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Theory of LTR for non-minimum phase systems, recoverable target loops, and recovery in a subspace

Part 1. Analysis

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A complete analysis of loop transfer recovery problem using full order observer based controllers for general not necessarily left invertible and not necessarily minimum phase systems is considered. The analysis here, while showing that neither exact nor asymptotic loop transfer recovery is in general possible, focuses on three fundamental issues. The first issue is concerned with what can and what canot be achieved for a given system and for an arbitrarily specified target loop transfer function, while the second issue is concerned with the development of necessary and/or sufficient conditions a target loop has to satisfy so that it can be either exactly or asymptotically be recovered for a given system. The third issue deals with the development of method(s) to test whether recovery is possible in a given subspace of the control space or not, i.e. to test whether projections of target and achievable sensitivity and complimentary sensitivity functions onto a given subspace match each other or not. Such an analysis pinpoints the limitations of the given system for the recovery of arbitrarily specified target loops via observer based controllers. These limitations are the consequences of the structural properties (i.e. finite and infinite zero structure, and invertibility) of the given system. Furthermore, the analysis discovers a multitude of ways in which freedom exists to shape the loops in a desired way as close as possible to the target shapes. Also, possible pole zero cancellations between the eigenvalues of the controller and the input and/or output decoupling zeros of the plant are characterized.

1. Introduction and problem statement

In multi-input and multi-output feedback control system design, performance specifications such as command following, disturbance rejection, closed-loop bandwidth, stability robustness with respect to unstructured dynamic uncertainties etc. are naturally posed in the frequency domain in terms of sensitivity and complementary sensitivity functions (Doyle and Stein 1981). These sensitivity and complementary sensitivity functions are related to the loop transfer matrices evaluated by breaking the control loop at critical points, commonly either the input or output point of the given plant. Thus typically, one is interested in designing a closed-loop control system to arrive at a specified loop transfer function. In this paper, we concentrate on a case when the uncertainties are modelled at the input point of a nominal plant model and hence the required loop transfer function is specified at the plant input point. However, our results can be dualized for the case when the required loop transfer function is specified at the plant output point. In recent years, a design procedure called LQG/LTR, originally proposed by Doyle and Stein

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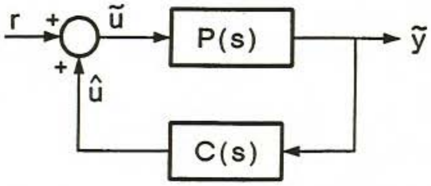

Figure 1. Plant–controller closed-loop configuration.

(1979), has gained some prominence. Essentially, LQG/LTR is a two step design procedure. In the first step of design, a standard state feedback design is done so that the resulting loop transfer function at the plant input point, here after called as a target loop transfer function, meets the given specifications. In the second step of design, one first assumes a closed loop configuration as in Fig. 1 where $C(s)$ and $P(s)$ are respectively the transfer functions of a controller and the given plant. Given $P(s)$ and the target loop transfer function $L(s)$, one seeks to design a $C(s)$ such that $C(j\omega)P(j\omega)$ is either exactly or 'approximately' equal to $L(j\omega)$ in the frequency region of interest. This second step of design is termed as LTR design and is the focus of this paper.

Let us consider a plant $\tilde{\Sigma}$,

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\tilde{\mathbf{u}}, \quad \tilde{\mathbf{y}} = \tilde{\mathbf{C}}\tilde{\mathbf{x}} \tag{1.1}$$

where the state vector $\tilde{\mathbf{x}} \in \mathbb{R}^n$, output vector $\tilde{\mathbf{y}} \in \mathbb{R}^p$ and input vector $\tilde{\mathbf{u}} \in \mathbb{R}^m$. Without loss of generality, assume that $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ are of maximal rank. Let us also assume that $\tilde{\Sigma}$ is stabilizable and detectable. Let $\tilde{\mathbf{F}}$ be a full state feedback gain matrix such that:

(*a*) the closed-loop system is asymptotically stable, i.e. eigenvalues of $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ lie in the left half s-plane; and

(*b*) the open-loop transfer function when the loop is broken at the input point of the plant meets the given frequency dependent specifications.

The state feedback control is

$$\tilde{\mathbf{u}} = -\tilde{\mathbf{F}}\tilde{\mathbf{x}} \tag{1.2}$$

and the loop transfer function evaluated when the loop is broken at the input point of the plant, the so-called target loop transfer function, is

$$L(s) = \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}} \tag{1.3}$$

where $\tilde{\mathbf{\Phi}} = (sI - \tilde{\mathbf{A}})^{-1}$. Instead of using the state feedback control law (1.2), if one uses an output feedback controller $C(s)$ as in Fig. 1, then the achieved loop transfer function evaluated when the loop is broken at the input point of the plant is

$$L_0(s) = C(s)P(s), \quad P(s) = \tilde{\mathbf{C}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}} \tag{1.4}$$

and thus our goal is to design a $C(s)$ such that the mismatch function $E(j\omega)$ with $E(s)$ defined as

$$E(s) = L(s) - L_0(s) \tag{1.5}$$

is either exactly zero or in some sense approximately zero over the frequency range of interest. More precisely, we say exact LTR (ELTR) is achieved if

$$C(s)P(s) = L(s) \text{ for all } s$$

Achieving ELTR is in general not possible. In an attempt to achieve 'approximate' LTR, one normally parameterizes $C(s)$ as a function of a scalar or a vector parameter σ and thus obtains a family of controllers $C(s, \sigma)$. We say asymptotic LTR (ALTR) is achieved if

$$C(s, \sigma)P(s) \rightarrow L(s) \quad \text{pointwise in } s$$

as the tuning parameter $\sigma \rightarrow \infty$, or equivalently $E(s, \sigma) \rightarrow 0$ pointwise in s as $\sigma \rightarrow \infty$. Achievability of ALTR enables the designer to choose a member of the family of controllers that corresponds to a particular value of σ which achieves a desired level of recovery. Traditionally, observer based controllers either full or reduced order type are used for LTR (an exception to this is Chen *et al.* 1990). In these controllers, observer gain $\tilde{\mathbf{K}}$ is the only design variable and hence a family of controllers is obtained by parameterizing it as a function of σ and $\tilde{\mathbf{K}}(\sigma)$ is designed so that ALTR is achieved. Such a design is in general possible only for left invertible and minimum phase systems. The purpose of this paper is to analyse what is and what is not possible for general non-minimum phase systems without imposing any assumptions either on them or on target loop transfer functions.

Ever since the seminal works of Kwakernaak (1969) and Doyle and Stein (1979), there have been many papers on LTR using observer based controllers (Athans 1986, Chen *et al.* 1989, Chen *et al.* 1990, Dowdle *et al.* 1982, Goodman 1984, Madiwale and Williams 1985, Matson and Maybeck 1987, Ridgely and Banda 1986, Saberi and Sannuti 1990, Sogaard-Andersen 1989, Sogaard-Andersen and Niemann 1989, Stein and Athans 1987, Zhang and Freudenberg 1987, 1990). All these papers attest to the fact that ALTR is always achievable for left invertible and minimum phase plants while providing some sufficient conditions under which ELTR is possible. In general, ALTR requires high observer gain and LTR is achieved asymptotically as the gain tends to infinity, i.e. $\tilde{\mathbf{K}}(\sigma) \rightarrow \infty$ as $\sigma \rightarrow \infty$. As is well known, in high-gain systems, some of the closed-loop eigenvalues tend to infinity at several rates and others to finite values corresponding to the finite zeros of the given plant. In other words, high-gain feedback in general induces a multiple time-scale structure to a closed-loop system. Thus observer design can be viewed as assigning a proper time-scale structure and an eigenstructure to the observer dynamic matrix. At this time, there exist three methods of determining the required observer gain for minimum phase plants: (1) Kalman filter formalism (Doyle and Stein 1979), (2) direct eigenstructure placement method (Sogaard-Andersen 1989), and (3) asymptotic eigenstructure and time-scale structure assignment (ATEA) method (Saberi and Sannuti 1990). Kalman filter formalism has been well studied and well understood for left invertible and minimum phase plants. In it, the observer eigenstructure is controlled by varying the intensity of the input process noise, i.e. the tuning parameter σ is the intensity of the input process noise. Here an appropriate high gain is obtained by solving a parameter dependent algebraic Riccati equation (ARE). Thus it is an asymptotic LQG method. The main computational effort here is spent in solving repeatedly ARE's for each σ. We will refer to such a design as ARE based design. In direct eigenstructure placement method, some of the eigenvalues of the observer are placed at the plant finite (invariant) zeros while the rest of them are placed far away in the negative half

s-plane. However, there is a fundamental difficulty in placing the far away eigenvalues. One has to make sure that the residues associated with the far away eigenvalues remain uniformly bounded as these eigenvalues are pushed to infinity. There is no direct way of assuring this. In ATEA method, observer gain is parameterized directly in terms of σ rather than being done indirectly via a parameter dependent ARE. The parameter σ comes into play only in changing the degree of fastness of various time-scales. It is fair to state that for the case of left invertible and minimum phase plants, there exists ample literature describing both the mechanism of either ELTR or ALTR and the methods of determining the required gain. However, in contrast, not much work has been done in discussing the issues that arise when general not necessarily left invertible, not necessarily minimum phase plants are considered.

In the existing literature, LTR for non-minimum phase plants is mainly handled by ignoring the non-minimum phase property of the given plant and using the ARE based design as is done for minimum phase plants and then accepting the consequent recovery error as it is. Such an approach is equivalent to performing the LTR procedure for the minimum phase image model which is extracted via an all-pass/minimum phase decomposition of the plant, since it is well known that the Kalman filter gain in an LQG controller for the non-minimum phase plant is equivalent to the corresponding gain for its minimum phase image model. Recently, Zhang and Freudenberg (1987, 1990) for the first time showed explicitly what happens when such an approach is taken. They develop expressions for the resulting asymptotic behaviour of the loop transfer as well as sensitivity functions. More recently, Niemann and Jannerup (1990) have expanded further on the results of Zhang and Freudenberg (1987, 1990). Thus one can conclude that the LTR research for non-minimum phase plants so far has concentrated on the *analysis of the outcome* of the application of ARE-based design which was developed earlier for left invertible and minimum phase systems. Consequently, all the design possibilities and constraints for non-minimum phase plants are not known in general when one does not keep in mind any particular design method. All one knows at this stage of research is that for general systems, even ALTR, let alone ELTR, is not possible. Because of this fact, several issues spring up. The first and foremost issue irrespective of the design methodology that can be used concerns with questions such as (1) What is feasible in general? (2) What are the design limitations that one encounters? (3) Is there any freedom to shape asymptotically or otherwise the part that which is not completely recoverable? (4) Can one develop meaningful bounds on the error between the target and the attainable loop transfer functions or sensitivity and complimentary sensitivity functions? (5) Can one characterize the manifolds or subspaces in which complete recovery is possible? etc. To seek answers to these questions in a systematic way, let us recall once again the conventional design methodology. Typically, the design is separated into two distinct tasks of first designing a state feedback gain which achieves a specified target loop transfer function and then designing an observer based controller to recover it as best as possible. In such a design methodology, one could first design an arbitrarily specified target loop transfer function without knowing whether it is recoverable or not by an observer based controller. This implies that the first issue of our analysis is to investigate what is and what is not feasible by observer based controllers for a given system irrespective of the properties of the target loop transfer functions. Such an analysis will point out the limitations of the given system as a consequence of its structural properties such as finite and infinite zero structure, and invertibility. On the other hand, it makes sense to characterize the required necessary and sufficient conditions on target loop transfer functions so that they are either exactly or asymptotically recoverable by means of observer based controllers for a given system. Such an analysis helps a designer to set meaningful goals at the onset of design. In other words, although the actual physical tasks of first designing a target loop and then designing an observer based controller are separable, one needs to bridge or link these two tasks philosophically by knowing ahead what kind of target loops are recoverable for a given system. Having thus developed the necessary analytical tools which can point out the limitations of the given system or the constraints on the recoverable target loops, one can move onto further analysis. In particular, since recovery in general in all control loops as desired by the designer is not feasible, one can then naturally look for methods to analyse whether recovery in a chosen subspace of the control space is feasible or not. That is, due to the directional behaviour of MIMO systems, one can begin to characterize the required conditions so that the projection of the target and attainable sensitivity and complimentary sensitivity functions onto a given subspace of $\mathbb{R}^m$ match each other either exactly or asymptotically. This in turn can lead to several other pertinent questions, e.g. What is the maximum dimension of a subspace in which the target sensitivity and complimentary sensitivity functions can be recovered? Thus the fact that complete recovery in general is not possible, releases a flood of issues for careful study. As alluded to by the above discussion, one can divide the analysis of these issues into three central parts. The first one is to analyse what can and what cannot be achieved for a given system when the target loop transfer function is arbitrarily specified. The second part is the development of necessary or/and sufficient conditions a target loop has to satisfy so that it can be either exactly or asymptotically be recovered for a given system. The third part is to develop method(s) to test whether recovery is possible in a given subspace of the control space or not. The recovery formulation in this part is in a sense a generalization of the notion of conventional LTR. Inherent in all the above three issues is the characterization of the resulting controller eigenvalues and possible pole zero cancellations. Such an investigation is important in view of the fact, controller eigenvalues become the invariant zeros of the closed-loop system and thus affect the performance with respect to command following and other design objectives. Our goal is to deal with all these issues systematically and explicitly in a direct way without being tied to any design methodology in the process of analysis.

The paper is organized as follows. Recognizing that finite and infinite zero structure of a given system plays a dominant role in LTR, in § 2, we recall a special coordinate basis (s.c.b.) of Sannuti and Saberi (1987) which displays clearly the required zero structure. Zero dynamics, invariant and variant zeros of a given system are defined and how s.c.b. portrays these zeros is clearly discussed. Connections between the s.c.b. and the various invariant and almost invariant subspaces of geometric theory as needed for our development are also given there. Section 3 deals with all the fundamental analysis. In particular, in § 3.1 we analyse the recovery mechanism for an arbitrarily given target loop. This analysis includes not only the recovery of target loop transfer function but also target sensitivity and complimentary sensitivity functions. We show that either ELTR or ALTR in general is not possible. Whenever LTR is not possible, we give explicit expressions for the asymptotic limits of loop transfer function and sensitivity and complimentary sensitivity functions. Moreover, we give explicit bounds on the attainable sensitivity and complimentary sensitivity functions in terms of the singular values of what is called a recovery error matrix $\bar{M}_e(s)$. These bounds can be used to analyse the inevitable trade-off between the good recovery as indicated by $\sigma_{\max}\bar{M}_e(j\omega)$ and robustness and performance as reflected in the sensitivity and complimentary sensitivity functions. We next move on to the characterization and possible shaping of a subspace in which the target sensitivity and complimentary sensitivity functions can be recovered. All the analysis given here treats the target loop transfer function $L(s)$ as an arbitrarily given matrix, i.e. no particular properties of $L(s)$ are exploited in the analysis. However, § 3.2 takes into account the specific characteristics $L(s)$ might have. Here the necessary and sufficient conditions under which $L(s)$ can either exactly or approximately be recovered are given. Interestingly enough, these constraints turn out to be constraints on the finite and infinite zero structure of it. Such an interpretation of the constraints reveals that either ELTR or ALTR is possible under a variety of conditions. For instance, LTR can be achieved even if the target loop transfer function does not contain non-minimum phase zero structure of the given system provided some other conditions are satisfied. An example illustrates this. Both § 3.1 and 3.2 stress recoverability in the entire control space $\mathbb{R}^m$. On the other hand, § 3.3 generalizes all the results developed in § 3.1 and § 3.2 to cover recoverability of the target sensitivity and complimentary sensitivity functions in a specified subspace and thus adds a considerable amount of flexibility to the process of design. It also shows that for left invertible systems irrespective of the number of non-minimum phase zeros and irrespective of the nature of the target loop transfer function, there exists at least one $m-1$ dimensional subspace of $\mathbb{R}^m$ in which the target sensitivity and complimentary sensitivity functions can always be recovered by an appropriate design of the controller. Also, in § 3 under all the analysis conditions given above, the resulting controller eigenvalues and possible pole zero cancellations are clearly discussed.

As mentioned earlier, this paper deals only with the issues concerning analysis. It shows the limitations of the given system as a consequence of its structural properties while discovering a multitude of ways in which freedom exists to shape the loops as close as possible to the target shapes. In a sequel to this paper, we will present a design methodology which is capable of utilizing the complete freedom a design can have. This is in contrast to the ARE based approach which chooses to shape the loops in a particular way among an array of available choices. For left invertible and minimum phase plants, since ALTR is always possible, the particular way the ARE based approach accomplishes the design does not play a critical role although it results in an unnecessarily high controller gain and band-width. However, for general systems, ability to utilize all the available design freedom is of paramount importance. The path taken by the ARE based design to shape the loop is not necessarily the best path and hence one needs to explore all the available design freedom; especially exploring such a freedom in the subspace in which complete recovery is not possible is a dire necessity. The design methodology proposed in our sequel paper allows the designer to utilize all the available freedom. It follows the asymptotic time-scale structure and eigenstructure assignment (ATEA) concepts of our earlier work (Saberi and Sannuti 1989, 1990). Also, in Part 2, a method of design to achieve ELTR whenever it is possible is given. These design methods have been implemented into a 'Matlab' software package and our experience with them shows that ARE based design has an abundant number of deficiencies.

Throughout this paper, A' denotes the transpose of A, A^H denotes the complex conjugate transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. $\lambda(A)$ and $\mathrm{Re}\,[\lambda(A)]$ respectively denote the set of eigenvalues and real parts of eigenvalues of A. Similarly, $\sigma_{\max}[A]$ and $\sigma_{\min}[A]$ respectively denote the maximum and minimum singular values of A. $\mathrm{Ker}\,[V]$ and $\mathrm{Im}\,[V]$ denote respectively the kernel and the image of V. The open left and closed right half s-planes are respectively denoted by $\mathscr{C}^-$ and $\mathscr{C}^+$.

2. Preliminaries

As we shall see throughout this paper, finite and infinite zero structures of both the given system and the target loop transfer function play a dominant role in the recovery analysis as well as design. In fact, the whole subject of LTR can be viewed as the study of assigning a zero structure to a closed-loop system within the constraints imposed by the zero structures of the given open-loop system and the target loop transfer function. Thus a good non-ambiguous understanding of zero structure is essential for our study. Keeping this in mind, we recall in this section a special coordinate basis (s.c.b.) of a linear time invariant system (Sannuti and Saberi 1987). Such a s.c.b. has a distinct feature of explicitly displaying the finite and infinite zero structure of a given system. Various definitions regarding zero dynamics, invariant and variant zeros of a given system are given and portrayal of these by the s.c.b. is discussed. Connections between the s.c.b. and the various invariant and almost invariant subspaces of geometric theory as needed for our development are also given. A reader who is not interested in the proofs of theorems in subsequent sections can either skip or read this section lightly. We have the following theorem.

Theorem 2.1: *Special coordinate basis*

Consider the system $\tilde{\boldsymbol{\Sigma}}$ characterized by the triple $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$. There exist non-singular transformations Γ_1, Γ_2 and Γ_3, an integer $m_u \leqslant m$, and integer indexes q_i, $i = 1$ to m_u, such that

$$\tilde{\mathbf{x}} = \Gamma_1 \tilde{x}, \quad \tilde{\mathbf{y}} = \Gamma_2 \tilde{y}, \quad \tilde{\mathbf{u}} = \Gamma_3 [\tilde{u}', \tilde{v}']'$$

$$\tilde{x} = [\tilde{x}_a', \tilde{x}_b', \tilde{x}_c', \tilde{x}_f']', \quad \tilde{x}_a = [(\tilde{x}_a^+)', (\tilde{x}_a^-)']'$$

$$\tilde{x}_f = [\tilde{x}_1', \tilde{x}_2', \ldots, \tilde{x}_{m_u}']'$$

$$\tilde{y} = [\tilde{y}_f', \tilde{y}_s']', \quad \tilde{y}_f = [\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{m_u}]'$$

$$\tilde{u} = [\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m_u}]'$$

$$\dot{\tilde{x}}_a^+ = A_{aa}^+ \tilde{x}_a^+ + L_{af}^+ \tilde{y}_f + L_{as}^+ \tilde{y}_s \tag{2.1}$$

$$\dot{\tilde{x}}_a^- = A_{aa}^- \tilde{x}_a^- + L_{af}^- \tilde{y}_f + L_{as}^- \tilde{y}_s \tag{2.2}$$

$$\dot{\tilde{x}}_b = A_{bb} \tilde{x}_b + L_{bf} \tilde{y}_f, \quad \tilde{y}_s = C_s \tilde{x}_b \tag{2.3}$$

$$\dot{\tilde{x}}_c = A_{cc} \tilde{x}_c + L_{cf} \tilde{y}_f + L_{cs} \tilde{y}_s + B_c [E_{ca}^+ \tilde{x}_a^+ + E_{ca}^- \tilde{x}_a^- + \tilde{v}] \tag{2.4}$$

and for each $i = 1$ to m_u

$$\dot{\tilde{x}}_i = A_{q_i}\tilde{x}_i + L_i\tilde{y}_f + B_{q_i}\left[\tilde{u}_i + E_{ia}\tilde{x}_a + E_{ib}\tilde{x}_b + E_{ic}\tilde{x}_c + \sum_{j=1}^{m_u} E_{ij}\tilde{x}_j\right] \tag{2.5}$$

$$\tilde{y}_i = C_{q_i}\tilde{x}_i, \quad \tilde{y}_f = C_f\tilde{x}_f \tag{2.6}$$

Here the states $\tilde{x}_a^+$, $\tilde{x}_a^-$, $\tilde{x}_b$, $\tilde{x}_c$ and $\tilde{x}_f$ are respectively of dimension n_a^+, n_a^-, n_b, n_c and

$$n_f = \sum_{i=1}^{m_u} q_i$$

while $\tilde{x}_i$ is of dimension q_i for each $i = 1$ to m_u. The control vectors $\tilde{u}$ and $\tilde{v}$ are respectively of dimension m_u and $m_v = m - m_u$ while the output vectors $\tilde{y}_f$ and $\tilde{y}_s$ are respectively of dimension $p_f = m_u$ and $p_s \leqslant n_b$. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the following form

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1 \quad 0 \quad \dots \quad 0] \tag{2.7}$$

(Obviously for the case when $q_i = 1$, $A_{q_i} = 0$, $B_{q_i} = 1$ and $C_{q_i} = 1$.) Furthermore, we have $\lambda(A_{aa}^+) \in \mathscr{C}^+$, $\lambda(A_{aa}^-) \in \mathscr{C}^-$, the pair (A_{cc}, B_c) is controllable and the pair (A_{bb}, C_s) is observable. Moreover, assuming that x_i are arranged such that $q_i \leqslant q_{i+1}$, the matrix L_i has the particular form

$$L_i = [L_{i1} \quad L_{i2} \quad \dots \quad L_{ii-1} \quad 0 \quad 0 \quad \dots \quad 0]$$

Also, the last row of each L_i is identically zero.

Proof

The proof follows from Theorem 2.1 of Sannuti and Saberi (1987).

We can rewrite the s.c.b. given by Theorem 2.1 in a more compact form:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}[\tilde{u}', \tilde{v}']', \quad \tilde{y} = \tilde{C}\tilde{x}$$

where $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are in the form

$$\tilde{A} = \begin{bmatrix} A_{aa}^+ & 0 & L_{as}^+C_s & 0 & L_{af}^+C_f \\ 0 & A_{aa}^- & L_{as}^-C_s & 0 & L_{af}^-C_f \\ 0 & 0 & A_{bb} & 0 & L_{bf}C_f \\ B_cE_{ca}^+ & B_cE_{ca}^- & L_{cs}C_s & A_{cc} & L_{cf}C_f \\ B_fE_a^+ & B_fE_a^- & B_fE_b & B_fE_c & A_f \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_f & 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & C_s & 0 & 0 \end{bmatrix} \tag{2.8}$$

We next define a dual system $\tilde{\mathbf{\Sigma}}_t$ characterized by the triple $(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)$ where

$$\tilde{\mathbf{C}}_t = \tilde{\mathbf{B}}', \quad \tilde{\mathbf{A}}_t = \tilde{\mathbf{A}}', \quad \tilde{\mathbf{B}}_t = \tilde{\mathbf{C}}'$$

In what follows, we state some important properties of the s.c.b. which are pertinent to our present work. These properties are stated without proofs, however the proofs are straightforward and simple.

Property 2.1

We note that (A_{bb}, C_s) and (A_{q_i}, C_{q_i}) form observable pairs. Unobservability could arise only in the variables $\tilde{x}_a$ and $\tilde{x}_c$. In fact, the system $\tilde{\Sigma}$ is observable (detectable) iff $(A_{\text{obs}}, C_{\text{obs}})$ is an observable (detectable) pair, where

$$A_{\text{obs}} = \begin{bmatrix} A_{aa} & 0 \\ B_c E_{ca} & A_{cc} \end{bmatrix}, \quad A_{aa} = \begin{bmatrix} A_{aa}^+ & 0 \\ 0 & A_{aa}^- \end{bmatrix}, \quad C_{\text{obs}} = [E_a \quad E_c]$$

$$E_a = [E'_{1a} \quad E'_{2a} \quad \dots \quad E'_{m_u a}]', \quad E_c = [E'_{1c} \quad E'_{2c} \quad \dots \quad E'_{m_u c}]'$$

Similarly, (A_{cc}, B_c) and (A_{q_i}, B_{q_i}) form controllable pairs. Uncontrollability could arise only in the variables $\tilde{x}_a$ and $\tilde{x}_b$. In fact, $\tilde{\Sigma}$ is controllable (stabilizable) iff $(A_{\text{con}}, B_{\text{con}})$ is a controllable (stabilizable) pair, where

$$A_{\text{con}} = \begin{bmatrix} A_{aa} & L_{as} C_s \\ 0 & A_{bb} \end{bmatrix}, \quad B_{\text{con}} = \begin{bmatrix} L_{af} \\ L_{bf} \end{bmatrix}, \quad L_{as} = \begin{bmatrix} L_{as}^+ \\ L_{as}^- \end{bmatrix}, \quad L_{af} = \begin{bmatrix} L_{af}^+ \\ L_{af}^- \end{bmatrix}$$

Property 2.2

The given system $\tilde{\Sigma}$ is right invertible iff $\tilde{x}_b$ and hence $\tilde{y}_s$ are non-existent $(n_b = 0, p_s = 0)$, left invertible iff $\tilde{x}_c$ and hence $\tilde{v}$ are non-existent $(n_c = 0, m_v = 0)$, invertible iff both $\tilde{x}_b$ and $\tilde{x}_c$ are non-existent. Moreover, $\tilde{\Sigma}$ is degenerate iff it is neither left nor right invertible.

There are interconnections between the s.c.b. and various invariant and almost invariant geometric subspaces (Wonham 1985). To show these interconnections, we shall use the following standard notation:

$\mathscr{V}^*$ the supremal $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$-invariant subspace 'contained' in Ker $(\tilde{\mathbf{C}})$

$\mathscr{R}^*$ the supremal $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$-controllability subspace 'contained' in Ker $(\tilde{\mathbf{C}})$

$\mathscr{S}^*$ the infimal $(\tilde{\mathbf{A}}, \tilde{\mathbf{C}})$-invariant subspace 'containing' Im $(\tilde{\mathbf{B}})$

$\mathscr{N}^*$ the infimal $(\tilde{\mathbf{A}}, \tilde{\mathbf{C}})$-observability subspace 'containing' Im $(\tilde{\mathbf{B}})$

$\mathscr{V}^*_{b,\,\text{Ker}\,\tilde{\mathbf{C}}}$ the supremal L_p-almost invariant subspace 'contained' in Ker $(\tilde{\mathbf{C}})$

$\mathscr{R}^*_{b,\,\text{Ker}\,\tilde{\mathbf{C}}}$ the supremal L_p-almost controllability subspace 'contained' in Ker $(\tilde{\mathbf{C}})$

$\mathscr{S}^*_{b,\,\text{Im}\,\tilde{\mathbf{B}}}$ the infimal L_p-almost conditional invariant subspace 'containing' Im $(\tilde{\mathbf{B}})$

$\mathscr{N}^*_{b,\,\text{Im}\,\tilde{\mathbf{B}}}$ the infimal L_p-almost complementary observability subspace 'containing' Im $(\tilde{\mathbf{B}})$

$\mathscr{V}^-_{\text{Ker}\,\tilde{\mathbf{C}}}$ $\{\tilde{\mathbf{x}}_0 \in \mathbb{R}^n \mid \tilde{\mathbf{x}}(t) : \mathbb{R} \to \mathbb{R}^n, \ \tilde{\mathbf{x}}(t)$ absolutely continuous such that $\dot{\tilde{\mathbf{x}}} - \tilde{\mathbf{A}}\tilde{\mathbf{x}} \in \text{Im}\,(\tilde{\mathbf{B}})$ for almost all t, $\tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0$, $\lim_{t\to\infty} \tilde{\mathbf{x}}(t) = 0$ and $\tilde{\mathbf{C}}\tilde{\mathbf{x}} = 0$ for all $t\}$

$$\mathscr{V}^-_{b,\,\text{Ker}\,\tilde{\mathbf{C}}} := \mathscr{V}^-_{\text{Ker}\,\tilde{\mathbf{C}}} + \mathscr{R}^*_{b,\,\text{Ker}\,\tilde{\mathbf{C}}}$$

$$\mathscr{V}^+_{\text{Ker}\,\tilde{\mathbf{C}}} := \mathscr{V}^*_{\text{Ker}\,\tilde{\mathbf{C}}} / \mathscr{V}^-_{\text{Ker}\,\tilde{\mathbf{C}}}$$

$$\mathscr{V}^+_{b,\,\text{Ker}\,\tilde{\mathbf{C}}} := \mathscr{V}^+_{\text{Ker}\,\tilde{\mathbf{C}}} + \mathscr{R}^*_{b,\,\text{Ker}\,\tilde{\mathbf{C}}}$$

$$\mathscr{S}^-_{b,\,\text{Im}\,\tilde{\mathbf{B}}}(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) := (\mathscr{V}^-_{b,\,\text{Ker}\,\tilde{\mathbf{C}}_t}(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t))^\perp = \mathscr{V}^+_{\text{Ker}\,\tilde{\mathbf{C}}}(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$$

$$\mathscr{S}^+_{b,\,\text{Im}\,\tilde{\mathbf{B}}}(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) := (\mathscr{V}^+_{b,\,\text{Ker}\,\tilde{\mathbf{C}}_t}(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t))^\perp = \mathscr{V}^-_{\text{Ker}\,\tilde{\mathbf{C}}}(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$$

$$\mathscr{L}^*(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) := \mathscr{V}^*(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)^\perp$$

$$\mathscr{T}^*(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) := \mathscr{R}^*(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)^\perp$$

$$\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) := \mathscr{V}^-_{\text{Ker}\,\tilde{\mathbf{C}}_t}(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)^\perp$$

$$\mathscr{L}_+(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) := \mathscr{V}^+_{\text{Ker}\,\tilde{\mathbf{C}}_t}(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)^\perp$$

Various components of the state vector of the s.c.b. have the following geometrical interpretations.

Properties 2.3

(1) $$\tilde{x}_a \oplus \tilde{x}_c \text{ spans } \mathscr{V}^*$$

(2) $$\tilde{x}_c \text{ spans } \mathscr{R}^*$$

(3) $$\tilde{x}_c \oplus \tilde{x}_f \text{ spans } \mathscr{S}^*$$

(4) $$\tilde{x}_a \oplus \tilde{x}_c \oplus \tilde{x}_f \text{ spans } \mathscr{N}^*$$

(5) $$\tilde{x}_a \oplus \tilde{x}_c \oplus \tilde{x}_f \text{ spans } \mathscr{V}^*_{b,\,\mathrm{Ker}\,\tilde{\mathbf{C}}}$$

(6) $$\tilde{x}_c \oplus \tilde{x}_f \text{ spans } \mathscr{R}^*_{b,\,\mathrm{Ker}\,\tilde{\mathbf{C}}}$$

(7) $$\tilde{x}_c \text{ spans } \mathscr{S}^*_{b,\,\mathrm{Im}\,\tilde{\mathbf{B}}}$$

(8) $$\tilde{x}_a \oplus \tilde{x}_c \text{ spans } \mathscr{N}^*_{b,\,\mathrm{Im}\,\tilde{\mathbf{B}}}$$

(9) $$\tilde{x}_a^- \oplus \tilde{x}_c \text{ spans } \mathscr{V}^-_{\mathrm{Ker}\,\tilde{\mathbf{C}}}$$

(10) $$\tilde{x}_a^- \oplus \tilde{x}_c \oplus \tilde{x}_f \text{ spans } \mathscr{V}^-_{b,\,\mathrm{Ker}\,\tilde{\mathbf{C}}}$$

(11) $$\tilde{x}_a^+ \oplus \tilde{x}_c \text{ spans } \mathscr{V}^+_{\mathrm{Ker}\,\tilde{\mathbf{C}}}$$

(12) $$\tilde{x}_a^+ \oplus \tilde{x}_c \oplus \tilde{x}_f \text{ spans } \mathscr{V}^+_{b,\,\mathrm{Ker}\,\tilde{\mathbf{C}}}$$

(13) $$\tilde{x}_a^+ \oplus \tilde{x}_c \text{ spans } \mathscr{S}^-_{b,\,\mathrm{Im}\,\tilde{\mathbf{B}}}(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$$

(14) $$\tilde{x}_a^- \oplus \tilde{x}_c \text{ spans } \mathscr{S}^+_{b,\,\mathrm{Im}\,\tilde{\mathbf{B}}}(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$$

(15) $$\tilde{x}_c \oplus \tilde{x}_f \text{ spans } \mathscr{L}^*(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$$

(16) $$\tilde{x}_a \oplus \tilde{x}_c \oplus \tilde{x}_f \text{ spans } \mathscr{T}^*(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$$

(17) $$\tilde{x}_a^+ \oplus \tilde{x}_c \oplus \tilde{x}_f \text{ spans } \mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$$

(18) $$\tilde{x}_a^- \oplus \tilde{x}_c \oplus \tilde{x}_f \text{ spans } \mathscr{L}_+(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$$

Remark 2.1

With the help of Properties 2.3, one can easily interpret the state vectors of s.c.b. in terms of various geometric subspaces.

(1) $$\tilde{x}_a \text{ spans } \mathscr{V}^*/\mathscr{R}^*$$

(2) $$\tilde{x}_a^+ \text{ spans } \mathscr{V}^+_{\mathrm{Ker}\,\tilde{\mathbf{C}}}/\mathscr{R}^*$$

(3) $$\tilde{x}_a^- \text{ spans } \mathscr{V}^-_{\mathrm{Ker}\,\tilde{\mathbf{C}}}/\mathscr{R}^*$$

(4) $$\tilde{x}_b \text{ spans } \mathscr{N}^{*\perp}$$

(5) $$\tilde{x}_c \text{ spans } \mathscr{R}^*$$

(6) $$\tilde{x}_f \text{ spans } \mathscr{S}^*/\mathscr{R}^*$$

Also, all the geometric subspaces defined earlier can be seen to be appropriate unions of five basic subspaces which are spans of states $\tilde{x}_a^+$, $\tilde{x}_a^-$, $\tilde{x}_b$, $\tilde{x}_c$ and $\tilde{x}_f$.

We next proceed to discuss the finite zero structure of $\tilde{\Sigma}$. It is traditional to define an invariant zero z and the associated right state and input zero directions x_R and w_R of $\tilde{\Sigma}$ as those which satisfy the equation

$$P(z)\begin{bmatrix} x_R \\ w_R \end{bmatrix} = 0 \tag{2.9}$$

where

$$P(z) = \begin{bmatrix} zI_n - \tilde{\mathbf{A}} & -\tilde{\mathbf{B}} \\ -\tilde{\mathbf{C}} & 0 \end{bmatrix}$$

When the invariant zeros are not simple, MacFarlane and Karcanias (1976), also Soroka and Shaked (1988), give the following definitions.

Definition 2.1

The rank deficiency σ_z of $P(z)$ is called the geometric multiplicity of the zero z and the degree ρ_z of the product of elementary divisors of $P(s)$ that correspond to z is called the algebraic multiplicity of this zero. Thus $\rho_z \geqslant \sigma_z$.

Definition 2.2

Let z, x_R^0 and w_R^0 be an invariant zero and the corresponding state and input zero directions of $\tilde{\Sigma}$. Then the vectors x^j_R and w^j_R, $j = 1, ..., \rho_z - \sigma_z$ are defined as the state and input pseudo zero directions associated with z if they satisfy

$$P(z)\begin{bmatrix} x^j_R \\ w^j_R \end{bmatrix} = -\begin{bmatrix} x^{j-1}_R \\ 0 \end{bmatrix}, \quad j = 1, ..., \rho_z - \sigma_z \tag{2.10}$$

The following examples illustrate the short comings of these definitions.

Example 2.1

Let $\tilde{\Sigma}$ be characterized by $\tilde{\mathbf{A}} = I_4$

$$\tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \tilde{\mathbf{C}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Any point in the entire s-plane is an invariant zero by the traditional definition with the corresponding x_R and w_R as

$$x_R = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad w_R = 0$$

where α and β are some constants. Having invariant zeros at all points in the entire s-plane is problematic. However, as is well known, the situation described in this example happens only when the given system is not left invertible.

Example 2.2

Let $\tilde{\Sigma}$ be given by

$$\tilde{\mathbf{A}} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\tilde{\mathbf{C}} = [0 \quad 0 \quad 0 \quad 1]$$

This system is square and invertible. It has three invariant zeros at $z = -1$ with $\rho_z = 3$ and $\sigma_z = 2$. It is straight forward to verify that

$$x_R^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad w_R^0 = -1$$

are the right state and input zero directions associated with $z = -1$. But no solution exists for pseudo zero directions as defined by (2.10). Thus the Definition 2.2 of pseudo zero directions is questionable even for square invertible systems.

As shown by the above examples, the definitions of algebraic and geometric multiplicities given above are not sufficient enough to define pseudo state and input zero directions. We now proceed to show that s.c.b. displays directly and precisely the finite zero structure of $\tilde{\Sigma}$. We first define the following.

Definition 2.3

Eigenvalues of A_{aa} are said to be the *invariant zeros* of $\tilde{\Sigma}$. Also, $\tilde{\Sigma}$ is said to be of *non-minimum phase* if any one of $\lambda(A_{aa})$ is in $\mathscr{C}^+$, otherwise it is said to be of *minimum phase*.

In order to define various multiplicities of invariant zeros, let X be a non-singular transformation matrix such that

$$X^{-1}A_{aa}X = J = \text{Block diag}\,[J_1, J_2, \ldots, J_k] \tag{2.11}$$

where J_i, $i = 1$ to k are some $n_i \times n_i$ Jordan blocks

$$J_i = \text{Diag}\,[z_i, z_i, \ldots, z_i] + \begin{bmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{bmatrix} \tag{2.12}$$

We note that

$$\sum_{i=1}^{k} n_i = n_a$$

Also, the geometric multiplicity of each invariant zero $z \in \lambda(A_{aa})$, is the number of Jordan blocks in (2.11) associated with z and the algebraic multiplicity is the total number of repetitions of z in $\lambda(A_{aa})$. The missing information in the literature is the sizes of the Jordan blocks associated with each z. It turns out that this missing information is crucial to define the state and input zero direction chains associated with each z. Hence, in what follows we define what is called the *multiplicity* *structure* of an invariant zero in such a way that it contains all the needed information.

Definition 2.4

For any given $z \in \lambda(A_{aa})$, let there be v_z Jordan blocks of A_{aa} as in (2.11) and (2.12) associated with z. Let $n_{z,1}, n_{z,2}, \ldots, n_{z,v_z}$ be the dimensions of the corresponding Jordan blocks. Then we say z is an invariant zero of $\tilde{\Sigma}$ with *multiplicity structure* S_z^*

$$S_z^* = \{n_{z,1}, n_{z,2}, \ldots, n_{z,v_z}\} \tag{2.13}$$

If $n_{z,1} = n_{z,2} = \ldots = n_{z,v_z} = 1$, then we say z is a *simple invariant zero* of $\tilde{\Sigma}$.

Remark 2.2

The geometric multiplicity σ_z of z is v_z and the algebraic multiplicity ρ_z of z is then given by

$$\rho_z = n_{z,1} + n_{z,2} + \ldots + n_{z,v_z}$$

For left invertible systems, geometric and algebraic multiplicities defined here coincide with those given by MacFarlane and Karcanias (1976) and Soroka and Shaked (1988).

Remark 2.3

Invariant zeros and their multiplicity structures can be defined in a coordinate free setting. Let $\mathscr{F}(\mathscr{V}^*/\mathscr{R}^*)$ denote the class of maps $\tilde{\mathbf{F}}: (\mathbb{R}^n \to \mathbb{R}^m)$ such that $(\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}})(\mathscr{V}^*/\mathscr{R}^*) \subset (\mathscr{V}^*/\mathscr{R}^*)$. Let $\tilde{\mathbf{A}}_F = (\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}})$ and $\bar{\mathbf{A}}_F$ be the map induced by $\tilde{\mathbf{A}}_F$ in $\mathscr{V}^*/\mathscr{R}^*$. We note that $\bar{\mathbf{A}}_F$ is independent of $\tilde{\mathbf{F}} \in \mathscr{F}(\mathscr{V}^*/\mathscr{R}^*)$ and that $\lambda(\bar{\mathbf{A}}_F) \subset \lambda(\tilde{\mathbf{A}}_F)$. Then it can be shown easily that $\bar{\mathbf{A}}_F$ and A_{aa} are related by a similarity transformation. Thus all our results can be transformed to a geometric space setting.

Now we can move on to define a state and input zero direction chain associated with an invariant zero z of $\tilde{\Sigma}$. We note that there exist $x_{11}^{z,a}, x_{21}^{z,a}, \ldots, x_{v_z 1}^{z,a}$ independent eigenvectors of A_{aa} which are associated with an eigenvalue z of A_{aa} having the multiplicity structure as in (2.13).

Definition 2.5: *Eigenvector chain*

For each $i = 1$ to v_z, a set of vectors in $\mathbb{R}^{n_a}$ which satisfy the following condition (2.14) is said to be the eigenvector chain of A_{aa} associated with the invariant zero z

$$(A_{aa} - zI_{n_a})x_{ij+1}^{z,a} = x_{ij}^{z,a}, \quad j = 1, \ldots, n_{z,i} - 1 \tag{2.14}$$

Definition 2.6: *Right state zero direction chain*

For each $i = 1$ to v_z, a set of vectors in $\mathbb{R}^n$ given in (2.15) is said to be the right state zero direction chain of $\tilde{\Sigma}$ associated with the invariant zero z

$$x_{ij}^z = \Gamma_1 \begin{bmatrix} x_{ij}^{z,a} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad j = 1, \ldots, n_{z,i} \tag{2.15}$$

Also, x_{i1}^z is said to be the *right state zero direction* of $\tilde{\Sigma}$ associated with z.

Definition 2.7: *Right input zero direction chain*

For each $i = 1$ to v_z, a set of vectors w_{ij}^z, $j = 1$ to $n_{z,i}$, in $\mathbb{R}^m$ as given in (2.16) is said to be the right input zero direction chain of $\tilde{\Sigma}$ associated with the invariant zero z

$$w_{ij}^z = -\Gamma_3 \begin{bmatrix} E_a \\ E_{ca} \end{bmatrix} x_{ij}^{z,a} \tag{2.16}$$

where $E_{ca} = [E_{ca}^+ \;\; E_{ca}^-]$ and E_a is as defined in Property 2.1. Also, w_{i1}^z is said to be the *right input zero direction* of $\tilde{\Sigma}$ associated with z.

We have the following propositions.

Proposition 2.1

Corresponding to each invariant zero z, the state zero direction chain satisfies

$$\tilde{\mathbf{C}} x_{ij}^z = 0 \text{ for all } i = 1 \text{ to } v_z,\; j = 1 \text{ to } n_{z,i}$$

Proof

The proof is obvious.

Proposition 2.2

The set of vectors comprising of all state zero direction chains of all invariant zeros of $\tilde{\Sigma}$ form a linearly independent set and span $\mathscr{V}^*/\mathscr{R}^*$.

Proof

The proof is obvious.

Proposition 2.3

Invariant zero, state and input zero directions, z, x_{i1}^z and w_{i1}^z as in Definitions 2.3 to 2.7, always satisfy (2.9). Also, when $\tilde{\Sigma}$ is left invertible and when all its invariant zeros are simple, z, x_{i1}^z and w_{i1}^z as in Definitions 2.3 to 2.7 imply and are implied as well by the traditional ones given by (2.9).

Proof

In view of (2.8)

$$(zI_n - \tilde{\mathbf{A}}) x_{i1}^z = \Gamma_1 \begin{bmatrix} zI_{n_a} - A_{aa} & -L_{as}C_s & 0 & -L_{af}C_f \\ 0 & zI_{n_b} - A_{bb} & 0 & -L_{bf}C_f \\ -B_c E_{ca} & -L_{cs}C_s & zI_{n_c} - A_{cc} & -L_{cf}C_f \\ -B_f E_a & -B_f E_b & -B_f E_c & zI_{n_f} - A_f \end{bmatrix} \begin{bmatrix} x_{i1}^{z,a} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= -\Gamma_1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_f & 0 \end{bmatrix} \begin{bmatrix} E_a \\ E_{ca} \end{bmatrix} x_{il}^{z,a}$$

$$= -\tilde{\mathbf{B}}\Gamma_3 \begin{bmatrix} E_a \\ E_{ca} \end{bmatrix} x_{i1}^{z,a}$$

$$= \tilde{\mathbf{B}} w_{i1}^z$$

Also, from Proposition 2.1, we have $\tilde{\mathbf{C}}x^z_{i1} = 0$. This proves the first point of this proposition. The second point can easily be verified.

The following proposition gives a physical meaning to the invariant zeros, and the state and input zero direction chain of a system.

Proposition 2.4

For a system $\tilde{\Sigma}$ which is not necessarily left invertible, given that the initial condition, $\tilde{\mathbf{x}}(0) = x^z_{i\alpha}$ for any $\alpha \leqslant n_{z,i}$ and the input

$$\tilde{\mathbf{u}} = \sum_{j=1}^{\alpha} \frac{w^z_{ij} t^{\alpha-j} \exp(zt)}{(\alpha - j)!} \quad \text{for all } t \geqslant 0 \tag{2.17}$$

where z is any invariant zero of the system and $n_{z,i} \in S^*_z$, we have

$$\tilde{\mathbf{y}} \equiv 0$$

and

$$\tilde{\mathbf{x}}(t) = \sum_{j=1}^{\alpha} \frac{x^z_{ij} t^{\alpha-j} \exp(zt)}{(\alpha - j)!} \quad \text{for all } t \geqslant 0 \tag{2.18}$$

Proof

Without loss of generality, we can assume that $\tilde{\Sigma}$ is in the form of s.c.b. Then it is straightforward to verify that $\tilde{\mathbf{y}} \equiv 0$, $\tilde{x}_b(t) \equiv 0$ and $\tilde{x}_f(t) \equiv 0$ for all $t \geqslant 0$. This implies that

$$\dot{\tilde{x}}_a = A^-_{aa} \tilde{x}_a$$

Then under the given initial condition, it is easy to verify (2.18). The rest of the proof also follows by direct verification.

One can define the left state and input zero direction chain associated with an invariant zero of $\tilde{\Sigma}$ as follows.

Definition 2.8

The left state and input zero direction chain associated with each invariant zero of $\tilde{\Sigma}$ are defined as the corresponding right state and input zero direction chain of the dual system $\tilde{\Sigma}_t$.

In view of Definitions 2.3–2.7, we can reconsider Examples 2.1 and 2.2.

Example 2.3

Consider the system given in Example 2.1. For this non-invertible system, Definitions 2.3 to 2.7 result in one invariant zero at $z = 1$ with right state and input zero directions x_R and w_R as

$$x_R = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_R = 0$$

where α is some constant.

Example 2.4

Consider the system given in Example 2.2. As we discussed earlier, this system is square and invertible and has three invariant zeros at $z=-1$. Following the Definitions 2.3–2.7, we have $S^*_{-1}=\{1,2\}$ and

$$x_{11}^z=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\quad x_{21}^z=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\quad x_{22}^z=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

and

$$w_{11}^z=-1,\quad w_{21}^z=-1\quad\text{and}\quad w_{22}^z=-1$$

We would like to discuss next what is called 'zero dynamics' of a given system. Without loss of generality, let us assume that the given system is in the form of a s.c.b. Now the trajectories of this system restricted to a manifold $\mathscr{M}\equiv\{y_f=0$ and $y_s=0\}$ are given by

$$\dot{\tilde{x}}_a^+=A_{aa}^+\tilde{x}_a^+ \tag{2.19}$$

$$\dot{\tilde{x}}_a^-=A_{aa}^-\tilde{x}_a^- \tag{2.20}$$

$$\dot{\tilde{x}}_c=A_{cc}\tilde{x}_c+B_c[E_{ca}^+\tilde{x}_a^++E_{ca}^-\tilde{x}_a^-]+B_c\tilde{v} \tag{2.21}$$

This is called output nulling dynamics or more commonly referred to as 'zero dynamics'. Obviously, in order to maintain the trajectories of $\tilde{\Sigma}$ on $\mathscr{M}$, one must use the feedback law,

$$\tilde{u}_i=-[E_{ia}^+\tilde{x}_a^++E_{ia}^-\tilde{x}_a^-+E_{ic}\tilde{x}_c]$$

for all $i=1$ to m_u. It is easy to see that the poles of zero dynamics are

$$\lambda(A_{aa}^+)\cup\lambda(A_{aa}^-)\cup\lambda(A_{cc})$$

Note that $\lambda(A_{aa}^+)$ and $\lambda(A_{aa}^-)$ are fixed and are the invariant zeros of $\tilde{\Sigma}$. However, $\lambda(A_{cc})$ are controllable via the input $\tilde{v}$. Thus we have the following definition:

Definition 2.9

The dynamics represented by (2.19) and (2.20) is called the invariant zero dynamics, in particular (2.19) is called the unstable invariant zero dynamics and (2.20) is called the stable invariant zero dynamics. Also, $\lambda(A_{cc})$ are called the variant zeros of $\tilde{\Sigma}$.

The following property deals with other types of zeros defined in the literature (Rosenbrock 1970).

Properties 2.4

(1) Output decoupling (o.d.) zeros of $\tilde{\Sigma}$ are the unobservable eigenvalues of the pair $(A_{\mathrm{obs}}, C_{\mathrm{obs}})$. Also, o.d. zeros of (A_{cc}, E_c) are contained in the set of o.d. zeros of $(A_{\mathrm{obs}}, C_{\mathrm{obs}})$. Some of the o.d. zeros of $\tilde{\Sigma}$ could be contained among its invariant zeros.

(2) Input decoupling (i.d.) zeros of $\tilde{\Sigma}$ are the uncontrollable eigenvalues of the pair $(A_{\text{con}}, B_{\text{con}})$. Also, i.d. zeros of (A_{bb}, L_{bf}) are contained in the set of i.d. zeros of $\tilde{\Sigma}$. Some of the i.d. zeros of $\tilde{\Sigma}$ could be contained among its invariant zeros.

(3) Input output decoupling (i.o.d.) zeros of $\tilde{\Sigma}$ are contained among its invariant zeros.

(4) System zeros of $\tilde{\Sigma} = \{\text{invariant zeros}\} + \{\text{uncontrollable eigenvalues of the pair } (A_{bb}, L_{bf})\} + \{\text{unobservable eigenvalues of the pair } (A_{cc}, E_c)\}$.

The above properties clearly show the finite zero structure of $\tilde{\Sigma}$. The s.c.b. can also reveal the infinite zero structure of $\tilde{\Sigma}$. Let us say that a rational matrix $H(s)$ possesses an infinite zero of order k when $H(1/z)$ has a finite zero of precisely that order at $z = 0$. Then the following property shows the structure at infinity of $\tilde{\Sigma}$ as displayed by the s.c.b.

Property 2.5

Let $\bar{q}_j$ be an integer such that exactly $\bar{q}_j$ elements of q_i, $i = 1$ to m_u, are equal to j. Also, let KI be an integer such that $\bar{q}_j = 0$ for all $j > KI$. Then there are $j\bar{q}_j$ number of infinite zeros of order j, for $j = 1$ to KI. Also, noting that

$$\sum_{j=1}^{KI} j\bar{q}_j = \sum_{i=1}^{m_u} q_i = n_f$$

the total number of infinite zeros of all orders is n_f.

We note that $\tilde{\Sigma}$ does not have an infinite zero of order j iff $\bar{q}_j = 0$. As discussed in Sannuti and Saberi (1987), the orders of infinite zeros are same as the $\mathscr{C}^*$ structural invariant indices (list I_4 of Morse 1973),

$$\mathscr{C}^* = \{\bar{n}_1, \bar{n}_2, \dots, \bar{n}_{m_u}\}.$$

Let this list be ordered so that $\bar{n}_1 \leqslant \bar{n}_2 \leqslant \dots \leqslant \bar{n}_{m_u}$. Also, assume that this list has KI distinct entries $e_1 < e_2 \dots < e_{KI}$ where e_i has multiplicity d_i, $i = 1$ to KI. That is

$$\sum_{i=1}^{KI} d_i e_i = \sum_{i=1}^{m_u} \bar{n}_i = n_f$$

Then we have

$$\begin{aligned} \mathscr{C}^* &= \{\bar{n}_1, \bar{n}_2, \dots, \bar{n}_{m_u}\} \\ &= \{\overbrace{e_1, e_1, \dots, e_1}^{d_1}, \overbrace{e_2, e_2, \dots, e_2}^{d_2}, \dots, \overbrace{e_{KI}, e_{KI}, \dots, e_{KI}}^{d_{KI}}\} \\ &= \{\overbrace{1, 1, \dots, 1}^{\bar{q}_1}, \overbrace{2, 2, \dots, 2}^{\bar{q}_2}, \dots, \overbrace{KI, KI, \dots, KI}^{\bar{q}_{KI}}\} \end{aligned} \tag{2.22}$$

3. General analysis

In this section, we consider a full order observer based controller as depicted in Fig. 2. As discussed in the introduction, we have three central issues that need to

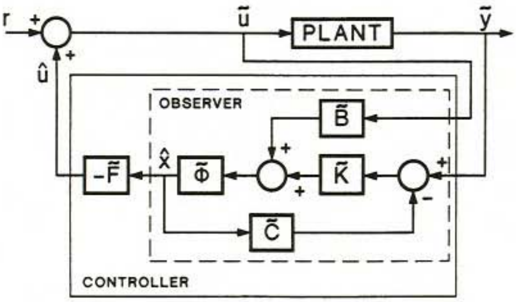

Figure 2. Plant with full order observer based controller.

be analysed here. The first issue concerns with the investigation of the available design freedom in characterizing the asymptotic behaviour of attainable loop transfer function and sensitivity and complimentary sensitivity functions. This issue does not concern itself with the nature of target loop transfer function which at this stage is assumed to be arbitrary. Thus the first issue purely deals with the limitations of the given system as a consequence of its structural properties in recovering an arbitrary target loop via a full order observer based controller. We discuss this issue in § 3.1. However, the second issue concerns with the characterization of target loop transfer functions which can either exactly or asymptotically be recovered for the given plant via an observer based controller. Such a characterization is of paramount importance for a designer in order to formulate meaningful goals at the onset of design. This issue is discussed in § 3.2 where the necessary and sufficient conditions on the recoverable target loop transfer functions are given. These conditions can be seen to be as constraints on the finite and infinite zero structure and the invertibility properties of target loop transfer functions. The third issue concerns with the first as well as the second issue when recovery is important solely in a given subspace of the entire control space. Recovery in a subspace means that the projection of the target and the achieved sensitivity and complimentary sensitivity functions onto a given subspace match each other exactly or asymptotically. Section 3.3 discusses this issue and gives every single result of § 3.1 and 3.2 when recovery is confined to a given subspace and thus generalizes the traditional notion of LTR. Inherently buried in all these three issues is the issue of analysing the controller eigenvalues, possible pole zero cancellations and the mechanism of such cancellations. This issue is also dealt with in this section.

We will now proceed with the analysis. $C(s)$, the transfer function of the observer based controller is given by

$$C(s) = \tilde{\mathbf{F}}[sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}\tilde{\mathbf{C}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}}]^{-1}\tilde{\mathbf{K}}$$

while $E(s)$, the error between the target loop transfer function $L(s)$ and that achievable by the controller, is given in (1.5). This expression (1.5) is not well suited for LTR analysis. Recognizing this, Goodman (1984) earlier related it to another transfer function $M(s)$. Although Goodman considers only square invertible systems, his results as given in the following lemma are equally valid for general systems.

Lemma 3.1

Consider any arbitrary $\tilde{\mathbf{F}}$ such that $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ is asymptotically stable. Then $E(s)$, the error between the target loop transfer function $L(s)$ and that realized by the controller of Fig. 2, is given by

$$E(s) = M(s)[I_m + M(s)]^{-1}(I_m + \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}) \tag{3.1}$$

where

$$M(s) = \tilde{\mathbf{F}}(sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}\tilde{\mathbf{C}})^{-1}\tilde{\mathbf{B}} \tag{3.2}$$

Furthermore for all $\omega \in \Omega$

$$E(j\omega) = 0 \quad \text{iff} \quad M(j\omega) = 0 \tag{3.3}$$

where Ω is the set of all $0 \leqslant |\omega| < \infty$ for which $L(j\omega)$ and $L_o(j\omega)$ are well defined (i.e. all required inverses exist).

Equations (3.2) and (3.3) present a clear perspective to study the basic mechanism of LTR. In fact, they facilitate the study of $E(s)$ in terms of the study of $M(s)$. Thus Lemma 3.1 and the expression for $M(s)$ as given by (3.2) form a basis for our study.

3.1. *Recovery analysis for an arbitrarily given target loop*

In a traditional LQG design, in view of the well known separation principle, a designer separates the designs of state feedback gain $\tilde{\mathbf{F}}$ and observer gain $\tilde{\mathbf{K}}$ into two different decoupled tasks. Keeping this in mind, our goal in this subsection is to analyse the LTR mechanism without taking into account any specific characteristics of $\tilde{\mathbf{F}}$. Thus the only freedom we have to achieve the needed recovery is in the selection of observer gain $\tilde{\mathbf{K}}$. First of all, in order to guarantee the closed-loop stability, $\tilde{\mathbf{K}}$ must be such that the observer dynamic matrix,

$$\tilde{\mathbf{A}}_0 = \tilde{\mathbf{A}} - \tilde{\mathbf{K}}\tilde{\mathbf{C}} \tag{3.4}$$

is an asymptotically stable matrix, i.e. $\lambda(\tilde{\mathbf{A}}_0) \in \mathscr{C}^-$. The remaining freedom in choosing $\tilde{\mathbf{K}}$ can then be used for the purpose of achieving LTR. Now in view of (3.2) and (3.3), ELTR is possible for an arbitrary $\tilde{\mathbf{F}}$ only if

$$\tilde{M}(j\omega) = (j\omega I_n - \tilde{\mathbf{A}}_0)^{-1}\tilde{\mathbf{B}} \equiv 0 \tag{3.5}$$

However, due to the nonsingularity of $(j\omega I_n - \tilde{\mathbf{A}}_0)^{-1}$, (3.5) implies that $\tilde{\mathbf{B}} \equiv 0$. But this is impossible in any real system and hence ELTR in general is impossible for an arbitrary $\tilde{\mathbf{F}}$. Thus one can only attempt to achieve ALTR, i.e. to render $\tilde{M}(j\omega)$ approximately zero in some sense. In order to analyse whether ALTR is possible, we parameterize the gain $\tilde{\mathbf{K}}$ with a tuning parameter σ and thus consider a family of controllers

$$C(s, \sigma) = \tilde{\mathbf{F}}[sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}}]^{-1}\tilde{\mathbf{K}}(\sigma) \tag{3.6}$$

Thus now $M(s)$ is also a function of σ and is denoted by $M(s, \sigma)$. To proceed with our analysis of $M(s, \sigma)$, for clarity of presentation we will temporarily assume that $\tilde{\mathbf{A}}_0$ is nondefective. This allows us to expand $M(s, \sigma)$ in a dyadic form

$$M(s, \sigma) = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i} \tag{3.7}$$

where the residue R_i is given by

$$R_i = \tilde{\mathbf{F}} W_i V_i^H \tilde{\mathbf{B}} \tag{3.8}$$

Here W_i and V_i are respectively the right and left eigenvectors associated with an eigenvalue λ_i of $\tilde{\mathbf{A}}_0$ and they are scaled so that $WV^H = V^H W = I_n$ where

$$W = [W_1, W_2, ..., W_n] \quad \text{and} \quad V = [V_1, V_2, ..., V_n] \tag{3.9}$$

In general, all λ_i, V_i and W_i are functions of σ. However, for economy of notation we will not show the dependence on σ explicitly unless it is needed for clarity.

Remark 3.1

The assumption that $\tilde{\mathbf{K}}(\sigma)$ is selected so that $\tilde{\mathbf{A}}_0$ is non-defective is not essential. It simplifies our presentation. A removal of this condition necessitates the use of generalized right and left eigenvectors of $\tilde{\mathbf{A}}_0$ instead of the right and left eigenvectors W_i and V_i and consequently the expansion of $M(s, \sigma)$ requires a double summation instead of (3.7).

We are looking for conditions under which for each $i = 1$ to n, the i-th term of $M(s, \sigma)$ in (3.7) can be made zero. There are only two possibilities to do so.

(*a*) The first possibility is by assigning λ_i to any finite value in $\mathscr{C}^-$ while simultaneously rendering the corresponding residue R_i zero. Since $\tilde{\mathbf{F}}$ is arbitrary, it necessitates rendering either $V_i^H(\sigma)\tilde{\mathbf{B}} = 0$ or $V_i^H(\sigma)\tilde{\mathbf{B}} \to 0$ as $\sigma \to \infty$. Thus this possibility deals with finite eigenstructure assignment of $\tilde{\mathbf{A}}_0$.

(*b*) The second possibility is to make

$$\frac{R_i}{s - \lambda_i} \to 0$$

pointwise in s as $\sigma \to \infty$. This can be done by placing the eigenvalue $\lambda_i(\sigma)$ asymptotically at infinity while making sure that the corresponding residue R_i is uniformly bounded as $\sigma \to \infty$. It is important to recognize that placing λ_i asymptotically at infinity alone is not beneficial unless the corresponding residue R_i is bounded. Since $\tilde{\mathbf{F}}$ is arbitrary, it amounts to assigning $W_i(\sigma)$ and $V_i(\sigma)$ such that $W_i(\sigma)V_i^H(\sigma)\tilde{\mathbf{B}}$ remains bounded while $\lambda_i \to \infty$ as $\sigma \to \infty$. Thus this possibility deals with infinite eigenstructure assignment of $\tilde{\mathbf{A}}_0$.

The above two possibilities of making a particular term of $M(s, \sigma)$ zero leads to two fundamental questions that need to be answered: (*a*) How many left eigenvectors of $\tilde{\mathbf{A}}_0$ can be assigned to the null space of $\tilde{\mathbf{B}}'$? and (*b*) How many eigenvalues of $\tilde{\mathbf{A}}_0$ can be placed at asymptotically infinite locations in $\mathscr{C}^-$ so that the corresponding residues are finite? The following two lemmas respectively answer these two questions.

Lemma 3.2

Let λ_i and V_i be an eigenvalue and the corresponding left eigenvector of $\tilde{\mathbf{A}}_0$ for any gain $\tilde{\mathbf{K}}(\sigma)$ such that $\tilde{\mathbf{A}}_0$ is stable. Then the maximum possible number of $\lambda_i \in \mathscr{C}^-$ which satisfy the condition $V_i^H \tilde{\mathbf{B}} = 0$ is $n_a^- + n_b$. A total of n_a^- of these λ_i

coincide with the plant invariant zeros which are in $\mathscr{C}^-$ (the so called minimum phase zeros) and the remaining n_b eigenvalues can be assigned arbitrarily to any locations in $\mathscr{C}^-$. All the eigenvectors V_i that correspond to these $n_a^- + n_b$ eigenvalues span the subspace $\mathscr{V}^-_{\mathrm{Ker}\,\tilde{\mathbf{C}}}/\mathscr{R}^* \oplus \mathscr{N}^{*\perp}$. Moreover, the n_a^- eigenvectors V_i which correspond to the eigenvalues which coincide with the plant invariant zeros in $\mathscr{C}^-$ coincide with the corresponding left state zero directions and span the subspace $\mathscr{V}^-_{\mathrm{Ker}\,\tilde{\mathbf{C}}}/\mathscr{R}^*$.

Proof

See Appendix A.

Lemma 3.3

Let λ_i, W_i and V_i be an eigenvalue and the corresponding right and left eigenvectors of $\tilde{\mathbf{A}}_0$ for any gain $\tilde{\mathbf{K}}(\sigma)$ such that $\tilde{\mathbf{A}}_0$ is stable. The maximum number of eigenvalues of $\tilde{\mathbf{A}}_0$ that can be assigned arbitrarily to asymptotically infinite locations in $\mathscr{C}^-$ so that the corresponding $W_i V_i^H \tilde{\mathbf{B}}$ are bounded as $|\lambda_i| \to \infty$ is $n_b + n_f$. Furthermore, all the corresponding left eigenvectors V_i of such eigenvalues asymptotically span the subspace $\mathscr{N}^{*\perp} \oplus \mathscr{S}^*/\mathscr{R}^*$.

Proof

See Appendix B.

As implied by Lemma 3.2, in addition to n_a^- eigenvalues which coincide with the plant minimum phase invariant zeros, there are n_b other eigenvalues which can be assigned arbitrarily to any locations in $\mathscr{C}^-$ such that $V_i^H \tilde{\mathbf{B}} \equiv 0$. This implies that $W_i V_i^H \tilde{\mathbf{B}}$ corresponding to these n_b eigenvalues are identically zero and hence are bounded. Thus these n_b eigenvalues are included among the $n_b + n_f$ eigenvalues indicated in Lemma 3.3. That is, there is a set of n_b eigenvalues which can be placed arbitrarily at either asymptotically finite locations in $\mathscr{C}^-$ as indicated by Lemma 3.2 or at asymptotically infinite locations in $\mathscr{C}^-$ as indicated by Lemma 3.3. Here after in order to conserve the controller band-width, we will assume that these n_b eigenvalues are always assigned to asymptotically finite locations.

Lemmas 3.2 and 3.3 together tell us all the possibilities of rendering various terms of $M(s, \sigma)$ zero either exactly or asymptotically. There are altogether $n_a^- + n_b + n_f$ eigenvalues which can be assigned either at finite or at asymptotically infinite locations so that the corresponding terms of $M(s, \sigma)$ in its dyadic expansion are either exactly or asymptotically zero. Thus a question arises as to under what conditions $n_a^- + n_b + n_f$ equals the dimension n of the given system. It is easy to see that $n_a^- + n_b + n_f = n$ iff $\tilde{\Sigma}$ is left invertible and of minimum phase. If $\tilde{\Sigma}$ is not left invertible or/and of non-minimum phase, there are $n_e \equiv n - n_a^- - n_b - n_f \equiv n_a^+ + n_c$ terms of $M(s, \sigma)$ which cannot in general be rendered zero. To emphasize explicitly the behaviour of each term of $M(s, \sigma)$, we partition it into four parts

$$M(s, \sigma) = M_-(s, \sigma) + M_b(s, \sigma) + M_\infty(s, \sigma) + M_e(s, \sigma) \tag{3.10}$$

where

$$M_-(s,\sigma) = \sum_{i=1}^{n_a^-} \frac{R_i}{s-\lambda_i}$$

$$M_b(s,\sigma) = \sum_{i=n_a^-+1}^{n_a^-+n_b} \frac{R_i}{s-\lambda_i}$$

$$M_\infty(s,\sigma) = \sum_{i=n_a^-+n_b+1}^{n_a^-+n_b+n_f} \frac{R_i}{s-\lambda_i}$$

and

$$M_e(s,\sigma) = \sum_{i=n-n_a^--n_b-n_f+1}^{n} \frac{R_i}{s-\lambda_i}$$

Let $\Lambda_-(\sigma)$, $\Lambda_b(\sigma)$, $\Lambda_\infty(\sigma)$ and $\Lambda_e(\sigma)$ be the sets of eigenvalues of $\tilde{\mathbf{A}}_0$ associated respectively with the parts $M_-(s,\sigma)$, $M_b(s,\sigma)$, $M_\infty(s,\sigma)$ and $M_e(s,\sigma)$. Similarly to correspond with this partition of eigenvalues, we partition the right and left eigenvectors of $\tilde{\mathbf{A}}_0$ into sets $W_-(\sigma)$, $W_b(\sigma)$, $W_\infty(\sigma)$, $W_e(\sigma)$, $V_-(\sigma)$, $V_b(\sigma)$, $V_\infty(\sigma)$ and $V_e(\sigma)$. Also, here after we will be using an over bar on a certain variable to denote its limit whenever it exists as $\sigma \to \infty$. For example, $\bar{M}_e(s)$ and $\bar{W}_e$ denote respectively the limits of $M_e(s,\sigma)$ and $W_e(\sigma)$ as $\sigma \to \infty$. We now note that various parts of $M(s,\sigma)$ have the following interpretation

(1) $M_-(s,\sigma)$ contains n_a^- terms. The n_a^- eigenvalues of $\tilde{\mathbf{A}}_0$ represented in it form a set $\Lambda_-(\sigma)$. In accordance with the Lemma 3.2, there exists a gain $\tilde{\mathbf{K}}(\sigma)$ such that $M_-(s,\sigma)$ can be rendered identically zero by assigning the elements of $\Lambda_-(\sigma)$ to coincide with the plant minimum phase invariant zeros while the corresponding set of left eigenvectors $V_-(\sigma)$ coincides with the corresponding set of left state zero directions. In fact, $\tilde{\mathbf{K}}(\sigma)$ can also be designed such that $\Lambda_-(\sigma)$ and $V_-(\sigma)$ approach asymptotically the set of plant minimum phase invariant zeros and the corresponding set of state zero directions as $\sigma \to \infty$. In this case, $M_-(s,\sigma) \to 0$ pointwise in s as $\sigma \to \infty$.

(2) $M_b(s,\sigma)$ contains n_b terms. The n_b eigenvalues of $\tilde{\mathbf{A}}_0$ represented in it form a set $\Lambda_b(\sigma)$. In accordance with the Lemmas 3.2 and 3.3, there exxists a gain $\tilde{\mathbf{K}}(\sigma)$ such that $M_b(s,\sigma)$ can be rendered identically zero by assigning the elements of $\Lambda_b(\sigma)$ arbitrarily to either asymptotically finite or infinite location in $\mathscr{C}^-$ as $\sigma \to \infty$. As discussed earlier, in order to conserve the controller band-width, we will assume here after that these eigenvalues are assigned to asymptotically finite locations. Also, $\tilde{\mathbf{K}}(\sigma)$ can be designed so that $M_b(s,\sigma) \to 0$ pointwise in s as $\sigma \to \infty$.

(3) $M_\infty(s,\sigma)$ contains n_f terms. The n_f eigenvalues of $\tilde{\mathbf{A}}_0$ represented in it form a set $\Lambda_\infty(\sigma)$. In accordance with the Lemma 3.3, there exists a gain $\tilde{\mathbf{K}}(\sigma)$ such that $M_\infty(s,\sigma) \to 0$ pointwise in s as $\sigma \to \infty$ by assigning the elements of $\Lambda_\infty(\sigma)$ arbitrarily to asymptotically infinite locations in $\mathscr{C}^-$.

(4) $M_e(s,\sigma)$ contains the remaining $n_e \equiv n_a^+ + n_c$ terms. It is non-existent, i.e. $n_e = 0$, iff $\tilde{\Sigma}$ is left invertible and of minimum phase. The n_e eigenvalues of $\tilde{\mathbf{A}}_0$ represented in $M_e(s,\sigma)$ form a set $\Lambda_e(\sigma)$. In view of Lemmas 3.2 and 3.3, $M_e(s,\sigma)$ cannot in general be rendered zero either asymptotically or otherwise by any assignment of $\Lambda_e(\sigma)$ and the associated sets of right and left

eigenvectors, $W_e(\sigma)$ and $V_e(\sigma)$. However, as will be discussed later on, $M_e(s, \sigma)$ can be shaped to have some desirable properties. Since $(\tilde{\mathbf{A}}, \tilde{\mathbf{C}})$ is assumed to be a detectable pair, except for the stable but unobservable eigenvalues of $\tilde{\mathbf{A}}$, others among the remaining eigenvalues of $\tilde{\mathbf{A}}_0$ which are in Λ_e can be assigned to arbitrary locations in $\mathscr{C}^-$. These arbitrary locations can either be asymptotically finite or infinite. But as will be shown by an example in Part 2 of our paper (Saberi *et al.* 1991), $M_e(j\omega, \sigma)$ can be unbounded as $\sigma \to \infty$ whenever any elements of $\Lambda_e(\sigma)$ are assigned to asymptotically infinite locations. Moreover, assigning elements of $\Lambda_e(\sigma)$ to asymptotically infinite locations increases unnecessarily controller bandwidth. Because of this, we assume Λ_e is confined to finite locations in $\mathscr{C}^-$.

Since both $M_-(s, \sigma)$ and $M_b(s, \sigma)$ can be rendered identically zero, for future use we can combine them into one term

$$M_0(s, \sigma) = M_-(s, \sigma) + M_b(s, \sigma)$$

and rewrite $M(s, \sigma)$ as

$$M(s, \sigma) = M_0(s, \sigma) + M_\infty(s, \sigma) + M_e(s, \sigma) \tag{3.11}$$

We define likewise, $\Lambda_0(\sigma) = \Lambda_-(\sigma) \cup \Lambda_b(\sigma)$, $W_0(\sigma) = W_-(\sigma) \cup W_b(\sigma)$ and $V_0(\sigma) = V_-(\sigma) \cup V_b(\sigma)$.

As the above discussion indicates, Lemmas 3.2 and 3.3 enable us to decompose the mechanism of LTR into several parts. They show clearly what is and what is not feasible under what conditions. Although they do not directly provide methods of obtaining the gain $\tilde{\mathbf{K}}(\sigma)$, they do provide structural guide lines as to how certain eigenvalues and eigenvectors are to be assigned while indicating a multitude of ways in which freedom exists in assigning the other eigenvalues and eigenvectors of $\tilde{\mathbf{A}}_0$. These guidelines, in turn, can appropriately be channeled to come up with a design method to obtain an appropriate gain $\tilde{\mathbf{K}}(\sigma)$ (see, Part 2 of our paper, Saberi *et al.* 1991). Thus in short, Lemmas 3.2 and 3.3 form the heart of the underlying mechanism of LTR.

Let $\mathscr{K}^*_{(\tilde{\mathbf{C}},\tilde{\mathbf{A}},\tilde{\mathbf{B}})}(\sigma)$ be the set of all gains $\tilde{\mathbf{K}}(\sigma)$ designed by following the guide lines given by the Lemmas 3.2 and 3.3. Namely, $\tilde{\mathbf{K}}(\sigma)$ is designed such that (1) $\tilde{\mathbf{A}}_0$ is asymptotically stable, (2) $M_0(s, \sigma) \to 0$ and (3) $M_\infty(s, \sigma) \to 0$ pointwise in s as $\sigma \to \infty$. Obviously $\mathscr{K}^*_{(\tilde{\mathbf{C}},\tilde{\mathbf{A}},\tilde{\mathbf{B}})}(\sigma)$ is a non-empty set. Theorem 3.1 given below characterizes the asymptotic behaviour of the achieved loop transfer function as well as sensitivity and complimentary sensitivity functions when $\tilde{\mathbf{K}}(\sigma) \in \mathscr{K}^*_{(\tilde{\mathbf{C}},\tilde{\mathbf{A}},\tilde{\mathbf{B}})}(\sigma)$. Let $S_0(s, \sigma)$ and $T_0(s, \sigma)$ be the achieved sensitivity and complementary sensitivity functions in the configuration of Fig. 2 when the loop is broken at the input point of the plant

$$S_0(s, \sigma) = [I_m + C(s, \sigma)P(s)]^{-1}$$

and

$$T_0(s, \sigma) = I_m - S_0(s, \sigma) = [I_m + C(s, \sigma)P(s)]^{-1}C(s, \sigma)P(s)$$

Also, let $S(s)$ and $T(s)$ be the sensitivity and complementary sensitivity functions corresponding to the target loop transfer function. We have the following theorem.

Theorem 3.1

Consider the closed-loop system $\tilde{\Sigma}^c$ comprising of the given plant $\tilde{\Sigma}$ and the controller as given in Fig. 2. Let $\tilde{\Sigma}$ be stabilizable and detectable. Then for any $\tilde{\mathbf{F}}$ such that $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ is asymptotically stable, and for any gain $\tilde{\mathbf{K}}(\sigma) \in \mathscr{K}^*_{(\tilde{\mathbf{C}},\tilde{\mathbf{A}},\tilde{\mathbf{B}})}(\sigma)$, the closed-loop system $\tilde{\Sigma}^c$ is asymptotically stable. Moreover, as $\sigma \to \infty$

$$E(s, \sigma) \to \bar{M}_e(s)[I_m + \bar{M}_e(s)]^{-1}(I_m + \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}), \text{ pointwise in } s \tag{3.12}$$

$$S_0(s, \sigma) \to S(s)[I_m + \bar{M}_e(s)], \text{ pointwise in } s \tag{3.13}$$

$$T_0(s, \sigma) \to T(s) - S(s)\bar{M}_e(s), \text{ pointwise in } s \tag{3.14}$$

$$\frac{|\sigma_i[S_0(j\omega, \sigma)] - \sigma_i[S(j\omega)]|}{\sigma_{\max}[S(j\omega)]} \leqslant \sigma_{\max}[\bar{M}_e(j\omega)] \tag{3.15}$$

and

$$\frac{|\sigma_i[T_0(j\omega, \sigma)] - \sigma_i[T(j\omega)]|}{\sigma_{\max}[S(j\omega)]} \leqslant \sigma_{\max}[\bar{M}_e(j\omega)] \tag{3.16}$$

Proof

Expressions (3.12), (3.13) and (3.14) follow from Lemmas 3.2 and 3.3. The bounds (3.15) and (3.16) are slight generalizations of similar results of Sogaard-Andersen and Niemann (1989) (See also Chen *et al.* 1990).

In view of Theorem 3.1, $\bar{M}_e(s)$ can be termed as the recovery error matrix. The following well known result can easily be deduced from Theorem 3.1.

Corollary 3.1

Consider the closed-loop system $\tilde{\Sigma}^c$ comprising of the given plant $\tilde{\Sigma}$ and the controller as given in Fig. 2. Let $\tilde{\Sigma}$ be stabilizable, detectable, left invertible and of minimum phase. Then for any $\tilde{\mathbf{F}}$ such that $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ is asymptotically stable, and for any gain $\tilde{\mathbf{K}}(\sigma) \in \mathscr{K}^*_{(\tilde{\mathbf{C}},\tilde{\mathbf{A}},\tilde{\mathbf{B}})}(\sigma)$, the closed-loop system $\tilde{\Sigma}^c$ is asymptotically stable. Moreover, ALTR can always be achieved, i.e. we have as $\sigma \to \infty$, pointwise in s

$$M(s, \sigma) \to 0$$

and hence

$$E(s, \sigma) \to 0$$

$$S_0(s, \sigma) \to S(s)$$

$$T_0(s, \sigma) \to T(s)$$

Proof

Since the recovery error matrix $\bar{M}_e(s)$ is non-existent for left invertible and minimum phase systems, the proof is obvious.

As implied by Theorem 3.1, the recovery error matrix $\bar{M}_e(s)$ plays a dominant role in the recovery process and hence it should be shaped to yield as best as possible the desired results. Shaping $\bar{M}_e(s)$ involves selecting the set of eigenvalues $\bar{\Lambda}_e$ represented in $\bar{M}_e(s)$ and the associated set of right and left eigenvectors $\bar{W}_e$ and $\bar{V}_e$. Such a selection can be done in a number of ways subject to the constraints imposed in selecting the eigenvectors (Moore 1976). However, note that though, $\bar{M}_e(s)$ would be small if all the non-minimum phase zeros of the given plant are far away in $\mathscr{C}^+$. Hence, in this case one may not need any shaping of $\bar{M}_e(s)$. The following observation formalizes this.

Observation 3.1

Let $\tilde{\Sigma}$ be left invertible and let all the n_a^+ non-minimum phase zeros be far away from the band-width of the target loop transfer function. Then the recovery error matrix $\bar{M}_e(s)$ is small. This is shown in Appendix C. A similar result has been obtained by Zhang and Freudenberg (1990) when a left invertible system $\tilde{\Sigma}$ has only one non-minimum phase zero.

In multivariable systems, one interesting aspect of Theorem 3.1 is that there could exist a subspace of the control space in which $\bar{M}_e(s)$ can be rendered zero. To pinpoint this, let

$$e_i = \tilde{\mathbf{B}}'\bar{V}_i, \ \bar{V}_i \in \bar{V}_e \tag{3.17}$$

and let $\mathscr{E}_e$ be the subspace of $\mathbb{R}^m$

$$\mathscr{E}_e = \text{Span}\,\{e_i \mid \bar{V}_i \in \bar{V}_e\} \tag{3.18}$$

Let the dimension of $\mathscr{E}_e$ be m_e. Now let

$$\mathscr{S}_e = \text{orthogonal complement of } \mathscr{E}_e \text{ in } \mathbb{R}^m \tag{3.19}$$

Let P_s be the orthogonal projection matrix onto $\mathscr{S}_e$. Then the following theorem pinpoints the directional behaviour of $M(s, \sigma)$ and consequently the behaviour of $S_0(s, \sigma)$ and $T_0(s, \sigma)$ as $\sigma \to \infty$.

Theorem 3.2

Consider the closed-loop system $\tilde{\Sigma}^c$ comprising of the given plant $\tilde{\Sigma}$ and the controller as given in Fig. 2. Let $\tilde{\Sigma}$ be stabilizable and detectable. Then for any $\tilde{\mathbf{F}}$ such that $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ is asymptotically stable, and for any gain $\tilde{\mathbf{K}}(\sigma) \in \mathscr{K}^*_{(\tilde{\mathbf{C}},\tilde{\mathbf{A}},\tilde{\mathbf{B}})}(\sigma)$, the closed-loop system $\tilde{\Sigma}^c$ is asymptotically stable. Moreover, we have as $\sigma \to \infty$, pointwise in s

$$M(s, \sigma)P_s \to 0,$$

$$S_0(s, \sigma)P_s \to S(s)P_s$$

$$T_0(s, \sigma)P_s \to T(s)P_s$$

Proof

In view of the definitions of the matrix P_s and the subspaces $\mathscr{E}_e$ and $\mathscr{S}_e$, Theorem 3.1 implies the results of Theorem 3.2.

In view of the directional behaviour of $\bar{M}_e(s)$ as given by Theorem 3.2, one could try to shape it in a particular way so as to obtain the recovery of sensitivity and complimentary sensitivity functions in certain desired directions or one could try to shape $\bar{M}_e(s)$ so that the subspace $\mathscr{S}_e$ has as large a dimension as possible, i.e. the subspace $\mathscr{E}_e$ has as small a dimension as possible. In this regard, we note that we have already selected Λ_0 and Λ_∞ and the corresponding sets of eigenvectors $\bar{V}_0$ and $\bar{V}_\infty$ so that $M_0(s, \sigma)$ and $M_\infty(s, \sigma)$ tend to zero pointwise in s as $\sigma \to \infty$. We also note that although all the $n_a^+ + n_c$ vectors $\bar{V}_i \in \bar{V}_e$ can be selected to be linearly independent, the corresponding $\mathbf{e}_i \equiv \tilde{\mathbf{B}}'\bar{V}_i$ need not be linearly independent. In fact for a given $\mathbf{e} \neq 0$, the equation

$$\mathbf{e} = \tilde{\mathbf{B}}'V$$

has $n - m + 1$ linearly independent solutions for V. Of course, not all such $n - m + 1$ vectors could be admissible eigenvectors of $\tilde{\mathbf{A}}_0$ for different eigenvalues of $\tilde{\mathbf{A}}_0$ in $\mathscr{C}^-$, and moreover some or all of these $n - m + 1$ vectors could also be linearly dependent on already selected eigenvectors in the sets $\bar{V}_0$ and $\bar{V}_\infty$. Thus the problem of shaping $\mathscr{E}_e$ is to find an admissible set of eigenvalues λ_i and vectors $\mathbf{e}_i$, $i = 1$ to $n_a^+ + n_c$, which are not necessarily linearly independent but the associated eigenvectors V_i of $\tilde{\mathbf{A}}_0$ satisfying $\mathbf{e}_i = \tilde{\mathbf{B}}'\bar{V}_i$, $i = 1$ to $n_a^+ + n_c$, together with the vectors in the sets $\bar{V}_0$ and $\bar{V}_\infty$ form n linearly independent vectors. This problem of selecting an admissible set $(\lambda_i, \mathbf{e}_i)$ is very much related to the traditional problem of distributing the modes of a closed-loop system to various output components by an appropriate selection of the closed-loop eigenstructure. This traditional problem of 'shaping the output response characteristics' of a closed-loop system has been studied first by Moore (1976) and Shaked (1977) and more recently by Sogaard-Andersen (1987) although to this date there exists no systematic design procedure.

The above discussion focuses how to shape the subspace $\mathscr{S}_e$ in which $M(s, \sigma)$, $S(s)$ and $T(s)$ are recovered. A practical problem of interest could be to achieve recovery of $M(s, \sigma)$, $S(s)$ and $T(s)$ in a prescribed subspace $\mathscr{S}_e$. We will discuss this aspect of the problem in § 3.3.

Remark 3.2

In general, although $M(s, \sigma)$ and hence $S(s)$ and $T(s)$ are recoverable in a subspace such as $\mathscr{S}_e$, the loop transfer function $L(s)$ is not necessarily recoverable in that subspace $\mathscr{S}_e$ as can be seen from the following example. However, this may not be as important as it seems since in most of the design schemes recovery of $L(s)$ is only a means to recover $S(s)$ and $T(s)$.

Example 3.1

Consider a system $\tilde{\Sigma}$ as given in Zhang and Freudenberg (1990), and characterized by

$$\tilde{\mathbf{A}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0{\cdot}2 & 0 \\ 0 & 0 & 0 & -0{\cdot}2 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} -0{\cdot}5 & -1{\cdot}25 \\ -2{\cdot}5 & -2{\cdot}5 \\ 0{\cdot}3 & 1{\cdot}25 \\ 1{\cdot}5 & 3{\cdot}5 \end{bmatrix}$$

and

$$\tilde{\mathbf{C}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Let the target loop be specified by giving

$$\tilde{\mathbf{F}} = \begin{bmatrix} -4{\cdot}9019 & -19{\cdot}6075 & -18{\cdot}0299 & -14{\cdot}9622 \\ 5{\cdot}5879 & 22{\cdot}3517 & -2{\cdot}7018 & 26{\cdot}4831 \end{bmatrix}$$

The given plant is of non-minimum phase and has two invariant zeros, one at $s = 1$ and another at $s = 2$. The right input zero directions associated with zeros at 1 and 2 are respectively

$$w_{R1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w_{R2} = \begin{bmatrix} 0{\cdot}9806 \\ 0{\cdot}1961 \end{bmatrix}$$

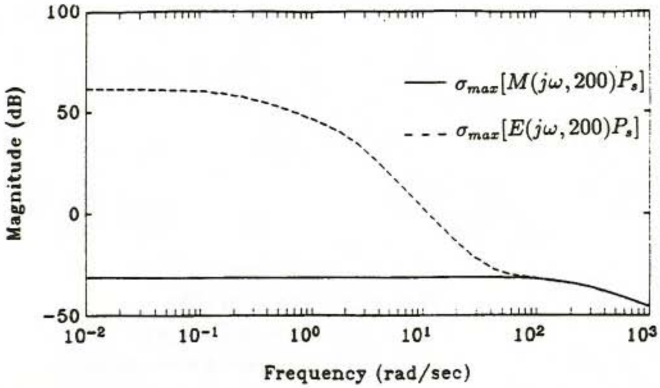

Figure 3. $\sigma_{\max}[M(j\omega, \sigma)P_s]$ and $\sigma_{\max}[E(j\omega, \sigma)P_s]$ of Example 3.1.

The vectors w_{R1} and w_{R2} together span $\mathbb{R}^2$. Following the analysis of Zhang and Freudenberg (1990), one can show that for this example ARE based design cannot produce any recoverable subspace of $\mathbb{R}^2$ since it can recover only in a subspace orthogonal to the space spanned by the right input zero directions. However, we will next give an observer gain $\tilde{\mathbf{K}}$ such that $\mathbf{e}_i \equiv \tilde{\mathbf{B}}'\bar{V}_i$, $i = 1, 2$, are along the direction $[-1, 1]'$. Let

$$\tilde{\mathbf{K}}(\sigma) = \begin{bmatrix} 8{\cdot}75(\sigma - 1) & -3{\cdot}5(\sigma + 6) \\ 25(\sigma - 1) & -10(\sigma + 6) \\ 1{\cdot}55(1 - 5\sigma) & 3{\cdot}5(\sigma + 6) \\ 5(1 - 5\sigma) & 11\sigma + 67{\cdot}8 \end{bmatrix}$$

so that the observer eigenvalues are placed at $-\sigma$, $-\sigma$, -2 and -1. Note that the last two eigenvalues are purposefully selected to be the mirror images of the non-minimum phase invariant zeros, although they can be placed at any locations. It is now straightforward to verify that $\mathbf{e}_1 = \alpha_1[-1, 1]'$ and $\mathbf{e}_2 = \alpha_2[-1, 1]'$ where α_1 and α_2 are some constants dependent on how one scales the vectors $\bar{V}_i$, $i = 1$ and 2. It is also simple to verify that a subspace $\mathscr{S}_e$ having an orthogonal projection matrix P_s

$$P_s = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}5 \\ 0{\cdot}5 & 0{\cdot}5 \end{bmatrix}$$

is recoverable. The resulting $M(j\omega, \sigma)P_s$, $E(j\omega, \sigma)P_s$, $S_0(j\omega, \sigma)P_s$ and $S(j\omega)P_s$ are plotted with respect to ω over a given range of ω in Figs 3 and 4 when $\sigma = 200$. It is easy to note that $M(j\omega, \sigma)P_s$ is approximately zero while $S_0(j\omega, \sigma)P_s$ is close to $S(j\omega)P_s$. (Note that the minimum singular values of $S_0(j\omega, \sigma)P_s$ and $S(j\omega)P_s$ are identically zero due to the singularity of P_s.) In view of the given expression for $\tilde{\mathbf{K}}(\sigma)$, one can easily calculate $E(j\omega, \sigma)P_s$ and show that it does not go to zero as $\sigma \to \infty$.

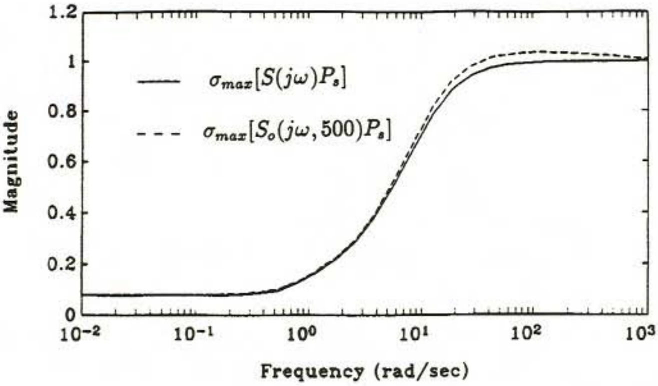

Figure 4. $\sigma_{\max}[S(j\omega)P_s]$ and $\sigma_{\max}[S_0(j\omega, \sigma)P_s]$ of Example 3.1.

We will next examine the asymptotic behaviour of open-loop eigenvalues of the full order observer based controller $C(s, \sigma)$ and the mechanism of pole–zero cancellation between the controller eigenvalues and the input or output decoupling zeros (Rosenbrock 1970) of the plant. It is important to know the eigenvalues of $C(s, \sigma)$ as they are included among the invariant zeros of the closed-loop system $\tilde{\Sigma}^c$ (Sannuti and Saberi 1987) and hence affect the performance of $\tilde{\Sigma}^c$, e.g. command following. The controller transfer function is given by (3.6) while the eigenvalues of it are

$$\lambda(\tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}})$$

To study the nature of these eigenvalues, let

$$\det [sI_n - \tilde{\mathbf{A}}_0] = \phi_0(s)\phi_\infty(s)\phi_e(s)$$

where $\phi_0(s)$, $\phi_\infty(s)$ and $\phi_e(s)$ are polynomials in s whose zeros are the eigenvalues of $\tilde{\mathbf{A}}_0$ that belong to the sets $\Lambda_0(\sigma)$, $\Lambda_\infty(\sigma)$ and $\Lambda_e(\sigma)$ respectively. Also, let

$$\bar{M}_e(s) = \frac{R_e(s)}{\phi_e(s)} \tag{3.20}$$

where $R_e(s)$ is a polynomial matrix in s. Now consider the following:

$$\begin{aligned} \det [sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}}] &= \det [sI_n - \tilde{\mathbf{A}}_0] \det [I_n + (sI_n - \tilde{\mathbf{A}}_0)^{-1}\tilde{\mathbf{B}}\tilde{\mathbf{F}}] \\ &= \phi_0(s)\phi_\infty(s)\phi_e(s) \det [I_m + \tilde{\mathbf{F}}(sI_n - \tilde{\mathbf{A}}_0)^{-1}\tilde{\mathbf{B}}] \\ &= \phi_0(s)\phi_\infty(s)\phi_e(s) \det [I_m + M(s, \sigma)] \\ &\to \phi_0(s)\phi_\infty(s)\phi_e(s) \det [I_m + \bar{M}_e(s)] \text{ as } \sigma \to \infty \\ &= \phi_0(s)\phi_\infty(s)\phi_e(s) \det \left[I_m + \frac{R_e(s)}{\phi_e(s)} \right] \\ &= \phi_0(s)\phi_\infty(s) \frac{\det [I_m \phi_e(s) + R_e(s)]}{[\phi_e(s)]^{m-1}} \end{aligned} \tag{3.21}$$

We note that the observer can be designed such that $\phi_0(s)$, $\phi_\infty(s)$ and $\phi_e(s)$ are coprime. Thus the open-loop eigenvalues of the controller are the zeros of $\phi_0(s)$,

$\phi_\infty(s)$ and

$$\frac{\det[I_m\phi_e(s) + R_e(s)]}{[\phi_e(s)]^{m-1}}$$

Thus Λ_0 and Λ_∞ are contained among the eigenvalues of the controller. Although Λ_0 and Λ_∞ are in $\mathscr{C}^-$, there is no guarantee that the zeros of

$$\frac{\det[I_m\phi_e(s) + R_e(s)]}{[\phi_e(s)]^{m-1}}$$

are in $\mathscr{C}^-$. Hence the controller may or may not be open-loop stable. In general, the loop transfer function $C(s, \sigma)P(s)$ has $2n$ eigenvalues, n of them coming from the plant and the other n coming from the controller. However, there are several cancellations among the input or output decoupling zeros (Rosenbrock 1970) of $C(s, \sigma)P(s)$ and the controller eigenvalues. The following Lemma 3.4 which is a slight generalization of a similar one in Goodman (1984), explores such a cancellation.

Lemma 3.4

Let λ be an eigenvalue of $\tilde{\mathbf{A}}_0 = \tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}}$ and the corresponding left eigenvector V be such that $V^H\tilde{\mathbf{B}} = 0$. Then λ is an eigenvalue of $\tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ with corresponding left eigenvector as V. Moreover, λ cancels an input decoupling zero of $C(s, \sigma)P(s)$.

Thus in view of Lemma 3.2, the above lemma implies that whatever may be the matrix $\tilde{\mathbf{F}}$, if observer is appropriately designed, there are $n_a^- + n_b$ cancellations among the eigenvalues of the controller and the input decoupling zeros of $C(s, \sigma)P(s)$. As will be seen in the next subsection, there may be additional cancellations if $\tilde{\mathbf{F}}$ satisfies certain properties.

3.2. *Analysis for recoverable target loops*

In this subsection, our aim is to characterize the class of target loops which are either exactly or asymptotically recoverable for non-minimum phase systems which are not necessarily invertible. Of course, in view of the previous subsection, such a characterization has to depend on specific properties of the target loop transfer function $L(s) = \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$ which is a function $\tilde{\mathbf{F}}$. We have the following result.

Theorem 3.3

Consider a system $\tilde{\Sigma}$ which is not necessarily of minimum phase and which is not necessarily left invertible. Then a target loop transfer function $L(s) = \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$ is exactly recoverable by the full order observer based controller iff $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) \subseteq \operatorname{Ker}\tilde{\mathbf{F}}$.

Proof

See Appendix D.

Several interpretations emerge from the recoverability conditions on the target loops given in Theorem 3.3. In fact the constraints given in Theorem 3.3 are nothing more than constraints on the finite and infinite zero structure and invertibility properties of $L(s)$. Some interesting interpretations in this regard can easily be exemplified. Let us first note that in view of Property 2.3, $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ is the span of $\tilde{x}_a^+ \oplus \tilde{x}_c \oplus \tilde{x}_f$. Thus whenever $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) \subseteq \operatorname{Ker} \tilde{\mathbf{F}}$, we have

(1) Span of $\tilde{x}_c \equiv \mathscr{R}^* \subseteq \operatorname{Ker} \tilde{\mathbf{F}}$.

(2) Span of $\tilde{x}_a^+ \oplus \tilde{x}_c \equiv \mathscr{V}^+_{\operatorname{Ker} \tilde{\mathbf{C}}} \subseteq \operatorname{Ker} \tilde{\mathbf{F}}$.

(3) Span of $\tilde{x}_f \equiv \mathscr{S}^*/\mathscr{R}^* \subseteq \operatorname{Ker} \tilde{\mathbf{F}}$.

Obviously, the first two of these conditions pertain to the finite zero structure of $\tilde{\Sigma}$ while the third one pertains to the infinite zero structure of $\tilde{\Sigma}$. One can easily interpret the above three conditions in terms of the invertibility and the finite and infinite zero structure of $L(s)$. We have the following interpretations.

(1) If $\tilde{\Sigma}$ is not left invertible, any exactly recoverable $L(s)$ is not left invertible. On the other hand, left invertibility of $\tilde{\Sigma}$ does not necessarily imply that an exactly recoverable $L(s)$ is left invertible. That is, whenever $\tilde{\Sigma}$ is left invertible, an exactly recoverable $L(s)$ could be either left invertible or not left invertible.

(2) Any left invertible and exactly recoverable $L(s)$ must contain the non-minimum phase zero structure of $\tilde{\Sigma}$. An exactly recoverable but not left invertible $L(s)$ does not necessarily contain the non-minimum phase zero structure of $\tilde{\Sigma}$. Example 3.2 given later on illustrates this.

(3) For simplicity of presentation, let us assume that $\tilde{\Sigma}$ is of uniform rank with relative degree q (i.e. all the infinite zeros of $\tilde{\Sigma}$ are of the same order q). Then the smallest order of infinite zero of exactly recoverable $L(s)$ is greater than q (See also, Corollary 3.2).

We have the following corollary to Theorem 3.3.

Corollary 3.2

Consider an invertible and non-minimum phase system $\tilde{\Sigma}$. Also, let $\tilde{\Sigma}$ be of uniform rank with relative degree q. Then any target loop transfer function $L(s)$ which is invertible with the smallest order of infinite zeros greater than q and which contains the non-minimum phase zero structure of $\tilde{\Sigma}$ is exactly recoverable.

Proof

See Appendix E.

Remark 3.3

A special case of Corollary 3.2 when $\tilde{\Sigma}$ is invertible and of minimum phase with relative degree $q = 1$ was given earlier by Goodman (1984). Thus Corollary 3.2 generalizes Goodman's result for both non-minimum phase systems and for systems with relative degree greater than unity.

Example 3.2

Consider an invertible system characterized by the triple

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 0 & 4 & 1 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -5 & 1 & 1 \\ -5 & 1 & 1 & -10 & 0 \\ -5 & 1 & 1 & 0 & -10 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\tilde{\mathbf{C}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The given system has three invariant zeros at $s = 1$, $s = -1$ and at $s = -5$. Let the target loop be defined by the triple $(\tilde{\mathbf{F}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ where

$$\tilde{\mathbf{F}} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Then, it is straightforward to show that $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ is asymptotically stable and $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$, the span of $\tilde{x}_a^+ \oplus \tilde{x}_f$, is a subset of $\operatorname{Ker} \tilde{\mathbf{F}}$. Thus ELTR can be achieved. In fact, the controller defined below having the eigenvalues at -1, -2, -3, -4 and -5 achieves ELTR

$$\hat{\mathbf{u}} = -\tilde{\mathbf{F}}\hat{\mathbf{x}}$$

where

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} 1 & 0 & 0 & 1{\cdot}5091 & 1{\cdot}4909 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ -5 & 0 & 0 & -5{\cdot}0110 & -1{\cdot}9894 \\ -5 & 0 & 0 & -2{\cdot}0106 & -4{\cdot}9890 \end{bmatrix} \hat{\mathbf{x}} + \begin{bmatrix} 2{\cdot}4909 & -0{\cdot}4909 \\ 1 & 1 \\ 1 & 1 \\ -4{\cdot}9890 & 1{\cdot}9894 \\ 2{\cdot}0106 & -5{\cdot}0110 \end{bmatrix} \tilde{\mathbf{y}}$$

The graphs of $\sigma_{\max}[S(j\omega, \sigma)]$ and $\sigma_{\max}[S_0(j\omega, \sigma)]$ shown in Fig. 5 attest to the fact that ELTR is achieved. However, it is simple to verify that the given $L(s)$ is right invertible and is of minimum phase with one invariant zero at $s = -3{\cdot}5$. Thus we can conclude that an exactly recoverable $L(s)$ need not contain the non-minimum phase zero structure of $\tilde{\Sigma}$.

Theorem 3.3 deals with ELTR. Since the required conditions for ELTR in general are severe, most often in practice one is interested only in ALTR. From its definition, it is easy to see that ALTR occurs, i.e. $\bar{M}_e(s) = 0$, iff $\tilde{\mathbf{F}}\bar{W}_e = 0$. We have the following results regarding ALTR.

Lemma 3.5

Im $\bar{W}_e$ coincides with $\mathscr{V}^+_{\operatorname{Ker} \tilde{\mathbf{C}}}$.

Proof

See Appendix F.

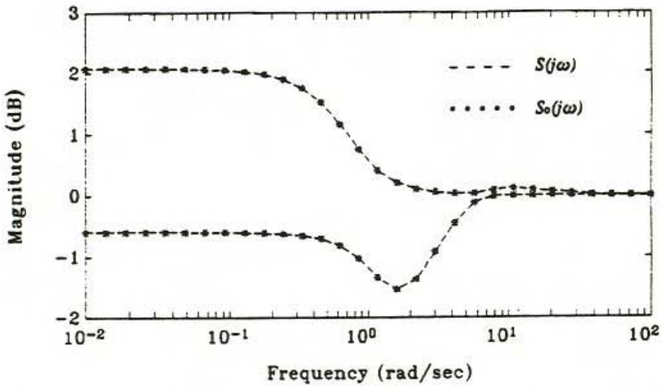

Figure 5. Maximum and minimum singular values of $S(j\omega)$ and $S_0(j\omega)$ of Example 3.2.

Theorem 3.4

Consider a system $\tilde{\Sigma}$ which is not necessarily of minimum phase and which is not necessarily left invertible. Then a target loop transfer function $L(s) = \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$ is asymptotically recoverable by the full order observer based controller iff $\mathscr{V}^+_{\mathrm{Ker}\,\tilde{\mathbf{C}}} \subseteq \mathrm{Ker}\,\tilde{\mathbf{F}}$.

Proof

If $\mathscr{V}^+_{\mathrm{Ker}\,\tilde{\mathbf{C}}} \subseteq \mathrm{Ker}\,\tilde{\mathbf{F}}$, Lemma 3.5 implies that $\tilde{\mathbf{F}}\bar{W}_e = 0$ and hence $\bar{M}_e(s) = 0$. Thus ALTR is achieved. On the other hand, if ALTR occurs, $\bar{M}_e(s) = 0$ and hence $\tilde{\mathbf{F}}\bar{W}_e = 0$ implying $\mathscr{V}^+_{\mathrm{Ker}\,\tilde{\mathbf{C}}} \subseteq \mathrm{Ker}\,\tilde{\mathbf{F}}$.

As in the case of ELTR, we can interpret the constraints imposed by Theorem 3.4 in terms of the invertibility and the finite zero structures of $L(s)$ and $\tilde{\Sigma}$. We have the following interpretations.

(1) If $\tilde{\Sigma}$ is not left invertible, any asymptotically recoverable $L(s)$ is not left invertible. On the other hand, left invertibility of $\tilde{\Sigma}$ does not necessarily imply that an asymptotically recoverable $L(s)$ is left invertible. That is, whenever $\tilde{\Sigma}$ is left invertible, an asymptotically recoverable $L(s)$ could be either left invertible or not left invertible.

(2) Any left invertible and asymptotically recoverable $L(s)$ must contain the non-minimum phase zero structure of $\tilde{\Sigma}$. An asymptotically recoverable but not left invertible $L(s)$ does not necessarily contain the non-minimum phase zero structure of $\tilde{\Sigma}$. Example 3.3 given later on illustrates this.

We have the following corollary to Theorem 3.4.

Corollary 3.3

Consider a left invertible and non-minimum phase system $\tilde{\Sigma}$. Then a target loop transfer function $L(s) = \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$ is asymptotically recoverable by the full order observer based controller if it contains the non-minimum phase zero structure of $\tilde{\Sigma}$.

Proof

Proposition E.1 (see Appendix E) and the left invertibility of $\tilde{\Sigma}$ together imply that $\mathscr{V}^+_{\mathrm{Ker}\,\tilde{\mathbf{C}}} \subseteq \mathrm{Ker}\,\tilde{\mathbf{F}}$. Hence the result.

Example 3.3

Consider an invertible system characterized by the triple,

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ -10 & 0 & -0{\cdot}5 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\tilde{\mathbf{C}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The given plant has a non-minimum phase invariant zero at $s = 1$. Let the target loop be defined by the triple $(\tilde{\mathbf{F}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ where

$$\tilde{\mathbf{F}} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The triple $(\tilde{\mathbf{F}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ forms a minimum phase right invertible system and hence it does not contain the non-minimum phase zero structure of $\tilde{\Sigma}$. However, for this example it can be easily seen that $\mathscr{V}^+_{\mathrm{Ker}\,\tilde{\mathbf{C}}}$ is the span of $[1 \;\; 0 \;\; 0]'$ and hence it is contained in $\mathrm{Ker}\,\tilde{\mathbf{F}}$. Thus in accordance with Theorem 3.4, there exists a controller which achieves ALTR. In fact, a full order observer based controller having the eigenvalues at -2, $-\sigma$ and $-\sigma$ and $\tilde{\mathbf{K}}(\sigma)$ as given below achieves ALTR,

$$\tilde{\mathbf{K}}(\sigma) = \begin{bmatrix} 3\sigma & 0 \\ \sigma - 1 & 0 \\ 0 & \sigma - 0{\cdot}5 \end{bmatrix}$$

The following Figs 6 and 7 pertaining to the case of $\sigma = 300$ illustrate that ALTR is achieved. Note that the minimum singular values of $L(j\omega)$ and $L_0(j\omega, \sigma)$ are identically zero for this example.

Now we proceed to discuss the possible cancellations between the eigenvalues of the controller and the input or output decoupling zeros of $C(s, \sigma)$ or $C(s, \sigma)P(s)$. Lemma 3.4 already discussed one such result which is a slight generalization of a similar one in Goodman (1984). The following lemma is also a slight generalization of a similar one in Goodman (1984).

Lemma 3.6

Let λ be an eigenvalue of $\tilde{\mathbf{A}}_0 = \tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}}$ and the corresponding right eigenvector W be such that $\tilde{\mathbf{F}}W = 0$. Then λ is an eigenvalue of $\tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ with corresponding right eigenvector as W. Moreover, λ cancels an output decoupling zero of $C(s, \sigma)$.

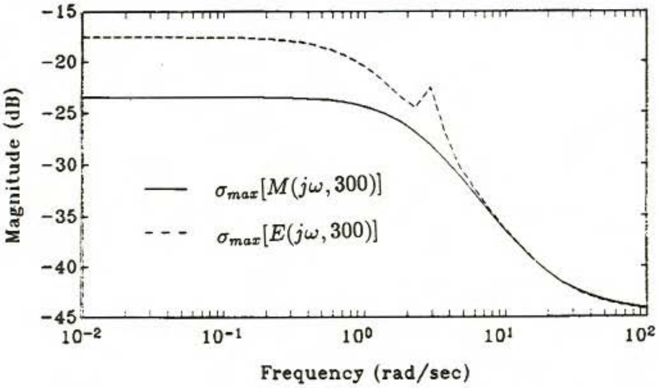

Figure 6. $\sigma_{\max}[M(j\omega, \sigma)]$ and $\sigma_{\max}[E(j\omega, \sigma)]$ of Example 3.3.

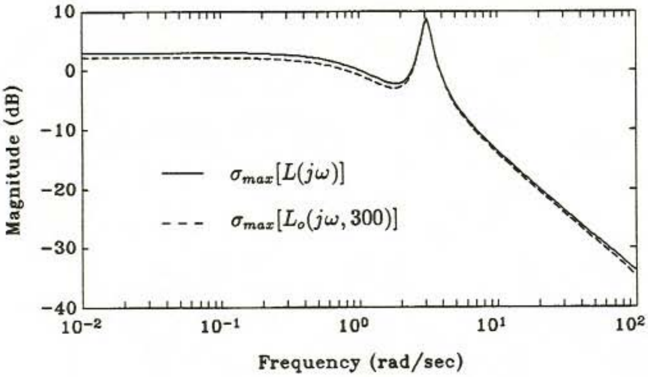

Figure 7. Maximum singular values of $L(j\omega)$ and $L_0(j\omega, \sigma)$ of Example 3.3.

We have the following theorems.

Theorem 3.5

If ELTR is achieved, i.e. if $E(j\omega, \sigma) = 0$ for all $0 \leqslant |\omega| < \infty$, then every eigenvalue of $\tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ cancels either an output decoupling zero of $C(s, \sigma)$ or an input decoupling zero of $C(s, \sigma)P(s)$.

Proof

ELTR is achieved iff either $\tilde{\mathbf{F}}W_i = 0$ or $V_i^H \tilde{\mathbf{B}} = 0$ or both. Hence the result follows from Lemmas 3.4 and 3.6.

Theorem 3.6

If ALTR is achieved, i.e. if $E(j\omega, \sigma) \to 0$ pointwise in ω as $\sigma \to \infty$ for all $0 \leqslant |\omega| < \infty$, then every asymptotically finite eigenvalue of $\tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ cancels either an output decoupling zero of $C(s, \sigma)$ or an input decoupling zero of $C(s, \sigma)P(s)$.

Proof

If ALTR is achieved, then every asymptotically finite eigenvalue of $\tilde{\mathbf{A}}_0$ with corresponding right and left eigenvectors W_i and V_i must be such that either $\tilde{\mathbf{F}} W_i = 0$ or $V_i^H \tilde{\mathbf{B}} = 0$ or both. Hence this result also follows from Lemmas 3.4 and 3.6.

In view of Lemmas 3.4 and 3.6, and Theorem 3.5, whenever ELTR occurs, there are n exact cancellations among the eigenvalues of the controller and the output decoupling zeros of $C(s)$ or the input decoupling zeros of $C(s)P(s)$.

3.3. *Recovery analysis in a given subspace*

In the last two subsections, we discussed recovery of loop transfer function and sensitivity and complimentary sensitivity functions for the general case when the recovery was required over the entire $\mathbb{R}^m$ and when the state feedback gain $\tilde{\mathbf{F}}$ was either arbitrary or a given fixed value. We found that such a recovery is in general not possible and characterized the recoverable target loop transfer functions. We also found that a matrix $M(s, \sigma)$ plays an important role in the recovery process. Also, we determined the subspace $\mathscr{S}_e$, called recoverable subspace, in which $M(s, \sigma)$ can be rendered zero. Rendering $M(s, \sigma)$ zero in a subspace $\mathscr{S}_e$ implies that the projections of target and achievable sensitivity and complimentary sensitivity functions onto $\mathscr{S}_e$ match each other. However, the projections of target and achievable loop transfer functions onto $\mathscr{S}_e$ do not necessarily match each other. This may not be a serious draw back since historically target loop transfer functions are formulated only to meet the required specifications on the sensitivity and complimentary sensitivity functions. Also, earlier in § 3.1, for a given design while discussing how to find $\mathscr{S}_e$, we discussed some aspects of shaping $\mathscr{S}_e$ to some extent. In this section, we take an alternate approach. Given a subspace $\mathscr{S}$ of $\mathbb{R}^m$, we first like to find whether exact or asymptotic recovery of target sensitivity and complimentary sensitivity functions in $\mathscr{S}$ is possible or not. In a sense, we are looking for a generalization of the traditional notion of LTR to cover recoverability in a subspace $\mathscr{S}$. An important and a natural issue that arises when one is interested in a recovery in a subspace $\mathscr{S}$ is the characterization of mismatch or error functions between the target and achieved sensitivity and complimentary sensitivity functions in the orthogonal complement of the subspace $\mathscr{S}$ or equivalently over the entire space $\mathbb{R}^m$. Thus our attention is focused next on this issue. Then we move on to find the maximum possible dimension of a recoverable subspace $\mathscr{S}$. Our results in this regard show that for a left invertible non-minimum phase system, whatever may be the given target sensitivity and complimentary sensitivity functions and whatever may be the number of non-minimum phase invariant zeros, there exists at least one $m-1$ dimensional subspace $\mathscr{S}$ of $\mathbb{R}^m$ in which complete recovery is possible.

We have the following formal definition.

Definition 3.1

A subspace $\mathscr{S}$ is said to be exactly (or asymptotically) recoverable if the projections of target and achievable sensitivity and complimentary sensitivity functions onto $\mathscr{S}$ match each other exactly (or asymptotically as the tuning parameter $\sigma \to \infty$). We say $\mathscr{S}$ is recoverable if $\mathscr{S}$ is either exactly or asymptotically recoverable.

Let $\mathbf{V}_s$ be a matrix whose columns form an orthogonal basis of the given subspace $\mathscr{S}$ of $\mathbb{R}^m$. Assume that the columns of $\mathbf{V}_s$ are scaled so that the norm of each column is unity. Let $P_s = \mathbf{V}_s\mathbf{V}_s'$ be the unique projection matrix onto $\mathscr{S}$. Obviously, $\mathscr{S}$ is exactly recoverable if

$$S_0(s, \sigma)P_s \equiv S(s)P_s \text{ and } T_0(s, \sigma)P_s \equiv T(s)P_s$$

Similarly, $\mathscr{S}$ is asymptotically recoverable if

$$S_{0,\sigma}(s)P_s \to S(s)P_s \text{ and } T_0(s, \sigma)P_s \to T(s)P_s \text{ pointwise in } s \text{ as } \sigma \to \infty.$$

We have the following observation.

Observation 3.2

$\mathscr{S}$ is exactly recoverable iff $M(s, \sigma)P_s \equiv 0$. On the other hand, $\mathscr{S}$ is asymptotically recoverable iff $M(s, \sigma)P_s \to 0$ pointwise in s as $\sigma \to \infty$.

Given the system $\tilde{\Sigma}$ characterized by the matrix triple $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$, let us now define an auxiliary system $\tilde{\Sigma}_s$ characterized by the matrix triple $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}\mathbf{V}_s)$. Obviously the auxiliary system $\tilde{\Sigma}_s$ differs from $\tilde{\Sigma}$ in its input distribution matrix $\tilde{\mathbf{B}}\mathbf{V}_s$. Now treating $\tilde{\Sigma}_s$ as the given system, one can rediscuss here *mutatis mutandis* all the results of the § 3.1 and 3.2. In particular, we have the following theorem.

Theorem 3.7

$\mathscr{S}$ is asymptotically recoverable for any arbitrarily specified target loop transfer function if the auxiliary system $\tilde{\Sigma}_s$ is left invertible and of minimum phase.

Proof

The proof is obvious.

Theorems 3.8 and 3.9 deal with the characterization of target loop transfer functions so that $\mathscr{S}$ is either exactly or asymptotically recoverable.

Theorem 3.8

Consider a system $\tilde{\Sigma}$ which is not necessarily of minimum phase and which is not necessarily left invertible. Let the target loop transfer function be specified as $L(s) = \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$. Then $\mathscr{S}$ is exactly recoverable by means of a full order observer based controller iff $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}\mathbf{V}_s) \subseteq \operatorname{Ker} \tilde{\mathbf{F}}$.

Proof

The proof is a consequence of Theorem 3.3.

Theorem 3.9

Consider a system $\tilde{\Sigma}$ which is not necessarily of minimum phase and which is not necessarily left invertible. Let the target loop transfer function be specified as $L(s) = \tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$. Then $\mathscr{S}$ is asymptotically recoverable iff $\mathscr{V}^+_{\operatorname{Ker}\tilde{\mathbf{C}}}(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}\mathbf{V}_s) \subseteq \operatorname{Ker} \tilde{\mathbf{F}}$.

Proof

The proof is a consequence of Theorem 3.4.

Given a subspace $\mathscr{S}$, Theorems 3.8 and 3.9 give the conditions under which it can either exactly or asymptotically be recovered. The necessary design for observer gain $\tilde{\mathbf{K}}(\sigma)$ to recover in a subspace $\mathscr{S}$ can be accomplished using the auxiliary system $\tilde{\Sigma}_s$ and any available design procedure. Knowing the gain $\tilde{\mathbf{K}}(\sigma)$, one can calculate $M(s, \sigma)$ from (3.2) and then easily characterize the behaviour of sensitivity and complimentary sensitivity functions, $S_0(s, \sigma)$ and $T_0(s, \sigma)$, over the orthogonal complement of $\mathscr{S}$ or over the entire space $\mathbb{R}^m$.

We next proceed to obtain the maximum possible dimension of a recoverable subspace $\mathscr{S}$. In this regard, our goal in what follows is to prove that whatever may be the given target loop transfer function and whatever may be the number of non-minimum phase zeros, there exists at least one $m-1$ dimensional subspace $\mathscr{S}$ of $\mathbb{R}^m$ which is always recoverable provided that the given system is left invertible. To prove this, for simplicity of presentation, we will make a technical assumption that all the non-minimum phase invariant zeros of $\tilde{\Sigma}$ have geometric multiplicity equal to unity. We next state two lemmas which lead to the intended result.

Lemma 3.7

Let the given system $\tilde{\Sigma}$ be left invertible and let z, x and w be respectively an invariant zero, the associated right state and input zero directions of $\tilde{\Sigma}$. Then we have the following properties.

(1) The auxiliary system $\tilde{\Sigma}_s$ is left invertible.
(2) Every invariant zero and the associated right state zero direction of $\tilde{\Sigma}_s$ are also the invariant zero and the associated right state zero direction of $\tilde{\Sigma}$.
(3) z and x are respectively an invariant zero and the associated right state zero direction of $\tilde{\Sigma}_s$ iff $w \in \mathscr{S}$.

Proof

See Appendix G.

Now let z_i, x_i and w_i, $i = 1$ to n_a^+, be respectively a non-minimum phase invariant zero and the associated right state and input zero directions of the given system $\tilde{\Sigma}$. Since $\tilde{\Sigma}$ is assumed to be stabilizable and detectable, we have $w_i \neq 0$ for all $i = 1$ to n_a^+. Because if $w_i = 0$, then by definition

$$(z_i I_n - \tilde{\mathbf{A}})x_i = \tilde{\mathbf{B}} w_i = 0, \quad \tilde{\mathbf{C}} x_i = 0$$

This implies that z_i is an output decoupling zero of $\tilde{\Sigma}$. But this contradicts the detectability of $\tilde{\Sigma}$ as $z_i \in \mathscr{C}^+$. Next let us define for each $i = 1$ to n_a^+

$$\mathscr{N}_i = \operatorname{Ker}[w_i']$$

Since $w_i \neq 0$, each $\mathscr{N}_i$ is an $m-1$ dimensional subspace. We have the following lemma.

Lemma 3.8

There exists at least one non-zero vector $\mathbf{e} \in \mathbb{R}^m$ such that

$$\mathbf{e} \notin \bigcup_{i=1}^{n_a^+} \mathcal{N}_i$$

Proof

See Appendix H. We are thankful for this proof to Mr Lin of Department of Electrical and Computer Engineering, Washington State University at Pullman.

Thus in view of Lemma 3.8, there exists at least one $\mathbf{e}$ such that

$$\mathbf{e}' w_i \neq 0 \quad \text{for all } i = 1 \text{ to } n_a^+ \tag{3.22}$$

We have the following theorem.

Theorem 3.10

Consider the closed-loop system $\tilde{\Sigma}^c$ comprising of the given plant $\tilde{\Sigma}$ and the controller as given in Fig. 2. Let $\tilde{\mathbf{F}}$ be such that it yields the required target loop transfer function. Let $\tilde{\Sigma}$ be left invertible with non-minimum phase invariant zeros having geometric multiplicity equal to unity. Then there exists at least one $m-1$ dimensional recoverable subspace $\mathcal{S}$ of $\mathbb{R}^m$.

Proof

Select $\mathbf{e}$ as in (3.22). Define $\mathcal{S}$ as

$\mathcal{S}$ = the orthogonal complement of the subspace spanned by $\mathbf{e}$ in $\mathbb{R}^m$.

Then it is trivial to see $\mathcal{S}$ has a dimension of $m-1$ and that $w_i \notin \mathcal{S}$ for all $i = 1$ to n_a^+. Because if $w_i \in \mathcal{S}$, say $w_i = \mathbf{V}_s v_i \in \mathcal{S}$, then $\mathbf{e}' w_i = 0$ which is a contradiction. In view of Lemma 3.7, this implies that $\tilde{\Sigma}_s$ is left invertible and of minimum phase. This in turn implies that $\mathcal{S}$ is recoverable.

Example 3.4

Consider a system $\tilde{\Sigma}$ characterized by

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\tilde{\mathbf{C}} = \tilde{\mathbf{B}}'$. This system has five non-minimum phase invariant zeros at $s = 1$, $s = 2$, $s = 3$, $s = 4$ and $s = 5$. It is simple to verify that the right input zero directions of these zeros are given by

$$[w_{R5} \quad w_{R4} \quad w_{R3} \quad w_{R2} \quad w_{R1}] = I_5$$

These vectors span the entire space $\mathbb{R}^5$. Following the analysis of Zhang and Freudenberg (1990), one can show then that for this example ARE based design cannot produce any recoverable subspace of $\mathbb{R}^5$ since it can recover only in a subspace orthogonal to the space spanned by the right input zero directions. However, considering a subspace spanned by column vectors of $\mathbf{V}_s$

$$\mathbf{V}_s = \begin{bmatrix} -0\cdot4472 & -0\cdot4472 & -0\cdot4472 & -0\cdot4472 \\ 0\cdot8618 & -0\cdot1382 & -0\cdot1382 & -0\cdot1382 \\ -0\cdot1382 & 0\cdot8618 & -0\cdot1382 & -0\cdot1382 \\ -0\cdot1382 & -0\cdot1382 & 0\cdot8618 & -0\cdot1382 \\ -0\cdot1382 & -0\cdot1382 & -0\cdot1382 & 0\cdot8618 \end{bmatrix}$$

it is straight forward to verify that the system $\tilde{\Sigma}_s$ characterized by the matrix triple $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}\mathbf{V}_s)$ is left invertible and of minimum phase. Hence $\mathscr{S}$ spanned by the columns of $\mathbf{V}_s$ is asymptotically recoverable. To exemplify this, let the target loop transfer function be specified by

$$\tilde{\mathbf{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 151\cdot71 & 21\cdot11 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 113\cdot57 & 0 & 0 & 19\cdot11 & 0 & 0 & 0 \\ 0 & 0 & 79\cdot43 & 0 & 0 & 0 & 0 & 17\cdot12 & 0 & 0 \\ 0 & 49\cdot28 & 0 & 0 & 0 & 0 & 0 & 0 & 15\cdot13 & 0 \\ 23\cdot14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13\cdot14 \end{bmatrix}$$

Let us choose $\tilde{\mathbf{K}}(\sigma)$ with $\sigma = 500$ as

$$\tilde{\mathbf{K}}(\sigma) = \begin{bmatrix} 15030 & 15030 & 15030 & 15030 & 15030 \\ -210840 & -210840 & -210840 & -210840 & -210840 \\ 845040 & 845040 & 845040 & 845040 & 845040 \\ -1270100 & -1270100 & -1270100 & -1270100 & -1270100 \\ 636300 & 636300 & 636300 & 636300 & 636300 \\ 507 & 6 & 6 & 6 & 6 \\ 6 & 507 & 6 & 6 & 6 \\ 6 & 6 & 507 & 6 & 6 \\ 6 & 6 & 6 & 507 & 6 \\ 6 & 6 & 6 & 6 & 507 \end{bmatrix}$$

The $\tilde{\mathbf{K}}(\sigma)$ given above places the observer eigenvalues at -500, -500, -500, -500, -500, -1, -2, -3, -4, and -5. Let the orthogonal projection matrix onto the subspace $\mathscr{S}$ be $P_s = \mathbf{V}_s \mathbf{V}_s'$. Then the resulting $M(j\omega, \sigma)P_s$, $E(j\omega, \sigma)P_s$, $S_0(j\omega, \sigma)P_s$ and $S(j\omega)P_s$ are plotted with respect to ω over a given range of ω in Figs 8 and 9. It is easy to note that $M(j\omega, \sigma)P_s$ is approximately zero while $S_0(j\omega, \sigma)P_s$ is close to $S(j\omega)P_s$. (Note that the minimum singular values of $S_0(j\omega, \sigma)P_s$ and $S(j\omega)P_s$ are identically zero due to the singularity of P_s.)

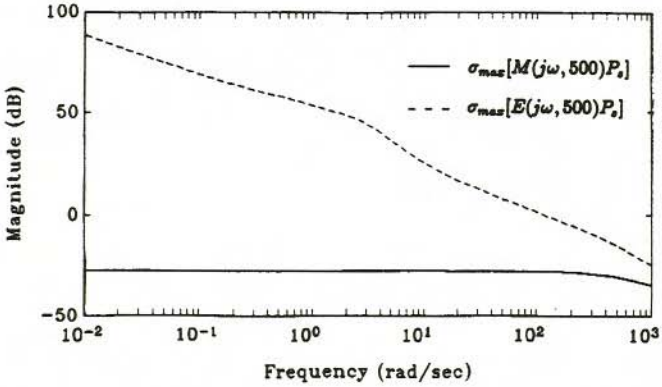

Figure 8. $\sigma_{\max}[M(j\omega, \sigma)P_s]$ and $\sigma_{\max}[E(j\omega, \sigma)P_s]$ of Example 3.4.

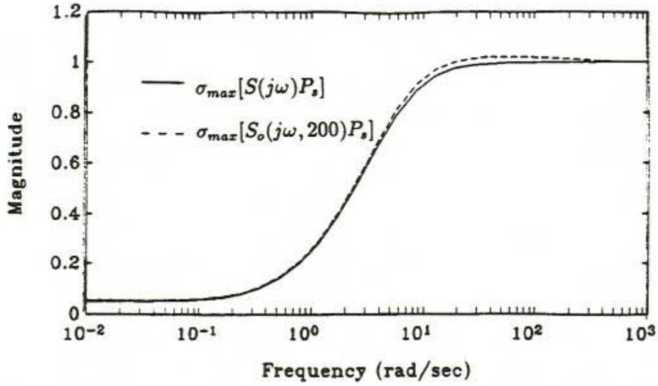

Figure 9. $\sigma_{\max}[S(j\omega)P_s]$ and $\sigma_{\max}[S_0(j\omega, \sigma)P_s]$ of Example 3.4.

4. Conclusions

The first of the paper deals only with the issues concerning analysis of loop transfer recovery problem using full order observer based controllers. All the analysis given here is independent of the methodology by which observers are designed. There are several fundamental results given here. At first for general systems, precise definitions and properties of invariant zeros and their state and input zero directions are presented. Also, what is called 'zero dynamics' of a given system is discussed. These structural properties of a given system lead us to decompose the recovery error between the target loop transfer function and that that can be achieved by the observer based controllers, into three distinct parts for any arbitrarily specified target loop transfer function. The first part of recovery error can be rendered exactly zero by an appropriate finite eigenstructure assignment of the observer dynamic matrix, while the second part can be rendered arbitrarily close to zero by an appropriate asymptotically infinite eigenstructure assignment. The third part in general cannot be rendered zero, either exactly or asymptotically, by any means although there exists a multitude of ways to shape it. Such a decomposition of loop transfer function recovery mechanism helps us to discover the subspace of the control space in which target sensitivity and complimentary sensitivity functions can either exactly or asymptotically be recovered.

Moreover, it helps to formulate explicit singular value bounds on the recovery error. All this analysis is given for an arbitrarily specified target loop transfer function. Thus it shows the limitations of the given system in recovering the target loop transfer functions as a consequence of its structural properties, namely finite and infinite zero structure and invertibility. On the other hand, the next issue of our analysis concentrates on characterizing the rquired necessary and sufficient conditions on the target loop transfer functions so that they are either exactly or asymptotically recoverable by means of observer based controllers for the given system. Interestingly enough, the developed conditions turn out to be constraints on the finite and infinite zero structure and invertibility properties of target loop transfer functions. Such an interpretation of the constraints reveals that either ELTR or ALTR is possible under a variety of conditions. For instance, LTR can be achieved even if the target loop transfer function does not contain non-minimum phase zero structure of the given system provided some other conditions are satisfied. Since recovery in all control loops in general is not feasible, our analysis next concentrates in developing the necessary or/and sufficient conditions under which either exact or asymptotic recovery of target sensitivity and complimentary sensitivity functions is possible in any specified subspace of the control space. In this connection, we prove that for left invertible systems irrespective of the number of non-minimum phase zeros and irrespective of the nature of the target loop transfer function, there exists at least one $m-1$ dimensional subspace of m dimensional control space, in which the target sensitivity and complimentary sensitivity functions can always be recovered by an appropriate design of the controller. Inherent in all the issues discussed here is the characterization of the resulting controller eigenvalues and possible pole zero cancellations. Such an investigation is important in view of the fact, controller eigenvalues become the invariant zeros of the closed-loop system and thus affect the performance with respect to command following and other design objectives.

To summarize, the analysis presented here adds a considerable amount of flexibility to the process of design and helps a designer to set meaningful goals at the onset of design. In other words, although the actual physical tasks of first designing a target loop and then designing an observer based controller are separable, one can link these two tasks philosophically by knowing ahead what is feasible and how. In a sequel, we will present a design methodology which is capable of utilizing the complete freedom a design can have as is discovered here.

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Appendix A

Proof of Lemma 3.2

Let λ_i and V_i be an eigenvalue and the corresponding left eigenvector of $\tilde{\mathbf{A}}-\tilde{\mathbf{K}}\tilde{\mathbf{C}}$ for any gain $\tilde{\mathbf{K}}$. To show that there are at most $n_a^- + n_b$ left eigenvectors of $\tilde{\mathbf{A}}-\tilde{\mathbf{K}}\tilde{\mathbf{C}}$ for any gain $\tilde{\mathbf{K}}$ such that the corresponding $\lambda_i \in \mathscr{C}^-$ and that $V_i^H \tilde{\mathbf{B}} = 0$, consider the dual system $\tilde{\Sigma}_t$ characterized by the triple $(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)$. Let $\mathscr{V}_t$ be the subspace of all right eigenvectors V_t of $(\tilde{\mathbf{A}}_t - \tilde{\mathbf{B}}_t\tilde{\mathbf{K}}_t)$ for some $\tilde{\mathbf{K}}_t$ such that $\tilde{\mathbf{C}}_t V_t = 0$. Observe that $\mathscr{V}_t$ is a stable $(\tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)$-invariant subspace. Furthermore, $\mathscr{V}_t$ is in the kernel of $\tilde{\mathbf{C}}_t$. Hence $\mathscr{V}_t$ is a subset of $\mathscr{V}^-_{\mathrm{Ker}\,\tilde{\mathbf{C}}_t}(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)$. The largest possible dimension of $\mathscr{V}^-_{\mathrm{Ker}\,\tilde{\mathbf{C}}_t}(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)$ is $n_a^- + n_b$. Hence, there are at most $n_a^- + n_b$ left eigenvectors of $\tilde{\mathbf{A}} - \tilde{\mathbf{K}}\tilde{\mathbf{C}}$ for any gain $\tilde{\mathbf{K}}$ such that the corresponding $\lambda_i \in \mathscr{C}^-$ and that $V_i^H \tilde{\mathbf{B}} = 0$.

We now proceed to determine the necessary gain $\tilde{\mathbf{K}}$ to assign such eigenvalues. Again without loss of generality, we can assume that the given system is represented by the s.c.b. as given by Theorem 2.1 and hence it is characterized by the triple $(\tilde{C}, \tilde{A}, \tilde{B})$ given in (2.8). Then consider a gain $\tilde{\mathbf{K}}$ of the form

$$\tilde{\mathbf{K}} = \begin{bmatrix} 0 & 0 \\ L_{af}^- & L_{as}^- \\ L_{bf} & K_{bb} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{A 1}$$

where K_{bb} is selected such that $\lambda(A_{bb} - K_{bb}C_s)$ are in $\mathscr{C}^-$. Let V_{a-} and V_b respectively be any left eigenvectors of A_{aa}^- and $A_{bb} - K_{bb}C_s$. It can easily be verified that $\lambda(A_{aa}^-)$ and $\lambda(A_{bb} - K_{bb}C_s)$ are among the eigenvalues of $\tilde{\mathbf{A}} - \tilde{\mathbf{K}}\tilde{\mathbf{C}}$ and that $[0 \;\; V_{a\_}^H \;\; 0 \;\; 0 \;\; 0]^H$ and $[0 \;\; 0 \;\; V_b^H \;\; 0 \;\; 0]^H$ are the associated left eigenvectors of $\tilde{\mathbf{A}} - \tilde{\mathbf{K}}\tilde{\mathbf{C}}$. Furthermore, it is easy to verify that

$$[0 \quad V_{a\_}^H \quad 0 \quad 0 \quad 0]\tilde{\mathbf{B}} = 0 \quad \text{and} \quad [0 \quad 0 \quad V_b^H \quad 0 \quad 0]\tilde{\mathbf{B}} = 0$$

while $[0 \;\; V_{a\_}^H \;\; 0 \;\; 0 \;\; 0]^H$ is a left state zero direction of $\tilde{\Sigma}$. Finally, in view of the properties of s.c.b., it is straightforward to see that such vectors $[0 \;\; V_{a\_}^H \;\; 0 \;\; 0 \;\; 0]^H$ and $[0 \;\; 0 \;\; V_b^H \;\; 0 \;\; 0]^H$ respectively span the subspaces $\mathscr{V}^-_{\mathrm{Ker}\,\tilde{\mathbf{C}}}/\mathscr{R}^*$ and $\mathscr{N}^{*\perp}$. □

Appendix B

Proof of Lemma 3.3

Consider

$$\tilde{M}(s, \sigma) = (sI_n - \tilde{\mathbf{A}} + \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}})^{-1}\tilde{\mathbf{B}}$$

Let $\tilde{X}[t, \tilde{\mathbf{K}}(\sigma)]$ be the Laplace inverse of $\tilde{\mathbf{M}}(s, \sigma)$. Then

$$\tilde{X}(t, \tilde{\mathbf{K}}(\sigma)) = \exp[(\tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}})t]\tilde{\mathbf{B}}$$

Thus the problem of requiring that the residue matrices $W_i(\sigma)V_i^H(\sigma)\tilde{\mathbf{B}}$ associated with the unbounded eigenvalues of $\tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}}$ be uniformly bounded as $|\lambda_i| \to \infty$ is equivalent to the requirement that

$$\operatorname*{Sup}_{\sigma} \operatorname*{Sup}_{t \geqslant 0} \|\tilde{X}[t, \tilde{\mathbf{K}}(\sigma)]\| < \infty \tag{B 1}$$

The problem as formulated in (B 1) is same as the bounded peaking problem which was treated earlier by Kimura (1981), except that now we need to consider the dual system $\tilde{\Sigma}_t$ characterized by the triple $(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)$ rather than the given system $\tilde{\Sigma}$. From Lemma 6 in Kimura (1981), it follows that the asymptotic eigenspace corresponding to the unbounded eigenvalues of $\tilde{\mathbf{A}}' - \tilde{\mathbf{C}}'\tilde{\mathbf{K}}(\sigma)'$ is restricted to the subspace $\mathscr{S}^*$ of the dual system $\tilde{\Sigma}_t$. It is evident from Property 2.3 that $\mathscr{S}^*$ of $\tilde{\Sigma}_t$ is $\mathscr{N}^{*\perp} \oplus \mathscr{S}^*/\mathscr{R}^*$ of $\tilde{\Sigma}$. Then it follows that the maximum allowable number of unbounded eigenvalues of $\tilde{\mathbf{A}} - \tilde{\mathbf{K}}(\sigma)\tilde{\mathbf{C}}$ is equal to $n_b + n_f$, the dimension of $N^{*\perp} \oplus \mathscr{S}^*/\mathscr{R}^*$. □

Appendix C

Proof of Observation 3.1

From (3.10), we note that as $\sigma \to \infty$, the limiting value of $M_e(s)$ can be written as

$$\bar{M}_e(s) = \sum_{i=1}^{n_a^+} \frac{\tilde{\mathbf{F}} \bar{W}_i \bar{V}_i^H \tilde{\mathbf{B}}}{s - \lambda_i}$$

Since $\tilde{\Sigma}$ is left invertible, as in Appendix E, it is easy to see that

$$\{\bar{W}_i, i = 1 \text{ to } n_a^+\} \quad \text{spans } \mathscr{V}^+_{\operatorname{Ker} \tilde{\mathbf{C}}} \tag{C 1}$$

Let z_i, x_i and w_i, $i = 1$ to n_a^+, be respectively the non-minimum phase invariant zeros and state and input zero directions. Then we know that

$$\{x_i, i = 1 \text{ to } n_a^+\} \quad \text{spans } \mathscr{V}^+_{\operatorname{Ker} \tilde{\mathbf{C}}} \tag{C 2}$$

and

$$(z_i I_n - \tilde{\mathbf{A}})x_i = \tilde{\mathbf{B}} w_i \quad \text{and} \quad \tilde{\mathbf{C}} x_i = 0 \tag{C 3}$$

Since $\tilde{\Sigma}$ is stabilizable and detectable, $(z_i I_n - \tilde{\mathbf{A}})$ is non-singular. Hence

$$x_i = (z_i I_n - \tilde{\mathbf{A}})^{-1} \tilde{\mathbf{B}} w_i = \tilde{\mathbf{\Phi}}(z_i) \tilde{\mathbf{B}} w_i$$

Thus from (C 1) to (C 3)

$$\bar{W}_i = \sum_{k=1}^{n_a^+} \alpha_{ik} x_k = \sum_{k=1}^{n_a^+} \alpha_{ik} \tilde{\mathbf{\Phi}}(z_k) \tilde{\mathbf{B}} w_k$$

for some constants α_{ik}, $i = 1$ to n_a^+ and $k = 1$ to n_a^+. Now $\bar{M}_e(s)$ can be rewritten as

$$\begin{aligned} \bar{M}_e(s) &= \sum_{i=1}^{n_a^+} \frac{1}{s - \lambda_i} \left[\sum_{k=1}^{n_a^+} \alpha_{ik} \tilde{\mathbf{F}} \tilde{\mathbf{\Phi}}(z_k) \tilde{\mathbf{B}} w_k \right] \bar{V}_i^H \tilde{\mathbf{B}} \\ &= \sum_{k=1}^{n_a^+} [\tilde{\mathbf{F}} \tilde{\mathbf{\Phi}}(z_k) \tilde{\mathbf{B}}] w_k \sum_{i=1}^{n_a^+} \frac{\alpha_{ik} \bar{V}_i^H \tilde{\mathbf{B}}}{s - \lambda_i} \end{aligned}$$

Thus if z_k, $k = 1$ to n_a^+, is far away from the band-width of the target loop, then $\tilde{\mathbf{F}} \tilde{\mathbf{\Phi}}(z_k) \tilde{\mathbf{B}}$ is small and hence the result. □

Appendix D

Proof of Theorem 3.3

As in (3.8), let W_i and V_i be respectively the right and left eigenvectors associated with an eigenvalue λ_i of $\tilde{\mathbf{A}}_0$. Define

$$V_0 = \{V_i \mid V_i^H \tilde{\mathbf{B}} = 0, \quad i = 1, 2, \ldots, \alpha\}$$

$$V_r = \{V_i \mid V_i^H \tilde{\mathbf{B}} \neq 0, \quad i = \alpha + 1, \alpha + 2, \ldots, n\}$$

$$W_0 = \{W_i \mid V_i^H \tilde{\mathbf{B}} = 0, \quad i = 1, 2, \ldots, \alpha\}$$

and

$$W_r = \{W_i \mid V_i^H \tilde{\mathbf{B}} \neq 0, \quad i = \alpha+1, \alpha+2, ..., n\}$$

Lemma 3.2 implies that $\alpha \leqslant n_a^- + n_b$. Also, it is apparent that $V_0 \subseteq \mathscr{V}^-_{\text{Ker}\,\tilde{\mathbf{C}}_t}(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)$. Without loss of generality, let us assume that the given system is represented by the s.c.b. as given by Theorem 2.1. Recalling that $\tilde{x}_a^- \oplus \tilde{x}_b$ spans $\mathscr{V}^-_{\text{Ker}\,\tilde{\mathbf{C}}_t}(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)$, V_0 is of the form

$$V_0 = \begin{bmatrix} V_0^1 \\ V_0^2 \\ V_0^3 \end{bmatrix}, \quad V_0^1 = 0, \quad V_0^3 = 0 \tag{D 1}$$

where V_0^1, V_0^2 and V_0^3 are respectively $n_a^+ \times \alpha$, $(n_a^- + n_b) \times \alpha$ and $(n_c + n_f) \times \alpha$ dimensional matrices while V_0^2 is of full rank α. Noting that

$$[V_0 \quad V_r]^{-1} = [W_0 \quad W_r]^H$$

we then have

$$V_0^H W_r \equiv 0 \tag{D 2}$$

This implies that $V_0 \subseteq \text{Ker}\,(W_r^H)$. In fact for $\alpha = n_a^- + n_b$, the column vectors of V_0 span $\text{Ker}\,(W_r^H)$. Now if ELTR is achievable, in view of (3.8), the only way it could happen is by letting $\tilde{\mathbf{F}} W_r = 0$, implying that $\tilde{\mathbf{F}}' \subseteq \text{Ker}\, W_r^H$ or equivalently $\tilde{\mathbf{F}}' \subseteq \text{Span}\, V_0$. Then $\tilde{\mathbf{F}}$ must be of the form

$$\tilde{\mathbf{F}} = [F_1 \quad F_2 \quad F_3], \quad F_1 = 0, \quad F_3 = 0 \tag{D 3}$$

where F_1, F_2 and F_3 are respectively $m \times n_a^+$, $m \times (n_a^- + n_b)$ and $m \times (n_c + n_f)$ dimensional matrices. Also, the feedback gains F_1, F_2 and F_3 are respectively associated with the states $\tilde{x}_a^+$, $[(\tilde{x}_a^-)' \; \tilde{x}_b']'$ and $[\tilde{x}_c' \; \tilde{x}_f']'$. Thus (D 3) reveals that $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ which is spanned by $\tilde{x}_a^+ \oplus \tilde{x}_c \oplus \tilde{x}_f$ is a subset of $\text{Ker}\,\tilde{\mathbf{F}}$. This proves that if ELTR is achieved, then $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) \subseteq \text{Ker}\,\tilde{\mathbf{F}}$. On the other hand, let us assume that $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) \subseteq \text{Ker}\,\tilde{\mathbf{F}}$. We first can assign V_i, $i = 1$ to $n_a^- + n_b$ such that $V_i^H \tilde{\mathbf{B}} = 0$. This implies that the columns of V_0 span $\mathscr{V}^-_{\text{Ker}\,\tilde{\mathbf{C}}_t}(\tilde{\mathbf{C}}_t, \tilde{\mathbf{A}}_t, \tilde{\mathbf{B}}_t)$ and hence V_0^2 of (D 1) is non-singular. Now let us partition W_r in the same way as V_0 was partitioned

$$W_r = \begin{bmatrix} W_r^1 \\ W_r^2 \\ W_r^3 \end{bmatrix}$$

Then in view of (D 2) and V_0^2 being non-singular, we see that $W_r^2 \equiv 0$. Thus if $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ which is spanned by $\tilde{x}_a^+ \oplus \tilde{x}_c \oplus \tilde{x}_f$ is a subset of $\text{Ker}\,\tilde{\mathbf{F}}$, then we have $\tilde{\mathbf{F}} W_r^2 = 0$ and thus ELTR is achievable as indicated by (3.8). □

Appendix E

Proof of Corollary 3.2

Again, without loss of generality, let us assume that the given system is represented by the s.c.b. as given by Theorem 2.1. First, we have the following propositions.

*Proposition E.*1

The fact that the target loop transfer function $\tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$ contains the non-minimum phase zero structure of $\tilde{\Sigma}$ implies that $\mathscr{V}^{+}_{\mathrm{Ker}\,\tilde{\mathbf{C}}}/\mathscr{R}^* \subseteq \mathrm{Ker}\,\tilde{\mathbf{F}}$, i.e. the span of $\tilde{x}_a^+ \subseteq \mathrm{Ker}\,\tilde{\mathbf{F}}$.

Proof

Let x_{ri} and w_{ri}, $i = 1$ to n_a^+, be the right state and input zero directions associated with the invariant zeros z_i, $i = 1$ to n_a^+, which are in $\mathscr{C}^+$. In view of Propositions 2.1 and 2.2, we note that

$$\mathrm{Span}\,\{x_{ri}, i = 1, 2, \ldots, n_a^+\} = \mathrm{Span\ of}\ \tilde{x}_a^+.$$

We also have for all $i = 1$ to n_a^+

$$[z_i I_n - \tilde{\mathbf{A}}]x_{ri} = \tilde{\mathbf{B}}w_{ri} \quad \text{and} \quad \tilde{\mathbf{C}}x_{ri} = 0$$

Since the target loop transfer function $\tilde{\mathbf{F}}\tilde{\mathbf{\Phi}}\tilde{\mathbf{B}}$ contains the same non-minimum phase zero structure as $\tilde{\Sigma}$, the above implies

$$[z_i I_n - \tilde{\mathbf{A}}]x_{ri} = \tilde{\mathbf{B}}w_{ri} \quad \text{and} \quad \tilde{\mathbf{F}}x_{ri} = 0$$

for all $i = 1$ to n_a^+. Hence the span of $\tilde{x}_a^+ \subseteq \mathrm{Ker}\,\tilde{\mathbf{F}}$. □

*Proposition E.*2

Let $\tilde{\Sigma}$ be invertible and of uniform rank with relative degree q. Then the fact that the smallest order of infinite zero of $L(s)$ is greater than q implies that $\mathscr{S}^*/\mathscr{R}^* \subseteq \mathrm{Ker}\,\tilde{\mathbf{F}}$, i.e. the span of $\tilde{x}_f \subseteq \mathrm{Ker}\,\tilde{\mathbf{F}}$.

Proof

Given that $\tilde{\Sigma}$ is invertible and of uniform rank with relative degree q implies that the matrices $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ are of the form,

$$\tilde{\mathbf{A}} = \begin{bmatrix} A_{aa} & L_a & 0 & 0 & \ldots & 0 \\ 0 & 0 & I & 0 & \ldots & 0 \\ 0 & 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & I \\ E_a & E_1 & E_2 & E_3 & \ldots & E_q \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

$$\tilde{\mathbf{C}} = [0 \quad I \quad 0 \quad 0 \quad \ldots \quad 0]$$

It is then straightforward to verify that

$$[\tilde{\mathbf{B}} \quad \tilde{\mathbf{A}}\tilde{\mathbf{B}} \quad \tilde{\mathbf{A}}^2\tilde{\mathbf{B}} \quad \ldots \quad \tilde{\mathbf{A}}^{q-1}\tilde{\mathbf{B}}] = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & I \\ 0 & 0 & 0 & \ldots & I & X \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & I & \ldots & X & X \\ 0 & I & X & \ldots & X & X \\ I & X & X & \ldots & X & X \end{bmatrix} \qquad \text{(E 1)}$$

where X denotes a non-zero element. The fact that the smallest order of infinite zero of $L(s)$ is greater than q implies that

$$\tilde{\mathbf{F}}\tilde{\mathbf{B}} = \tilde{\mathbf{F}}\tilde{\mathbf{A}}\tilde{\mathbf{B}} = \tilde{\mathbf{F}}\tilde{\mathbf{A}}^2\tilde{\mathbf{B}} = \dots = \tilde{\mathbf{F}}\tilde{\mathbf{A}}^{q-1}\tilde{\mathbf{B}} = 0$$

Then in view of (E 1), we can conclude that $\tilde{\mathbf{F}}$ is of the form,

$$\tilde{\mathbf{F}} = [X \quad 0 \quad 0 \quad \dots \quad 0]$$

This implies that the span of $\tilde{x}_f \subseteq \operatorname{Ker} \tilde{\mathbf{F}}$.

Now to prove the Corollary 3.2, we observe that invertibility of $\tilde{\Sigma}$ implies that $\tilde{x}_c$ is non-existent. Thus in view of the above propositions, $\tilde{x}_a^+ \oplus \tilde{x}_c \oplus \tilde{x}_f$ is a subset of Ker $\tilde{\mathbf{F}}$, i.e. $\mathscr{L}_-(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}) \subseteq \operatorname{Ker} \tilde{\mathbf{F}}$. This proves the Corollary 3.2. □

Appendix F

Proof of Lemma 3.5

From the work of Kimura (1981), we know that Im $\bar{V}_0$ coincides with $\mathscr{V}^-_{\operatorname{Ker}\tilde{\mathbf{C}}}/\mathscr{R}^* \oplus \mathscr{N}^{*\perp}$ while Im $\bar{V}_\infty$ coincides with $\mathscr{S}^*/\mathscr{R}^*$. This implies that assuming $\tilde{\Sigma}$ is in the form of s.c.b. $\bar{V}_0$ and $\bar{V}_\infty$ have the special forms

$$\bar{V}_0 = \begin{bmatrix} \bar{V}_0^1 \\ \bar{V}_0^2 \\ \bar{V}_0^3 \\ \bar{V}_0^4 \end{bmatrix}, \quad \bar{V}_0^1 = 0, \quad \bar{V}_0^3 = 0, \quad \bar{V}_0^4 = 0 \tag{F 1}$$

and

$$\bar{V}_\infty = \begin{bmatrix} \bar{V}_\infty^1 \\ \bar{V}_\infty^2 \\ \bar{V}_\infty^3 \\ \bar{V}_\infty^4 \end{bmatrix}, \quad \bar{V}_\infty^1 = 0, \quad \bar{V}_\infty^2 = 0, \quad \bar{V}_\infty^3 = 0 \tag{F 2}$$

where $\bar{V}_0^2$ and $\bar{V}_\infty^4$ are non-singular matrices of dimension $(n_a^- + n_b) \times (n_a^- + n_b)$ and $n_f \times n_f$ respectively. Also, $\bar{V}_0^1$, $\bar{V}_0^3$, $\bar{V}_0^4$, $\bar{V}_\infty^1$, $\bar{V}_\infty^2$ and $\bar{V}_\infty^3$ are of dimension $n_a^+ \times (n_a^- + n_b)$, $n_c \times (n_a^- + n_b)$, $n_f \times (n_a^- + n_b)$, $n_a^+ \times n_f$, $(n_a^- + n_b) \times n_f$ and $n_c \times n_f$ respectively. Now noting that

$$[\bar{V}_0 \quad \bar{V}_e \quad \bar{V}_\infty]^{-1} = [\bar{W}_0 \quad \bar{W}_e \quad \bar{W}_\infty]^H$$

we have

$$[\bar{W}_e]^H \bar{V}_0 = 0 \quad \text{and} \quad [\bar{W}_e]^H \bar{V}_\infty = 0$$

Hence, in view of (F 1) and (F 2), we have

$$[\bar{W}_e]^H = [\bar{W}_e^1 \quad 0 \quad \bar{W}_e^3 \quad 0]$$

where $[\bar{W}_e^1 \ \bar{W}_e^3]$ is a $(n_a^+ + n_c) \times (n_a^+ + n_c)$ non-singular matrix. Hence the result. □

Appendix G

Proof of Lemma 3.7

Assume that $\tilde{\Sigma}_s$ is not left invertible. Then it is well known that for any complex number z_1, there exist $0 \neq x_1 \in \mathbb{R}^n$ and $v_1 \in \mathbb{R}^m$ such that

$$\begin{bmatrix} z_1 I_n - \tilde{\mathbf{A}} & -\tilde{\mathbf{B}}\mathbf{V}_s \\ -\tilde{\mathbf{C}} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \end{bmatrix} = 0$$

This implies that

$$\begin{bmatrix} z_1 I_n - \tilde{\mathbf{A}} & -\tilde{\mathbf{B}} \\ -\tilde{\mathbf{C}} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \mathbf{V}_s v_1 \end{bmatrix} = 0$$

Since $\tilde{\Sigma}$ is left invertible, this then in turn implies that z_1 is an invariant zero of $\tilde{\Sigma}$. This is a contradiction and hence $\tilde{\Sigma}_s$ is left invertible. To prove the second property of the lemma, let z_s, x_s and w_s be respectively an invariant zero, the associated right state and input zero directions of $\tilde{\Sigma}_s$. Then by definition, we have

$$\begin{bmatrix} z_s I_n - \tilde{\mathbf{A}} & -\tilde{\mathbf{B}}\mathbf{V}_s \\ -\tilde{\mathbf{C}} & 0 \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix} = 0$$

Thus we note that

$$\begin{bmatrix} z_s I_n - \tilde{\mathbf{A}} & -\tilde{\mathbf{B}} \\ -\tilde{\mathbf{C}} & 0 \end{bmatrix} \begin{bmatrix} x_s \\ \mathbf{V}_s w_s \end{bmatrix} = 0$$

This proves the second property of the Lemma. Let us next prove the sufficiency part of Property (3) of the lemma. Let $w = \mathbf{V}_s v$, then

$$\begin{bmatrix} z I_n - \tilde{\mathbf{A}} & -\tilde{\mathbf{B}} \\ -\tilde{\mathbf{C}} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0$$

implies that

$$\begin{bmatrix} z I_n - \tilde{\mathbf{A}} & -\tilde{\mathbf{B}}\mathbf{V}_s \\ -\tilde{\mathbf{C}} & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = 0$$

As $\tilde{\Sigma}_s$ is left invertible, the above implies that z and x are an invariant zero and the associated right state zero direction of $\tilde{\Sigma}_s$. To prove necessity, assume that z and x are an invariant zero and the associated right state zero direction of $\tilde{\Sigma}_s$. Then there exists a w_s such that

$$(zI_n - \tilde{\mathbf{A}})x = \tilde{\mathbf{B}}\mathbf{V}_s w_s$$

In view of this and by the definition of z, x and w, we have

$$\tilde{\mathbf{B}}\mathbf{V}_s w_s = \tilde{\mathbf{B}} w$$

Since $\tilde{\mathbf{B}}$ is of full rank, it implies then that $w \in \mathscr{S}$. □

Appendix H

Proof of Lemma 3.8

The proof is by induction. The lemma is trivially true when $n_a^+ = 1$. Assume that the given lemma is true for $n_a^+ = k$. Then there exists a vector $0 \neq v \in \mathbb{R}^m$ such that

$$v \notin \bigcup_{i=1}^{k} \mathcal{N}_i$$

To proceed with the proof, let us first assume that $v \notin \mathcal{N}_{k+1}$. Then

$$\mathbf{e} = v \notin \bigcup_{i=1}^{k+1} \mathcal{N}_i$$

and hence the result.

On the other hand, assume that $v \in \mathcal{N}_{k+1}$. First select non-zero scalar numbers α_i, $i = 1$ to $k+1$, such that $\alpha_i \neq \alpha_j$ if $i \neq j$. Then we have for all $i = 1$ to $k+1$, $\alpha_i v \in \mathcal{N}_{k+1}$. Since $\mathcal{N}_{k+1}$ has only a dimension of $m-1$, there exists a vector $0 \neq w \in \mathbb{R}^m$ such that $w \notin \mathcal{N}_{k+1}$. Now define for each $i = 1$ to $k+1$

$$x_i = \alpha_i v + w \neq 0$$

Because of the fact that $\alpha_i \neq \alpha_j$ if $i \neq j$, we note that $x_i \neq x_j$ if $i \neq j$. Moreover $x_i \notin \mathcal{N}_{k+1}$ for all $i = 1$ to $k+1$ since $w \notin \mathcal{N}_{k+1}$. Now if

$$x_i \in \bigcup_{j=1}^{k} \mathcal{N}_j \quad \text{for all } i = 1 \text{ to } k+1$$

then there exists two distinct vectors among x_i, $i = 1$ to $k+1$, say x_s and x_t for some integers s and t, such that both are contained in some $\mathcal{N}_\beta$ for some $\beta \leqslant k$. Thus $0 \neq (\alpha_s - \alpha_t)v = (x_s - x_t) \in \mathcal{N}_\beta$. This implies that $v \in \mathcal{N}_\beta$ and thus contradicts the inductive hypothesis. Hence there exists at least one x_i for some $i \leqslant k+1$, such that

$$x_i \notin \bigcup_{j=1}^{k} \mathcal{N}_j \quad \text{and} \quad \mathbf{e} = x_i \notin \bigcup_{j=1}^{k+1} \mathcal{N}_j$$

Hence the result. □

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