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MATH 544/CPT S 531 MIDTERM EXAM

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SHOW ALL WORK

1. a) (12 pts.) Determine the permutation matrix P, unit lower triangular matrix L and the upper triangular matrix U that satisfy PA = LU and are produced when Gaussian elimination with partial pivoting is applied to the matrix A given by $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 2 & 0 \\ 1 & 3 & 2 \end{bmatrix}$.

Answer:

$A = P_1 M_1^{-1} P_2 M_2^{-1} U$

$PA = (P_2 P_1) P_1 M_1^{-1} P_2 M_2^{-1} U$

$\therefore P = P_2 P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ .5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & .5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ .5 & 1 & 0 \\ 0 & .5 & 1 \end{bmatrix}$

$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad P_1 A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$

$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -.5 & 0 & 1 \end{bmatrix}$ $\quad M_1 P_1 A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad P_2 M_1 P_1 A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -.5 & 1 \end{bmatrix}$ $\quad M_2 P_2 M_1 P_1 A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} = U$

b) (7 pts.) Briefly describe how P, L and U could be used to solve a linear system Ax = b if b was given.

$Ax = b$

$\Rightarrow PAx = Pb$

$PA = LU \Rightarrow PAx = LUx = Pb$

(1) solve $Ly = Pb$

(2) solve $Ux = y$

c) (3 pts.) How many flops are required to reduce an nxn matrix to upper triangular form using Gauss transformations?

$\frac{n^3}{3}$ flops

2. a) (10 pts.) Define the singular value decomposition of an mxn matrix A, and describe how the SVD can be used to determine the rank of A.

For any given $m \times n$ A. $p \triangleq \min(m, n)$.

$\exists$ an othogonal matrix U $(m \times m)$ and an orthogonal matrix V $(n \times n)$, and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_p)$ where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$. such that

$$A = U \Sigma V^T$$

The number of non-zero singular values is the rank of matrix A.

b) (6 pts.) Show that $||A||_2$ is equal to the largest singular value of A.

Suppose A has S.V.D.

$$A = U \Sigma V^T$$

by properties of of orthogonal matixes, we have $\|A\|_2 = \|\Sigma\|_2$

$$\|\Sigma\|_2 = \sup_{\|X\|_2 = 1} \|\Sigma X\|_2 = \sigma_1 \quad \text{when } X = e_1$$

because $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$

3. a) (4 pts.) Define the condition number for an nxn matrix.

$$k(A) = \|A\| \cdot \|A^{-1}\|$$

$$k_2(A) = \|A\|_2 \|A^{-1}\|_2 = \sigma_1 / \sigma_n$$

b) (6 pts.) If $\hat{x}$ is a computed solution to Ax = b when b is exact but A has errors contained in an error matrix E, how can the condition number be used to provide a bound for the relative error in x?

not in this bound

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq k(A) \cdot c_n \cdot \left(\frac{\|E\|}{\|A\|} + 0 \right)$$

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if b is exact.

where c_n is a function related to n.

4. a) (9 pts.) Define a Gauss transformation M, state why using Gauss transformations can increase errors, and why partial pivoting helps to reduce the possible increase in errors during Gaussian elimination.

Gauss transformation

$$M_K = I - \alpha e_K^T \quad \text{where}$$

$$\alpha^T = \{0, 0, \cdots, 0, x_{K+1}/x_K, \cdots, x_n/x_K]$$

If x_K is small, this will increase the errors in computing $x_{K+1}/x_K, \cdots, x_n/x_K$.

If we apply partial pivoting that is find $x_K' = \max(x_K, \cdots, x_n)$, then it may reduce the errors in divisions

b) (5 pts.) If M is a Gauss transformation, how can M^{-1} be directly determined?

If M is Gauss transformation: $M = I - \alpha e_K^T$

Then $M^{-1} = I + \alpha e_K^T$

c) (3 pts.) Why don't orthogonal transformations increase errors in the way that Gauss transformations sometimes do?

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Orthogonal transformations are much stable than non-orthogonal transfomations.

5. (14 pts.) Write down the general form for i) Householder and ii) Givens transformations and show that both transformations are orthogonal.

Householder transformation:

$$P = I - 2VV^T/V^TV$$

$$PP^T = (I - 2VV^T/V^TV)(I - 2VV^T/V^TV)$$

$$= I - 4VV^T/V^TV + 4V\cancel{(V^TV)}V^T/\cancel{(V^TV)^2}$$

$$= I \Rightarrow P \text{ is orthogonal.}$$

Givens transformation:

$$J(i,k,\theta) = \begin{bmatrix} 1 & & & & 0 \\ & c & & -s & \\ & & \ddots & & \\ & -s & & c & \\ 0 & & & & 1 \end{bmatrix} \begin{array}{l} \\ i \\ \\ k \\ \\ \end{array} \qquad c^2 + s^2 = 1$$

$$\begin{array}{cc} i & k \end{array}$$

$$J^T(i,k,\theta) * J(i,k,\theta) = \begin{bmatrix} 1 & & & \\ & c & -s & \\ & s & c & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & c & s & \\ & -s & c & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & c^2+s^2 & 0 & \\ & 0 & s^2+c^2 & \\ & & & 1 \end{bmatrix} = I$$

Givens is orthogonal.

6. (9 pts.) Determine the Cholesky decomposition for the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

$$G = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \quad G^T = \begin{bmatrix} g_{11} & g_{21} & g_{31} \\ 0 & g_{22} & g_{32} \\ 0 & 0 & g_{33} \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$GG^T = \begin{bmatrix} g_{11}^2 & g_{11} \cdot g_{21} & g_{11} \cdot g_{31} \\ \times & g_{21}^2 + g_{22}^2 & g_{21} \cdot g_{31} + g_{22} \cdot g_{32} \\ \times & \times & g_{31}^2 + g_{32}^2 + g_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ & 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$g_{11} = 1$, $g_{21} = 2$, $g_{31} = 0$

$g_{22} = 1$ $\quad g_{32} = 1$

$g_{33} = 0$.

7. (12 pts.) Assume P is mxm and P = I - vv^T, with the first k components of v equal to zero. If A is mxn, show that number of flops required to compute PA is 2n(m-k).

$$PA = A - VV^TA$$

$$V^TA = [0, \cdots, 0, V_{k+1}, \cdots, V_m] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$\underbrace{VV^TA = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ V_{k+1} \\ \vdots \\ V_m \end{bmatrix} \underbrace{\Big[\underbrace{\sum_{p=k+1}^{m} V_p \cdot a_{p1}}_{(m-k)\text{ flops}}, \cdots, \underbrace{\sum_{p=k+1}^{m} V_p \cdot a_{pn}}_{(m-k)\text{ flops}} \Big]}_{n(m-k)\text{ flops}}}_{(m-k) \cdot n}$$

Total flops $n(m-k) + n(m-k)$

$\approx 2n(m-k) + O(\;)$

checked