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A New Stable Compensator Design for Exact and Approximate Loop Transfer Recovery*

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For minimum phase MIMO systems, a LTR design is presented via a new compensator which is stable and which allows much lower gain and thus lower controller band-width than the conventional observer based controller.

Key Words—Loop transfer recovery; robust control, linear quadratic Gaussian theory.

Abstract—In this paper, a new compensator structure for loop transfer recovery (LTR) is proposed. The proposed compensator (a) is open-loop stable, (b) guarantees closed-loop stability and above all (c) requires much smaller values of gain than the conventional observer-based controller for the same degree of loop-transfer recovery. The fact that the new compensator requires much smaller values of gain than the conventional controller results in several practical advantages, the most important among them being the reduction in controller band-width and freedom from the woes of saturation. The trade-off between the value of gain and the degree of loop transfer recovery as well as the bounds on singular values of sensitivity and complementary sensitivity functions is shown clearly. Both full and reduced order compensators for LTR when the design specifications are reflected either at the input or at the output point of the given plant are considered. Numerical examples illustrate the advantages of the new compensator structure.

The new compensator structure is inspired by a careful and clear understanding of how loop transfer recovery occurs when conventional observer based controllers are used. To motivate and deduce our new compensator structure, a unified treatment of observer theory for LTR is presented. In the context of such a unification, some new results are also given.

1. INTRODUCTION AND PROBLEM STATEMENT

In multi-input and multi-output (MIMO) feedback control system design, performance specifications such as command following, disturbance rejection, closed-loop band-width, stability robustness with respect to unstructured dynamic uncertainties etc., are naturally posed in the frequency domain in terms of sensitivity and complementary sensitivity functions. These sensitivity and complementary sensitivity functions are related to the loop transfer matrices evaluated by breaking the control loop at critical points, commonly either the input or output point of the given plant. Recent results have shown that the formal mathematical synthesis procedures based on linear quadratic Gaussian (LQG) with loop transfer recovery (LTR), the so called LQG/LTR techniques, provide a broad flexibility in achieving the necessary loop transfer matrices. LQG/LTR technique is essentially a two-step approach and involves two separate designs of a linear quadratic regulator and either an observer or a Kalman filter based controller (Athans, 1986; Stein and Athans, 1987). The exact design procedure depends on the point, either the input or the output point of the plant, where the loop is broken to evaluate the open-loop transfer matrices. We will first concentrate our discussion on the case when the loop is broken at the input point of the plant. Dual discussion can be given for the case when the loop is broken at the output point. Thus in the two step procedure of LQG/LTR, the first step of design involves loop shaping by state feedback design to obtain an appropriate loop transfer function, called the target loop transfer function. Such a loop shaping is an engineering art and often involves the use of linear quadratic regulator (LQR) design in which the cost matrices are used as free design parameters to generate the target loop transfer function and thus the desired sensitivity and complementary sensitivity functions. However, when such a feedback design is implemented via an observer (or Kalman filter) based controller that uses only the output feedback, the obtained loop transfer function, in general, is not the same as the target loop transfer function, unless proper care is taken in designing the observers. This is when

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the second step of LQG/LTR design philosophy comes into picture. In this step, the required observer is designed so as to recover either exactly or approximately the loop transfer function of the full state feedback controller. In this paper, we focus our attention on this second step of design.

LTR has been the subject of many papers. The heart of the LTR problem can easily be understood by considering the closed-loop structure depicted in Fig. 1 where $\mathbf{C}(s)$ and $P(s)$ are respectively the transfer functions of a controller and the given plant. Given $P(s)$ and the target loop transfer function $L(j\omega)$, the goal is to design a $\mathbf{C}(s)$ such that $\mathbf{C}(j\omega)P(j\omega)$ is either exactly or approximately equal to $L(j\omega)$, when the performance specifications are reflected at the input point. All the existing literature chooses either a full or reduced order observer based structure for the controller in which the state feedback gain F and the observer gain K are incorporated and K is a free design parameter. In this paper, we propose a new compensator structure for the controller. This structure is inspired by a clear and careful understanding of the error or mismatch function $E_0(s) \equiv L(s) - \mathbf{C}_0(s)P(s)$ where $\mathbf{C}_0(s)$ is the transfer function of an observer based controller.

We now briefly review the observer based LTR theory to motivate the need for new controller structures other than observers. Consider a given plant,

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{1.1}$$

where x, u and y are respectively n-, m- and p-dimensional state, input and output vectors. Assume that B and C are of maximal rank and that (1.1) is stabilizable and detectable. Let F be a stabilizing full state feedback gain matrix such that (a) the closed-loop system is asymptotically stable and (b) the open-loop transfer function when the loop is broken at the input point of the plant meets the given frequency dependent specifications. The state feedback control is

$$u = -Fx \tag{1.2}$$

and the open-loop transfer function when the loop is broken at the input point is

$$L(s) = F\Phi B \tag{1.3}$$

where $\Phi = (sI - A)^{-1}$.

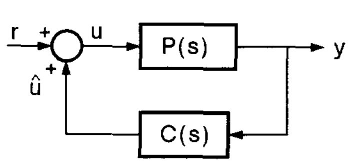

FIG. 1. Plant–controller closed-loop configuration.

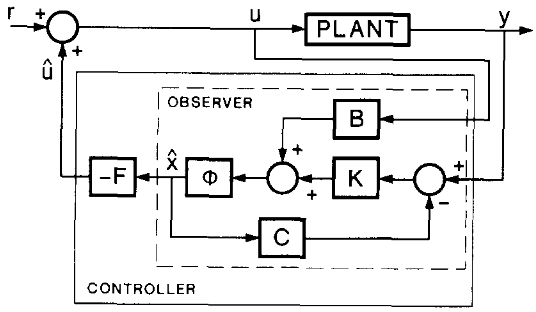

FIG. 2. Plant with full orders observer based controller.

The interest next is to design an observer which implements (1.2) using only the output y, and at the same time recovers the target loop transfer function $L(s)$. Let us first consider a controller based on a full order observer,

$$u = \hat{u} = -F\hat{x} \tag{1.4}$$

where

$$\dot{\hat{x}} = (A - KC)\hat{x} + Bu + Ky. \tag{1.5}$$

Here K is the observer gain matrix. The block diagram of Fig. 2 illustrates the controller implementation. It is important to note that the framework of observer theory requires that the control input to the plant be also an input to the observer so that the stability of the observer error dynamics is guaranteed. The transfer function of the observer based controller, i.e. the transfer function from the output y of the plant to $-\hat{u}$ is

$$\mathbf{C}_0(s) = F[\Phi^{-1} + KC + BF]^{-1}K, \tag{1.6}$$

and the open-loop transfer function when the loop is broken at the input point of the plant is

$$L_0(s) = \mathbf{C}_0(s)P(s) \tag{1.7}$$

where $P(s) \equiv C\Phi B$. Thus the error or mismatch between the target loop transfer function $L(s)$ and that realized by the observer implementation is

$$E_0(s) = L(s) - L_0(s). \tag{1.8}$$

For square invertible systems, Goodman (1984) showed that $E_0(j\omega) = 0$ iff

$$M(j\omega) \equiv 0 \tag{1.9}$$

for all ω where

$$M(s) = F(\Phi^{-1} + KC)^{-1}B. \tag{1.10}$$

Goodman's result can easily be generalized to cover left invertible systems as well. Thus exact loop transfer recovery at the input point (ELTRI) is possible iff (1.9) is satisfied while approximate LTR at the input point (ALTRI) is possible if the size of $M(j\omega)$ (in some norm sense) can be made arbitrarily small for all ω.

Equation (1.9) can be given a physical interpretation. Considering the observer based controller as a device with its output as $\hat{u}$ and inputs as u and y, it is easy to see that

$$\hat{x}(s) = (\Phi^{-1} + KC)^{-1}Bu(s) + (\Phi^{-1} + KC)^{-1}Ky(s),$$

and

$$\hat{u}(s) = -F\hat{x}(s) = -M(s)u(s) - F(\Phi^{-1} + KC)^{-1}Ky(s). \quad (1.11)$$

In view of the condition (1.9), equation (1.11) implies that the output $\hat{u}$ of the observer based controller does not entail the feedback from the control signal u. However, the transfer function from u to $\hat{x}$ is in general non-zero even when $M(s)$ is zero.

In practice, the condition $M(j\omega) = 0$ cannot always be satisfied exactly. The only recourse is then to make the size of $M(j\omega)$ in some sense small for all ω. Let the gain K be parameterized in terms of a scalar or a vector parameter σ and be denoted by $K(\sigma)$. Thus for ALTRI, one needs to obtain a $K(\sigma)$ such that,

$$M(s) = F(\Phi^{-1} + K(\sigma)C)^{-1}B \to 0 \quad \text{pointwise in } s \text{ as } \sigma \to \infty. \quad (1.12)$$

The condition (1.12) involves the state feedback gain F. However, in order to have the state feedback and observer designs to be independent of one another, one needs to require that

$$(\Phi^{-1} + K(\sigma)C)^{-1}B \to 0 \text{ pointwise in } s \text{ as } \sigma \to \infty, \quad (1.13)$$

which is a sufficient condition for (1.12). Essentially there exists three methods of obtaining such a $K(\sigma)$. In their seminal work Doyle and Stein (1979) and others later on (Madiwale and Williams, 1985; Sogaard-Andersen, 1987b; Matson and Maybeck, 1987) explored Kalman filter formalism (or asymptotic LQG theory) in which additional ficticious process noise of intensity proportional to σ is injected into the system through the input into the plant and then the gain $K(\sigma)$ is calculated by solving the resulting filter Riccati equations. Sogaard-Andersen (1987a) proposed observer eigenstructure assignment techniques. More recently, Saberi and Sannuti (1988) proposed an asymptotic pole placement method which generalizes the earlier methods while simplifying the computational task in obtaining $K(\sigma)$. In all these ALTRI design methods, $\|K(\sigma)\| \to \infty$ as $\sigma \to \infty$ so that (1.13) can be satisfied asymptotically. Thus all these are high-gain schemes. This implies that any ALTRI design scheme should include a trade-off between the required

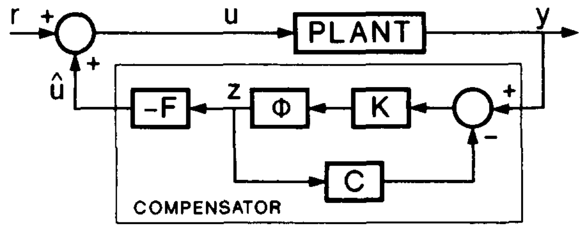

FIG. 3. Plant with full order compensator.

robustness properties of the closed-loop system and the size of the feedback gain. The size of a feedback gain is very critical in many circumstances due to unavoidable controller bandwidth constraints. Thus a cursory exploration of the literature indicates a definite need to develop dynamic compensators which would preserve closed-loop stability and at the same time achieve ALTRI without requiring large amounts of gain. In this paper, we propose such a scheme of developing dynamic compensators of order either n (full order) or $n - m$ (reduced order).

Our central observation is this. When one is restricted to the framework of observer theory, the link from the control signal u to the observer via the control distribution matrix B is always present in the design configuration such as the one depicted in Fig. 2. In these observers, when $K(\sigma)$ is appropriately designed to achieve ALTRI, the effect of the above control-link on the output of observer based controller (namely $\hat{u}$) vanishes asymptotically as $\sigma \to \infty$. However, the effect of above link on $\hat{x}$ in general is nonzero and hence the need for the above link in the conventional observers. Based on this discussion, we are inspired to remove the above mentioned link structurally right from the beginning of the design. In other words, to develop an appropriate compensator of order n, we consider the configuration illustrated in Fig. 3. Once the link from the control input to the controller or what is now called a compensator, is removed we embark on a new design philosophy which is outside the realm of observer theory and hence the separation principle is no longer valid. Without the backing or blessing of the separation principle, one has to prove that the design objectives of closed-loop stability and recovering the target loop shape can both be simultaneously achieved. We intend to do exactly this.

Our design philosophy is deceptively very simple. Except for structurally omitting the link mentioned earlier, our compensator is exactly the same as the conventional observer-based controller. We plan to obtain a $K(\sigma)$ such that (a) $A - K(\sigma)C$ has all its eigenvalues in the left half s plane (i.e. the compensator is open-loop stable) and that (b) the condition (1.12) is satisfied asymptotically as $\sigma \to \infty$. For this purpose, we can use any of the existing methods of obtaining such a $K(\sigma)$. Thus our compensator design parallels in all respects the conventional observer design except for omitting the link mentioned earlier. Although our compensator structurally differs from the observer in a very simple way, it has a profound effect on the gain required for closed-loop stability and for ALTRI. We show theoretically that for the same gain, the difference between the target loop transfer function and the one achieved by our compensator is always much smaller than that that can be achieved by the observer based controller. But since our design method is also an asymptotic method, the above theoretical result does not reveal the whole story. The proof that our method works is evident from our examples. We have solved numerically many examples that appeared in the open literature, and noticed that the amount of gain required for the same degree of recovery by our compensator is orders of magnitude less than what is required by an observer-based controller. This obviously has a profound impact on the practical implementation of LQG/LTR schemes. Some specific attributes of our compensator are as follows:

1. Low values of gain obviously results in low compensator band-width, and hence much of the output noise that occurs at relatively high frequencies is filtered out. Furthermore, low values of gain relieves the design from ever present woes of saturation. To emphasize this, we refer to Sogaard-Andersen and Niemann (1989) who recently studied the design trade-offs between the level of loop transfer recovery and the necessary gain required by an observer-based controller. A major conclusion of their study is that the target loop transfer recovery design cannot always be achieved even when modest and practically meaningful constraints are imposed on the size of the observer gain. Furthermore, contrary to what has been discussed in the literature (e.g. Friedland, 1986; Baumgartner *et al.*, 1986), their study indicates that a high-gain from controller input to controller output affects the entire control-loop and in particular the control–noise signal ratio and the control–command signal ratio.

2. Since the given plant is of minimum phase, it is always possible to design an open-loop stable compensator to guarantee the over all closed-loop stability (Vidyasagar, 1985b). Our design results in an open-loop stable compensator. The advantages of having such a compensator cannot be over-emphasized. As is known (Shaw, 1971), open-loop unstable compensators result in poor overall system sensitivity to plant parameter variations. Furthermore, physical realizability of open-loop unstable compensators is rather difficult.

Our discussion so far has been concerned with a full order observer-based controller and that too only with the case when the target open-loop transfer matrix is specified at the plant input point. Similar discussion pertains as well to other cases: (a) when a reduced order observer-based controller is considered and (b) when a target open-loop transfer matrix is specified at the plant output point. Even when a reduced order observer-based controller is used, we observe that exact or approximate LTR is possible if and only if the transfer function from the point where the input u of the plant is fed to the controller to the output point $\hat{u}$ of the controller is either exactly or approximately zero. Thus again, our compensator structurally (i.e. physically) omits the link from the input point of the plant to the controller right from the beginning.

The paper is organized as follows. In Section 2, we consider conventional observers and while reviewing the existing theory, some clarifications and generalizations of it are presented. This section motivates the work that follows. Sections 3 and 4 respectively develop the full and reduced order compensators when the target open-loop transfer matrices are specified at the plant input point, while Section 5 dualizes the results of Sections 3 and 4 for the case when the target open-loop transfer matrices are specified at the plant output point. Section 6 deals with numerical results on some representative examples from the literature. Throughout this paper, A' denotes the transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. $\lambda(A)$ and $\mathrm{Re}\,[\lambda(A)]$ respectively denote the set of eigenvalues and real parts of eigenvalues of A. Similarly, $\sigma_{\max}[A]$ and $\sigma_{\min}[A]$ respectively denote the maximum and minimum singular values of A. The open left half plane is denoted by $\mathscr{C}^-$.

2. REVIEW OF LTR VIA OBSERVERS

The purpose of this section is to re-examine how LTR occurs when either full or reduced order observer-based controllers are used. This is done in order to obtain a better intuition and understanding of the theory of observer-based controllers for LTR so that our new compensator design, discussed in Sections 3 and 4, can easily be motivated and inspired. In this process of review, the conditions for achieving either exact or approximate LTR (ELTR or ALTR), when either full or reduced order observer based controllers are used, are brought to the same frame work, i.e. the observer-based controller theory is unified into a single frame work. Such a review and unification leads to several new results. For example, no method of determining gain for full order observer-based controllers exists in the literature for the case of achieving ELTR. Here an explicit method of determining such a gain is given. Also, both the cases when the loop is broken at either the input or the output point of the plant are considered. In short, this section summarizes, generalizes, unifies and in some cases clarifies the existing results on LTR using conventional observer-based controllers.

2.1. *Full order observers—ELTRI and ALTRI*

We will first consider the full order observer-based controller when the target open-loop transfer matrix is evaluated when the loop is broken at the input point of the plant. $E_0(s)$, the error between the target loop transfer function $L(s)$ and that achievable by the observer based controller of Fig. 2 is given by (1.8). In the observer design, K is the only free design parameter. First of all in order to guarantee the closed-loop stability, K must be such that $A - KC$ is an asymptotically stable matrix, i.e.

$$\mathrm{Re}\,[\lambda(A - KC)] < 0. \tag{2.1}$$

The remaining freedom in choosing K can then be used to achieve LTR. In an attempt to find such a K, Doyle and Stein (1979) first gave a sufficient condition,

$$K(I + C\Phi K)^{-1}C\Phi B = B, \tag{2.2}$$

under which $E_0(j\omega) \equiv 0$ for all ω. To understand the implications of (2.2), following Friedland (1986), we rewrite it in an equivalent way. In view of the identity,

$$\Phi K(I_p + C\Phi K)^{-1}C\Phi = \Phi - (\Phi^{-1} + KC)^{-1},$$

we have

$$\Phi K(I_p + C\Phi K)^{-1}C\Phi B = \Phi B - (\Phi^{-1} + KC)^{-1}B.$$

Thus, (2.2) implies that

$$(\Phi^{-1} + KC)^{-1}B = 0. \tag{2.3}$$

However, due to the nonsingularity of $(\Phi^{-1} + KC)^{-1}$, (2.3) implies that $B \equiv 0$. But this is impossible in any real system. Thus the sufficient condition (2.2) cannot exactly be satisfied at all. However, (2.3) can asymptotically be satisfied for large gain without requiring $B = 0$.

The above discussion reveals that the condition (2.2) is poorly suited to study the loop transfer recovery problem. Realizing this, Goodman (1984) proceeded to look at directly the error or mismatch function $E_0(s)$. Goodman studied square invertible systems. The following two lemmas represent minor extensions of Goodman's results and cover as well left invertible systems.

Lemma 1. $E_o(s)$, the error between the target loop transfer function $L(s)$ and that realized by the full order observer-based controller of Fig. 2 is given by

$$E_0(s) = M(s)(I_m + M(s))^{-1}(I_m + F\Phi B) \tag{2.4}$$

where

$$M(s) = F(\Phi^{-1} + KC)^{-1}B. \tag{2.5}$$

Lemma 2.

$$E_0(j\omega) = 0 \quad \text{iff} \quad M(j\omega) = 0 \quad \text{for all} \quad \omega \in \Omega \tag{2.6}$$

where Ω is the set of all $0 \le \omega < \infty$ for which $L_0(j\omega)$ and $L(j\omega)$ are well defined (i.e. all required inverses exist).

Thus equation (2.4) presents a clear perspective to study the basic mechanism by which both exact and approximate LTR occurs. It is clear that ELTRI is achievable if $M(j\omega) = 0$ exactly and on the other hand ALTRI is achievable if $\sigma_{\max}[M(j\omega)]$ can be made arbitrarily small for all ω. In order to investigate when $\sigma_{\max}[M(j\omega)]$ can be made either zero or arbitrarily small, assuming $A - KC$ is nondefective, Goodman [see also, Sogaard-Andersen, (1987c)] expands $M(s)$ in a dyadic form,

$$M(s) = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i} \tag{2.7}$$

where

$$R_i = FW_iV_i^H B.$$

Here superscript H indicates the complex conjugate transpose. Also, W_i and V_i are respectively the right and left eigenvectors associated with an eigenvalue λ_i of $A - KC$ and they are scaled so that $WV^H = V^H W = I_n$ where

$$W = [W_1, W_2, \ldots, W_n]$$

and

$$V = [V_1, V_2, \ldots, V_n].$$

In view of Lemma 2, ELTRI is possible iff $M(j\omega) = 0$ for all ω. This is the case iff for each $i = 1$ to n, either $FW_i = 0$ or $V_i^H B = 0$ or both. Since F is designed to satisfy the required loop transfer function, $FW_i = 0$ is generically not satisfied. One can try to satisfy $V_i^H B = 0$ for as many indexes i as possible. However, it is not possible to design so that $V_i^H B = 0$ for all indexes, $i = 1$ to n. Let us investigate how many left eigenvectors of $A - KC$ can satisfy $V_i^H B = 0$. Let n_a and n_f be respectively the number of invariant zeros and infinite zeros (Sannuti and Saberi, 1987) of the given plant. In general $n_a \leq n - n_f$. We have the following result.

Lemma 3. For a left invertible plant, there exists a gain matrix K such that at most a total of $n - n_f$ left eigenvectors V_i, $i = 1$ to $n - n_f$, of $A - KC$ can satisfy the condition $V_i^H B = 0$. Also, n_a of these $n - n_f$ eigenvalues are same as the invariant zeros of the plant while $n_b \equiv n - n_f - n_a$ eigenvalues can be assigned freely. Furthermore, those left eigenvectors of $A - KC$ which are associated with the invariant zeros are same as the left zero directions of the plant.

Proof. See Shaked and Karcanias (1976); Saberi and Sannuti (1988).

Lemma 3 implies that the maximum number of indexes i that satisfy $V_i^H B = 0$ is equal to the difference between the dynamic order and the number of infinite zeros of the given plant. The minimum number of infinite zeros is m and this happens for left invertible systems when all the infinite zeros are of order one, i.e. when CB is of maximum rank (Sannuti and Saberi, 1987). Then in view of Lemmas 2 and 3, ELTRI in general is impossible. It is possible only under some special circumstances. Goodman gives the following result.

Lemma 4. Let $A - KC$ be a nondefective matrix with left eigenvectors V_i, $i = 1$ to n, such that $V_i^H B = 0$ for i equal to any $n - m$ distinct indexes among 1 to n. Then $E_0(j\omega) \equiv 0$ for all $\omega \in \Omega$ is equivalent to $FB \equiv 0$.

Lemmas 3 and 4 culminate in the following theorem.

Theorem 1. Consider the closed-loop system comprising of the plant and the full order observer-based controller as in Fig. 2. Then both asymptotic stability of the closed-loop system and ELTRI can be achieved under the following conditions:

1. $FB = 0$.
2. The given plant has all its infinite zeros of order one (i.e. CB is of maximal rank).
3. The given plant is left invertible and has all its invariant zeros in the left half s plane (i.e. of minimum phase).

Moreover, a constructive method of obtaining a gain K to achieve both closed-loop stability and ELTRI can be given under the above three conditions. Such a gain K in general is nonunique and belongs to a class of gains denoted by $\mathcal{K}_e$.

Proof. Under the conditions given in the theorem, a method of calculating the class of gains $\mathcal{K}_e$ is given in Appendix A.

Remark 1. There is no method whatsoever in the literature to obtain the observer gain K that achieves ELTRI. Although this paper is not intended in general to give methods of obtaining K, the constructive proof of Theorem 1 yields one such method.

Remark 2. It is important to realize the implications of the condition $FB = 0$. Apparently, it restricts the class of loop transfer functions $L(s)$ that are attainable by full state feedback. In particular, under the condition $FB = 0$,

$$L(s) = F\Phi B \sim \frac{FAB}{s^2},$$

implying that $\|L(j\omega)\|$ must have at least a roll-off of 40 dB per decade with respect to ω. It is well known that whenever the state feedback gain F is calculated by LQR theory, $\|L(j\omega)\|$ has only a roll-off of 20 dB per decade with respect to ω. Thus the use of LQR theory is then ruled out to generate the target loop transfer function.

Since $FB = 0$ severely restricts the class of loop transfer functions that are achievable, most of the existing literature focuses attention on ALTRI methods. In these ALTRI methods, one tries to find a gain K such that (1.13) is satisfied. As we discussed earlier, the gain K in this case is parameterized in terms of a tuning parameter σ. Satisfying (1.13) is a sufficient condition to render $\sigma_{\max}[M(j\omega)]$ arbitrarily small for all ω. At first, Doyle and Stein (1979) gave a sufficient condition under which (1.13) is true. Their condition is as follows: Let $K(\sigma)$ be chosen such that as $\sigma \to \infty$, $K(\sigma)/\sigma \to BW$ for some nonsingular matrix W. Then, (1.13) is true and consequently ALTRI is achieved as $\sigma \to \infty$. There were several attempts later on to weaken the Doyle–Stein condition (Madiwale and Williams, 1985; Matson and Maybeck, 1987; Saberi and Sannuti, 1988). It is well known that in order to satisfy the Doyle–Stein condition, one requires only that the plant be left invertible and be of minimum phase. Thus in comparison with the sufficient conditions for ELTRI as stated in Theorem 1, one finds a drastic relaxation of the required conditions for ALTRI.

As far as the design of $K(\sigma)$ is concerned, presently there exists three different methods: (1) asymptotic LQG methods, (2) asymptotic pole placement methods and (3) eigenstructure assignment methods. An exhaustive comparison of all these three methods is given in Saberi and Sannuti (1988). All these procedures aim at obtaining a gain K such that (a) some of the observer eigenvalues either coincide or are close to the zeros of the given plant and that the associated left eigenvectors satisfy the condition $V_i^H B = 0$ either exactly or approximately, and (b) the remaining observer eigenvalues are placed far in the left half s plane so that the corresponding $R_i/(s-\lambda_i)$ in the dyadic expansion (2.7) are approximately zero. In other words, all these methods find a gain K such that $\sigma_{\max}[M(j\omega)]$ is arbitrarily small for all ω. Thus the term $M(s)$ plays a dominant role in LTR. The following result summarizes this discussion.

Theorem 2. Consider the closed-loop system comprising of the plant and the full order observer based controller as in Fig. 2. Let the given plant be left invertible and be of minimum phase. Then a gain $K(\sigma)$ can be designed such that both asymptotic stability of the closed-loop system and ALTRI can be achieved. Such a gain $K(\sigma)$ in general is nonunique and belongs to a class of gains denoted by $\mathcal{K}_a(\sigma)$.

Proof. See Saberi and Sannuti (1988).

Let us next examine the eigenvalues of the observer based controller. These eigenvalues are given by

$$\lambda(A-KC-BF).$$

These eigenvalues are not necessarily in the left half s plane for all K. To study the nature of these eigenvalues, consider the following:

$$\begin{aligned}\det[sI_n - A + KC + BF] &= \det[sI_n - A + KC] \\ &\quad \times \det[I_n + (\Phi^{-1} + KC)^{-1}BF] \\ &= \det[sI_n - A + KC] \\ &\quad \times \det[I_m + F(\Phi^{-1} + KC)^{-1}B] \\ &= \det[sI_n - A + KC] \\ &\quad \times \det[I_m + M(s)]. \qquad (2.8)\end{aligned}$$

Thus whenever ELTRI is achieved, i.e. whenever $M(s)=0$, the controller eigenvalues are given by $\lambda(A-KC)$. Hence the observer-based controller is asymptotically stable. On the other hand, in the case of ALTRI, the eigenvalues of the full order observer-based controller obviously approach $\lambda(A-KC)$ as $\sigma \to \infty$. However, in practice the value of σ needed for the desired degree of recovery might not yield an asymptotically stable controller. In fact, this is the case in most practical problems.

As discussed above, most often one opts for ALTRI design as it requires less stringent conditions than ELTRI design. In ALTRI, the level of recovery depends on $\sigma_{\max}[M(j\omega)]$. However in order to render $\sigma_{\max}[M(j\omega)]$ small, one needs to increase the tuning parameter σ which itself increases the gain $K(\sigma)$. Thus as discussed thoroughly by Sogaard-Andersen and Niemann (1989), there is a fundamental trade-off between the level of recovery and the size of gain. This trade-off can be visualized in a natural way in terms of the trade-off between the singular values of sensitivity and complementary sensitivity functions and the singular values of $M(j\omega)$. The reason for this is that the robust stability and nominal performance of a system are directly reflected in the singular values of sensitivity and complementary sensitivity functions; whereas the level of recovery (i.e. the size of $E_0(j\omega)$) is directly dependent on the singular values of $M(j\omega)$. With this point in view, Sogaard-Andersen and Niemann (1989) derive some analytical expressions for the discrepancy between the desired and the achieved sensitivity and complementary sensitivity functions. Let $S_0(s)$ and $T_0(s)$ be the achieved sensitivity and complementary sensitivity functions in the configuration of Fig. 2 when the loop is broken at the input point of the plant

$$S_0(s) = [I_m + \mathbf{C}_0(s)P(s)]^{-1}$$

and

$$T_0(s) = I_m - S_0(s) = [I_m + \mathbf{C}_0(s)P(s)]^{-1}\mathbf{C}_0(s)P(s)$$

where $\mathbf{C}_0(s)$ is as given in (1.6). Let $S_F(s)$ and $T_F(s)$ be the sensitivity and complementary sensitivity functions corresponding to the target loop-shape. The following lemma is a slight generalization of the results of Sogaard-Andersen and Niemann (1989).

Lemma 5. Consider the configuration of Fig. 2. We have the following bounds on all singular values $i = 1$ to m of $S_0(j\omega)$ and $T_0(j\omega)$:

$$\frac{|\sigma_i[S_0(j\omega)] - \sigma_i[S_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \le \sigma_{\max}[M(j\omega)],$$

and

$$\frac{|\sigma_i[T_0(j\omega)] - \sigma_i[T_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \le \sigma_{\max}[M(j\omega)].$$

The expressions given above can be used to analyze the inevitable trade-off between good

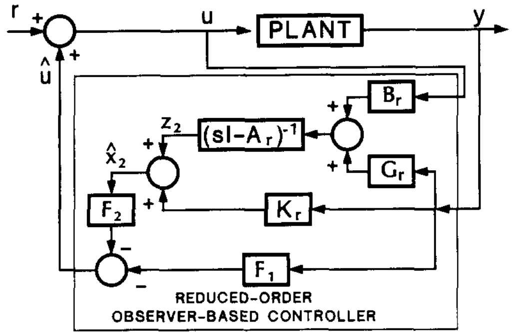

FIG. 4. Plant with reduced order observer based controller.

recovery as indicated by $\sigma_{max}[M(j\omega)]$ and robustness and performance as reflected in the sensitivity and complementary sensitivity functions. To do this, Sogaard-Andersen and Niemann (1989) developed some recovery diagrams.

Before closing this section, let us note that $M(s)$ plays a central role in every single result given in this section.

2.2. *Reduced order observers—ELTRI and ALTRI*

Now let us consider a reduced order observer based controller as in Fig. 4. Without loss of generality, let us assume that

$$C = [I_p, 0]$$

and hence the plant (1.1) is in the form,

$$\begin{aligned}\dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1u \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2u,\end{aligned} \qquad (2.9)$$

$$y = x_1. \qquad (2.10)$$

Also, let the state feedback gain matrix F which achieves the target loop transfer function $L(s)$ be partitioned in conformity with (2.9) as

$$F = [F_1, F_2]. \qquad (2.11)$$

Let $\Phi_{11} = (sI_p - A_{11})^{-1}$ and $\Phi_{22} = (sI_{n-p} - A_{22})^{-1}$. It is then straightforward to derive the following relationships:

$$P(s) = C\Phi B = \Phi_{11}B_1 + \Phi_{11}A_{12}H_2(s), \qquad (2.12)$$

$$L(s) = F\Phi B = F_1P(s) + F_2H_2(s), \qquad (2.13)$$

where $H_2(s)$ is the transfer function between u and x_2, i.e. $x_2(s) = H_2(s)u(s)$, where

$$H_2(s) = (\Phi_{22}^{-1} - A_{21}\Phi_{11}A_{12})^{-1}(A_{21}\Phi_{11}B_1 + B_2). \qquad (2.14)$$

The reduced order observer equations (O'Reilly, 1983; Madiwale and Williams, 1985) are given by

$$\dot{z}_2 = A_rz_2 + G_ry + B_ru, \qquad (2.15)$$

with

$$x_1 = y \quad \text{and} \quad \hat{x}_2 = K_ry + z_2, \qquad (2.16)$$

and with the reduced order observer based feedback control law as

$$u = \hat{u} = -F_1x_1 - F_2\hat{x}_2. \qquad (2.17)$$

Here K_r is the reduced order observer gain and the matrices A_r, B_r and G_r are given by

$$\begin{aligned}A_r &= A_{22} - K_rA_{12},\ B_r = B_2 - K_rB_1, \\ G_r &= A_{22}K_r - K_rA_{12}K_r + A_{21} - K_rA_{11}.\end{aligned} \qquad (2.18)$$

Now in order to bring the theory of full and reduced order observers to the same frame work and to understand the conditions for either ELTRI or ALTRI clearly, we present the following results which are analogous to Lemmas 1 and 2.

Lemma 6. $E_{or}(s)$, the error between the target loop transfer function $L(s)$ and that realized by the reduced order observer-based controller of Fig. 4 is given by

$$E_{or}(s) = M_r(s)(I_m + M_r(s))^{-1}(I_m + F\Phi B), \qquad (2.19)$$

where

$$M_r(s) = F_2(\Phi_{22}^{-1} + K_rA_{12})^{-1}B_r. \qquad (2.20)$$

Proof. See Appendix B.

Remark 3. The expression for $E_{or}(s)$ is identical to the corresponding one when full order observer-based controller is used; see (2.4), except that now $M_r(s)$ takes the place of $M(s)$.

Lemma 7.

$$E_{or}(j\omega) = 0 \quad \text{iff} \quad M_r(j\omega) = 0 \quad \text{for all} \quad \omega \in \Omega_r, \qquad (2.21)$$

where Ω_r is the set of all $0 \le \omega < \infty$ for which $L_{or}(j\omega)$ and $L(j\omega)$ are well defined (i.e. all required inverses exist).

Proof. The proof is obvious in view of Lemma 2.

As in the case of a full order observer, a physical interpretation can be given to the term $M_r(s)$. It is straightforward to show that

$$-\hat{u}(s) = M_r(s)u(s) + [F_1 + F_2(\Phi_{22}^{-1} + K_rA_{12})^{-1}G_r + K_r(s)]y(s).$$

Thus we note that $M_r(s)$ is the transfer function from u to $-\hat{u}$. Hence as in the case of a full order observer, whenever the size of $M_r(s)$ is small, the effect of the link from the input point of the plant to the observer, on $\hat{u}$ is small. In view of Lemma 7, the question now is when and how $M_r(j\omega)$ can be made either exactly or approximately zero for all ω. We note that,

$$B_r \equiv B_2 - K_r B_1 = 0, \quad (2.22)$$

is a sufficient condition for $[M_r(j\omega)]$ to be identically zero. Unlike in full order observers, the condition (2.22) involves K_r and hence there is a possibility of solving for K_r from it. Sogaard-Andersen (1987b) under the conditions that (a) the given system is square and invertible and (b) B_1 is nonsingular, solves for K_r,

$$K_r = B_2 B_1^{-1}. \quad (2.23)$$

It turns out that a gain K_r which satisfies (2.22) and thus achieves ELTR can be obtained under much relaxed conditions. We have the following result analogous to Theorem 1.

Theorem 3. Consider the closed-loop system comprising of the plant and the reduced order observer based controller as in Fig. 4. Then both the asymptotic stability of the closed-loop system and ELTRI can be achieved under the following conditions:

1. The given plant has all its infinite zeros of order one.
2. The given plant is left invertible and is of minimum phase.

Moreover, a constructive method of obtaining a gain K_r to achieve both ELTRI and asymptotic stability of the closed-loop system can be given under the above two conditions. Such a gain K_r in general is nonunique and belongs to a class of gains denoted by $\mathcal{K}_{er}$.

Proof. Under the conditions given in the theorem, a method of calculating the class of gains $\mathcal{K}_{er}$ is given in Appendix C.

Remark 4. When reduced order observer based controllers are used, the condition $FB = 0$ is not necessary. However, while full order observer always results in a strictly proper controller transfer function, the reduced order observer based controller has a nonstrictly proper transfer function. As discussed by Khalil (1981, 1984) and by Vidyasagar (1985a), a closed-loop system with a nonstrictly proper controller is not robust under unmodelled high frequency dynamics.

Remark 5. Madiwale and Williams (1985) gave a sufficient condition for ELTRI,

$$K_r(I_p + A_{12}\Phi_{22}K_r)^{-1}A_{12}\Phi_{22}B_r = B_r. \quad (2.24)$$

There is a one to one correspondence between (2.2) and (2.24). In the same way as we showed (2.3) is equivalent to $B \equiv 0$, we can show that (2.24) is equivalent to $B_r \equiv 0$.

Since in general $M_r(s)$ cannot exactly be made zero, one focuses attention on ALTRI. That is, one needs

$$M_r(s) = F_2(\Phi_{22}^{-1} + K_r(\sigma)A_{12})^{-1}B_r \to 0 \quad \text{pointwise in } s \text{ as } \sigma \to \infty,$$

where the gain $K_r(\sigma)$ is now parameterized in terms of a tuning parameter σ. However, as in the previous section, in order to have the state feedback and observer designs to be independent of one another, one needs to require that

$$(\Phi_{22}^{-1} + K_r(\sigma)A_{12})^{-1}B_r \to 0 \text{ pointwise in } s \text{ as } \sigma \to \infty. \quad (2.25)$$

To design such a $K_r(\sigma)$, Dowdle *et al.* (1982) study a restrictive class of systems where all the first Markov parameters of the given plant are zero. Such a severe restriction on the given plant is not imposed in Madiwale and Williams (1985), instead they require that certain matrices are of full rank and a certain subsystem of the given system is of minimum phase. Based on asymptotic LQG methods, Sogaard-Andersen (1985b) studies the general case without any restrictions. Since Sogaard-Andersen divides it into three different cases, his analysis and design besides being not unified becomes unnecessarily involved. Saberi and Sannuti (1988) give an explicit method of calculating the gain $K_r(\sigma)$ which satisfies the condition (2.25). In fact, they convert the problem of designing the reduced order observer for the given plant into that of a full order observer, however, for a reduced order subsystem of the given plant.

The above discussion can be summarized as Theorem 4 which is analogous to Theorem 2.

Theorem 4. Consider the closed-loop system comprising of the plant and the reduced order observer based controller as in Fig. 4. Let the given plant be left invertible and be of minimum phase. Then a gain $K_r(\sigma)$ can be designed such that both asymptotic stability of the closed-loop system and ALTRI can be achieved. Such a gain $K_r(\sigma)$ in general is nonunique and belongs to a class of gains denoted by $\mathcal{K}_{ar}(\sigma)$.

Proof. See Saberi and Sannuti (1988).

Let us next examine the eigenvalues of the reduced order observer based controller. These eigenvalues are given by

$$\lambda(A_{22} - K_r A_{12} - B_2 F_2).$$

These eigenvalues are not necessarily in the left half s plane for all K_r. As in (2.8), we can show that

$$\det[sI_{n-p} - A_{22} + K_rA_{12} + B_2F_2] = \det[sI_{n-p} - A_{22} + K_rA_{12}]\det[I_m + M_r(s)]. \tag{2.26}$$

Thus whenever ELTRI is achieved, i.e. whenever $M_r(s) = 0$, the controller eigenvalues are given by $\lambda(A_r)$ and hence the reduced order observer based controller is asymptotically stable. On the other hand in the case of ALTRI, the eigenvalues of the reduced order observer based controller tend to $\lambda(A_r)$ as $\sigma \to \infty$. However, as in full order observers, the value of σ needed for the desired degree of recovery might not yield an asymptotically stable controller. In fact, this is the case in most practical problems.

Now as in Lemma 5, we would like to develop bounds on the sensitivity and complementary sensitivity functions generated by the use of reduced order observer based controllers. Let $S_{or}(s)$ and $T_{or}(s)$ be the generated sensitivity and complementary sensitivity functions in the configuration of Fig. 4 when the loop is broken at the input point of the plant,

$$S_{or}(s) = [I_m + \mathbf{C}_{or}(s)P(s)]^{-1}$$

and

$$T_{or}(s) = I_m - S_{or}(s) = [I_m + \mathbf{C}_{or}(s)P(s)]^{-1}\mathbf{C}_{or}(s)P(s)$$

where $\mathbf{C}_{or}(s)$ is the transfer function of the reduced order observer based controller. We have the following result analogous to Lemma 5.

Lemma 8. Consider the configuration of Fig. 4. We have the following bounds on all singular values $i = 1$ to m of $S_{or}(j\omega)$ and $T_{or}(j\omega)$:

$$\frac{|\sigma_i[S_{or}(j\omega)] - \sigma_i[S_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \le \sigma_{\max}[M_r(j\omega)],$$

and

$$\frac{|\sigma_i[T_{or}(j\omega)] - \sigma_i[T_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \le \sigma_{\max}[M_r(j\omega)].$$

Proof. See Appendix D.

2.3. *Full and reduced order observers—ELTRO and ALTRO*

The target open-loop transfer functions can be designed when the loop is broken at either the input or the output point of the given plant depending upon the given specifications. We have discussed so far LTR recovery at the input point, either exact or approximate type (ELTRI or ALTRI), using either full or reduced order observer-based controllers. Now we would like to consider LTR recovery when the loop is broken at the output point (Kwakernaak, 1969). This method is used when the designer specifications and the modelling of uncertainties are reflected at the output point of the plant. In the literature, it is commonly said that LTR recovery at the input and output points (LTRI and LTRO) are dual to one another. This duality is well understood in the case when full order observers or Kalman filters are used in the controllers. That is, in the case of LTRO, the first step is to design a Kalman filter, via loop shaping techniques, whose loop transfer function meets the design specifications. The next step is to recover this Kalman filter loop transfer function via LTR technique. However, this kind of duality is not well understood when reduced order observer based controllers are used. For instance, Sogaard-Andersen (1987b) who has contributed much to the development of reduced or minimal order observers for LTRI makes the following comment: "The loop-shape formulation used here requires that the uncertainties and performance specifications are reflected to the plant input. Unfortunately similar results for the plant output cannot be derived since the minimal-order observer and the plant model are not dual." The confusion arises here because the duality is sought between the plant and the observer. The proper way is to seek the duality in the design methodology and when this is done, contrary to the statement of Sogaard-Andersen, minimal order observer based controllers can be designed for LTRO as well. In other words, one needs first to clearly define the duality in a mathematical way and then needs to interpret the implications of it as to the controller implementation. In order to avoid any confusion, we give below a formal step by step algorithm to show how duality arises for LTR recovery at the input and output points.

1. Let the given plant model Σ be characterized by the triple (A, B, C) where A, B and C are respectively $n \times n$, $n \times m$ and $p \times n$ matrices. Let Σ be of minimum phase and be right invertible implying $p \le m$. Also, let $P(s)$ be the transfer function of the plant Σ,

$$P(s) = C(sI_n - A)^{-1}B.$$

Let $L(s)$ be the required target open-loop transfer function when the loop is broken at the output point of the given plant. Thus, in the configuration of Fig. 1, we are seeking a controller $\mathbf{C}(s)$ such that $L(\mathrm{j}\omega)$ is either exactly or approximately equal to $P(\mathrm{j}\omega)\mathbf{C}(\mathrm{j}\omega)$.

2. Define a transposed system model Σ_t characterized by the triple (A_t, B_t, C_t) where

$$A_t \equiv A', \qquad B_t \equiv C', \qquad C_t \equiv B'.$$

Note that since Σ is of minimum phase and right invertible, Σ_t is of minimum phase and left invertible. Also, note that $P_t(s)$, the transfer function of the plant Σ_t is $P'(s)$. Let $L_t(s)$ be defined as

$$L_t(s) \equiv L'(s).$$

3. For the purpose of design alone, consider the fictitious plant Σ_t as given in step 2. Then design a controller $\mathbf{C}_t(s)$ such that $\mathbf{C}_t(\mathrm{j}\omega)P_t(\mathrm{j}\omega)$ is either exactly or approximately equal to $L_t(\mathrm{j}\omega)$. For this purpose one can use either a full or reduced order observer based controller design of Sections 2.1 or 2.2. In fact one can also use any other compensator design schemes such as those to be described in Sections 3 and 4. We note that the dynamic order of $\mathbf{C}_t(s)$ is either n or $n-m$ depending upon either full or reduced order observer is used for the controller design.

4. Define a controller $\mathbf{C}(s)$:

$$\mathbf{C}(s) \equiv \mathbf{C}_t'(s).$$

We note that the dynamic order of $\mathbf{C}(s)$ is same as that of $\mathbf{C}_t(s)$.

Then it can be shown trivially that the controller $\mathbf{C}(s)$ designed above and implemented as in Fig. 1 achieves either ELTRO or ALTRO depending upon whether $\mathbf{C}_t(s)$ in step 3 is designed to achieve ELTRI or ALTRI for the fictitious plant Σ_t.

3. FULL ORDER COMPENSATOR—ELTRI AND ALTRI

In this section and the next, we present our main contributions. To start with, let us recall the concept of LTRI. Given the plant transfer function $P(s) \equiv C\Phi B$ and the target loop transfer function $L(s) \equiv F\Phi B$, one wants to design a controller with transfer function $\mathbf{C}(s)$ such that $\mathbf{C}(\mathrm{j}\omega)P(\mathrm{j}\omega)$ is either exactly or approximately equal to $L(\mathrm{j}\omega)$. The only controller that is available so far for this purpose is observer based. As reviewed in the last section, in observer based controllers, $M(s)$ plays a central role in the recovery procedure. Numerical experience shows that in order to achieve a satisfactory degree of recovery, large values of gain in general are required by the observer based controllers. In an attempt to reduce the size of required gains, one then naturally seeks new structures for the controllers. The physical meaning of the transfer function $M(s)$ as explained in the last section, leads us to examine the observer based controller structure in which the link from the input point of the plant via the control distribution matrix B to the controller is removed. Such an omission of the link generates a new structure for the controller which we now call a compensator. Because of the omission of the link mentioned earlier, the celebrated separation principle is no longer valid and hence the properties of the compensator as to closed-loop stability and achieving LTR have to be examined carefully. This is the purpose of this section and the next.

Consider the dynamic compensator,

$$\dot{z} = (A - KC)z + Ky, \tag{3.1}$$

$$u = \hat{u} = -Fz. \tag{3.2}$$

The only unknown matrix in (3.1) is K which is considered as a free design parameter. The compensator transfer function (i.e., the transfer function from y to $-\hat{u}$) is given by

$$\mathbf{C}_c(s) = F(\Phi^{-1} + KC)^{-1}K. \tag{3.3}$$

We would like to design K to satisfy the following conditions:

1. *Stability of the closed-loop system.* The closed-loop system as depicted in Fig. 3 and characterized by (1.1), (3.1) and (3.2), is asymptotically stable, i.e.

$$\mathrm{Re}\,[\lambda(A_{cl})] < 0, \tag{3.4}$$

where

$$A_{cl} = \begin{bmatrix} A - KC & KC \\ -BF & A \end{bmatrix}. \tag{3.5}$$

2. *ELTRI or ALTRI.* The achieved loop transfer function $L_c(\mathrm{j}\omega)$,

$$L_c(\mathrm{j}\omega) = \mathbf{C}_c(\mathrm{j}\omega)P(\mathrm{j}\omega), \tag{3.6}$$

is either exactly or approximately equal to $L(\mathrm{j}\omega)$.

3. *Open-loop stability of the compensator.* The compensator is open-loop asymptotically stable, i.e.

$$\mathrm{Re}\,[\lambda(A - KC)] < 0. \tag{3.7}$$

The above three conditions are important from technical point of view. However, merely determining K to satisfy the above conditions is not enough because our primary goal as stated earlier is to come up with a scheme which requires smaller values of gain than the observer based controller to achieve the same level of LTR. In what follows, we show that our new compensator structure does exactly this. We first give the following lemma analogous to Lemma 1.

Lemma 9. $E_c(s)$, the error between the target open-loop transfer function $L(s)$ and $L_c(s)$, the one realized by the compensator, is given by

$$E_c(s) = M(s), \tag{3.8}$$

where

$$M(s) = F(\Phi^{-1} + KC)^{-1}B. \tag{3.9}$$

Proof.

$$\begin{aligned} E_c(s) &= L(s) - L_c(s) \\ &= F[I_n - (\Phi^{-1} + KC)^{-1}KC]\Phi B \\ &= F(\Phi^{-1} + KC)^{-1}B. \end{aligned}$$

Remark 6. Observe that $M(s)$ defined here is exactly the same as the one defined earlier for the full order observer-based controllers, see (2.5). In view of this, the two expressions for the error between the required and the achieved loop transfer functions, one for the conventional observer-based design (2.4) and the other for the new compensator design (3.8), differ significantly. This as we shall see later on in Theorems 7 and 8 leads to an overwhelming advantage in favor of the new compensator approach.

Since $M(s)$ defined here and in the case of full order observer based design is one and the same, we naturally see that ELTRI or ALTRI is achievable by the new compensator under exactly the same conditions as in the previous case. That is K has to be an element of either $\mathcal{K}_e$ for ELTRI or an element of $\mathcal{K}_a(\sigma)$ for ALTRI. However, the stability of the closed-loop system has to be separately examined. We have the following theorem.

Theorem 5. Consider the closed-loop system comprising of the plant along with the compensator as in Fig. 3. Then both the asymptotic stability of the closed-loop system and ELTRI can be achieved under the following conditions:

1. $FB = 0$.
2. The given plant has all its infinite zeros of order one.
3. The given plant is left invertible and is of minimum phase.

Moreover, under the above conditions, K can be selected as an element of $\mathcal{K}_e$. Also, the eigenvalues of A_{cl} are given by $\lambda(A - KC)$ and $\lambda(A - BF)$. Furthermore, the developed compensator is always open-loop asymptotically stable.

Proof. Under the conditions given and in view of the Theorem 1, it is obvious that K can be selected as an element of $\mathcal{K}_e$ and hence $M(s) \equiv 0$. It is also evident that the compensator is open-loop asymptotically stable. Next, the closed-loop stability can be proved as follows. The dynamic matrix of the closed-loop system is given by (3.5). Then consider the following reductions:

$$\begin{aligned} \det[sI_{2n} - A_{cl}] &= \det\begin{bmatrix} sI_n - A + KC & -KC \\ BF & sI_n - A \end{bmatrix} \\ &= \det\begin{bmatrix} \Phi^{-1} & -KC \\ \Phi^{-1} + BF & \Phi^{-1} \end{bmatrix} \\ &= \det\begin{bmatrix} \Phi^{-1} & -KC \\ BF & \Phi^{-1} + KC \end{bmatrix} \\ &= \det\begin{bmatrix} \Phi^{-1} + BF & \Phi^{-1} \\ BF & \Phi^{-1} + KC \end{bmatrix}. \end{aligned} \tag{3.10}$$

Now using Schur's formula for calculating the determinant of a partitioned matrix, we have

$$\begin{aligned} &\det[sI_{2n} - A_{cl}] \\ &\quad = \det[\Phi^{-1} + KC] \\ &\qquad \times \det[\Phi^{-1} + BF - \Phi^{-1}(\Phi^{-1} + KC)^{-1}BF] \\ &\quad = \det[\Phi^{-1} + KC]\det[\Phi^{-1}] \\ &\qquad \times \det[I_n + \Phi BF - (\Phi^{-1} + KC)^{-1}BF] \\ &\quad = \det[\Phi^{-1} + KC]\det[\Phi^{-1}] \\ &\qquad \times \det\{I_n + [\Phi B - (\Phi^{-1} + KC)^{-1}B]F\}. \end{aligned} \tag{3.11}$$

Now using the identity

$$\det[I_n + A_1A_2] = \det[I_m + A_2A_1] \tag{3.12}$$

for any $n \times m$ and $m \times n$ matrices A_1 and A_2,

$$\begin{aligned} &\det[sI_{2n} - A_{cl}] \\ &\quad = \det[\Phi^{-1} + KC]\det[\Phi^{-1}] \\ &\qquad \times \det\{I_m + F[\Phi B - (\Phi^{-1} + KC)^{-1}B]\} \\ &\quad = \det[\Phi^{-1} + KC]\det[\Phi^{-1}] \\ &\qquad \times \det\{I_m + F\Phi B - F(\Phi^{-1} + KC)^{-1}B\} \\ &\quad = \det[\Phi^{-1} + KC]\det[\Phi^{-1}] \\ &\qquad \times \det\{I_m + F\Phi B - M(s)\}. \end{aligned} \tag{3.13}$$

Noting that $M(s) \equiv 0$, (3.13) reduces to

$$\begin{aligned} &\det[sI_{2n} - A_{cl}] \\ &\quad = \det[\Phi^{-1} + KC]\det[\Phi^{-1}]\det[I_n + \Phi BF] \\ &\quad = \det[\Phi^{-1} + KC]\det[\Phi^{-1} + BF]. \end{aligned}$$

Since by design $A - KC$ and $A - BF$ are asymptotically stable matrices, the closed-loop system of Fig. 3 is then asymptotically stable.

As in the case of full order observer based controllers, the conditions given in Theorem 5, especially the conditions 1 and 2 are very restrictive and hence are not true for many practical systems. To broaden the class of systems, one abandons the goal of achieving ELTRI and instead seeks ALTRI. For this purpose, as in the previous section, we parameterize K in terms of a tuning parameter σ. We have the following theorem dealing with ALTRI.

Theorem 6. Consider the closed-loop system comprising of the plant along with the compensator as in Fig. 3. Assume that the given plant is left invertible and is of minimum phase. Select the gain K which is parameterized in terms of a tuning parameter σ, as an element of $\mathcal{K}_a(\sigma)$. Then ALTRI is achieved as $\sigma \to \infty$. Furthermore, there exists a σ_1 such that the closed-loop system is asymptotically stable for all $\sigma > \sigma_1$. More specifically, as $\sigma \to \infty$, eigenvalues of A_{cl} are given by

$$\lambda(A - K(\sigma)C) + O(1/\sigma)$$

and

$$\lambda(A - BF) + O(1/\sigma).$$

Also, the developed compensator is always open-loop asymptotically stable.

Proof. The results of achieving ALTRI and open-loop asymptotic stability of the compensator are obvious. The proof of closed-loop stability of Fig. 3 can be seen as follows. In view of (3.13) and noting that $M(s)$ tends to zero point wise in s as $\sigma \to \infty$, we have

$$\begin{aligned} \det\,[sI_{2n} - A_{cl}] &\to \det\,[\Phi^{-1} + K(\sigma)C]\det\,[\Phi^{-1}] \\ &\quad \times \det\,[I_n + \Phi BF] \quad \text{as} \quad \sigma \to \infty \\ &= \det\,[\Phi^{-1} + K(\sigma)C]\det\,[\Phi^{-1} + BF]. \quad (3.14) \end{aligned}$$

This completes the proof of Theorem 6.

Remark 7. The full order observer is not in general open-loop stable while open-loop stability of the compensator is always guaranteed.

As discussed earlier, one often opts for ALTRI design as it requires less stringent conditions than ELTRI design. However, ALTRI is fundamentally an asymptotic result. In practice, the degree of recovery depends on the size of gain. Both conventional observer-based controller and our new compensator are capable of achieving ALTRI. The following theorem however shows that for the same value of gain, the new compensator achieves much better degree of recovery than the observer-based controller.

Theorem 7. Let $K(\sigma)$ be an element of $\mathcal{K}_a(\sigma)$. Assume also that the same gain $K(\sigma)$ is used for both the observer-based controller and for the new compensator. Let σ be such that $\sigma_{\max}[M(\mathrm{j}\omega)]$ is small (say, $\ll 1$) for all ω. Furthermore, assume that

$$\begin{aligned} &\sigma_{\min}[L(j\omega)] \\ &\quad = \sigma_{\min}[F(\mathrm{j}\omega - A)^{-1}B] \gg 1 \quad \text{for all} \quad \omega \in D_c, \end{aligned} \tag{3.15}$$

for some frequency region of interest, D_c. Then for all $\omega \in D_c$, the mismatch between the target loop transfer function and the one achieved by the compensator is always less than the corresponding one achieved by the full order observer-based controller. More specifically, we have

$$\sigma_{\max}[E_0(\mathrm{j}\omega)] \gg \sigma_{\max}[E_c(\mathrm{j}\omega)] \quad \text{for all} \quad \omega \in D_c, \tag{3.16}$$

where $E_c(s)$ is as in (3.8) and $E_0(s)$ is as in (2.4).

Proof. Recalling the expression for $E_0(\mathrm{j}\omega)$ from (2.4), we have

$$\begin{aligned} &\sigma_{\max}[E_0(\mathrm{j}\omega)] \\ &\quad = \sigma_{\max}\{M(\mathrm{j}\omega)[I_m + M(\mathrm{j}\omega)]^{-1}(I_m + F\Phi(\mathrm{j}\omega)B)\} \\ &\quad \geq \sigma_{\max}[M(\mathrm{j}\omega)]\sigma_{\min}\{[I_m + M(\mathrm{j}\omega)]^{-1}\} \\ &\qquad \times \sigma_{\min}[I_m + F\Phi(\mathrm{j}\omega)B] \\ &\quad = \frac{\sigma_{\max}[M(\mathrm{j}\omega)]\sigma_{\min}[I_m + F\Phi(\mathrm{j}\omega)B]}{\sigma_{\max}[I_m + M(\mathrm{j}\omega)]} \\ &\quad \geq \sigma_{\max}[E_c(\mathrm{j}\omega)]\alpha(\mathrm{j}\omega), \end{aligned} \tag{3.17}$$

where

$$\alpha(\mathrm{j}\omega) = \frac{\sigma_{\min}[F\Phi(\mathrm{j}\omega)B] - 1}{1 + \sigma_{\max}[M(\mathrm{j}\omega)]}.$$

Now by our assumption, $\sigma_{\max}[M(\mathrm{j}\omega)]$ is $\ll 1$ and $\sigma_{\min}[F\Phi(\mathrm{j}\omega)B]$ is $\gg 1$ for all $\omega \in D_c$ and hence $\alpha(\mathrm{j}\omega)$ is $\gg 1$ for all $\omega \in D_c$. Thus

$$\sigma_{\max}[E_0(\mathrm{j}\omega)] \gg \sigma_{\max}[E_c(\mathrm{j}\omega)] \quad \text{for all} \quad \omega \in D_c.$$

Remark 8. It is well known (Doyle and Stein, 1981) that in order to have good command following and disturbance rejection properties, the loop transfer function matrix $L(\mathrm{j}\omega)$ has to be large and consequently, the minimum singular value $\sigma_{\min}[L(\mathrm{j}\omega)]$ should be large in the appropriate frequency region. Thus the condition (3.15) is always satisfied in all practical situations.

Remark 9. Theorem 7 is intuitively evident. In our compensator $E_c(s)$, the error between the required and the achieved loop transfer function is equal to $M(s)$ which is designed to be small in some sense. On the other hand, in conventional observer-based design, the corresponding error

$E_0(s)$, is a multiple of $M(s)$, $[I_m + M(s)]^{-1}$ and $I_m + F\Phi B$. But in any good design, the loop transfer function $F\Phi B$ is large in the frequency region of interest. Thus for the same gain $K(\sigma)$, $E_0(s)$ differs from $E_c(s)$ by a large factor ($\approx \|F\Phi B\|$) making $E_0(s)$ much worse than $E_c(s)$.

Once again, as in Lemma 5, we now develop bounds on sensitivity and complementary sensitivity functions when the new compensator is used. Let $S_c(s)$ and $T_c(s)$ be the generated sensitivity and complementary sensitivity functions in the configuration of Fig. 3 when the loop is broken at the input point of the plant,

$$S_c(s) = [I_m + \mathbf{C}_c(s)P(s)]^{-1}$$

and

$$T_c(s) = I_m - S_c(s) = [I_m + \mathbf{C}_c(s)P(s)]^{-1}\mathbf{C}_c(s)P(s)$$

where $\mathbf{C}_c(s)$ is as in (3.3). We have the following result analogous to Lemma 5.

Theorem 8. Consider the configuration of Fig. 3. Assume that (3.15) is true. We have the following bounds on all singular values $i = 1$ to m of $S_c(j\omega)$ and $T_c(j\omega)$:

$$\frac{|\sigma_i[S_c(j\omega)] - \sigma_i[S_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \le \frac{\sigma_{\max}[M(j\omega)]}{\sigma_{\min}[F\Phi(j\omega)B] - \sigma_{\max}[M(j\omega)] - 1} \ll \sigma_{\max}[M(j\omega)] \quad \text{for all} \quad \omega \in D_c \tag{3.18}$$

and

$$\frac{|\sigma_i[T_c(j\omega)] - \sigma_i[T_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \le \frac{\sigma_{\max}[M(j\omega)]}{\sigma_{\min}[F\Phi(j\omega)B] - \sigma_{\max}[m(j\omega)] - 1} \ll \sigma_{\max}[M(j\omega)] \quad \text{for all} \quad \omega \in D_c. \tag{3.19}$$

Proof. See Appendix E.

Remark 10. It is evident that due to the presence of the sign $\ll$ in the expressions (3.18) and (3.19), the new compensator yields much better sensitivity and complimentary sensitivity recovery than the conventional full order observer based controller.

4. REDUCED ORDER COMPENSATOR—ELTRI AND ALTRI

In the previous section, we studied a new compensator whose dynamic order is the same as that of the given plant. Corresponding to the reduced order observer-based controllers, one naturally would like to investigate also the possibility of a reduced order compensator for the purpose of achieving either ELTRI or ALTRI. Motivated by the results of the previous section, in this section we study such a reduced order compensator structure (see Fig. 5).

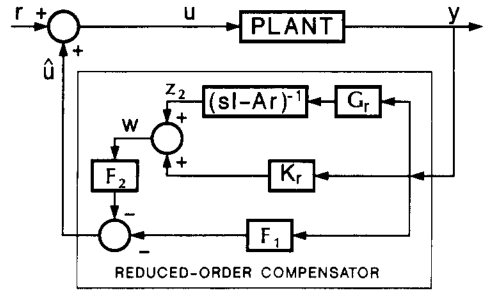

FIG. 5. Plant with reduced order compensator.

The structure shown in Fig. 5 corresponds to that of the reduced order observer-based controller except that, as in the case of full order compensator, the link from the plant input point via the matrix B to the controller (or what is now called a compensator) is omitted. Because of this omission of the link, again the separation principle is no longer valid and hence we need to study and establish the necessary properties of the reduced order compensator for LTRI.

As in Section 2, without loss of generality, we will assume that the given plant is described by (2.9) and (2.10) while the state feedback gain matrix F which achieves the target loop transfer function $L(s)$ be partitioned as in (2.11). The reduced order compensator is dynamically described by

$$\dot{z}_2 = A_r z_2 + G_r y, \tag{4.1}$$

$$u = \hat{u} = -F_1 x_1 - F_2 w, \tag{4.2}$$

$$w = K_r y + z_2. \tag{4.3}$$

The matrices A_r and G_r are as in (2.18). Here K_r is a free design parameter which is to be selected to satisfy the following conditions:

1. *Stability of the closed-loop system.* The closed-loop system as depicted in Fig. 5 and characterized by (2.9), (2.18), (4.1) to (4.3), is asymptotically stable, i.e.

$$\mathrm{Re}\,[\lambda(A_{clr})] < 0, \tag{4.4}$$

where

$$A_{clr} = \begin{bmatrix} A_{22} - K_r A_{12} - K_r B_1 F_2 & A_{21} - K_r B_1 F_1 & K_r A_{12} \\ -B_1 F_2 & A_{11} - B_1 F_1 & A_{12} \\ -B_2 F_2 & A_{21} - B_2 F_1 & A_{22} \end{bmatrix}. \tag{4.5}$$

2. *ELTRI or ALTRI.* The achieved loop

transfer function $L_{cr}(j\omega)$,

$$L_{cr}(j\omega) = \mathbf{C}_{cr}(j\omega)P(j\omega), \tag{4.6}$$

is either exactly or approximately equal to $L(j\omega)$, where $\mathbf{C}_{cr}(s)$ denotes the transfer function of the compensator (i.e. the transfer function from y to $-\hat{u}$).

3. *Open-loop stability of the compensator.* The compensator is open-loop asymptotically stable, i.e.

$$\operatorname{Re}[\lambda(A_r)] < 0. \tag{4.7}$$

Besides satisfying the above technical conditions, as in the previous section, one expects that the value of gain needed for a certain degree of LTR is much smaller than that required by the reduced order observer-based controller. We have the following lemma:

Lemma 10. $E_{cr}(s)$, the error between the target loop transfer function $L(s)$ and that realized by the reduced order compensator, is given by

$$E_{cr}(s) = L(s) - L_{cr}(s) = M_r(s), \tag{4.8}$$

where

$$M_r(s) = F_2(\Phi_{22}^{-1} + K_r A_{12})^{-1} B_r. \tag{4.9}$$

Proof. See Appendix F.

Remark 11. The expression for $E_{cr}(s)$ is identical to the corresponding one for the full order compensator, see (3.8), except that now $M_r(s)$ takes the place of $M(s)$. Also $M_r(s)$ is the same as defined in (2.20) for the case of reduced order observer-based controller. We also note that the two expressions for the error between the required and the achieved loop transfer functions, one for the conventional reduced order observer design (2.19) and the other for the new compensator design (4.8), again differ significantly. Thus as we expect from Theorems 7 and 8, this leads to an overwhelming advantage in favor of the new reduced order compensator in contrast to a reduced order observer-based controller.

We have the following two theorems.

Theorem 9. Consider the closed-loop system as depicted in Fig. 5. Then both the asymptotic stability of the closed-loop system and ELTRI can be achieved under the following conditions:

1. The given plant has all its infinite zeros of order one.
2. The given plant is left invertible and is of minimum phase.

Moreover, under the above conditions, K_r can be selected as an element of $\mathcal{K}_{er}$. Also, the eigenvalues of A_{clr} are given by

$$\lambda(A_{22} - K_r A_{12}) \quad \text{and} \quad \lambda(A - BF).$$

Furthermore, the developed compensator is always open-loop asymptotically stable.

Proof. Under the given conditions, K_r can be selected as an element of $\mathcal{K}_{er}$ and hence $M_r(s) \equiv 0$. Thus ELTRI is achieved. Also, it is evident that the compensator is open-loop asymptotically stable. The closed-loop stability of Fig. 5 is given in Appendix G.

Theorem 10. Consider the closed-loop system as depicted in Fig. 5. Assume that the given plant is left invertible and is of minimum phase. Select the gain K_r which is parameterized in terms of a tuning parameter σ, as an element of $\mathcal{K}_{ar}(\sigma)$. Then ALTRI is achieved as $\sigma \to \infty$. Furthermore, there exists a σ_2 such that the closed-loop system is asymptotically stable for all $\sigma > \sigma_2$. More specifically, as $\sigma \to \infty$, eigenvalues of A_{clr} are given by

$$\lambda(A_{22} - K_r(\sigma)A_{12}) + O(1/\sigma)$$

and

$$\lambda(A - BF) + O(1/\sigma).$$

Also, the developed compensator is always open-loop asymptotically stable.

Proof. The results of achieving ALTRI and open-loop asymptotic stability of the compensator are obvious. The proof of closed-loop stability of Fig. 5 is given in Appendix H.

As is clear by now, one often seeks an ALTRI design which as we know is asymptotic where the degree of recovery depends on the size of gain. Both the conventional reduced order observer-based controller and our new reduced order compensator are capable of achieving ALTRI. As expected, the following theorem, however, shows that for the same value of gain, the new compensator achieves a much better degree of recovery than the observer-based controller.

Theorem 11. *Let* $K_r(\sigma)$ be an element of $\mathcal{K}_{ar}(\sigma)$. Assume also that the same gain $K_r(\sigma)$ is used for both the reduced order observer-based controller and the reduced order compensator. Let σ be such that $\sigma_{\max}[M_r(j\omega)]$ is small (say, $\ll 1$) for all ω. Furthermore, assume that (3.15) is true. Then for all $\omega \in D_c$, the mismatch between the target loop transfer function and the one achieved by the reduced order compensator is always less than the corresponding one achieved by the reduced order observer-based controller.

More specifically, we have

$$\sigma_{\max}[E_{or}(j\omega)] \gg \sigma_{\max}[E_{cr}(j\omega)] \quad \text{for all} \quad \omega \in D_c, \tag{4.10}$$

where $E_{cr}(s)$ is as in (4.8) and $E_{or}(s)$ is as in (2.19).

Proof. The proof follows along the same lines as that of Theorem 7.

As in the previous section, we now turn our attention to developing bounds on sensitivity and complementary sensitivity functions. Let $S_{cr}(s)$ and $T_{cr}(s)$ be the generated sensitivity and complementary sensitivity functions in the configuration of Fig. 5 when the loop is broken at the input point of the plant,

$$S_{cr}(s) = [I_m + \mathbf{C}_{cr}(s)P(s)]^{-1}$$

and

$$T_{cr}(s) = I_m - S_{cr}(s) = [I_m + \mathbf{C}_{cr}(s)P(s)]^{-1}\mathbf{C}_{cr}(s)P(s).$$

We have the following result analogous to Theorem 8.

Theorem 12. Consider the configuration of Fig. 5. Assume that (3.15) is true. We have the following bounds on all singular values $i = 1$ to m of $S_{cr}(j\omega)$ and $T_{cr}(j\omega)$:

$$\frac{|\sigma_i[S_{cr}(j\omega)] - \sigma_i[S_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \le \frac{\sigma_{\max}[M_r(j\omega)]}{\sigma_{\min}[F\Phi(j\omega)B] - \sigma_{\max}[M_r(j\omega)] - 1} \ll \sigma_{\max}[M_r(j\omega)] \text{ for all } \omega \in D_c \tag{4.11}$$

and

$$\frac{|\sigma_i[T_{cr}(j\omega)] - \sigma_i[T_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \le \frac{\sigma_{\max}[M_r(j\omega)]}{\sigma_{\min}[F\Phi(j\omega)B] - \sigma_{\max}[M_r(j\omega)] - 1} \ll \sigma_{\max}[M_r(j\omega)] \text{ for all } \omega \in D_c. \tag{4.12}$$

Proof. It follows along the same lines as that of Theorem 8.

Remark 12. Remarks similar to 7–10 are obviously true even for reduced order compensators.

5. FULL AND REDUCED ORDER COMPENSATORS—ELTRO AND ALTRO

The results for the case when the target open-loop transfer functions are specified at the output point of the plant can be obtained by dualizing those for the case when the target open-loop transfer functions are specified at the input point of the plant. However, one has to interpret duality in a proper manner and this was discussed earlier in Section 2.3. All this discussion also applies to compensator design. All one has to do is to use either full or reduced order compensator design of Sections 3 and 4 to achieve LTR in the third step of the design algorithm discussed in Section 2.3. The remaining steps of the design algorithm given in Section 2.3 remain intact.

6. EXAMPLES

Examples are presented in this section comparing the new compansator with the conventional observer approach. These examples are worked out using the software reported by Chen *et al.* (1989). Clearly all the examples support the theoretical development given earlier and demonstrate that the new compensator approach is much better than the conventional observer approach in all cases, namely, (a) when the performance specifications are reflected either at the input or at the output point of the plant, and (b) whether the full or reduced order compensator is used.

Most often in the literature, the maximum and minimum singular value graphs of the target and achieved loop transfer matrices are drawn with respect to ω and are then compared. These graphs could be misleading. Although the singular values of target and achieved loop transfer matrices may match perfectly, the difference or mismatch between them could be very high owing to the phase difference between them. This has been pointed out by Ridgely and Banda (1986) in an example. The best way is to check the singular values of the *mismatch function* between the target and achieved loop transfer matrices.

To show the effects of the phase difference, consider the example of Ridgely and Banda (1986)

$$\dot{x} = Ax + Bu + \Gamma\zeta = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 35 \\ -61 \end{bmatrix} \zeta \tag{6.1}$$

and

$$y = Cx + \eta = [2 \quad 1]x + \eta. \tag{6.2}$$

The state feedback law is selected as

$$u = -Fx = -[50 \quad 10]x. \tag{6.3}$$

The full order observer gain $K(\sigma)$ is obtained by solving the filter Riccati equation,

$$A\Sigma + \Sigma A' + Q(\sigma) - \Sigma C'C\Sigma = 0 \tag{6.4}$$

and

$$K(\sigma) = \Sigma C'$$

where

$$Q(\sigma) = \Gamma\Gamma' + \sigma^2 BB'.$$

The magnitude plots of the target and achieved loop transfer function when σ^2 takes values 0; 500; 2500; 3600; 8100; and 250,000 are presented in Fig. 6(a). As σ^2 begins to increase, the low frequency region of the achieved loop transfer function begins to approach the target loop, while the high frequency region remains virtually unchanged. As σ^2 takes the value 3000, the low frequency region also almost matches the target loop. At $\sigma^2 = 3600$ as shown in Fig. 6(a), the target and achieved loop transfer function magnitudes are almost "tight" together. However, as shown in Fig. 6(b), the phases are about 180° apart in the low frequency region. This shows that no recovery has been achieved

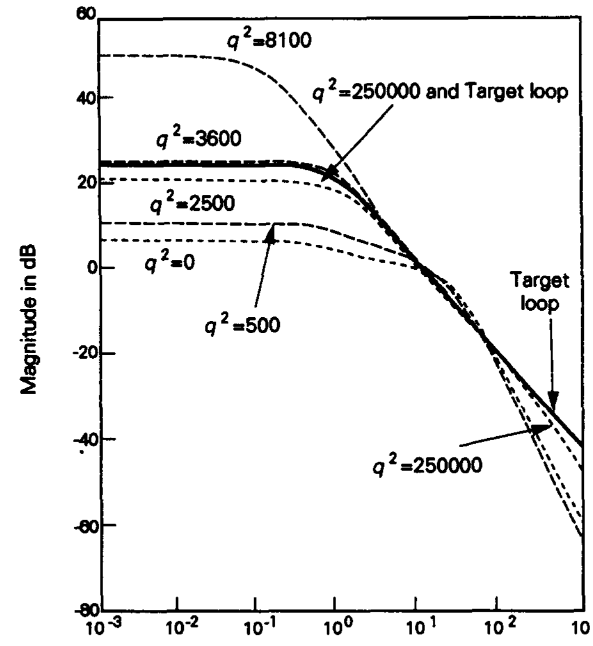

FIG. 6(a). Singular values of target loop and design loops.

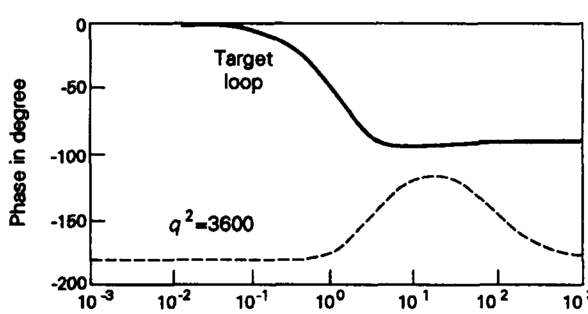

FIG. 6(b). Phase responses of target loop and design loop with $q^2 = 3600$.

TABLE 1(a). SUPREMUM OF MAXIMUM SINGULAR VALUES OF MISMATCH FUNCTIONS OVER FREQUENCIES PLOTTED

	Tuning parameter	Supremum $\sigma_{max}\{E_0(j\omega)\}$	Supremum $\sigma_{max}\{E_c(j\omega)\}$
Case 1	$\sigma^2 = 500$	20.1627	8.0747
Case 2	$\sigma^2 = 10^3$	21.6324	5.4534
Case 3	$\sigma^2 = 10^4$	55.7910	0.7910

TABLE 1(b). COMPARISON OF FULL ORDER OBSERVER-BASED CONTROLLER *VS* FULL ORDER COMPENSATOR FOR THE SAME DEGREE OF RECOVERY

Degree of recovery
sup $\sigma_{max}[E_0(j\omega)] \cong$ sup $\sigma_{max}[E_c(j\omega)] \cong 0.7910$ for $10^{-2} \leq \omega < \infty$ rad s^{-1}.

	Observer-based controller	Full order compensator
Gain norm	353.4295	84.8997
Eigenvalues	−1.8664 −370.9527	−2.0603 −100.4328
Bandwidth	5170 rad/s^{-1}	1682 rad/s^{-1}

at $\sigma^2 = 3600$. It takes a σ^2 of 250,000 to achieve the needed recovery.

In what follows, for each example, we present the traditional maximum and minimum singular value graphs of the target and achieved loop transfer matrices. However, in view of the above discussion, the maximum singular value graphs of the mismatch function are also separately given. Also, a tabular column presents the supremum of the maximum singular value of the mismatch function with respect to ω over the frequency range of interest. All of the above data relate to the comparison between the observer-based controller and our new compensator when both of them use the same value of gain. Another method of comparison is to give the value of gain, eigenvalues and bandwidth of both the observer-based controller and the compensator in order that both of them achieve the same degree of recovery as measured by the supremum of the maximum singular value of the correspondingly generated mismatch function. Another tabular column shows this information. Also, for a chosen supremum of maximum singular value, a graph shows the variation of maximum singular value of the observer-based controller and that of the compensator with respect to ω over the frequency range of interest. From all these data, it is easy to see that the new compensator approach has better recovery properties than the conventional observer approach.

Example 1 (Full order ALTRI). Consider the example in Doyle and Stein (1979) [and see Table 1 (a, b) and Fig. 7 (a, b)].

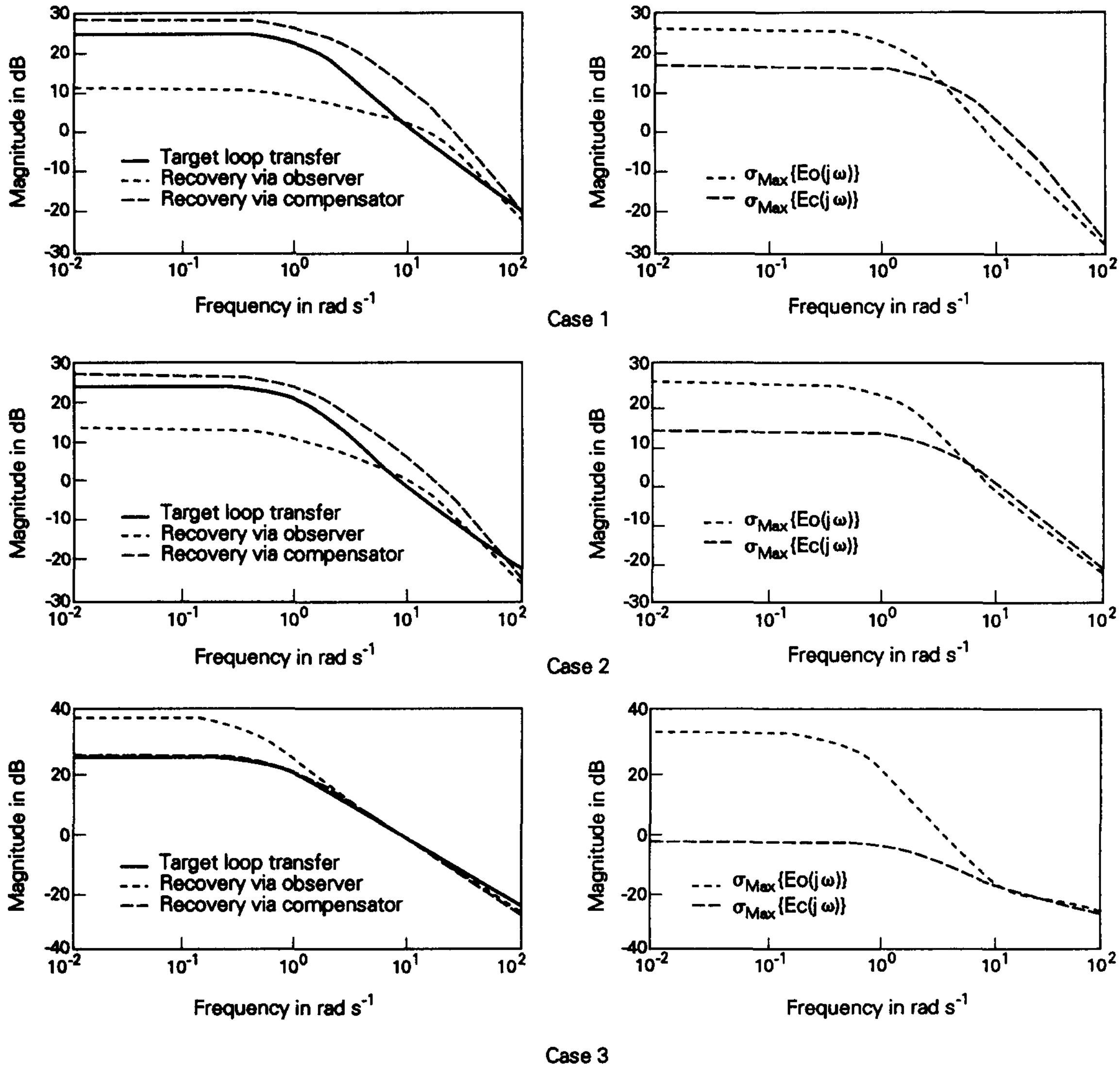

FIG. 7(a). Frequency responses for all the cases given in Table 1(a).

Plant:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

$$y = [2 \quad 1]x.$$

State feedback gain:

$$F = [50 \quad 10].$$

Example 2 (Full order ALTRO). Consider the following example in Ridgely and Banda (1986) [and see Table 2 (a, b) and Fig. 8 (a, b)]

$$\mathbf{A} = \begin{bmatrix} -0.08527 & -0.0001423 & -0.9994 & 0.04142 & 0 & 0.1862 \\ -46.86 & -2.757 & 0.3896 & 0 & -124.3 & 128.6 \\ -0.4248 & -0.06224 & -0.06714 & 0 & -8.792 & -20.46 \\ 0 & 1 & 0.0523 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 \end{bmatrix}$$

TABLE 2(a). SUPREMUM OF MAXIMUM SINGULAR VALUES OF MISMATCH FUNCTIONS OVER FREQUENCIES PLOTTED

Supremum $\sigma_{max}\{E_0(j\omega)\}$	Supremum $\sigma_{max}\{E_c(j\omega)\}$
471.9951	0.2398

TABLE 2(b). COMPARISON OF FULL ORDER OBSERVER-BASED CONTROLLER *VS* FULL ORDER COMPENSATOR FOR THE SAME DEGREE OF RECOVERY

Degree of recovery
$\sup \sigma_{max}[E_0(j\omega)] \cong \sup \sigma_{max}[E_c(j\omega)] \cong 0.2398$ for $10^{-3} \leq \omega < \infty$ rad s^{-1}.

	Observer-based controller	Full order compensator
Gain norm	3.1623×10^9	317.4681
Eigenvalues	-158.15	-104.69
	-19923	-51.17
	$-76736 \pm j76737$	$-52.5 \pm j89.62$
	$-9956 \pm j17244$	$-26.8 \pm j42.71$
Bandwidth	4.78×10^{10} rad/s^{-1}	5288 rad/s^{-1}

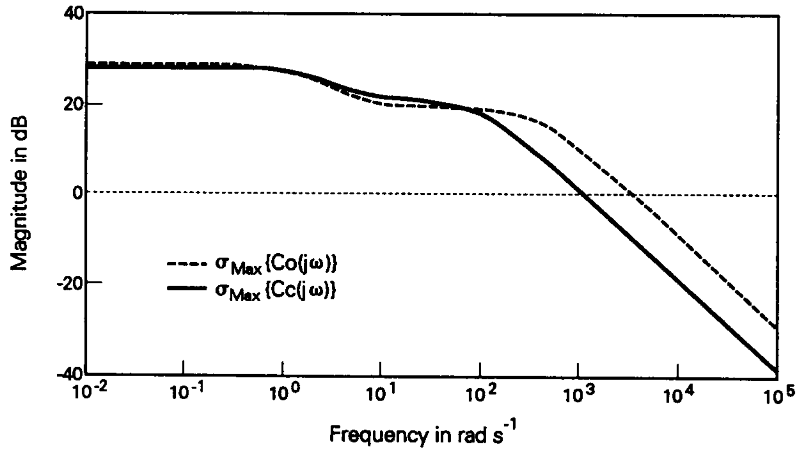

FIG. 7(b). Maximum singular values of full order observer based controller and compensator given in Table 1(b).

$$\mathbf{B}=\begin{bmatrix}0&0&0&0&20&0\\0&0&0&0&0&20\end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{C}=\begin{bmatrix}1&0&0&0&0&0\\0&0&0&1&0&0\end{bmatrix}.$$

For this example, the observer gain is given, instead of the state feedback gain, to meet all the design specifications for the loop broken at the output of plant.

$$\mathbf{K}=\begin{bmatrix}4.20&-18.17&-9.92&-1.19&0.0181&0.1149\\-1.19&55.81&-0.60&10.49&-0.3330&0.3380\end{bmatrix}^{\mathrm{T}}.$$

Example 3 (reduced order ALTRI). Consider the example in Sogaard-Andersen (1987b) [and see Table 3 and Fig. 9]

$$\mathbf{A}=\begin{bmatrix}2&0&1&0&0&1&1\\-1&2&0&0&0&0&0\\3&2&-1&0&0&-2&1\\2&-2&0&-4&2&0&-1\\0&2&3&0&-2&1&-1\\1&0&2&-3&2&2&0\\-1&-1&1&0&0&-1&1\end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix}2&0&0\\1&0&0\\1&0&0\\0&2&0\\0&3&0\\0&0&1\\0&-1&0\end{bmatrix}$$

$$\mathbf{C}=[I_3\quad 0_{3\times 4}].$$

The state feedback gain is an LQ-design with weights $Q=I_7$ and $R=10^{-3}I_3$.

TABLE 3. SUPREMUM OF MAXIMUM SINGULAR VALUES OF MISMATCH FUNCTIONS OVER FREQUENCIES PLOTTED

	Tuning parameter	Supremum $\sigma_{\max}\{E_0(j\omega)\}$	Supremum $\sigma_{\max}\{E_c(j\omega)\}$
Case 1	$\sigma=10$	203.0387	19.9622
Case 2	$\sigma=100$	136.1517	2.1660

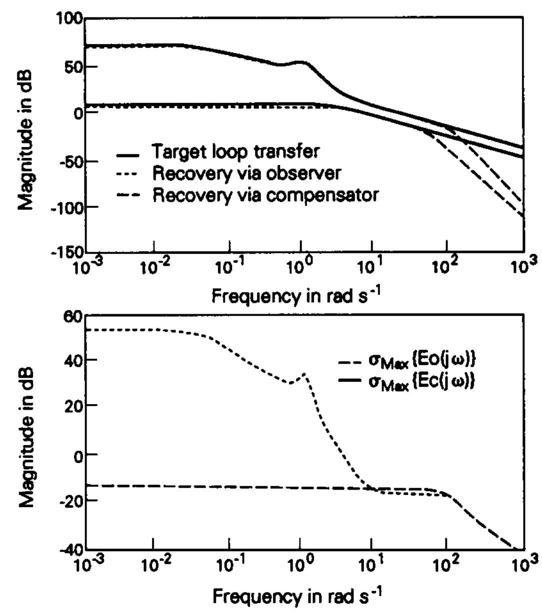

FIG. 8(a). Frequency responses for the case given in Table 2(a).

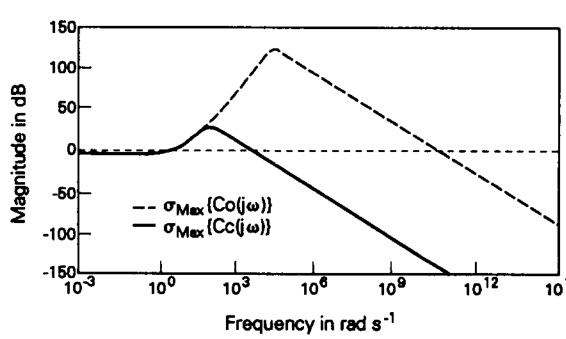

FIG. 8(b). Maximum singular values of full order observer based controller and compensator given in Table 2(b).

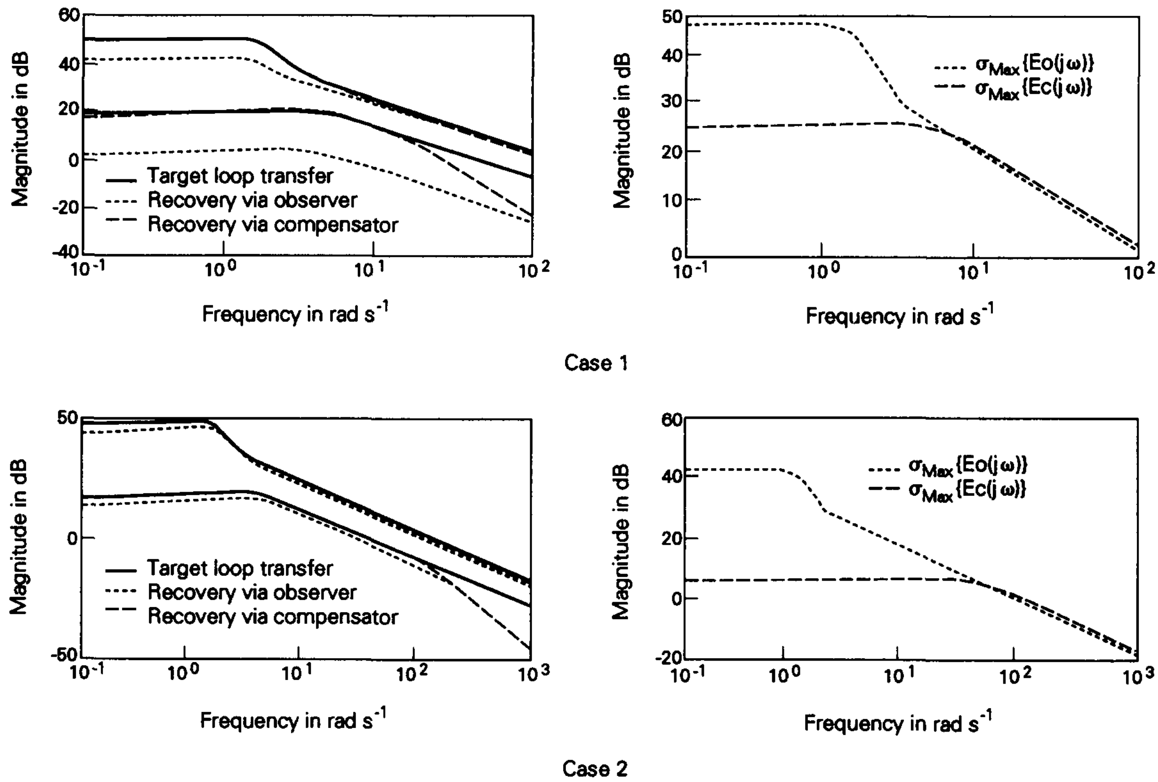

FIG. 9. Frequency responses for both the case given in Table 3(a).

7. CONCLUSIONS

The loop transfer recovery (LTR) methods using observer or Kalman filter-based controllers are streamlined and the theory of both full and reduced order observers is brought to the same framework. It is shown that either exact or approximate LTR can be accomplished iff $M(j\omega)$ [or equivalently $M_r(j\omega)$ for the reduced order observers] either exactly or approximately zero for all ω. The term $M(s)$ [or $M_r(s)$] has a physical interpretation; it is the transfer function from the point where the input u of the plant is fed to the observer based controller to the output point $-\hat{u}$ of the controller. Also, the conditions for ELTR are presented directly in terms of the given system matrices A, B and C and the state feedback gain F. The methods of calculating the needed observer gain for both full and reduced order observer based controllers to achieve LTR are presented. The singular value bounds on the difference between the achieved and the target sensitivity and complementary sensitivity functions are developed when both full and reduced order observer based controllers are used. The duality between the two cases when the design specifications reflect at the input or at the output point of the plant is discussed. One has to interpret this duality carefully. From the view point of design methodology, the two cases are completely dual and this duality holds for either full or reduced order observer-based controllers or in fact for any other controllers or compensators.

A new compensator structure for loop transfer recovery either at the input or at the output point of the plant is proposed. It could be either full or reduced order type. The compensator is structurally different from the observer in the sense that no link from the input point of the plant to the controller is used. This omission of the link from the input point of the plant to the controller has a profound effect on all aspects of the loop transfer recovery. It results in an open-loop stable compensator. Also, the closed-loop stability can be guaranteed. More importantly, the value of gain required for a given degree of LTR is orders of magnitude less than what is required in the conventional approach. Also, singular value bounds on sensitivity and complementary sensitivity functions illustrate that the proposed compensator has better recovery properties than the conventional observer based controller. These advantages reflect in various ways. First, the woes of saturation are either eliminated or at least dampened. The controller band-width is reduced and consequently the control signal to noise ratio at the input point of the plant is increased. All these claims are theoretically obvious from our development. Also, numerical examples illustrate the same.

A fundamental assumption throughout this paper has been that the given plant is of minimum phase. We are presently in the process of developing compensators for nonminimum phase plants where obviously, the structures given in this paper will not work out since in general nonminimum phase plants might not be stabilizable by stable compensators. Hence we are looking at some other appropriate structures to deal with nonminimum phase systems. These results will be reported in a forthcoming paper.

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APPENDIX A: PROOF OF THEOREM 1 AND CALCULATION OF GAIN K TO ACHIEVE ELTRI

Under the conditions (2) and (3) of Theorem 1, a theorem of Sannuti and Saberi (1987) implies that there exist nonsingular transformations Γ_1, Γ_2 and Γ_3 such that

$$
\begin{gathered}
x = \Gamma_1 \bar{x}, \quad y = \Gamma_2 [y_s', y_f']', \quad u = \Gamma_3 \bar{u} \\
\bar{x} = [x_a', x_b', x_f']', \\
\dot{x}_a = A_{aa} x_a + L_{as} y_s + L_{af} y_f, \\
\dot{x}_b = A_{bb} x_b + L_{bf} y_f, \quad y_s = C_s x_b, \\
\dot{x}_f = E_a x_a + E_b x_b + E_f x_f + \bar{u}, \quad y_f = x_f.
\end{gathered} \tag{A.1}
$$

Here, the pair (A_{bb}, C_s) is observable. Furthermore, $\lambda(A_{aa})$ are the invariant zeros of the given plant and hence in view of condition (2) of Theorem 1, they are in the left half s plane, $\mathscr{C}^-$.

Since (A_{bb}, C_s) is observable, one can select a gain K_{bb} such that $\lambda(A_{bb} - K_{bb} C_s)$ are in the desired locations in $\mathscr{C}^-$. Also, one can always choose a gain K_{ff} such that $\lambda(E_f - K_{ff})$ are in the desired locations in $\mathscr{C}^-$. Now choose a gain $\bar{K}$ as,

$$
\bar{K} = \begin{bmatrix} L_{as} & L_{af} \\ K_{bb} & L_{bf} \\ K_{fb} & K_{ff} \end{bmatrix}, \tag{A.2}
$$

where K_{fb} is an arbitrary matrix with appropriate dimensions. Finally let

$$
K = \Gamma_1 \bar{K} \Gamma_2^{-1}.
$$

All such gains K with K_{fb} arbitrary form the class of gains $\mathscr{K}_e$. Due to the special structure of matrices in (A.1) and in view of (A.2), it is straightforward to verify that $A - KC$ has eigenvalues in $\mathscr{C}^-$ and that

$$
F(\Phi^{-1} + KC)^{-1} B \equiv 0
$$

whenever $FB \equiv 0$. Hence in view of Lemmas 1 and 2, ELTRI is achieved.

APPENDIX B: PROOF OF LEMMA 6

In the reduced order observer-based feedback control system of Fig. 4, at first we want to evaluate the loop transfer function $L_{or}(s)$ when the loop is broken at the input point of the plant. For this purpose, consider the plant input u and the controller output $\hat{u}$ as two separate variables. Then from (2.15) to (2.17),

$$\dot{\hat{x}}_2 = (A_{22} - K_r A_{12})\hat{x}_2 + A_{21}x_1 + B_r\hat{u} + K_r(\dot{x}_1 - A_{11}x_1).$$

Hence

$$\hat{x}_2(s) = (\Phi_{22}^{-1} + K_r A_{12})^{-1}[(A_{21} + K_r\Phi_{11}^{-1})x_1(s) + B_r\hat{u}(s)],$$

$$-\hat{u}(s) = F_1x_1(s) + F_2(\Phi_{22}^{-1} + K_rA_{12})^{-1} \times [(A_{21} + K_r\Phi_{11}^{-1})x_1(s) + B_r\hat{u}(s)].$$

Thus

$$[I_m + F_2(\Phi_{22}^{-1} + K_rA_{12})^{-1}B_r][F_1x_1(s) + F_2\hat{x}_2(s)] = F_1x_1(s) + F_2(\Phi_{22}^{-1} + K_rA_{12})^{-1}(A_{21} + K_r\Phi_{11}^{-1})x_1(s),$$

and therefore

$$L_{or}(s) = [I_m + M_r(s)]^{-1}[F_1 + F_2(\Phi_{22}^{-1} + K_rA_{12})^{-1}(K_r\Phi_{11}^{-1} + A_{21})]P(s), \quad \text{(B.1)}$$

where

$$M_r(s) = F_2(\Phi_{22}^{-1} + K_rA_{12})^{-1}B_r.$$

We will next simplify some expressions. Using (2.12),

$$\begin{aligned}(K_r\Phi_{11}^{-1} + A_{21})P(s) &= (K_r\Phi_{11}^{-1} + A_{21})(\Phi_{11}B_1 + \Phi_{11}A_{12}H_2(s)) \\ &= K_rB_1 - B_2 + A_{21}\Phi_{11}B_1 + B_2 + (K_r + A_{21}\Phi_{11})A_{12}H_2(s).\end{aligned} \quad \text{(B.2)}$$

But from (2.14)

$$A_{21}\Phi_{11}B_1 + B_2 = (\Phi_{22}^{-1} - A_{21}\Phi_{11}A_{12})H_2(s). \quad \text{(B.3)}$$

Thus (B.2) and (B.3) imply that

$$(K_r\Phi_{11}^{-1} + A_{21})P(s) = -B_r + (\Phi_{22}^{-1} + K_rA_{12})H_2(s). \quad \text{(B.4)}$$

Using (B.4) and (2.13),

$$\begin{aligned}F_2(\Phi_{22}^{-1} + K_rA_{12})^{-1}(K_r\Phi_{11}^{-1} + A_{21})P(s) &= -F_2(\Phi_{22}^{-1} + K_rA_{12})^{-1}B_r + F_2H_2(s) \\ &= L(s) - F_1P(s) - M_r(s).\end{aligned} \quad \text{(B.5)}$$

Now in view of (B.1) and (B.5),

$$L_{or}(s) = [I_m + M_r(s)]^{-1}[L(s) - M_r(s)].$$

Thus we have

$$\begin{aligned}E_{or}(s) &= L(s) - L_{or}(s) \\ &= [I_m + M_r(s)]^{-1}[(I_m + M_r(s))L(s) - L(s) + M_r(s)] \\ &= [I_m + M_r(s)]^{-1}M_r(s)(I_m + L(s)) \\ &= M_r(s)[I_m + M_r(s)]^{-1}(I_m + L(s)).\end{aligned}$$

APPENDIX C: PROOF OF THEOREM 3

Without loss of generality, we will assume that the given plant is in the form of a special coordinate basis as in (A.1) (see Appendix A). Then partitioning the state variable x_b as $x_b = [x_{b1}', x_{b2}']'$ with $y_s = x_{b1}$, we can write the matrices A, B and C characterizing (A.1) as

$$A = \begin{bmatrix} E_f & E_{b1} & E_{b2} & E_a \\ L_{bf1} & A_{b11} & A_{b12} & 0 \\ L_{bf2} & A_{b21} & A_{b22} & 0 \\ L_{af} & L_{as} & 0 & A_{aa}\end{bmatrix}, \quad \text{(C.1)}$$

$$B = \begin{bmatrix} I_m \\ 0 \\ 0 \\ 0\end{bmatrix}, \quad C = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & I_{p-m} & 0 & 0\end{bmatrix}.$$

The triple (A, B, C) in (C.1) assumes that the condition 1 of Theorem 3 is true. Then in view of (C.1), (2.9) and (2.10), we have

$$A_{22} = \begin{bmatrix} A_{b22} & 0 \\ 0 & A_{aa}\end{bmatrix}, \quad A_{12} = \begin{bmatrix} E_{b2} & E_a \\ A_{b12} & 0\end{bmatrix},$$

$$B_1 = \begin{bmatrix} I_m \\ 0\end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0\end{bmatrix}.$$

Also, it can be easily seen that the pair (A_{b22}, A_{b12}) is observable (Saberi and Sannuti, 1988). Hence there exists a K_{b2} such that $\lambda(A_{b22}^c)$ are in the desired locations in $\mathscr{C}^-$ where $A_{b22}^c = A_{b22} - K_{b2}A_{b12}$. Now consider a reduced order observer gain matrix K_r as

$$K_r = \begin{bmatrix} 0 & K_{b2} \\ 0 & K_{as}\end{bmatrix} \quad \text{(C.2)}$$

where K_{as} is arbitrary. It is then simple to verify that A_r,

$$A_r = A_{22} - K_rA_{12} = \begin{bmatrix} A_{b22}^c & 0 \\ -K_{as}A_{b12} & A_{aa}\end{bmatrix},$$

is a stable matrix provided that the given plant is of minimum phase, i.e. Re $\lambda(A_{aa}) < 0$. Furthermore, we note that

$$B_r = B_2 - K_rB_1 = 0.$$

This in view of (2.22) shows that ELTRI is achieved. Also, we note that all gains K_r as in (C.2) with K_{as} arbitrary form the class of gains $\mathscr{K}_{er}$.

APPENDIX D: PROOF OF LEMMA 8

From (2.19), we have

$$\begin{aligned}E_{or}(s) &\equiv F\Phi B - C_{or}(s)P(s) \\ &= M_r(s)(I_m + M_r(s))^{-1}(I_m + F\Phi B),\end{aligned}$$

and hence

$$\begin{aligned}I_m + C_{or}(s)P(s) &= I_m + F\Phi B - E_{or}(s) \\ &= I_m + F\Phi B - M_r(s)[I_m + M_r(s)]^{-1}(I_m + F\Phi B) \\ &= [I_m + M_r(s)]^{-1}(I_m + F\Phi B).\end{aligned}$$

Thus

$$S_{or}(s) = S_F(s)[I_m + M_r(s)]. \quad \text{(D.1)}$$

Then using singular value inequalities, we have for each $i = 1$ to m,

$$\sigma_i[S_{or}(j\omega)] \leq \sigma_i[S_F(j\omega)] + \sigma_{\max}[S_F(j\omega)M_r(j\omega)],$$

and thus

$$\sigma_i[S_{or}(j\omega)] - \sigma_i[S_F(j\omega)] \leq \sigma_{\max}[S_F(j\omega)]\sigma_{\max}[M_r(j\omega)]. \quad \text{(D.2)}$$

Now rewriting (D.1) as,

$$S_F(s) = S_{or}(s) - S_F(s)M_r(s),$$

we have for each $i = 1$ to m,

$$\sigma_i[S_F(j\omega)] - \sigma_i[S_{or}(j\omega)] \leq \sigma_{\max}[S_F(j\omega)]\sigma_{\max}[M_r(j\omega)]. \quad \text{(D.3)}$$

Then in view of (D.2) and (D.3), we get

$$\frac{|\sigma_i[S_{or}(j\omega)] - \sigma_i[S_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \leq \sigma_{\max}[M_r(j\omega)].$$

Next in view of (D.1),

$$T_{or}(s) = I_m - S_{or}(s) = T_F(s) - S_F(s)M_r(s).$$

Now using singular value inequalities and proceeding as above, we get

$$\frac{|\sigma_i[T_{or}(j\omega)] - \sigma_i[T_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \leq \sigma_{\max}[M_r(j\omega)].$$

APPENDIX E: PROOF OF THEOREM 8

From (3.8), we have

$$E_c(s) \equiv F\Phi B - \mathbf{C}_c(s)P(s) = M(s),$$

and hence

$$I_m + \mathbf{C}_c(s)P(s) = I_m + F\Phi B - M(s)$$
$$= \{I_m - M(s)[I_m + F\Phi B]^{-1}\}[I_m + F\Phi B].$$

Thus

$$S_c(s) = S_f(s)\{I_m - M(s)[I_m + F\Phi B]^{-1}\}^{-1}$$
$$= S_F(s)\{I_m + M(s)[I_m + F\Phi B - M(s)]^{-1}\}$$
$$= S_F(s) + S_F(s)M(s)[I_m + F\Phi B - M(s)]^{-1}. \quad (E.1)$$

Then using singular value inequalities, we have for each $i = 1$ to m,

$$\sigma_i[S_c(j\omega)] \le \sigma_i[S_F(j\omega)] + \sigma_{\max}\{S_F(j\omega)M(j\omega)[I_m + F\Phi(j\omega)B - M(j\omega)]^{-1}\},$$

or equivalently

$$\sigma_i[S_c(j\omega)] - \sigma_i[S_F(j\omega)] \le \sigma_{\max}\{S_F(j\omega)M(j\omega)[I_m + F\Phi(j\omega)B - M(j\omega)]^{-1}\}. \quad (E.2)$$

Now rewriting (E.1) as

$$S_F(s) = S_c(s) - S_F(s)M(s)[I_m + F\Phi B - M(s)]^{-1},$$

we have for each $i = 1$ to m,

$$\sigma_i[S_F(j\omega)] \le \sigma_i[S_c(j\omega)] + \sigma_{\max}\{S_F(j\omega)M(j\omega)[I_m + F\Phi(j\omega)B - M(j\omega)]^{-1}\},$$

or equivalently

$$\sigma_i[S_F(j\omega)] - \sigma_i[S_c(j\omega)] \le \sigma_{\max}\{S_F(j\omega)M(j\omega)[I_m + F\Phi(j\omega)B - M(j\omega)]^{-1}\}. \quad (E.3)$$

Combining (E.2) and (E.3), we get for each $i = 1$ to m,

$$\frac{|\sigma_i[S_c(j\omega)] - \sigma_i[S_f(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \le \sigma_{\max}[M(j\omega)]\sigma_{\max}\{[I_m + F\Phi(j\omega)B - M(j\omega)]^{-1}\}$$
$$\le \frac{\sigma_{\max}[M(j\omega)]}{\sigma_{\min}[I_m + F\Phi(j\omega)B - M(j\omega)]}$$
$$\le \frac{\sigma_{\max}[M(j\omega)]}{\sigma_{\min}[f\Phi(j\omega)B] - \sigma_{\max}[M(j\omega)] - 1}$$
$$\ll \sigma_{\max}[M(j\omega)] \text{ for all } \omega \in D_c. \quad (E.4)$$

The last step in (E.4) follows from (3.15). Next, in view of (E.1),

$$T_c(s) = I_m - S_c(s)$$
$$= I_m - S_F(s) - S_F(s)M(s)[I_m + F\Phi B - M(s)]^{-1}$$
$$= T_F(s) - S_F(s)M(s)[I_m + F\Phi B - M(s)]^{-1}.$$

Now using singular value inequalities and proceeding as above, we get

$$\frac{|\sigma_i[T_c(j\omega)] - \sigma_i[T_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \ll \sigma_{\max}[M(j\omega)] \text{ for all } \omega \in D_c.$$

APPENDIX F: PROOF OF LEMMA 10

In the reduced order compensator-based feedback control system of Fig. 5, at first we want to evaluate the loop transfer function $L_{cr}(s)$ when the loop is broken at the input point of the plant. For this purpose, consider the plant input u and the controller output $\hat{u}$ as two separate variables. Then in view of (4.1) to (4.3),

$$\dot{w} = (A_{22} - K_r(\sigma)A_{12})w + A_{21}x_1 + K_r(\sigma)(\dot{x}_1 - A_{11}x_1). \quad (F.1)$$

Hence

$$w(s) = (\Phi_{22}^{-1} + K_r(\sigma)A_{12})^{-1}(A_{21} + K_r(\sigma)\Phi_{11}^{-1})P(s)u(s). \quad (F.2)$$

Thus in view of (F.2),

$$-\hat{u}(s) = F_1x_1(s) + F_2w(s)$$
$$= [F_1 + F_2(\Phi_{22}^{-1} + K_r(\sigma)A_{12})^{-1}(A_{21} + K_r(\sigma)\Phi_{11}^{-1})]P(s)u(s). \quad (F.3)$$

Now using (B.5),

$$L_{cr}(s) = L(s) - M_r(s). \quad (F.4)$$

Hence

$$L(s) - L_{cr}(s) = M_r(s).$$

APPENDIX G: PROOF OF THEOREM 9

We first note the following:

$$sI_n - A \equiv \Phi^{-1} = \begin{bmatrix} \Phi_{11}^{-1} & -A_{12} \\ -A_{21} & \Phi_{22}^{-1} \end{bmatrix},$$

and hence

$$\Phi^{-1}\begin{bmatrix} 0 \\ I_{n-p} \end{bmatrix} = \Phi_a,$$

where

$$\Phi_a = \begin{bmatrix} -A_{12} \\ \Phi_{22}^{-1} \end{bmatrix}.$$

Thus

$$F\Phi\Phi_a = F_2. \quad (G.1)$$

Using A_{clr} as in (4.5), we have the following series of reductions:

$$\det[sI_{2n-p} - A_{clr}]$$
$$= \det\begin{bmatrix} \Phi_{22}^{-1} + K_rA_{12} + K_rB_1F_2 & -A_{21} + K_rB_1F_1 & -K_rA_{12} \\ B_1F_2 & \Phi_{11}^{-1} + B_1F_1 & -A_{12} \\ B_2F_2 & -A_{21} + B_2F_1 & \Phi_{22}^{-1} \end{bmatrix}$$
$$= \det\begin{bmatrix} \Phi_{22}^{-1} + K_rB_1F_2 & -A_{21} + K_rB_1F_1 & -K_rA_{12} \\ -A_{12} + B_1F_2 & \Phi_{11}^{-1} + B_1F_1 & -A_{12} \\ \Phi_{22}^{-1} + B_2F_2 & -A_{21} + B_2F_1 & \Phi_{22}^{-1} \end{bmatrix}$$
$$= \det\begin{bmatrix} \Phi_{22}^{-1} + K_rB_1F_2 & -A_{21} + K_rB_1F_1 & -K_rA_{12} \\ -A_{12} + B_1F_2 & \Phi_{11}^{-1} + B_1F_1 & -A_{12} \\ (B_2 - K_rB_1)F_2 & (B_2 - K_rB_1)F_1 & \Phi_{22}^{-1} + K_rA_{12} \end{bmatrix}$$
$$= \det\begin{bmatrix} \Phi_{22}^{-1} + B_2F_2 & -A_{21} + B_2F_1 & \Phi_{22}^{-1} \\ -A_{12} + B_1F_2 & \Phi_{11}^{-1} + B_1F_1 & -A_{12} \\ (B_2 - K_rB_1)F_2 & (B_2 - K_rB_1)F_1 & \Phi_{22}^{-1} + K_rA_{12} \end{bmatrix}$$
$$= \det\begin{bmatrix} \Phi_{11}^{-1} + B_1F_1 & -A_{12} + B_1F_2 & -A_{12} \\ -A_{21} + B_2F_1 & \Phi_{22}^{-1} + B_2F_2 & \Phi_{22}^{-1} \\ (B_2 - K_rB_1)F_1 & (B_2 - K_rB_1)F_2 & \Phi_{22}^{-1} + K_rA_{12} \end{bmatrix}$$
$$= \det\begin{bmatrix} \Phi^{-1} + BF & \Phi_a \\ (B_2 - K_rB_1)F & \Phi_{22}^{-1} + K_rA_{12} \end{bmatrix}$$
$$= \det[\Phi_{22}^{-1} + K_rA_{12}] \cdot \det\{\Phi^{-1} + BF - \Phi_a(\Phi_{22}^{-1} + K_rA_{12})^{-1}(B_2 - K_rB_1)F\}$$
$$= \det[\Phi_{22}^{-1} + K_rA_{12}]\det[\Phi^{-1}] \cdot \det\{I_n + [\Phi B - \Phi\Phi_a(\Phi_{22}^{-1} + K_rA_{12})^{-1}(B_2 - K_rB_1)]F\}$$
$$= \det[\Phi_{22}^{-1} + K_rA_{12}]\det[\Phi^{-1}] \cdot \det\{I_m + F\Phi B - F\Phi\Phi_a(\Phi_{22}^{-1} + K_rA_{12})^{-1}(B_2 - K_rB_1)\} \quad (G.2)$$
$$= \det[\Phi_{22}^{-1} + K_rA_{12}]\det[\Phi^{-1}] \cdot \det\{I_m + F\Phi B - F_2(\Phi_{22}^{-1} + K_rA_{12})^{-1}(B_2 - K_rB_1)\} \quad (G.3)$$
$$= \det[\Phi_{22}^{-1} + K_rA_{12}]\det[\Phi^{-1}]\det\{I_m + F\Phi B - M_r(s)\}. \quad (G.4)$$

We used (G.1) in order to get (G.3) from (G.2). Noting that since $K_r \in \mathcal{K}_{er}$, $M_r(s) \equiv 0$ and hence

$$\begin{aligned}\det[sI_{2n-p} - A_{clr}] &= \det[\Phi_{22}^{-1} + K_r A_{12}] \det[\Phi^{-1}] \det\{I_m + F\Phi B\} \\ &= \det[\Phi_{22}^{-1} + K_r A_{12}] \det[\Phi^{-1}] \det\{I_m + \Phi BF\} \\ &= \det[\Phi_{22}^{-1} + K_r A_{12}] \det[\Phi^{-1} + BF].\end{aligned}$$

This proves the theorem.

APPENDIX H: PROOF OF THEOREM 10

Since $K_r(\sigma) \in \mathcal{K}_{ar}(\sigma)$, $M_r(s)$ tends to zero point-wise in s as $\sigma \to \infty$. Then from (G.4),

$$\begin{aligned}\det[sI_{2n-p} - A_{clr}] &= \det[\Phi_{22}^{-1} + K_r(\sigma)A_{12}] \det[\Phi^{-1}] \det\{I_m + F\Phi B - M_r(s)\} \\ &\to \det[\Phi_{22}^{-1} + K_r(\sigma)A_{12}] \det[\Phi^{-1}] \\ &\quad \cdot \det\{I_m + F\Phi B\} \text{ as } M_r(s) \to 0 \\ &= \det[\Phi_{22}^{-1} + K_r(\sigma)A_{12}] \det[\Phi^{-1}] \det\{I_n + \Phi BF\} \\ &= \det[\Phi_{22}^{-1} + K_r(\sigma)A_{12}] \det[\Phi^{-1} + BF].\end{aligned}$$

This proves the theorem.