and hence on the controller band-width for any prescribed recovery error.

2.2. *Limitations of ARE-based design*

As discussed in the introduction, one of the existing and prominent methods of design to achieve ALTR for arbitrary specified target loops is the ARE based design. It was developed originally for left invertible and minimum phase systems. However, as discussed earlier, ARE based design could be used for non-minimum phase systems as well provided one is content with the recovery error it results in. In ARE based design method, observer gain $\tilde{\mathbf{K}}(\sigma)$ is given by

$$\tilde{\mathbf{K}}(\sigma) = P\tilde{\mathbf{C}}'Q_2^{-1} \tag{2.10}$$

where P is the symmetric non-negative definite solution to the algebraic Riccati equation (ARE)

$$\tilde{\mathbf{A}}P + P\tilde{\mathbf{A}}' - P\tilde{\mathbf{C}}'Q_2^{-1}\tilde{\mathbf{C}}P + \sigma^2\tilde{\mathbf{B}}Q_1\tilde{\mathbf{B}}' = 0 \tag{2.11}$$

Here Q_1 and Q_2 are the covariance matrices of fictitious input and output noises and σ is a tuning parameter which controls the intensity of input process noise. Thus $\tilde{\mathbf{K}}(\sigma)$ is implicitly parameterized via ARE. Although, ARE based design has become common, the underlining mechanism of what it does and what it does not do is not well understood except for some recent study by Zhang and Freudenberg (1987, 1990) and Niemann and Jannerup (1990). The purpose of this section is to conduct such a study and there by find its advantages and limitations. A clear advantage of it is that at the onset of design, it does not require much systematic planning. By simply choosing some Q_1 and Q_2 (normally as identity matrices), one somehow solves (2.11) repeatedly for several values of σ and then checks to see whether $\tilde{\mathbf{K}}(\sigma)$ as calculated by (2.10) serves as an appropriate gain or not. For left invertible and minimum phase plants, one eventually obtains an observer gain by simply making σ larger and larger until the required level of loop transfer recovery is achieved. As seen in Part 1 (Saberi *et al.* 1991), for general systems such a recovery is not possible. Whether complete recovery is possible or not, so far one does not know in general what kind of freedom ARE based design has in shaping the attainable loop transfer function and sensitivity and complimentary sensitivity functions. As reviewed earlier, recovery of loop transfer function and sensitivity and complimentary sensitivity functions is tied with rendering $M(s, \sigma)$ zero. It turns out that in general ARE based design minimizes asymptotically, i.e. as $\sigma \to \infty$, the H_2 norm of $\tilde{M}(j\omega, \sigma) \equiv (j\omega I_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}})^{-1}\tilde{\mathbf{B}}$. The minimum asymptotic H_2 norm of $\tilde{M}(j\omega, \sigma)$ is zero for left invertible minimum phase systems and is non-zero otherwise. However, the path taken by ARE based design in minimizing the above H_2 norm may not always be the best path. To understand this and other aspects, we now proceed with a systematic investigation of ARE based design. Earlier $M(s, \sigma)$ has been partitioned into four parts. We will examine what ARE based design does to each one of these parts. In general, eigenvalues and eigenvectors of observer dynamic matrix are functions of σ. However, whenever there is no ambiguity, our notation suppresses the dependence on σ.

We first examine what ARE based design does to $M_-(s, \sigma)$. Let z_i^- and x_{Li}^-, $i = 1$ to n_a^-, be the minimum phase invariant zeros and the corresponding left state zero directions of the given system $\tilde{\Sigma}$. Let λ_i^- and V_i^-, $i = 1$ to n_a^-, be the eigenvalues and the associated left eigenvectors of $\tilde{\mathbf{A}}_0$ represented in $M_-(s, \sigma)$. Then following the results of Saberi and Sannuti (1987), one can show easily that for all $i = 1$ to n_a^-

$$\lambda_i^- \to z_i^- \text{ as } \sigma \to \infty$$

Moreover, since whenever an eigenvalue of $\tilde{\mathbf{A}}_0$ coincides with an invariant zero of $\tilde{\Sigma}$, the corresponding left eigenvector of $\tilde{\mathbf{A}}_0$ coincides with a corresponding left state zero direction of $\tilde{\Sigma}$, we have for all $i = 1$ to n_a^-

$$V_i^- \to x_{Li}^- \text{ as } \sigma \to \infty$$

Naturally then $M_-(s, \sigma) \to 0$ as $\sigma \to \infty$. The limitation of ARE based design in view of $M_-(s, \sigma)$ is that it always places some eigenvalues of $\tilde{\mathbf{A}}_0$ at the minimum phase invariant zeros only asymptotically as $\sigma \to \infty$ where as in general one has the freedom to place them either exactly or asymptotically at the minimum phase invariant zeros. However, this is not a major limitation.

Let us next examine what ARE based design does to $M_b(s, \sigma)$. Let λ_i^b and V_i^b, $i = 1$ to n_b, be the eigenvalues and the associated left eigenvectors of $\tilde{\mathbf{A}}_0$ represented in $M_b(s, \sigma)$. Then again following the results of Saberi and Sannuti (1987), one can show easily that some of the λ_i^b, $i = 1$ to n_b, coincide with stable but uncontrollable eigenvalues of $\tilde{\Sigma}$ while the rest of them tend to what are called 'compromise' zeros (Saberi and Sannuti 1987) as $\sigma \to \infty$. Also, it can be shown easily that the corresponding eigenvectors satisfy either $\tilde{\mathbf{B}}' V_i^b \equiv 0$ or $\tilde{\mathbf{B}}' V_i^b \to 0$ so that $M_b(s, \sigma) \to 0$ as $\sigma \to \infty$. The limitation of ARE based design in view of $M_b(s, \sigma)$ is that the locations of the eigenvalues of $\tilde{\mathbf{A}}_0$ represented in $M_b(s, \sigma)$ are fixed where as in general one has complete freedom to place them arbitrarily. However, this is not a major limitation either.

We now examine what ARE based design does to $M_e(s, \sigma)$. For simplicity of presentation, let us now assume that $\tilde{\Sigma}$ is left invertible but of non-minimum phase. Let z_i^+, x_{Ri}^+ and w_{Ri}^+, $i = 1$ to n_a^+, be the non-minimum phase invariant zeros and the corresponding right state and input zero directions of the given system $\tilde{\Sigma}$. Let

λ_i^+ and V_i^+, $i = 1$ to n_a^+, be the eigenvalues and the associated left eigenvectors of $\tilde{\mathbf{A}}_0$ represented in $M_e(s, \sigma)$. Then we have the following result.

Lemma 2.1

For each $i = 1$ to n_a^+, as $\sigma \to \infty$

$$\lambda_i^+ \to -(z_i^+)^*$$

where superscript * denotes complex conjugation. Furthermore

$$\tilde{\mathbf{B}}' V_i^+ \to c_i w_{Ri}^+ \tag{2.12}$$

for some constant c_i.

Proof

The behaviour of eigenvalues follows from (Saberi and Sannuti 1987, Zhang and Freudenberg 1990). For the behaviour of eigenvectors, see Appendix A.

The limitation of ARE based design regarding $M_e(s, \sigma)$ can now be seen easily in view of Lemma 2.1. The eigenvalues of $\tilde{\mathbf{A}}_0$ represented in $M_e(s, \sigma)$ are asymptotically assigned at the mirror images of the non-minimum invariant zeros while the corresponding error vectors, $e_i^+ = \tilde{\mathbf{B}}' V_i^+$, $i = 1$ to n_a^+, are assigned to the corresponding right input zero directions. This implies that there is no freedom in ARE based design to shape $M_e(s, \sigma)$ or $\bar{M}_e(s)$ directionally or otherwise. In fact as mentioned earler, it is straightforward to show that as $\sigma \to \infty$, ARE based design minimizes the H_2 norm of $\tilde{M}(j\omega, \sigma)$. However, such a mathematical minimization may not yield desired results. Moreover, the path taken by ARE based design in minimizing the H_2 norm of $\tilde{M}(j\omega)$ may not always be desirable from an engineering point of view. That is, for any fixed σ, the controller that results from ARE based design may not be acceptable suboptimal controller. One needs flexibility to shape the path and the eventual $\bar{M}_e(s)$ by appropriate selection of $\bar{\Lambda}_e$ and $\bar{W}_e$. Indeed, in general one has some freedom to shape $\bar{M}_e(s)$ and thus $M_e(s, \sigma)$ as discussed in Part 1 (Saberi *et al.* 1991) where an example was also given to illustrate what can be done with such a freedom. The lack of such a freedom in ARE based design is an important limitation.

Before we examine what ARE based design does to $M_\infty(s, \sigma)$, let us recall the infinite zero structure of $\tilde{\Sigma}$ as given by the list of structural invariant indices $\mathscr{C}^*$ [see (2.22) of Part 1],

$$\mathscr{C}^* = \{\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_{m_u}\}$$

$$= \{\overbrace{1, 1, \ldots, 1}^{\bar{q}_1}, \overbrace{2, 2, \ldots, 2}^{\bar{q}_2}, \ldots, \overbrace{KI, KI, \ldots, KI}^{\bar{q}_{KI}}\} \tag{2.13}$$

Let λ_i^∞, W_i^∞ and V_i^∞, $i = 1$ to n_f, be the eigenvalues and the associated right and left eigenvectors of $\tilde{\mathbf{A}}_0$ represented in $M_\infty(s, \sigma)$. As is well known, the eigenvalues λ_i^∞, $i = 1$ to n_f, go to infinity in m_u Butterworth patterns as $\sigma \to \infty$. Each pattern, say the ith pattern, has $\bar{n}_i$ eigenvalues and the radii of them vary at the rate of $\sigma^{1/\bar{n}_i}$. The limitation of ARE based design in view of $M_\infty(s, \sigma)$ can now be seen in the way the eigenvalues are pushed to infinity. As discussed above, the radii of far away eigenvalues vary with respect to σ at different rates. The fastest and slowest varying radii are respectively proportional to $\sigma^{1/\bar{n}_1}$ and $\sigma^{1/\bar{n}_{m_u}}$. Thus unless $\bar{n}_{m_u} = \bar{n}_1$, there is a wide spread among the radii of far away eigenvalues which vary at different rates and this rate of variation and hence the spread is dictated by the integers in $\mathscr{C}^*$. On the other hand, a given desired level of recovery $M_\infty(s, \sigma)$ requires the smallest far away eigenvalue to be of certain value, say $|\lambda_d|$, which fixes the value of the tuning parameter at some σ_d. Although for the given level of recovery, the smallest required radii of the far away eigenvalues need to be $|\lambda_d|$, one is forced by ARE based design to place some of the eigenvalues at much higher radii than $|\lambda_d|$. This forces the controller to have higher gain and hence higher band-width than is necessary. This can best be illustrated by an example. Let $\mathscr{C}^* = [1\ 1\ 1\ 1\ 1\ 4]$. Then in ARE based design, there are five eigenvalues whose radii vary at the rate of σ while there is one Butterworth pattern (having four eigenvalues) whose radius varies at the rate of $\sigma^{\frac{1}{4}}$. Suppose for the desired level of recovery, the smallest radius of far away eigenvalue is 10 implying that the desired value of σ is in the order of 10^4. Thus altogether there are nine far away eigenvalues among which five of them, owing to their rate of variation being proportional to σ, have radii in the order of 10^4 while the remaining four which vary at the rate of $\sigma^{\frac{1}{4}}$ have radii in the order of 10. On the other hand, if one is free to design the far away eigenvalues in any manner instead of being constrained by $\mathscr{C}^*$, one could design all these nine eigenvalues to vary at the same rate, say, all of them proportional to σ. Then for the desired level of recovery all the eigenvalues can have a radius of the order of 10 while the value of the tuning parameter is also of the order of 10. We will show shortly by means of examples, that this conserves the controller gain and band-width. In summary of this discussion, we note that the fast time-scale structure of the observer in ARE based design is dictated by the infinite zero structure of the given plant. On the other hand, as discussed earlier, there is in general plenty of freedom available in assigning a fast time-scale structure to the observer. This lack of freedom in ARE based design is the source of it requiring a much higher gain and hence band-width than is necessary. This is one of the major limitations of ARE based design.

In what follows we give four examples to illustrate this limitation. All the four examples consider minimum phase left invertible plants so than only $M_\infty(s, \sigma)$ plays a dominating role in the behaviour of mismatch function. For each example, two design methods, ARE based design and another ATEA design of § 2.3 are considered. Observer gain and observer eigenvalues are given for both the design methods so that they result in the same value for the supremum of maximum singular value of the mismatch function over a given frequency range. Also, for each example and for both the designs, the maximum singular value of the mismatch function is plotted with respect to the frequency over a given range. In each of the first three examples, there are different orders of infinite zeros and hance as expected ARE based design results in a much wider spread among different groups of eigenvalues than is necessary. Consequently, ARE based design requires higher gain and hence higher band-width than ATEA design. Example 2.5 illustrates another aspect of ARE based design. For this example $\mathscr{C}^* = \{1, 1\}$ and hence in ARE based design, there are two Butterworth patterns for fast eigenvalues, the radius of each pattern being asymptotically proportional to σ. However, the two proportionality constants are vastly different in magnitude since the given system matrices $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ contain elements having different orders of magnitude. Because of this ARE based design again results in a wide spread among the two fast eigenvalues. On the other hand, in ATEA design, one is not constrained by Butterworth patterns and is free to assign any eigenstructure so that the required gain and controller band-width are conserved.

Example 2.2

Let $\tilde{\Sigma}$ be characterized by

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -4 & -4 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{\mathbf{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

This system has no invariant zeros while the infinite zero structure is given by $\mathscr{C}^* = \{1, 4\}$. The target loop transfer function $\tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$ is specified by giving

$$\tilde{\mathbf{F}} = \begin{bmatrix} 8{\cdot}1125 & -0{\cdot}7311 & -0{\cdot}0391 & -0{\cdot}7926 & 0{\cdot}6427 \\ 0{\cdot}6427 & 12{\cdot}0911 & 11{\cdot}9185 & 8{\cdot}7115 & 4{\cdot}2773 \end{bmatrix}$$

Table 1 and Fig. 1 compare ARE based design and ATEA design.

Example 2.3

Consider the example of Doyle and Stein (1981) characterized by

$$\tilde{\mathbf{A}} = \begin{bmatrix} -0{\cdot}02 & 0{\cdot}005 & 2{\cdot}4 & -32 \\ -0{\cdot}14 & 0{\cdot}44 & -1{\cdot}3 & -30 \\ 0 & 0{\cdot}018 & -1{\cdot}6 & 1{\cdot}2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0{\cdot}14 & -0{\cdot}12 \\ 0{\cdot}36 & -8{\cdot}6 \\ 0{\cdot}35 & -0.009 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{\mathbf{C}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57{\cdot}3 \end{bmatrix}$$

	ARE based	ATEA
Observer gain	$\begin{pmatrix} 10001 & 0{\cdot}9994 \\ 0{\cdot}9994 & 25{\cdot}904 \\ 25{\cdot}904 & 336{\cdot}02 \\ 337{\cdot}02 & 2552{\cdot}1 \\ 2573{\cdot}0 & 9658{\cdot}2 \end{pmatrix}$	$\begin{pmatrix} 51 & 0 \\ 0 & 16 \\ 0 & 175{\cdot}6916 \\ 0 & 893{\cdot}5328 \\ -4 & 3030{\cdot}954 \end{pmatrix}$
Gain norm 2	11370·62	3164·84
Eigenvalues of observer	-10000 $-3{\cdot}83 + j9{\cdot}23$ $-3{\cdot}83 - j9{\cdot}23$ $-9{\cdot}25 + j3{\cdot}82$ $-9{\cdot}25 - j3{\cdot}82$	-50 $-4 + j7$ $-4 - j7$ $-4 + j5{\cdot}54$ $-4 - j5{\cdot}54$
$\sup \{\sigma_{\max}[E(j\omega)]\}$	7·8263	7·8262

Table 1. Comparison of ARE based design and ATEA design (over the frequency range: 0·01 to 100 rad s^{-1}).

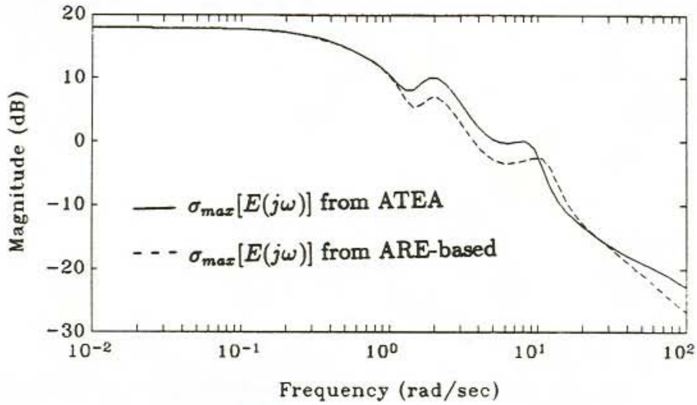

Figure 1. Maximum singular values of $E(j\omega)$ of Example 2.2.

This system has one invariant zero at $s = -0{\cdot}018$ while the infinite zero structure is given by $\mathscr{C}^* = \{1, 2\}$. Unlike in Doyle and Stein (1981), the required target loop transfer function is specified by

$$\tilde{\mathbf{F}} = \begin{bmatrix} -0{\cdot}0033 & 0{\cdot}0472 & 14{\cdot}6420 & 60{\cdot}8890 \\ 0{\cdot}0171 & -1{\cdot}0515 & 0{\cdot}2927 & 3{\cdot}2469 \end{bmatrix}$$

Table 2 and Fig. 2 compare ARE based design and ATEA design.

	ARE based	ATEA
Observer gain	$\begin{bmatrix} 1{\cdot}2334 & 1{\cdot}5730 \\ 86{\cdot}5144 & -0{\cdot}1090 \\ 0{\cdot}0425 & 3{\cdot}0041 \\ -0{\cdot}0019 & 0{\cdot}3238 \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}5083 & 1{\cdot}4804 \\ 36{\cdot}0000 & 0{\cdot}0000 \\ 0{\cdot}0000 & 2{\cdot}8272 \\ 0{\cdot}0000 & 0{\cdot}3141 \end{bmatrix}$
Gain norm 2	86·5232	36·0036
Eigenvalues of observer	$-0{\cdot}018$ $-10{\cdot}10 + j9{\cdot}95$ $-10{\cdot}10 - j9{\cdot}95$ $-86{\cdot}0759$	$-0{\cdot}018$ $-9{\cdot}8 + j9{\cdot}67$ $-9{\cdot}8 - j9{\cdot}67$ $-35{\cdot}56$
$\sup\{\sigma_{\max}[E(j\omega)]\}$	5·2405	5·7665

Table 2. Comparison of ARE based design and ATEA design (over frequency range: 0·1 to 1000 rad s^{-1}).

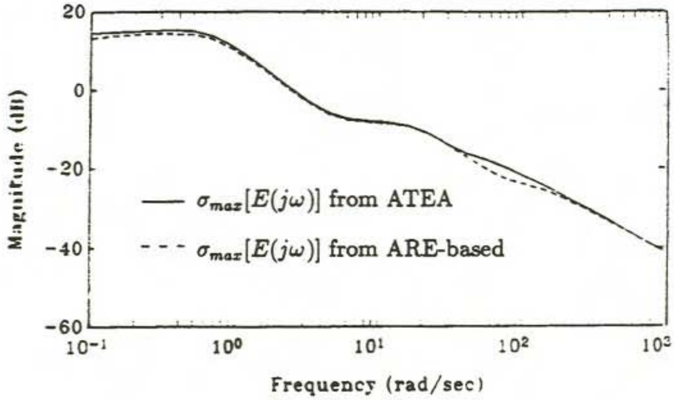

Figure 2. Maximum singular values of $E(j\omega)$ of Example 2.3.

Example 2.4

Consider the example in (Sogaard-Anderson 1987, Saberi and Sannuti 1990) and characterized by

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 0{\cdot}9945 & 0{\cdot}1044 & 0 \\ 0 & -1{\cdot}5250 & 0{\cdot}0678 & -30{\cdot}02 \\ 0 & -0{\cdot}0166 & -0{\cdot}1502 & 5{\cdot}159 \\ 0{\cdot}035 & 0{\cdot}0689 & -0{\cdot}9920 & -0{\cdot}0903 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 \\ 11{\cdot}51 & 5{\cdot}241 \\ 0{\cdot}1894 & -1{\cdot}968 \\ -0{\cdot}003 & 0{\cdot}135 \end{bmatrix}$$

$$\tilde{\mathbf{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

This system has one invariant zero at $s = -0{\cdot}6$ while the infinite zero structure is given by $\mathscr{C}^* = \{1, 2\}$. The required target loop transfer function is specified by

$$\tilde{\mathbf{F}} = \begin{bmatrix} 1{\cdot}14 & 0{\cdot}410 & 0.27 & -1{\cdot}02 \\ -0.53 & 0.058 & -2.21 & 6{\cdot}85 \end{bmatrix}$$

Table 3 and Fig. 3 compare ARE based design and ATEA design.

	ARE based	ATEA
Observer gain	$\begin{pmatrix} 6{\cdot}1989 & 1{\cdot}9529 \\ 19{\cdot}9046 & 127{\cdot}13 \\ 12{\cdot}6888 & -6{\cdot}776 \\ -5{\cdot}2629 & -0{\cdot}2596 \end{pmatrix}$	$\begin{pmatrix} 10{\cdot}000 & 0{\cdot}0000 \\ 46{\cdot}448 & 0{\cdot}3694 \\ 36{\cdot}465 & -3{\cdot}5193 \\ -9{\cdot}983 & 1{\cdot}1113 \end{pmatrix}$
Gain norm 2	128·8000	60·7523
Eigenvalues of observer	$-0{\cdot}65$ $-3{\cdot}04 + j3{\cdot}41$ $-3{\cdot}04 - j3{\cdot}41$ $-134{\cdot}89$	$-0{\cdot}6$ $-3{\cdot}53 + j3{\cdot}26$ $-3{\cdot}53 - j3{\cdot}26$ $-9{\cdot}1$
$\sup\{\sigma_{\max}[E(j\omega)]\}$	352·93	364·40

Table 3. Comparison of ARE based design and ATEA design (over frequency range: 0·01 to 100 rad s^{-1}).

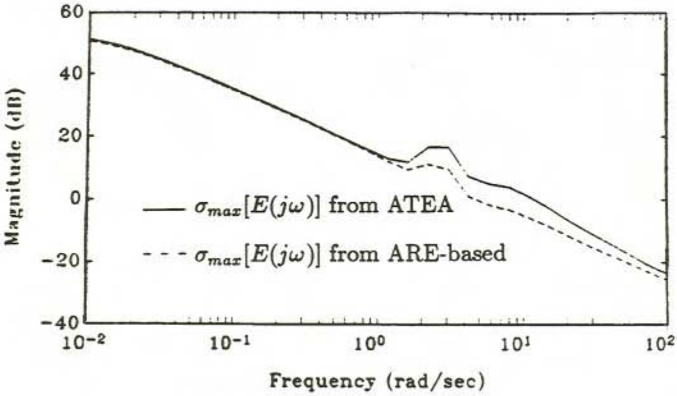

Figure 3. Maximum singular values of $E(j\omega)$ of Example 2.4.

Example 2.5

Consider the example of Kazerooni and Houpt (1986) characterized by

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 76 & -105 \\ -105 & 280 \end{bmatrix}, \quad \tilde{\mathbf{C}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

This system has invariant zeros at $s = -0{\cdot}25$ and $s = -1$ while the infinite zero structure is given by $\mathscr{C}^* = \{1, 1\}$. The required target loop transfer function is specified by

$$\tilde{\mathbf{F}} = \begin{bmatrix} 4{\cdot}7234 & 3{\cdot}4265 & 0{\cdot}9923 & 0{\cdot}6631 \\ 1{\cdot}1497 & 0{\cdot}8579 & 0{\cdot}2633 & 0{\cdot}1952 \end{bmatrix}$$

Table 4 and Fig. 4 compare ARE based design and ATEA design.

	ARE based	ATEA
Observer gain	$\begin{pmatrix} 0{\cdot}9987 & -0{\cdot}0001 \\ -0{\cdot}0000 & 0{\cdot}2500 \\ 503{\cdot}52 & -1302{\cdot}1 \\ -325{\cdot}52 & 3204{\cdot}3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0{\cdot}25 \\ 300 & 0 \\ 0 & 82 \end{pmatrix}$
Gain norm 2	3493·7	300·0
Eigenvalues of observer	−0·25 −1·00 −367·34 −12953·0	−0·25 −1·00 −300·00 −328·00
$\sup\{\sigma_{\max}[E(j\omega)]\}$	$4{\cdot}1809 \times 10^5$	$4{\cdot}1713 \times 10^5$

Table 4. Comparison of ARE based design and ATEA design (over frequency range: 0·01 to 100 rad s^{-1}).

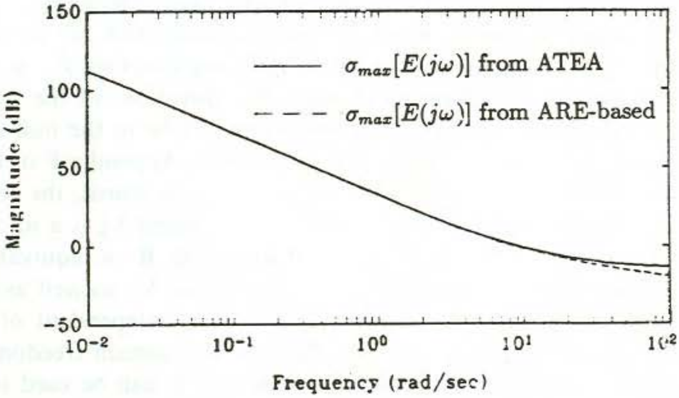

Figure 4. Maximum singular values of $E(j\omega)$ of Example 2.5.

To summarize this subsection, we note that the major limitations of ARE based design lie in the way it arrives at $M_e(s, \sigma)$ and $M_\infty(s, \sigma)$. $M_e(s, \sigma)$ cannot be rendered zero by any method. However, in general there is some freedom available in shaping it directionally or otherwise. ARE based design shapes it in a particular way dictated by the finite zero structure of the given system. Although $M_\infty(s, \sigma)$ can be rendered asymptotically zero as $\sigma \to \infty$, the way it is accomplished in ARE based design requires a very high gain and consequently it results in a higher controller band-width than is really necessary. This is because the fast time-scale structure induced in the observer dynamics by ARE based design is fixed and is dictated by the infinite zero structure of the given system. This is by far a very severe limitation of ARE based design as conservation of controller band-width is sought in almost all practical designs. Besides these limitations, due to the implicit parameterization of gain, a non-linear algebraic Riccati equation (2.11) has to be solved repeatedly. Such a solution is numerically cumbersome as (2.11) is 'stiff', especially for large σ owing to the interaction of several slow and fast time-scales. This can be brushed off as being a numerical problem. Nevertheless, it is an important limitation in practice.

2.3. *Observer design by ATEA*

As discussed above, the conventional ARE based design has several serious limitations. Here we present a new design method which overcomes these limitations. The new design method follows the asymptotic time-scale and eigenstructure assignment (ATEA) concepts proposed originally in Saberi and Sannuti (1989). Following those concepts, we developed earlier an observer design for left invertible and minimum phase plants in Saberi and Sannuti (1990). In what follows, we will present a step by step algorithm for general systems. The method is decentralized in nature. It uses the special coordinate basis (s.c.b.) of the given system $\tilde{\Sigma}$ (see theorem 2.1 of Part 1, Saberi *et al.* 1991 and Sannuti and Saberi 1987). The specified finite eigenstructure of the observer dynamic matrix $\tilde{\mathbf{A}}_0 = \tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}}$ is assigned appropriately by working with subsystems which represent the finite zero structure of the given system (see (2.1) to (2.4) of Part 1). Similarly the specified asymptotically infinite eigenstructure of $\tilde{\mathbf{A}}_0$ is assigned appropriately by working with subsystems which represent the infinite zero structure of the given system (see (2.5) of Part 1 for each $i = 1$ to m_u).

As discussed earlier, regarding the asymptotically finite eigenvalues, designer has the freedom to specify $\bar{\Lambda}_b$ and $\bar{\Lambda}_e$ while $\bar{\Lambda}_-$ must coincide with the set of minimum phase invariant zeros of the plant. The set of right eigenvectors $\bar{V}_-$ is constrained to coincide with the corresponding set of state zero directions of the plant. On the other hand, the set of eigenvectors $\bar{V}_b$ is constrained to be in the null space of $\tilde{\mathbf{B}}'$. In view of the special structure of s.c.b. and as shown in Appendix F of Part 1 every element $\bar{V}_i^b$ of $\bar{V}_b$ has the form $[0\ 0\ (V_i^b)'\ 0\ 0]'$. In other words, the set $\bar{V}_b$ can be represented in a matrix notation as $[0\ 0\ (V_b^b)'\ 0\ 0]'$ where V_b^b is a $n_b \times n_b$ matrix. Thus the selection of $\bar{V}_b$ to be in the null space of $\tilde{\mathbf{B}}'$ is equivalent to any appropriate selection of V_b^b. The eigenvalue sets, $\bar{\Lambda}_-$ and $\bar{\Lambda}_b$, as well as eigenvector sets, $\bar{V}_-$ and $\bar{V}_b$, could either be assigned exactly, i.e. independent of the tuning parameter σ, or asymptotically as $\sigma \to \infty$. There is also certain freedom to specify $\bar{W}_e$. This freedom along with the freedom in selecting $\bar{\Lambda}_e$ can be used to shape the recovery error matrix $\bar{M}_e(s)$ so that it has certain desired directional properties or it is as small as it could be. Again due to the special structure of s.c.b. and as shown in Appendix F of Part 1, $\bar{W}_e$ has the special matrix form $[(W_e^+)'\ 0\ 0\ (W_e^c)'\ 0]'$ where $W_{ee} = [(W_e^+)', (W_e^c)']'$ is a $n_e \times n_e$ matrix. Thus an appropriate selection of $\bar{W}_e$ is equivalent to a similar selection of W_{ee}. Next, the freedom that exists in specifying the asymptotically infinite eigenstructure of $\tilde{\mathbf{A}}_0$ reflects itself in specifying an appropriate fast time-scale structure. The asymptotic directions of asymptotically infinite eigenvalues can be specified by the sets Λ_l, $l = 1$ to r, where r is an integer less than or equal to n_f. The relative fastness of time-scales is specified by specifying the small positive parameters μ_l, $l = 1$ to r, which are appropriate functions of the tuning parameter σ so that (2.9) is true as $\sigma \to \infty$. There is a constraint on the infinite eigenstructure, namely, for every asymptotically infinite eigenvalue $\lambda_i(\sigma)$, the corresponding right and left eigenvectors $W_i(\sigma)$ and $V_i(\sigma)$ of $\tilde{\mathbf{A}}_0$ must be such that $W_i(\sigma)V_i^H(\sigma)\tilde{\mathbf{B}}$ is uniformly bounded as $\sigma \to \infty$.

In what follows, we give a step by step design algorithm. In view of the above discussion, the input parameters of the algorithm are $\bar{\Lambda}_b$, V_b^b, $\bar{\Lambda}_e$, W_{ee}, Λ_l and μ_l, $l = 1$ to r, as well as the integer r. In fact, the primary inputs to the algorithm are (1) $\bar{\Lambda}_e$ and W_{ee} which shape the resulting $\bar{M}_e(s)$ and (2) Λ_l and μ_l, $l = 1$ to r, which control the time-scale structure of the observer and thus have a strong impact on the resulting gain of the controller. The rest of the input parameters, namely $\bar{\Lambda}_b$ and V_b^b are secondary inputs to the algorithm. Our algorithm can be divided into three steps. Steps 1 and 2 deal respectively with subsystem designs to assign the asymptotically finite and infinite eigenstructures. In Step 3, subsystem designs of Steps 1 and 2 are put together to form a composite design for the given system.

Step 1. This step deals with the assignment of asymptotically finite eigenstructure (i.e. slow time-scale structure) and makes use of subsystems (2.1)–(2.4) of Part 1. $\lambda(A_{aa}^-)$ are the minimum phase invariant zeros of the given system $\tilde{\Sigma}$ and these are left alone to form some of the eigenvalues of $\tilde{\mathbf{A}}_0$, namely the set $\bar{\Lambda}_-$, while the corresponding left eigenvectors of $\tilde{\mathbf{A}}_0$ coincide with the corresponding left state zero directions of $\tilde{\Sigma}$. To place the set of eigenvalues $\bar{\Lambda}_b$ and left eigenvectors $\bar{V}_b$, choose a gain K_b such that $\lambda(A_{bb}^c)$ coincides with $\bar{\Lambda}_b$ while V_b^b coincides with the set of left eigenvectors of A_{bb}^c where

$$A_{bb}^c = A_{bb} - K_b C_s \tag{2.14}$$

Note that the existence of such a K_b is guaranteed by property 2.1 of § 2 of Part 1 (Saberi *et al.* 1991) as long as the eigenvector set $\bar{V}_b$ is consistent with the freedom available in assigning it (Moore 1976). Next, in order to place the set of eigenvalues $\bar{\Lambda}_e$ and right eigenvectors W_{ee}, let us first form matrices A_{ee} and C_e as follows:

$$A_{ee} = \begin{bmatrix} A_{aa}^{+} & 0 \\ B_c E_{ca}^{+} & A_{cc} \end{bmatrix}, \quad C_e = [E_a^{+} \quad E_c] \tag{2.15}$$

where

$$E_a^{+} = [(E_{1a}^{+})' \quad (E_{2a}^{+})' \quad \dots \quad (E_{m_u a}^{+})'], \quad E_{ia} = [E_{ia}^{+} \quad E_{ia}^{-}],$$
$$E_c = [E'_{1c} \quad E'_{2c} \quad \dots \quad E'_{m_u c}]'$$

Now select a gain K_e such that the set of eigenvalues and right eigenvectors of A_{ee}^{c} coincide with $\bar{\Lambda}_e$ and W_{ee} respectively where

$$A_{ee}^{c} = A_{ee} = K_e C_e \tag{2.16}$$

Again note that the existence of such a K_e is guaranteed by property 2.1 of § 2 of Saberi *et al.* (1991) as long as the eigenvector set W_{ee} is consistent with the freedom available in assigning it (Moore 1976). For future use, let us partition K_e as

$$K_e = [K_{e1} \quad K_{e2} \quad \dots \quad K_{em_u}] \tag{2.17}$$

where K_{ei} is a $n_e \times 1$ dimensional vector.

Step 2. This step deals with the assignment of asymptotically infinite eigenstructure (i.e. the fast time-scale structure) and makes use of subsystems, $i = 1$ to m_u, represented by (2.5) of Part 1. As discussed earlier, there is complete freedom to specify any $r \leqslant n_f$ fast time-scales. In particular, one can always choose $r = 1$. For generality, we will keep r as arbitrarily given. The freedom in assigning the fast time-scales is reflected in specifying the sets Λ_l, and the small positive parameters μ_l, $l = 1$ to r. Our design to assign an appropriate fast time-scale structure is again decentralized. We deal with one single input single output system at a time as represented by (2.5) of Part 1 for a particular value of i, $i = 1$ to m_u. Thus to proceed with our design, we need to distribute the designer specified elements of the sets Λ_l, and the parameters μ_l, $l = 1$ to r, among m_u subsystems. There exists a complete freedom in such a distribution and hence it can be done in a number of ways. Let subsystem i be assigned r_i time-scales for some $r_i \leqslant q_i$. Let

$$\frac{\Lambda_{ij}}{\mu_{ij}}, \quad j = 1 \text{ to } r_i$$

be the asymptotically infinite eigenvalues that need to be assigned to subsystem i. Let n_{ij} be the number of eigenvalues corresponding to the time-scale t/μ_{ij}. That is, let Λ_{ij} contain n_{ij} elements. As usual, the set Λ_{ij} is assumed to be closed under complex conjugation. Also, in order to have a well defined separation of time-scales in subsystem i, we will assume that

$$\mu_{ij}/\mu_{ij+1} \to 0 \quad \text{as } \mu_{ij+1} \to 0 \text{ for all } j = 1 \text{ to } r_i - 1 \tag{2.18}$$

We note that when $r = 1$, all μ_{ij} are equal to a single parameter μ and all r_i are equal to unity. That is, there is only one time-scale to be assigned to all subsystems. In this case, σ can be taken as $1/\mu$. With these preliminaries, we are now ready to design the i-th subsystem. At first, we will design a gain matrix K_{ij} for each time-scale t/μ_{ij}, $j = 1$ to r_i. Define a $n_{ij} \times n_{ij}$ dimensional matrix G_{ij} and a $1 \times n_{ij}$ dimensional matrix C_{ij} having the following structure

$$G_{ij} = \begin{bmatrix} 0 & I_{n_{ij}-1} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_{ij} = [1 \quad 0]$$

Choose a $n_{ij} \times 1$ dimensional gain vector K_{ij} such that $\lambda(G_{ij}^c)$ coincides with Λ_{ij} where $G_{ij}^c = G_{ij} - K_{ij}C_{ij}$. Owing to the special structure of G_{ij} and C_{ij}, such a K_{ij} always exists. Let K_{ij} be partitioned as

$$K_{ij} = \begin{bmatrix} K_{ijc} \\ K_{ijd} \end{bmatrix}$$

where K_{ijd} is a scalar. Moreover, the non-singularity of G_{ij}^c implies that K_{ijd} is non-zero. Next, the gains K_{ij}, $j = 1$ to r_i, obtained above are put together to form a composite gain vector which will induce the required fast time-scales in the i-th subsystem. Define the scalar numbers J_{ij} as

$$J_{i1} = 1, \quad J_{ij} = \prod_{l=1}^{j-1} K_{ild}, \quad j = 2 \text{ to } r_i$$

Let $\alpha_{i0} = 0$ and

$$\alpha_{ij} = \sum_{k=1}^{j} n_{ik}, \quad j = 1 \text{ to } r_i$$

Note that $\alpha_{ir_i} = q_i$. Also, let for each $j = 1$ to r_i

$$\varepsilon_{i\alpha_{ij-1}+1} = \varepsilon_{i\alpha_{ij-1}+2} = \dots = \varepsilon_{i\alpha_{ij}} = \mu_{ij}$$

and

$$\eta_i = \prod_{k=1}^{q_i} \varepsilon_{ik} \tag{2.19}$$

Also, define a scaling matrix S_{ij} as

$$S_{ij} = \text{Diag}\left[\prod_{l=\alpha_{ij-1}+2}^{q_i} \varepsilon_{il}, \prod_{l=\alpha_{ij-1}+3}^{q_i} \varepsilon_{il}, \dots, \prod_{l=\alpha_{ij}+1}^{q_i} \varepsilon_{il} \right] \tag{2.20}$$

In (2.20), for $j = r_i$, the product $\prod_{l=q_i+1}^{q_i} \varepsilon_{il}$ is taken as unity. Now let

$$\tilde{K}_{ij}(\sigma) = \frac{1}{\eta_i} J_{ij} S_{ij} K_{ij}$$

and

$$\tilde{K}_i(\sigma) = [\tilde{K}_{i1}' \quad \tilde{K}_{i2}' \quad \dots \quad \tilde{K}_{ir_i}']' \tag{2.21}$$

The above design is rather simple when $r_i = 1$. For this case, let $\bar{\mu}_i$ denote the small parameter. Then

$$\tilde{K}_i(\sigma) = \frac{1}{(\bar{\mu}_i)^{q_i}} [(\bar{\mu}_i)^{q_i-1}\hat{K}_{i1} \quad (\bar{\mu}_i)^{q_i-2}\hat{K}_{i2} \quad \dots \quad \hat{K}_{iq_i}]' \tag{2.22}$$

where $\hat{K}_{ij}$, $j=1$ to q_i, are selected such that $\lambda(G_i^c)$ are as desired where

$$G_i^c = -\begin{bmatrix} \hat{K}_{i1} & \hat{K}_{i2} & \dots & \hat{K}_{iq_i-1} & \hat{K}_{iq_i} \\ & & -I_{q_i-1} & & 0 \end{bmatrix}'$$

Here we did not discuss any eigenvector assignment. However, it turns out that our eventual design is such that the eigenvectors corresponding to the asymptotically infinite eigenvalues are naturally assigned to appropriate locations so that $M_\infty(j\omega, \sigma) \to 0$ as $\sigma \to \infty$.

Step 3. In this step, various gains calculated in Steps 1 and 2 are put together to form a composite observer gain for the given system $\tilde{\Sigma}$. Define $\tilde{K}_e$ as

$$\tilde{K}_e(\sigma) = \begin{bmatrix} \tilde{K}_a^+(\sigma) \\ \tilde{K}_c(\sigma) \end{bmatrix} = [\tilde{K}_{e1} \quad \tilde{K}_{e2} \quad \dots \quad \tilde{K}_{em_u}], \quad \tilde{K}_{ei} = \frac{1}{\eta_i} J_{ir_i} K_{ir_id} K_{ei} \tag{2.23}$$

For the case when $r_i = 1$, K_{i1d} is the same as $\hat{K}_{iq_i}$ and η_i is same as $(\bar{\mu}_i)^{q_i}$. Finally define the observer gain $\tilde{\mathbf{K}}(\sigma)$ as

$$\tilde{\mathbf{K}}(\sigma) = \Gamma_1 \tilde{K}(\sigma) \Gamma_2^{-1} \tag{2.24}$$

where

$$\tilde{K}(\sigma) = \begin{bmatrix} L_{af}^+ + \tilde{H}_{af}^+ + \tilde{K}_a^+(\sigma) & L_{as}^+ + \tilde{H}_{as}^+ \\ L_{af}^- + \tilde{H}_{af}^- & L_{as}^- + \tilde{H}_{as}^- \\ L_{bf} + \tilde{H}_{bf} & K_b \\ L_{cf} + \tilde{H}_{cf} + \tilde{K}_c(\sigma) & L_{cs} + \tilde{H}_{cs} \\ L_f + \tilde{K}_f(\sigma) & 0 \end{bmatrix} \tag{2.25}$$

and where

$$\tilde{K}_f(\sigma) = \mathrm{Diag}\,[\tilde{K}_1(\sigma) \quad \tilde{K}_2(\sigma) \quad \dots \quad \tilde{K}_{m_u}(\sigma)]$$

$$L_f = [L_1' \quad L_2' \quad \dots \quad L_{m_u}']'$$

while the gains $\tilde{H}_{af}^+$, $\tilde{H}_{as}^+$, $\tilde{H}_{af}^-$, $\tilde{H}_{as}^-$, $\tilde{H}_{bf}$, $\tilde{H}_{cf}$ and $\tilde{H}_{cs}$ are arbitrary but finite. We have the following theorem.

Theorem 2.1

Consider an observer with its gain given by (2.24). Then we have the following properties.

(1) There exists a σ^* such that for all $\sigma > \sigma^*$, the designed observer is asymptotically stable. Furthermore, it has the time-scale structure t, t/μ_{ij}, $j = 1$ to r_i, $i = 1$ to m_u. That is, the eigenvalues of the observer as $\mu_r \to 0$ are given by

$$\bar{\Lambda}_- + 0(\mu_r), \quad \bar{\Lambda}_b + 0(\mu_r), \quad \bar{\Lambda}_e + 0(\mu_r)$$

$$\frac{\Lambda_{ij}}{\mu_{ij}} + 0(1) \text{ for } j = 1 \text{ to } r_i \text{ and } i = 1 \text{ to } m_u$$

Moreover, if $\tilde{H}_{af}^- = 0$ and $\tilde{H}_{bf} = 0$, some finite eigenvalues of $\tilde{\mathbf{A}}_0$ are exactly equal to $\bar{\Lambda}_-$ and $\bar{\Lambda}_b$ for all σ rather than asymptotically tending to $\bar{\Lambda}_-$ and $\bar{\Lambda}_b$.

(2) LTR is achieved as intended in the sense that as $\sigma \to \infty$

$$M(s, \sigma) \to \bar{M}_e(s) \text{ pointwise in } s$$

Proof

See Appendix B.

As can be easily seen, ATEA design is decentralized. Required time-scale structure and eigenstructure is assigned to the subsystems of the given system $\tilde{\Sigma}$. The calculations involved in subsystem designs do not explicitly require the value of tuning parameter σ. σ enters only in (2.21) or (2.22) where subsystem designs are put together to form a composite gain which assigns the required time-scale structure. Thus σ truely and directly acts as a tuning parameter and controls the degree of fastness of fast time-scales. We present next an example to illustrate ATEA design algorithm.

Example 2.6

Let $\tilde{\Sigma}$ be characterized by

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 4 & 4 & 4 & 0 & 0 \\ 0 & 5 & 5 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \tilde{\mathbf{C}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

This system is left invertible and is of non-minimum phase with two invariant zeros at $s = 1$ and 2. Also, $\tilde{\Sigma}$ is already in the form of s.c.b. We note that

$$A_{ee} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad C_e = \begin{bmatrix} 4 & 4 \\ 0 & 5 \end{bmatrix}$$

Let the target loop transfer function $\tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$ be specified by

$$\tilde{\mathbf{F}} = \begin{bmatrix} 65 & 15 & 23 & 50 & 0 \\ 65 & 15 & 23 & 0 & 50 \end{bmatrix}$$

Since the given system is of non-minimum phase, we cannot completely recover the given target loop. Let

$$\bar{\Lambda}_e = \{-1{\cdot}5 \quad -2{\cdot}5\} \quad \text{and} \quad W_{ee} = [W_{ee1} \quad W_{ee2}] = \begin{bmatrix} 0{\cdot}4856 & -0{\cdot}3940 \\ -0{\cdot}8742 & 0{\cdot}9191 \end{bmatrix}$$

so that the recovery error $\bar{M}_e(s)$ is prescribed as

$$\bar{M}_e(s) = -\frac{15{\cdot}3086(s + 8{\cdot}2973)}{(s + 1{\cdot}5)(s + 2{\cdot}5)} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Then K_e is given by

$$K_e = \begin{bmatrix} -0{\cdot}2188 & 0 \\ 1{\cdot}9687 & 0 \end{bmatrix}$$

Also, in this example

$$A_{bb} = 0 \quad \text{and} \quad C_s = 1$$

Let the required $\bar{\Lambda}_b$ be -3. Then $K_b = 3$. Let us require only one fast time-scale and let the required Λ_1 be

$$\Lambda_1 = \{-1 \quad -1\}$$

Then following the ATEA algorithm, the gain $\tilde{\mathbf{K}}(\sigma)$ is given by

$$\tilde{\mathbf{K}}(\sigma) = \begin{bmatrix} -0\cdot2188\sigma & 1 & 1 \\ 1\cdot9687\sigma & 2 & 2 \\ 3 & 3 & 3 \\ \sigma & 0 & 4 \\ 0 & \sigma & 5 \end{bmatrix}$$

This $\tilde{\mathbf{K}}(\sigma)$ places one observer eigenvalue exactly at -3 and the remaining eigenvalues asymptotically at $-1\cdot5$, $-2\cdot5$, $-\sigma$ and $-\sigma$. Figures 5 and 6 show the plots of maximum and minimum singular values of the target and the achieved loop transfer functions as well as the sensitivity functions for $\sigma = 1000$.

3. Design for exactly recoverable target loops

In this section, we consider the design of observer based controllers for exactly recoverable target loops. As stated in Theorem 3.3 of Part 1 (Saberi *et al.* 1991), a target loop transfer function, $L(s) = \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$, is exactly recoverable iff $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) \subseteq \operatorname{Ker} \tilde{\mathbf{F}}$. Also, in view of Properties 2.3 of Part 1, $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ is the span of $\tilde{x}_a^+ \oplus \tilde{x}_c \oplus \tilde{x}_f$. This implies that $L(s)$ is exactly recoverable if $\tilde{\mathbf{F}}$ is in the form

$$\tilde{\mathbf{F}} = \Gamma_3 F \Gamma_1^{-1}, \quad F = \begin{bmatrix} 0 & F_{a1}^- & F_{b1} & 0 & 0 \\ 0 & F_{a2}^- & F_{b2} & 0 & 0 \end{bmatrix} \tag{3.1}$$

where Γ_3 and Γ_1 are non-singular transformation matrices as defined in Theorem 2.1 of Part 1. Now in view of (*a*) Lemmas 3.2 and 3.3 of Part 1, (*b*) the form of F as in (3.1) and (*c*) interpretations of different partitions of $M(s, \sigma)$ as in § 2.1, it is easy to note the following.

(1) A set of n_a^- eigenvalues of $\tilde{\mathbf{A}}_0$, namely Λ_-, must be chosen to coincide exactly with the set of plant minimum phase invariant zeros while the corresponding left eigenvectors of $\tilde{\mathbf{A}}_0$ must coincide exactly with the corresponding left state zero directions of $\tilde{\Sigma}$ so that $M_-(s, \sigma)$ is rendered zero.

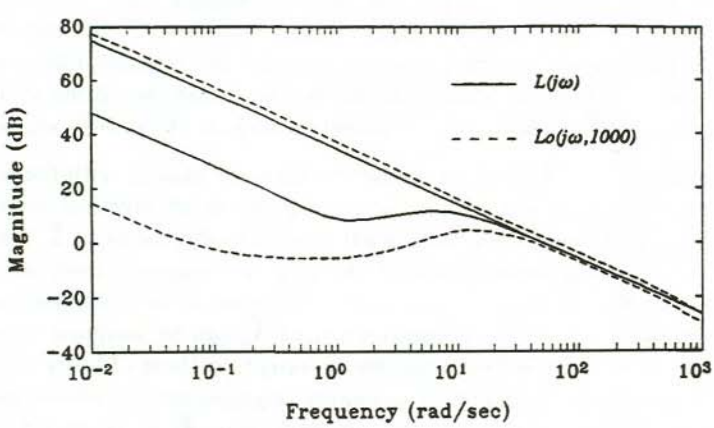

Figure 5. Maximum and minimum singular values of $L(j\omega)$ and $L_0(j\omega, 1000)$ for Example 2.6.

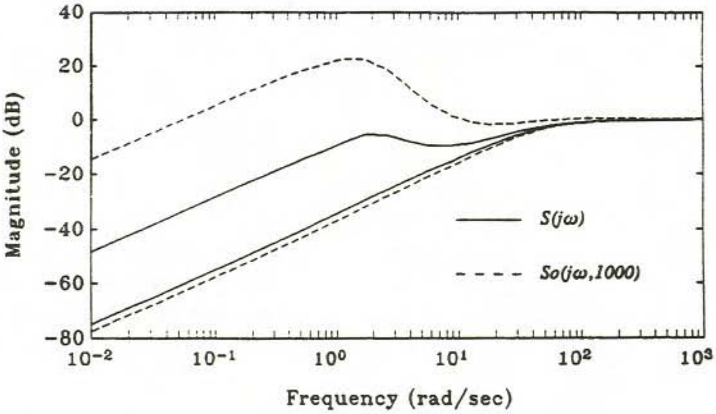

Figure 6. Maximum and minimum singular values of $S(j\omega)$ and $S_0(j\omega, 1000)$ for Example 2.6.

(2) A set of n_b eigenvalues of $\tilde{\mathbf{A}}_0$, namely Λ_b, can be assigned arbitrarily at finite locations in $\mathscr{C}^-$. Moreover, the eigenvector set V_b correspoding to these eigenvalues can be selected freely within the constraints defined in Moore (1976). However, V_b must be selected to be in the null space of $\tilde{\mathbf{B}}'$ so that $M_b(s, \sigma)$ is rendered zero.

(3) A set of $n_a^+ + n_c$ eigenvalues of $\tilde{\mathbf{A}}_0$, can be assigned arbitrarily at finite locations in $\mathscr{C}^-$ subject to the condition that any unobservable but stable eigenvalues of the given system must be included among Λ_e. Moreover, the eigenvector set W_{ee} corresponding to these eigenvalues can be selected freely within the constraints defined in Moore (1976). We note that due to the structure of $\tilde{\mathbf{F}}$ as in (3.1), $M_e(s, \sigma)$ is zero irrespective of how Λ_e and W_{ee} are selected. Also, we note that $n_a^+ + n_c = 0$ if the given system is of minimum phase and left invertible.

(4) A set of n_f eigenvalues of $\tilde{\mathbf{A}}_0$, namely Λ_f, can be assigned arbitrarily at any *finite* locations in $\mathscr{C}^-$. (The set Λ_∞ is renamed here as Λ_f due to the finiteness of the involved eigenvalues.) Moreover, the eigenvector set V_f corresponding to these eigenvalues can be selected freely within the constraints defined in Moore (1976). We note that due to the structure of $\tilde{\mathbf{F}}$ as in (3.1), $M_\infty(s, \sigma)$ is zero irrespective of how Λ_f and V_f are selected.

Thus n_a^- eigenvalues of $\tilde{\mathbf{A}}_0$ must be chosen to coincide exactly with the set of plant minimum phase invariant zeros while the corresponding left eigenvectors of $\tilde{\mathbf{A}}_0$ must coincide exactly with the corresponding left state zero directions of $\tilde{\mathbf{\Sigma}}$. On the other hand, $n_b + n_a^+ + n_c + n_f$ eigenvalues of $\tilde{\mathbf{A}}_0$ can be assigned freely at any finite locations in $\mathscr{C}^-$. Also, n_b eigenvectors must be selected to be in the null space of $\tilde{\mathbf{B}}'$ while the remaining $n_a^+ + n_c + n_f$ eigenvectors of $\tilde{\mathbf{A}}_0$ can be assigned in any chosen way consistent with the freedom available in assigning them (Moore 1976). Moreover, since there is no necessity of assigning asymptotically infinite eigenvalues, there is no need to parameterize the observer gain $\tilde{\mathbf{K}}$ in terms of the tuning parameter σ. Thus the design for ELTR consists of assigning only an appropriate finite eigenstructure of $\tilde{\mathbf{A}}_0$.

We now move on to give the design steps to obtain $\tilde{\mathbf{K}}$ which assigns an appropriate finite eigenstructure to $\tilde{\mathbf{A}}_0$ so that the observer based controller achieves ELTR.

Step 1a. This step deals with the assignment of finite eigenstructure to the subsystem (2.3) of Part 1. Choose a gain K_b such that $\lambda(A_{bb}^c)$ coincides with Λ_b, a set of n_b designer specified eigenvalues all in $\mathscr{C}^-$, where

$$A_{bb}^c = A_{bb} - K_b C_s \tag{3.2}$$

Note that the existence of such a K_b is guaranteed by Property 2.1 of § 2 of Part 1 (Saberi *et al.* 1991). Also, in our design, the eigenvectors of A_{bb}^c can be assigned in any chosen way consistent with the freedom available in assigning them (Moore 1976). Owing to the properties of s.c.b., our design always results in the eigenvector set V_b corresponding to the eigenvalues Λ_b of $\mathbf{A}_0$, in the null space of $\tilde{\mathbf{B}}'$ so that $M_b(s) = 0$.

Step 1b. This step deals with the assignment of finite eigenstructure to the subsystems (2.1), (2.4) and (2.5) of Part 1. Let A_x and C_x be defined as

$$A_x = \begin{bmatrix} A_{aa}^+ & 0 & L_{af}^+ C_f \\ B_c E_{ca}^+ & A_{cc} & L_{cf} C_f \\ B_f E_a^+ & B_f E_c & A_f \end{bmatrix}, \quad C_x = [0 \quad 0 \quad C_f] \tag{3.3}$$

Also, let $\Lambda_x \equiv \Lambda_e \cup \Lambda_f$ be a set of $n_a^+ + n_c + n_f$ designer specified eigenvalues all in $\mathscr{C}^-$ subject to the condition that any unobservable but stable eigenvalues of the given system must be included among Λ_x. Now select a gain K_x such that $\lambda(A_x^c)$ coincides with Λ_x where

$$A_x^c = A_x - K_x C_x \tag{3.4}$$

Again note that the existence of such a K_x is guaranteed by Property 2.1 of § 2 of Part 1. Also, the eigenvectors of A_x^c can be assigned in any chosen way consistent with the freedom available in assigning them (Moore 1976). Let us next partition K_x as

$$K_x = \begin{bmatrix} K_a^+ \\ K_c \\ K_f \end{bmatrix}$$

where K_a^+, K_c and K_f are respectively of dimension $n_a^+ \times p_f$, $n_c \times p_f$ and $n_f \times p_f$.

Step 2. In this step, K_b and K_x calculated in Step 1 are put together into a composite matrix. Let

$$K = \begin{bmatrix} K_a^+ & L_{as}^+ \\ L_{af}^- & L_{as}^- \\ L_{bf} & K_b \\ K_c & L_{cs} \\ K_f & 0 \end{bmatrix}$$

Finally define the observer gain $\tilde{\mathbf{K}}$ as

$$\tilde{\mathbf{K}} = \Gamma_1 K \Gamma_2^{-1} \tag{3.5}$$

We have the following theorem.

Theorem 3.1

Consider an observer with its gain as given by (3.5). Then the eigenvalues of the observer are given by Λ_-, Λ_b and Λ_x. Moreover, the observer based controller which uses the gain as in (3.5) achieves ELTR.

Proof

See Appendix C.

Remark 3.1

We note that in general the observer gain which leads to ELTR is not unique.

Example 3.1

Let $\tilde{\Sigma}$ be characterized by

$$\tilde{\mathbf{A}} = \begin{bmatrix} 7\cdot8234 & 0\cdot2582 & -2\cdot2012 & -4\cdot2041 & -4\cdot7312 \\ 8\cdot7309 & -12\cdot6825 & -1\cdot2351 & -0\cdot3500 & 0\cdot0711 \\ -2\cdot8977 & -2\cdot6133 & -1\cdot6768 & 2\cdot5151 & 3\cdot8657 \\ 9\cdot0282 & -2\cdot9915 & -1\cdot7178 & -6\cdot4098 & -2\cdot4021 \\ 18\cdot0111 & -1\cdot1576 & -3\cdot2050 & -5\cdot7940 & -12\cdot0543 \end{bmatrix}$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} 0\cdot5007 & 0\cdot4644 \\ 0\cdot3841 & 0\cdot9410 \\ 0\cdot2771 & 0\cdot0501 \\ 0\cdot9138 & 0\cdot7615 \\ 0\cdot5297 & 0\cdot7702 \end{bmatrix}$$

$$\tilde{\mathbf{C}} = \begin{bmatrix} -2\cdot4336 & -2\cdot8360 & -0\cdot0746 & 1\cdot8284 & 3\cdot1294 \\ -0\cdot2668 & 1\cdot5733 & -0\cdot5524 & -0\cdot2347 & -0\cdot1949 \end{bmatrix}$$

This system is invertible and of non-minimum phase with invariant zeros at $s = -5$, $s = -1$ and $s = 1$. The target loop transfer function is specified by giving

$$\tilde{\mathbf{F}} = \begin{bmatrix} 6\cdot0447 & 1\cdot8718 & -0\cdot6686 & -1\cdot1307 & -4\cdot7705 \\ 13\cdot4444 & 3\cdot9393 & -0\cdot8832 & -3\cdot0416 & 9\cdot8552 \end{bmatrix}$$

It is straightforward to verify that the target loop specified by $\tilde{\mathbf{F}}$ is exactly recoverable. In fact, as illustrated in Fig. 7, the following observer gain does achieve ELTR

$$\tilde{\mathbf{K}} = \begin{bmatrix} -2\cdot1408 & -2\cdot1504 \\ -0\cdot9402 & -6\cdot2707 \\ 0\cdot9038 & 1\cdot1040 \\ -3\cdot4980 & -3\cdot9161 \\ -3\cdot0080 & -5\cdot0407 \end{bmatrix}$$

Figure 7 shows the plots of maximum and minimum singular values of the target and the achieved sensitivity functions, $S(j\omega)$ and $S_0(j\omega)$.

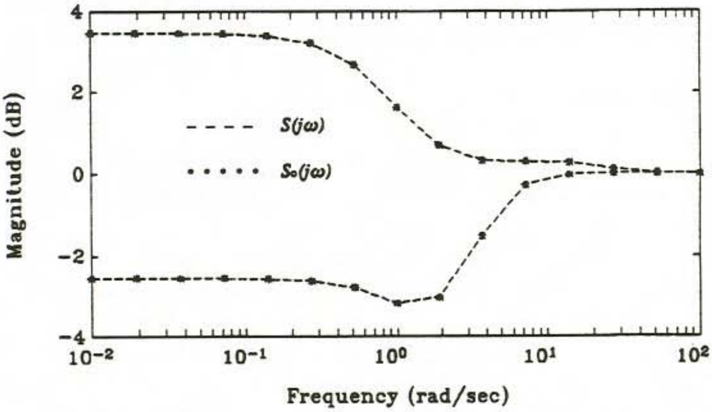

Figure 7. Maximum and minimum singular values of $S(j\omega)$ and $S_0(j\omega)$ of Example 3.1.

4. Design for asymptotically recoverable target loops

In this section, we consider the design of observer based controllers for asymptotically recoverable target loops. As stated in Theorem 3.4 of Part 1 (Saberi *et al.* 1991), a target loop transfer function, $L(s) = \tilde{\mathbf{F}}\mathbf{\Phi}\tilde{\mathbf{B}}$, is asymptotically recoverable iff $\mathscr{V}^+_{\mathrm{Ker}\tilde{\mathbf{C}}} \subseteq \mathrm{Ker}\,\tilde{\mathbf{F}}$. Also, in view of Properties 2.3 of Part 1, $\mathscr{V}^+_{\mathrm{Ker}\tilde{\mathbf{C}}}$ is the span of $\tilde{x}_a^+ \oplus \tilde{x}_c$. This implies that $L(s)$ is asymptotically recoverable iff $\tilde{\mathbf{F}}$ is in the form

$$\tilde{\mathbf{F}} = \Gamma_3 F \Gamma_1^{-1}, \quad F = \begin{bmatrix} 0 & F_{a1}^- & F_{b1} & 0 & F_{f1} \\ 0 & F_{a2}^- & F_{b2} & 0 & F_{f2} \end{bmatrix} \tag{4.1}$$

Thus in view of (*a*) Lemmas 3.2 and 3.3 of Part 1, (*b*) the form of F as in (4.1) and (*c*) interpretations of different partitions of $M(s,\sigma)$ as in § 2.1, it is easy to note the following.

(1) A set of n_a^- eigenvalues of $\tilde{\mathbf{A}}_0$, namely $\bar{\Lambda}_-$, must be chosen to coincide either exactly or asymptotically with the set of plant minimum phase invariant zeros while the corresponding left eigenvectors of $\tilde{\mathbf{A}}_0$ must coincide either exactly or asymptotically with the corresponding left state zero directions of $\tilde{\Sigma}$ so that $M_-(s,\sigma)$ is rendered zero.

(2) A set of n_b eigenvalues of $\tilde{\mathbf{A}}_0$, namely $\bar{\Lambda}_b$, can be assigned arbitrarily at finite locations in $\mathscr{C}^-$. However, V_b must be selected to be in the null space of $\tilde{\mathbf{B}}'$ so that $M_b(s,\sigma)$ is rendered zero either exactly or asymptotically.

(3) A set of $n_a^+ + n_c$ eigenvalues of $\tilde{\mathbf{A}}_0$, namely $\bar{\Lambda}_e$, can be assigned arbitrarily at finite locations in $\mathscr{C}^-$ subject to the conditions that any unobservable but stable eigenvalues of the given system must be included among $\bar{\Lambda}_e$. Moreover, the eigenvector set $\bar{W}_{ee}$ can be selected freely within the constraints defined in Moore (1976). We note that due to the structure of $\tilde{\mathbf{F}}$ as in (4.1), $M_e(s,\sigma)$ is zero irrespective of how $\bar{\Lambda}_e$ and $\bar{W}_e$ are selected. Also, we note that $n_a^+ + n_c = 0$ if the given systm is of minimum phase and left invertible.

(4) A set of n_f eigenvalues of $\tilde{\mathbf{A}}_0$, namely Λ_∞, can be assigned arbitrarily to asymptotically *infinite* locations in $\mathscr{C}^-$ as $\sigma \to \infty$. The right and left eigenvectors W_i and V_i are selected such that $W_i V_i^H \tilde{\mathbf{B}}$ corresponding to each $\lambda_i \in \Lambda_\infty$ is uniformly bounded as $|\lambda_i| \to \infty$.

Thus n_a^- eigenvalues of $\tilde{\mathbf{A}}_0$ must be chosen to coincide exactly or asymptotically with the set of plant minimum phase invariant zeros while the corresponding left eigenvectors of $\tilde{\mathbf{A}}_0$ must coincide exactly or asmptotically with the corresponding left state zero directions of $\tilde{\Sigma}$. On the other hand, $n_b + n_a^+ + n_c$ eigenvalues of $\tilde{\mathbf{A}}_0$ can be assigned freely at any finite locations in $\mathscr{C}^-$. Also $\bar{V}_b$, a set of n_b eigenvectors must be selected to be in the null space of $\tilde{\mathbf{B}}'$ while $\bar{W}_e$, a set of $n_a^+ + n_c$ eigenvectors of $\tilde{\mathbf{A}}_0$ can be assigned in any chosen way consistent with the freedom available in assigning them (Moore 1976). Moreover, there exists a freedom to assign n_f asymptotically infinite eigenvalues. The right and left eigenvectors W_i and V_i are selected such that $W_i V_i^H \tilde{\mathbf{B}}$ corresponding to each $\lambda_i \in \Lambda_\infty$ is uniformly bounded as $|\lambda_i| \to \infty$. Thus the design freedom that exists is exactly same as described in § 2.1 and hence ATEA design method developed in § 2.3 can be used here as well. The essential difference between this section and § 2 is that § 2 treats an arbitrarily given target loop transfer function $L(s)$ where as this section deals with an $L(s)$ which is specified by an $\tilde{\mathbf{F}}$ satisfying (4.1) so that $M_e(s, \sigma)$ is zero. However, the same method of design can be used for both cases.

Example 4.1

Consider the system given in Example 3.1 except that now the target loop transfer function $L(s)$ is specified by

$$\tilde{\mathbf{F}} = \begin{bmatrix} -16{\cdot}2010 & -18{\cdot}2158 & -1{\cdot}1275 & 12{\cdot}4677 & 20{\cdot}4128 \\ 4{\cdot}3114 & 7{\cdot}1386 & -1{\cdot}8405 & -2{\cdot}6013 & -2{\cdot}8678 \end{bmatrix}$$

It is straightforward to verify that the target loop specified by $\tilde{\mathbf{F}}$ is asymptotically recoverable. Using ATEA design method, an observer gain $\tilde{\mathbf{K}}(\sigma)$ with $\sigma = 200$ is given by

$$\tilde{\mathbf{K}}(\sigma) = \begin{bmatrix} 83{\cdot}8784 & 89{\cdot}8098 \\ 36{\cdot}7824 & 180{\cdot}0434 \\ -14{\cdot}9693 & 11{\cdot}0206 \\ 105{\cdot}7923 & 146{\cdot}8637 \\ 97{\cdot}1095 & 147{\cdot}4598 \end{bmatrix}$$

The singular value plots given in Figs 8 and 9 show that ALTR is achieved.

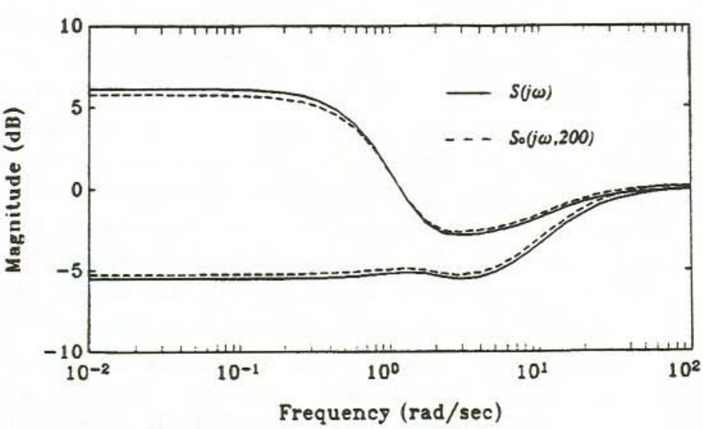

Figure 8. Maximum and minimum singular values of $S(j\omega)$ and $S_0(j\omega, 200)$ of Example 4.1.

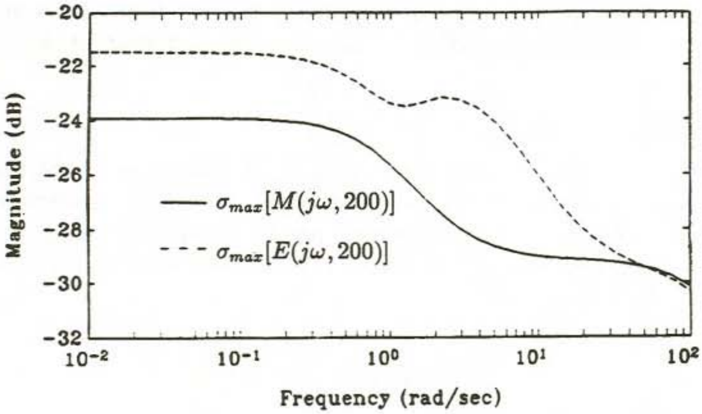

Figure 9. Maximum singular values of $M(j\omega, 200)$ and $E(j\omega, 200)$ of Example 4.1.

5. Design to recover over a specified subspace

The design task considered in this section is the following. Given a subspace $\mathscr{S}$ of $\Re^m$, we are interested in designing an observer so that the achieved and target sensitivity and complimentary sensitivity functions projected onto the subspace $\mathscr{S}$ match each other either exactly or asymptotically. The conditions under which such a design is possible are given in Part 1. To recapitulate these conditions, let $\mathbf{V}_s$ be a matrix whose columns form an orthogonal basis of the given subspace $\mathscr{S}$ of $\Re^m$. Also, given the system $\tilde{\Sigma}$ characterized by the matrix triple $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$, let us define an auxiliary system $\tilde{\Sigma}_s$ characterized by the matrix triple $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}\mathbf{V}_s)$. Thus the auxiliary system $\tilde{\Sigma}_s$ differs from $\tilde{\Sigma}$ in its input distribution matrix $\tilde{\mathbf{B}}\mathbf{V}_s$. Also, let $L(s) = \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$ be the specified target loop transfer function. Then the analysis given in Part 1 (see Theorems 3.8 and 3.9) imply the following:

(1) The projections of achievable and target sensitivity and complimentary sensitivity functions onto the subspace $\mathscr{S}$ match each other exactly iff $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}\mathbf{V}_s) \subseteq \operatorname{Ker} \tilde{\mathbf{F}}$.

(2) The projections of achievable and target sensitivity and complimentary sensitivity functions onto the subspace $\mathscr{S}$ match each other asymptotically iff $\mathscr{V}^+_{\operatorname{Ker}\tilde{\mathbf{C}}}(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}\mathbf{V}_s) \subseteq \operatorname{Ker} \tilde{\mathbf{F}}$.

Thus the task of designing observers for either exact or asymptotic recovery over a subspace collapses to the task discussed either in § 3 or § 4 except that one needs to use $\tilde{\Sigma}_s$ instead of $\tilde{\Sigma}$. The following example illustrates this.

Example 5.1

Consider a system $\tilde{\Sigma}$ characterized by

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 4 & 4 & 4 & 4 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\tilde{\mathbf{C}} = \tilde{\mathbf{B}}'$. This system has two non-minimum phase invariant zeros at $s = 1$ and at $s = 2$. Now consider a specified subspace $\mathscr{S}$ which is a span of the vector

$$\mathbf{V}_s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is simple to verify that the auxiliary system $\tilde{\Sigma}_s$ characterized by the matrix triple $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}\mathbf{V}_s)$ is left invertible and of minimum phase. Hence the projections of target and achievable sensitivity and complimentary sensitivity functions onto $\mathbf{V}_s$ can match each other asymptotically. To exemplify this, let the target loop be specified by

$$\tilde{\mathbf{F}} = \begin{bmatrix} 0 & 178 & 53 & 0 \\ 204 & 0 & 0 & 54 \end{bmatrix}$$

Let us choose $\tilde{\mathbf{K}}(\sigma)$ as

$$\tilde{\mathbf{K}}(\sigma) = \begin{bmatrix} 1 & 2{\cdot}5 \\ 2 & 7{\cdot}5 \\ 3+\sigma & 1{\cdot}5 \\ 4 & 13 \end{bmatrix}$$

so that the observer eigenvalues are placed at $-\sigma$, -1, -2 and -3 for all σ. Let the orthogonal projection matrix onto the subspace $\mathscr{S}$ be $P_s = \mathbf{V}_s\mathbf{V}_s'$. Then the resulting $M(j\omega, \sigma)P_s$, $S_0(j\omega, \sigma)P_s$ and $S(j\omega)P_s$ are plotted with respect to ω over a given range of ω in Figs 10 and 11 when $\sigma = 1000$. It is easy to note that $M(j\omega, \sigma)P_s$ is approximately zero while $S_0(j\omega, \sigma)P_s$ is close to $S(j\omega)P_s$. Also, note that the minimum singular values of $S_0(j\omega, \sigma)P_s$ and $S(j\omega)P_s$ are identically zero due to the singularity of P_s.

6. Conclusions

Full order observer design for loop transfer recovery is considered. Four different design tasks are pursued depending upon the nature of the target loop transfer functions. For two design tasks, namely when the target loop transfer

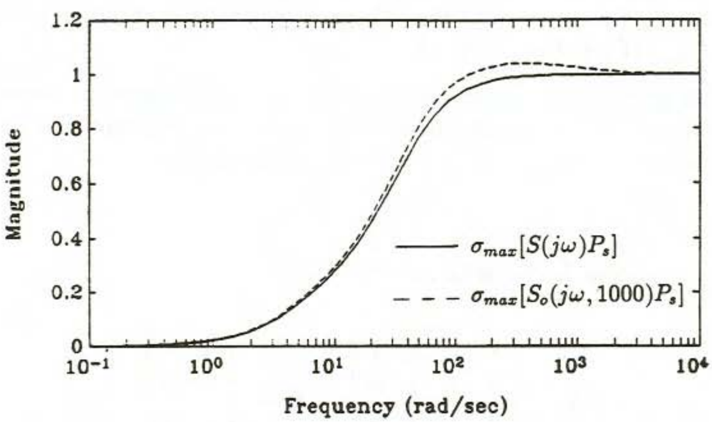

Figure 10. Maximum singular values of $S(j\omega)P_s$ and $S_0(j\omega, 1000)P_s$ of Example 5.1.

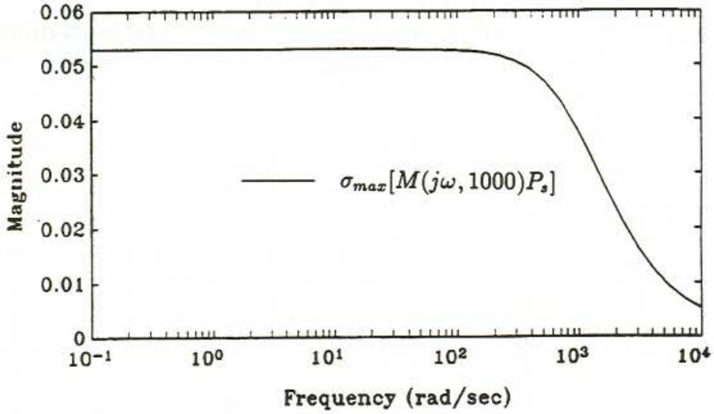

Figure 11. Maximum singular values of $M(j\omega, 1000)P_s$ of Example 5.1.

function is arbitrarily specified or when it is asymptotically recoverable, a new method of design based on asymptotic time-scale and eigenstructure assignment (ATEA) is developed. This method is capable of utilizing all the available design freedom to shape the loops as desired. On the other hand, the traditional ARE (algebraic Riccati equation) based design is shown to have several limitations. Prominent among these limitations are:

(1) its inability to shape the recovery error, for example, in the subspace in which the error cannot be rendered zero either exactly or asymptotically;

(2) its inability to assign any arbitrary fast time-scale structure (infinite eigenstructure) to the observer; and

(3) numerical difficulties due to 'stiffness' of design equations and 'repetitive' nature of design.

On the other hand, the ATEA design method:

(1) can utilize all the available design freedom to shape the recovery error as desired;

(2) allows arbitrary assignment of observer fast time-scale structure; and

(3) is free of 'stiffness' problems.

Also, because of the above mentioned limitations, ARE based design results in a higher controller gain and band-width than is necessary. A bank of numerical examples illustrate this. For the design task when the given target loop is exactly recoverable, no design method is yet available in the literature. To fill this gap, a new finite eigenstructure assignment method is also developed here. Another design task of recovering the target sensitivity and complimentary sensitivity functions over any specified subspace of the control space is also considered. This task generalizes the notion of traditional loop transfer recovery. All the design methods developed here are implemented in a 'Matlab' software package. A number of design examples show various capabilities of the developed design methods.

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Appendix A

Proof of Lemma 2.1

We assume that the given system $\tilde{\Sigma}$ has simple non-minimum phase invariant zeros so as to avoid complexity in presenting the proof. In our proof, we also use the all-pass/minimum-phase decomposition as in Zhang and Freudenberg (1990) of $\tilde{\Sigma}$. Let $B_m^0 = \tilde{\mathbf{B}}$ and for each $i = 1$ to n_a^+,

$$B_m^i = B_m^{i-1} - 2\mathrm{Re}\,(z_i^+)\zeta_i \eta_i^H$$

where ζ_i and η_i are the right state and input zero directions associated with an invariant zero z_i^+ of a system characterized by the triple $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, B_m^{i-1})$ and where $\eta_i^H \eta_i = 1$. We have the following properties:

Properties A.1

(1) $-(z_i^+)^*$ is an invariant zero of a system characterized by the triple $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, B_m^i)$ along with the right state and input zero directions ζ_i and η_i respectively.

(2) Let ψ_i be the left state zero direction associated with the invariant zero $-(z_i^+)^*$ of $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, B_m^i)$, then $\psi_i^H B_m^{i-1} = c_i \eta_i^H$ where $c_i = 2\mathrm{Re}\,(z_i^+)\psi_i^H \zeta_i$ is a constant.

Proof

Consider

$$\begin{aligned} B_m^i \eta_i &= [B_m^{i-1} - 2\mathrm{Re}\,(z_i^+)\zeta_i \eta_i^H]\eta_i \\ &= B_m^{i-1}\eta_i - 2\mathrm{Re}\,(z_i^+)\zeta_i \\ &= (z_i^+ I_n - \tilde{\mathbf{A}})\zeta_i - 2\mathrm{Re}\,(z_i^+)\zeta_i \\ &= (-(z_i^+)^* I_n - \tilde{\mathbf{A}})\zeta_i \end{aligned}$$

Now to prove the second point, we have by definition

$$\begin{aligned} 0 = \psi_i^H B_m^i &= \psi_i^H [B_m^{i-1} - 2\mathrm{Re}\,(z_i^+)\zeta_i \eta_i^H] \\ &= \psi_i^H B_m^{i-1} - c_i \eta_i^H \end{aligned}$$

Hence the result □

Property A.2

Consider a left invertible plant $\tilde{\Sigma}$ with simple non-minimum phase invariant zeros. Let z_i^+, x_{Ri}^+ and w_{Ri}^+, $i = 1$ to n_a^+, be the non-minimum phase invariant zeros and the corresponding right state and input zero directions of the given system $\tilde{\Sigma}$. Let λ_i^+ and V_i^+, $i = 1$ to n_a^+, be the eigenvalues and the associated left eigenvectors of $\tilde{\mathbf{A}}_0$ represented in $M_e(s, \sigma)$. Then the minimum phase image model of $\tilde{\Sigma}$, namely the sytem characterized by $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, B_m^{n_a^+})$, has invariant zeros at $-(z_i^+)^*$, $i = 1$ to n_a^+, with the corresponding left state zero directions ψ_i, $i = 1$ to n_a^+, satisfying the following property as $\sigma \to \infty$:

$$e_i^+ = \tilde{\mathbf{B}}' V_i^+ \to \tilde{\mathbf{B}}' \psi_i = c_i w_{Ri}^+$$

where c_i is a constant.

Proof

Consider a non-minimum phase zero z_1^+ of $\tilde{\Sigma}$. Then it follows from Property A1 that $-(z_1^+)^*$ is an invariant zero of $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, B_m^1)$. Let ψ_1 and v_1 be respectively the left state and input zero directions associated with the invariant zero $-(z_1^+)^*$ of $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, B_m^1)$. It follows from the results of Appendix B in Chen *et al.* (1990 b) that the left state and input zero directions associated with the minimum phase invariant zeros of a left invertible system remain unchanged in its minimum phase image model. Since $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, B_m^{n_a^+})$ is a minimum phase image of $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, B_m^1)$ as well, we note that ψ_1 and v_1 are also the left state and input zero directions associated with the invariant zero $-(z_1^+)^*$ of $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, B_m^{n_a^+})$. Hence

$$0 = \psi_1^H B_m^{n_a^+} = \psi_1^H B_m^1 = \psi_1^H[\tilde{\mathbf{B}} - 2\mathrm{Re}\,(z_1^+) x_{R1}^+ (w_{R1}^+)^H]$$

In view of the above and in view of the well known result $V_1^+ \to \psi_1$ as $\sigma \to \infty$, we have

$$e_1^+ = \tilde{\mathbf{B}}' V_1^+ \to \tilde{\mathbf{B}}' \psi_1 = [2\mathrm{Re}\,(z_1^+) \psi_1^H x_{R1}^+] w_{R1}^+ \tag{A 1}$$

On the other hand, it is quite easy to verify that the all-pass/minimum phase decomposition and in particular $B_m^{n_a^+}$ does not depend on the order of naming the invariant zeros. Thus we may rearrange any of the invariant zeros to be z_1^+ to yield the desired result as in (A 1).

Now the proof of Lemma 2.1 is evident from Property A.2 since LQG controller for a non-minimum phase plant is same as the corresponding one for the minimum phase image model of the plant. □

Appendix B

Proof of Theorem 2.1

Without loss of generality, we will assume that the given system is in the form of s.c.b. Then by renaming the variables $\tilde{x}_0 = [(\tilde{x}_a^-)', \tilde{x}_b']'$ and $\tilde{x}_e = [(\tilde{x}_a^+)', \tilde{x}_c']'$, we can rewrite the observer dynamic matrix $\tilde{\mathbf{A}}_0$ as

$$\tilde{\mathbf{A}}_0 = \begin{bmatrix} A_{00} & 0 & -\tilde{H}_{0f} C_f \\ A_{e0} & A_{ee} & -[\tilde{H}_{ef} + \tilde{K}_e(\sigma)] C_f \\ B_f E_0 & B_f C_e & A_f - \tilde{K}_f(\sigma) C_f - L_f C_f \end{bmatrix} \tag{B 1}$$

where

$$A_{00} = \begin{bmatrix} A_{aa}^- & -\tilde{H}_{as}^- C_s \\ 0 & A_{bb}^c \end{bmatrix}, \quad A_{e0} = \begin{bmatrix} 0 & -\tilde{H}_{as}^+ C_s \\ B_c E_{ca}^- & -\tilde{H}_{cs} C_s \end{bmatrix}, \quad A_{ee} = \begin{bmatrix} A_{aa}^+ & 0 \\ B_c E_{ca}^+ & A_{cc} \end{bmatrix}$$

$$\tilde{H}_{0f} = \begin{bmatrix} \tilde{H}_{af}^- \\ \tilde{H}_{bf} \end{bmatrix}, \quad \tilde{H}_{ef} = \begin{bmatrix} \tilde{H}_{af}^+ \\ \tilde{H}_{cf} \end{bmatrix}$$

We prove the TSS properties of the observer on a transposed system $\tilde{\Sigma}_t$ whose closed-loop dynamic matrix is a transpose of $\tilde{\mathbf{A}}_0$. Consider $\tilde{\Sigma}_t$,

$$\dot{\tilde{x}}_0 = A'_{00}\tilde{x}_0 + A'_{e0}\tilde{x}_e + \sum_{l=1}^{m_u} E'_{l0}\tilde{x}_{lq_l} \tag{B 2}$$

$$\dot{\tilde{x}}_e = A'_{ee}\tilde{x}_e + \sum_{l=1}^{m_u} E'_{le}\tilde{x}_{lq_l} \tag{B 3}$$

and for each $i = 1$ to m_u

$$\dot{\tilde{x}}_i = A'_{q_i}\tilde{x}_i - \begin{bmatrix}1\\0\end{bmatrix}[\tilde{H}'_{0i}\tilde{x}_0 + (\tilde{H}'_{ei} + \tilde{K}'_{ei}(\sigma))\tilde{x}_e + \tilde{K}'_i(\sigma)\tilde{x}_i] + \sum_{l=1}^{m_u} E'_{l_i}\tilde{x}_{lq_l} \tag{B 4}$$

where

$$E_{i0} = [E^-_{ia} \quad E_{ib}], \quad E_{ie} = [E^+_{ia} \quad E_{ic}]$$
$$\tilde{H}_{0f} = [\tilde{H}_{01} \quad \tilde{H}_{02} \quad \dots \quad \tilde{H}_{0m_u}], \quad \tilde{H}_{ef} = [\tilde{H}_{e1} \quad \tilde{H}_{e2} \quad \dots \quad \tilde{H}_{em_u}]$$
$$\tilde{x} = [\tilde{x}'_0 \quad \tilde{x}'_e \quad \tilde{x}'_1 \quad \dots \quad \tilde{x}'_{m_u}]', \quad \tilde{x}_i = [\tilde{x}_{i1} \quad \tilde{x}_{i2} \quad \dots \quad \tilde{x}_{iq_i}]'$$

Let us adopt the following scaling and transformation of variables,

$$\mathbf{x}_0 = \tilde{x}_0, \quad \mathbf{x}_e = \tilde{x}_e, \quad \mathbf{x}_{iq_i} = \tilde{x}_{iq_i} + K'_{ei}\tilde{x}_e, \quad \mathbf{x}_{ik} = \prod_{l=k+1}^{q_i} \varepsilon_{il}\tilde{x}_{ik}, \quad i = 1 \text{ to } q_i - 1$$

We next define

$$\tilde{X}_{ij} = [\tilde{x}_{i\alpha_{ij-1}+1} \quad \tilde{x}_{i\alpha_{ij-1}+2} \quad \dots \quad \tilde{x}_{i\alpha_{ij}}]'$$

and

$$\mathbf{X}_{ij} = [\mathbf{x}_{i\alpha_{ij-1}+1} \quad \mathbf{x}_{i\alpha_{ij-1}+2} \quad \dots \quad \mathbf{x}_{i\alpha_{ij}}]'$$

so that

$$\mathbf{X}_{ij} = S_{ij}\tilde{X}_{ij} \text{ for } j = 1 \text{ to } r_i - 1$$

and

$$\mathbf{X}_{ir_i} = S_{ir_i}\tilde{X}_{ir_i} + \begin{bmatrix}0\\1\end{bmatrix} K'_{ei}\tilde{x}_e$$

where S_{ij} is as defined in (2.20). Then (B 2) to (B 4) can be rewritten as

$$\dot{\mathbf{x}}_0 = A'_{00}\mathbf{x}_0 + D_{0e}\mathbf{x}_e + \sum_{l=1}^{m_u} D_{0l}[0 \quad 1]\mathbf{X}_{lr_l} \tag{B 5}$$

$$\dot{\mathbf{x}}_e = (A^c_{ee})'\mathbf{x}_e + \sum_{l=1}^{m_u} D_{el}[0 \quad 1]\mathbf{X}_{lr_l} \tag{B 6}$$

for each $i = 1$ to m_u

$$\mu_{i1}\dot{\mathbf{X}}_{i1} = G'_{i1}\mathbf{X}_{i1} - H_{i1}\sum_{j=1}^{r_i} J_{ij}K'_{ij}\mathbf{X}_{ij} + \sum_{l=1}^{m_u} D_{i1l}[0 \quad 1]\mathbf{X}_{lr_l} + D_{i10}\mathbf{x}_0 + D_{i1e}\mathbf{x}_e \tag{B 7}$$

$$\mu_{ij}\dot{\mathbf{X}}_{ij} = G'_{ij}\mathbf{X}_{ij} + H_{ij}\mathbf{X}_{ij-1} + \sum_{l=1}^{m_u} D_{ijl}[0 \quad 1]\mathbf{X}_{ir_l} + D_{ije}\mathbf{x}_e \text{ for } j = 2 \text{ to } r_i \tag{B 8}$$

where

$$H_{ij} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The zero elements in $n_{ij} \times n_{ij-1}$ dimensional matrix H_{ij} are of appropriate dimension and may or may not exist depending upon the values of n_{ij} and n_{ij-1}. Also, various coefficient matrices in the above equations are as follows

$$D_{0e} = A'_{e0} - \sum_{l=1}^{m_u} E'_{l0} K'_{el}, \quad D_{0l} = E'_{l0}, \quad D_{el} = E'_{le}$$

$$D_{i10} = -\eta_i H_{i1} \tilde{H}'_{0i}, \quad D_{i1e} = -\eta_i H_{i1} \tilde{H}'_{ei} - \sum_{l=1}^{m_u} \mathscr{E}'_{li1} K'_{el}$$

$$[\mathscr{E}_{li1} \quad \mathscr{E}_{li2} \quad \dots \quad \mathscr{E}_{lir_i}] = E_{li} \operatorname{Diag}\left[\prod_{k=1}^{q_i} \varepsilon_{ik}, \prod_{k=2}^{q_i} \varepsilon_{ik}, \dots, \varepsilon_{iq_i}\right]$$

$$D_{ije} = -\sum_{l=1}^{m_u} \mathscr{E}'_{lij} K'_{el} \quad \text{for } j = 2 \text{ to } r_i - 1$$

$$D_{ijl} = \mathscr{E}'_{lij} \quad \text{for } j = 1 \text{ to } r_i - 1$$

$$D_{ir_ie} = -\sum_{l=1}^{m_u} \mathscr{E}'_{lir_i} K'_{el} + \mu_{ir_i} \begin{bmatrix} 0 \\ 1 \end{bmatrix} K'_{ei} (A^c_{ee})'$$

and

$$D_{ir_il} = \mathscr{E}'_{lir_i} + \mu_{ir_i} \begin{bmatrix} 0 \\ 1 \end{bmatrix} K'_{ei} D_{el} \tag{B 9}$$

Although (B 5) to (B 8) are in singularly perturbed form, their time-scale structure properties are not transparent. In order to bring various time-scales into focus, we adopt another transformation of variables. Let for each $i = 1$ to m_u

$$X_{ir_i} = \mathbf{X}_{ir_i} \quad \text{and} \quad X_{ij} = \mathbf{X}_{ij} + \mathscr{K}_{ij+1} X_{ij+1} \quad \text{for } j = 1 \text{ to } r_i - 1 \tag{B 10}$$

where $n_{ij-1} \times n_{ij}$ dimensional matrix

$$\mathscr{K}_{ij} = \begin{bmatrix} 0 & 0 \\ K'_{ijc} & K_{ijd} \end{bmatrix}$$

Then it is straightforward to verify that (B 7) and (B 8) can be rewritten as

$$\mu_{i1} \dot{X}_{i1} = (G^c_{i1})' X_{i1} + \sum_{l=1}^{m_u} D_{i1l} [0 \quad 1] X_{lr_l} + D_{i10} \mathbf{x}_0 + D_{i1e} \mathbf{x}_e$$
$$+ \sum_{k=2}^{r_i} \frac{\mu_{i1}}{\mu_{ik}} \mathscr{K}_{i2} \mathscr{K}_{i3} \quad \dots \quad \mathscr{K}_{ik}$$
$$\times \left[(G^c_{ik})' X_{ik} + H_{ik} X_{ik-1} + \sum_{l=1}^{m_u} D_{ikl} [0 \quad 1] X_{lr_l} + D_{ike} \mathbf{x}_e \right]$$

$$\mu_{ij}\dot{X}_{ij} = (G_{ij}^c)'X_{ij} + H_{ij}X_{ij-1} + \sum_{l=1}^{m_u} D_{ijl}[0 \quad 1]X_{lr_l} + D_{ije}\mathbf{x}_e$$

$$+ \sum_{k=j+1}^{r_i} \frac{\mu_{ij}}{\mu_{ik}} \mathscr{K}_{ij+1}\mathscr{K}_{ij+2} \quad \cdots \quad \mathscr{K}_{ik}$$

$$\times\left[(G_{ik}^c)'X_{ik} + H_{ik}X_{ik-1} + \sum_{l=1}^{m_u} D_{ikl}[0 \quad 1]X_{lr_l} + D_{ike}\mathbf{x}_e\right] \quad \text{for } j = 2 \text{ to } r_i - 1$$

$$\mu_{ir_i}\dot{X}_{ir_i} = (G_{ir_i}^c)'X_{ir_i} + H_{ir_i}X_{ir_i-1} + \sum_{l=1}^{m_u} D_{ir_il}[0 \quad 1]X_{lr_l} + D_{ir_ie}\mathbf{x}_e \tag{B 11}$$

Since the interconnection matrices in the coupled equations (B 11) tend to null matrices as $\sigma \to \infty$, the time-scale structure property of the observer follows directly from singular perturbation theory. To show this more explicitly, we next do Lyapunov analysis of the above dynamic system. For this purpose all the small parameters are defined as

$$\mu_{ij} = \varepsilon^{a_{ij}} \tag{B 12}$$

for some positive scalars a_{ij} where

$$\varepsilon = \frac{1}{\sigma}$$

Then in view of the property (2.18), we note that

$$a_{ij} > a_{ij+1} \quad \text{for all } j = 1 \text{ to } r_i - 1$$

Also, we can rewrite (B 11) as

$$\varepsilon^{a_{i1}}\dot{X}_{i1} = (G_{i1}^c)'X_{i1} + \varepsilon^{d_{i1}^*}\left[\mathscr{D}_{i10}^*\mathbf{x}_0 + \mathscr{D}_{i1e}^*\mathbf{x}_e + \sum_{l=1}^{m_u}\mathscr{D}_{i1l}^*X_{lr_l} + \sum_{k=1}^{r_i} K_{1ik}^*X_{ik}\right]$$

$$\varepsilon^{a_{ij}}\dot{X}_{ij} = (G_{ij}^c)'X_{ij} + H_{ij}X_{ij-1} + \varepsilon^{d_{ij}^*}\left[\mathscr{D}_{ije}^*\mathbf{x}_e + \sum_{l=1}^{m_u}\mathscr{D}_{ijl}^*X_{lr_l} + \sum_{k=j}^{r_i} K_{jik}^*X_{ik}\right]$$

for $j = 2$ to $r_i - 1$

$$\varepsilon^{a_{ir_i}}\dot{X}_{ir_i} = (G_{ir_i}^c)'X_{ir_i} + H_{ir_i}X_{ir_i-1} + \varepsilon^{d_{ir_i}^*}\left[\mathscr{D}_{ir_ie}^*\mathbf{x}_e + \sum_{l=1}^{m_u}\mathscr{D}_{ir_il}^*X_{lr_l}\right] \tag{B 13}$$

for some positive scalars d_{ij}^* and for some appropriately defined interconnection coefficient matrices. It is important to note that all the interconnection matrices are bounded as $\varepsilon \to 0$. Let

$$d = \tfrac{1}{2}\min\{d_{ij}^*;\, i = 1 \text{ to } m_u \text{ and } j = 1 \text{ to } r_i\}$$

Then

$$d_{ij} \equiv d_{ij}^* - d > 0 \quad \text{for all } i \text{ and } j$$

Also, let us define

$$X_0 = \varepsilon^d \mathbf{x}_0 \quad \text{and} \quad X_e = \varepsilon^d \mathbf{x}_e$$

Then we can rewrite (B 5), (B 6) and (B 13) as

$$\dot{X}_0 = A'_{00} X_0 + D_{0e} X_e + \varepsilon^d \sum_{l=1}^{m_u} D_{0l} [0 \quad 1] \mathbf{X}_{lr_l}$$

$$\dot{X}_e = (A^c_{ee})' X_e + \varepsilon^d \sum_{l=1}^{m_u} D_{el} [0 \quad 1] \mathbf{X}_{lr_l}$$

and for each $i = 1$ to m_u

$$\varepsilon^{a_{i1}} \dot{X}_{i1} = (G^c_{i1})' X_{i1} + \varepsilon^{d_{i1}} \left[\mathscr{D}_{i10} X_0 + \mathscr{D}_{i1e} X_e + \sum_{l=1}^{m_u} \mathscr{D}_{i1l} X_{ir_l} + \sum_{k=1}^{r_i} K_{1ik} X_{ik} \right]$$

$$\varepsilon^{a_{ij}} \dot{X}_{ij} = (G^c_{ij})' X_{ij} + H_{ij} X_{ij-1} + \varepsilon^{d_{ij}} \left[\mathscr{D}_{ije} X_e + \sum_{l=1}^{m_u} \mathscr{D}_{ijl} X_{lr_l} + \sum_{k=1}^{r_i} K_{jik} X_{ik} \right]$$

for $j = 2$ to $r_i - 1$

$$\varepsilon^{a_{ir_i}} \dot{X}_{ir_i} = (G^c_{ir_i})' X_{ir_i} + H_{ir_i} X_{ir_i - 1} + \varepsilon^{d_{ir_i}} \left[\mathscr{D}_{ir_ie} X_e + \sum_{l=1}^{m_u} \mathscr{D}_{ir_il} X_{lr_l} \right] \tag{B 14}$$

To proceed with a Lyapunov analysis of the above system, let us select positive definite matrices P_0, P_e and P_{ij}, $i = 1$ to m_u and $j = 1$ to r_i, satisfying the following Lyapunov equations

$$P_0 A'_{00} + A_{00} P_0 = -I$$

$$P_e A'_{ee} + A_{ee} P_e = -I$$

$$P_{ij} (G^c_{ij})' + G^c_{ij} P_{ij} = -I$$

We next define a Lyapunov function

$$V(X) = X'_0 P_0 X_0 + c_e X'_e P_e X_e + \sum_{l=1}^{m_u} \sum_{j=1}^{r_i} c_{ij} X'_{ij} P_{ij} X_{ij} \tag{B 15}$$

where c_e and c_{ij} are some positive scalars that are yet to be selected. It is then easy to show that dV/dt calculated along the trajectory of (B 14) satisfies the following:

$$\frac{dV}{dt} \leqslant -\|X_0\|^2 + 2\|P_0\|\,\|D_{0e}\|\,\|X_0\|\,\|X_e\| + 2\varepsilon^d \sum_{k=1}^{m_u} \|P_0\|\,\|D_{0k}\|\,\|X_0\|\,\|X_{kr_k}\|$$

$$- c_e\|X_e\|^2 + 2c_e\varepsilon^d \sum_{k=1}^{m_u} \|P_e\|\,\|D_{ek}\|\,\|X_e\|\,\|X_{kr_k}\|$$

$$+ \sum_{i=1}^{m_u} \left\{ -\frac{c_{i1}}{\varepsilon^{a_{i1}}}\|X_{i1}\|^2 + 2\frac{c_{i1}}{\varepsilon^{a_{i1}}}\varepsilon^{d_{i1}}\|X_{i1}\|\,\|P_{i1}\| \right.$$

$$\times \left[\|\mathscr{D}_{i10}\|\,\|X_0\| + \|\mathscr{D}_{i1e}\|\,\|X_e\| + \sum_{k=1}^{m_u} \|\mathscr{D}_{i1k}\|\,\|X_{kr_k}\| + \sum_{k=1}^{r_i} \|K_{1ik}\|\,\|X_{ik}\| \right]$$

$$+ \sum_{j=2}^{r_i-1} \left[-\frac{c_{ij}}{\varepsilon^{a_{ij}}}\|X_{ij}\|^2 + 2\frac{c_{ij}}{\varepsilon^{a_{ij}}}\|P_{ij}\|\,\|X_{ij}\|\,\|X_{ij-1}\| + 2\frac{c_{ij}}{\varepsilon^{a_{ij}}}\varepsilon^{d_{ij}}\|X_{ij}\|\,\|P_{ij}\| \right.$$

$$\left. \times \left(\|\mathscr{D}_{ije}\|\,\|X_e\| + \sum_{k=1}^{m_u} \|\mathscr{D}_{ijk}\|\,\|X_{kr_k}\| + \sum_{k=j}^{r_i} \|K_{jik}\|\,\|X_{ik}\| \right) \right]$$

$$-\frac{c_{ir_i}}{\varepsilon^{a_{ir_i}}}\|X_{ir_i}\|^2 + 2\frac{c_{ir_i}}{\varepsilon^{a_{ir_i}}}\|P_{ir_i}\|\,\|X_{ir_i}\|\,\|X_{ir_i-1}\| + 2\frac{c_{ir_i}}{\varepsilon^{a_{ir_i}}}\varepsilon^{d_{ir_i}}\|X_{ir_i}\|\,\|P_{ir_i}\|$$

$$\left. \times \left[\|\mathscr{D}_{ir_ie}\|\,\|X_e\| + \sum_{k=1}^{m_u} \|\mathscr{D}_{ir_ik}\|\,\|X_{kr_k}\| \right] \right\}$$

$$= -[\|X_0\|, \|X_e\|, \|X_{11}\|, \dots, \|X_{1r_1}\|, \dots, \|X_{m_u1}\|, \dots, \|X_{m_ur_{m_u}}\|]R(\varepsilon)$$

$$\times [\|X_0\|, \|X_e\|, \|X_{11}\|, \dots, \|X_{1r_1}\|, \dots, \|X_{m_u1}\|, \dots, \|X_{m_ur_{m_u}}\|]' \qquad \text{(B 16)}$$

Let us next choose

$$c_e > \|P_0\|^2\|D_{0e}\|^2 \qquad \text{(B 17)}$$

In order to facilitate the selection of coefficients c_{ij}, let

$$d_i = \frac{1}{r_i+1}\min\{d_{ij};\quad j=1 \text{ to } r_i\}$$

and define

$$b_{ij} = (r_i + 1 - j)d_i < d_{ij}, \quad j = 1 \text{ to } r_i \qquad \text{(B18)}$$

Then for each $i = 1$ to m_u and $j = 1$ to r_i, select

$$c_{ij} = \varepsilon^{a_{ij} - b_{ij}}$$

Here we note that for $j = 1$ to $r_i - 1$

$$b_{ij} > b_{ij+1}$$

Then the matrix $R(\varepsilon)$ is given as

$$R(\varepsilon) = \begin{bmatrix} R_{0e} & \star & \star & \dots & \star \\ \star & R_1 & \star & \dots & \star \\ \star & \star & R_2 & \dots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \dots & R_{m_u} \end{bmatrix} \qquad \text{(B 19)}$$

where ⋆'s represent appropriate dimensional submatrices which tend to null matrices as $\varepsilon \to 0$. Also

$$R_{0e} = \begin{bmatrix} 1 & -\|P_0\|\,\|D_{0e}\| \\ -\|P_0\|\,\|D_{0e}\| & c_e \end{bmatrix} > 0 \tag{B 20}$$

whenever c_e is as in (B 17). Furthermore, for each $i = 1$ to m_u

$$R_i = \begin{bmatrix} \frac{1}{\varepsilon^{b_{i1}}} - \star & \frac{\|P_{i2}\|}{\varepsilon^{b_{i2}}} & \cdots & \star & \star \\ -\frac{\|P_{i2}\|}{\varepsilon^{b_{i2}}} & \frac{1}{\varepsilon^{b_{i2}}} - \star & \ddots & \star & \star \\ \vdots & & \ddots & \ddots & \vdots \\ \star & \star & & \frac{1}{\varepsilon^{b_{ir_i-1}}} - \star & -\frac{\|P_{ir_i}\|}{\varepsilon^{b_{ir_i}}} \\ \star & \star & \cdots & -\frac{\|P_{ir_i}\|}{\varepsilon^{b_{ir_i}}} & \frac{1}{\varepsilon^{b_{ir_i}}} - \star \end{bmatrix} \tag{B 21}$$

Now in view of (B 18), it is straightforward to verify that R_i, for each $i = 1$ to m_u, is positive definite for ε sufficiently small. Then in view of the special structure of $R(\varepsilon)$ as in (B 19), there exists an ε^* such that for any $\varepsilon < \varepsilon^*$, $R(\varepsilon)$ is indeed a positive definite matrix and thus the stability of the observer dynamics is guaranteed. This completes our Lyapunov analysis.

So far we proved that ATEA algorithm yields an admissible observer gain $\tilde{\mathbf{K}}(\sigma)$ in the sense that $\tilde{\mathbf{A}}_0$ is a stable matrix for sufficiently large σ and that it has the required time-scale structure. In what follows, we will show that $\tilde{\mathbf{K}}(\sigma)$ achieves LTR in the sense that

$$M(s, \sigma) = \tilde{\mathbf{F}}(sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}})^{-1}\tilde{\mathbf{B}} \to \bar{M}_e(\sigma) \quad \text{pointwise in } s \tag{B 22}$$

as $\sigma \to \infty$. In view of (2.25), it can be seen easily that $\tilde{\mathbf{K}}(\sigma)$ has the following form,

$$\tilde{\mathbf{K}}(\sigma) = T(\sigma)\Gamma(\sigma)N + Q \tag{B 23}$$

where

$$\Gamma(\sigma) = \mathrm{Diag}\begin{bmatrix} \frac{1}{\eta_1} & \frac{1}{\eta_2} & \cdots & \frac{1}{\eta_{m_u}} \end{bmatrix}, \quad N = [I_{m_u} \quad 0]$$

$$Q = \begin{bmatrix} L_{af}^{+} + \tilde{H}_{af}^{+} & L_{as}^{+} + \tilde{H}_{as}^{+} \\ L_{af}^{-} + \tilde{H}_{af}^{-} & L_{as}^{-} + \tilde{H}_{as}^{-} \\ L_{bf} + \tilde{H}_{bf} & K_b \\ L_{cf} + \tilde{H}_{cf} & L_{cs} + \tilde{H}_{cs} \\ L_f & 0 \end{bmatrix} \tag{B 24}$$

While $T(\sigma)$ satisfies

$$T(\sigma) \to B_m T \tag{B 25}$$

as $\sigma \to \infty$ where

$$B_m = \begin{bmatrix} K_a^+ \\ 0 \\ 0 \\ K_c \\ B_f \end{bmatrix}, \quad T = \text{Diag}\,[J_{1r_1}K_{1r_1d} \quad J_{2r_2}K_{2r_2d} \quad \dots \quad J_{m_ur_{m_u}}K_{m_ur_{m_u}d}] \tag{B 26}$$

It is shown in Chen *et al.* (1990 b) that the triple $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, B_m)$ forms a left invertible and a minimum phase system. Thus it follows from the results of Saberi and Sannuti (1990) that

$$(sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}})^{-1}B_m \to 0 \quad \text{pointwise in } s \tag{B 27}$$

as $\sigma \to \infty$. Next let

$$\tilde{\mathbf{B}} = [B_m \quad 0] + B_e \tag{B 28}$$

where

$$\tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_f & 0 \end{bmatrix} \quad \text{and} \quad B_e = \begin{bmatrix} -K_a^+ & 0 \\ 0 & 0 \\ 0 & 0 \\ -K_c & B_c \\ 0 & 0 \end{bmatrix} \tag{B 29}$$

Thus we have

$$\begin{aligned} M(s, \sigma) &= \tilde{\mathbf{F}}(sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}})^{-1}\tilde{\mathbf{B}} \\ &= \tilde{\mathbf{F}}(sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}})^{-1}([B_m, 0] + B_e) \\ &\to \tilde{\mathbf{F}}(sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}})^{-1}B_e \end{aligned} \tag{B 30}$$

as $\sigma \to \infty$. We will next show that $M(s, \sigma) \to \bar{M}_e(s)$ as $\sigma \to \infty$. To simplify the notation, we reorder some variables and rewrite $\tilde{\mathbf{A}}_0$ as in (B 1). We note that

$$\tilde{K}_e(\sigma)\Gamma^{-1}(\sigma) = K_e T \tag{B 31}$$

and

$$\tilde{K}_f(\sigma)\Gamma^{-1}(\sigma) \to B_f T \tag{B 32}$$

as $\sigma \to \infty$. Let $\lambda_{ei}(\sigma)$ and $W_{ei}(\sigma)$ respectively be an eigenvalue and eigenvector of $\tilde{\mathbf{A}}_0$ represented in $M_e(s, \sigma)$. Let us partition $W_{ei}(\sigma)$ as

$$W_{ei}(\sigma) = [W'_{e0i}(\sigma) \quad W'_{eei}(\sigma) \quad W'_{e\infty i}(\sigma)]' \tag{B 33}$$

It is then easy to show that as $\sigma \to \infty$

$$\lambda_{ei}(\sigma) \to \bar{\lambda}_{ei} \in \bar{\Lambda}_e$$

$$W_{e0i}(\sigma) \to 0, \quad W_{eei}(\sigma) \to \bar{W}_{ei}, \quad W_{e\infty i}(\sigma) \to C_f'\Gamma^{-1}(\sigma)T^{-1}C_e\bar{W}_{eei} \to 0 \tag{B 34}$$

where $\bar{\lambda}_{ei}$ and $\bar{W}_{eei}$ are respectively an eigenvalue and eigenvector of A_{ee}^c. Now in view of the fact

$$[\bar{V}_0 \quad \bar{V}_e \quad \bar{V}_\infty]^H[\bar{W}_0 \quad \bar{W}_e \quad \bar{W}_\infty] = I_n$$

we note that $\bar{V}_0$ and $\bar{V}_\infty$ are of the form

$$\bar{V}_0 = [* \quad 0 \quad *]' \quad \text{and} \quad \bar{V}_\infty = [* \quad 0 \quad *]'$$

where * denotes some finite value not necessarily zero. Hence

$$\bar{V}_0 B_e = 0 \quad \text{and} \quad \bar{V}_\infty B_e = 0 \tag{B 35}$$

Thus we can rewrite (B 30) as

$$\begin{aligned} M(s,\sigma) &\to \tilde{\mathbf{F}}(sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}})^{-1} B_e \\ &= \sum_{i=1}^{n} \frac{\tilde{\mathbf{F}} W_i(\sigma) V_i^H(\sigma) B_e}{s - \lambda_i} \\ &\to \sum_{i=1}^{n_e} \frac{\tilde{\mathbf{F}} \bar{W}_{ei} \bar{V}_{ei}^H B_e}{s - \bar{\lambda}_{ei}} \\ &= \bar{M}_e(s) \end{aligned} \tag{B 36}$$

Next by partitioning $\tilde{\mathbf{F}}$ as

$$\tilde{\mathbf{F}} = [F_0 \quad F_e \quad F_\infty]$$

and letting

$$B_{ee} = \begin{bmatrix} -K_a^+ & 0 \\ -K_c & B_c \end{bmatrix}$$

we note that

$$M(s,\sigma) \to \bar{M}_e(s) = \sum_{i=1}^{n_e} \frac{F_e \bar{W}_{eei} \bar{V}_{eei}^H B_{ee}}{s - \bar{\lambda}_{ei}} = F_e(sI_{n_e} - A_{ee}^c)^{-1} B_{ee} \tag{B 37}$$

where

$$[\bar{V}_{ee1} \quad \bar{V}_{ee2} \quad \cdots \quad \bar{V}_{een_e}] = [\bar{W}_{ee1} \quad \bar{W}_{ee2} \quad \cdots \quad \bar{W}_{een_e}]^{-H}$$

This completes the LTR analysis of ATEA algorithm. □

Appendix C

Proof of Theorem 3.1

We assume that the given system $\tilde{\Sigma}$ is in the form of s.c.b. (see Theorem 2.1 of Part 1). Then for the gain $\tilde{\mathbf{K}}$ given by (3.5), we note that

$$\tilde{\mathbf{A}}_0 = \tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}} = \begin{bmatrix} A_{aa}^+ & 0 & 0 & 0 & (L_{af}^+ - K_a^+)C_f \\ 0 & A_{aa}^- & 0 & 0 & 0 \\ 0 & 0 & A_{bb}^c & 0 & 0 \\ B_c E_{ca}^+ & B_c E_{ca}^- & 0 & A_{cc} & (L_{cf} - K_c)C_f \\ B_f E_a^+ & B_f E_a^- & B_f E_b & B_f E_c & A_f - K_f C_f \end{bmatrix}$$

Then it is simple to verify that the eigenvalues of $\tilde{\mathbf{A}}_0$ are given by $\Lambda_- \cup \Lambda_b \cup \Lambda_x$, and moreover $M(s) \equiv \tilde{\mathbf{F}}(sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}\tilde{\mathbf{C}})^{-1}\tilde{\mathbf{B}} \equiv 0$. Hence ELTR is achieved. □

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