- <u>Isometry</u> : (X, d) and $(\tilde{X}, \tilde{d})$ = metric spaces.

  $T: X \to \tilde{X}$ is called an isometry if $(\forall x, y \in X)$

  $d(x,y) = \tilde{d}(Tx, Ty)$.

  * Isometry is 1−1 map.

- (X, d) and $(\tilde{X}, \tilde{d})$ are isometric if $\exists$ an isometry $T: X \to \tilde{X}$ onto $\tilde{X}$.

▲ Let (X, d) = metric space. Then [$\exists$ $(\hat{X}, \hat{d})$ which is complete metric space] and a subspace W such that W is isometric to X and W is dense in $\hat{X}$. Furthermore, $\hat{X}$ is unique up to isometry. And $(\hat{X}, \hat{d})$ is called the completion of (X, d).

- <u>Normed Space</u> : Let X be a vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), i.e., $x + y \in X$, $\forall x, y \in X$ and $\alpha x \in X$, $\forall x \in X$ and $\alpha \in \mathbb{F}$. A norm on X is a function $X \longmapsto \mathbb{R}$ $(x \longmapsto \|x\|)$ satisfying:

  (N1) $\quad \|x\| > 0 \quad$ if $\quad x \neq 0$

  (N2) $\quad \|x\| = 0 \quad$ if $\quad x = 0$

  (N3) $\quad \|\alpha x\| = |\alpha| \cdot \|x\| \quad (\forall x \in X)$ and $(\forall \alpha \in \mathbb{F})$

  (N4) $\quad \|x + y\| \leq \|x\| + \|y\| \quad (\forall x, y \in X)$.

  $(X, \|\cdot\|)$ is called a normed space, which is also a metric space with usual metric $d(x,y) = \|x - y\|$.

- <u>Banach Space</u> is a complete normed space.

▲ $\big|\|x\| - \|y\|\big| \leq d(x,y) \quad (\forall x, y \in X)$

▲ $\|\cdot\|$ is a Lipchitz continue function with $L = 1$.

▲ Hölder Inequality for Sequences:

$x = (x_i)$, $y = (y_i)$, $\frac{1}{p} + \frac{1}{q} = 1$. $\Longrightarrow$ $(p=1, q=\infty)$

$$\sum_{i=1}^{\infty} |x_i y_i| \le \|x\|_p \cdot \|y\|_q = \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p} \cdot \sqrt[q]{\sum_{i=1}^{\infty} |y_i|^q}.$$

(*) $$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}.$$ (Finite sume)

▲ Minkowski Inequality for Sequences

Let $1 \le p \le \infty$, $x, y \in l_p$. Then $x + y \in l_p$ and

$$\|x+y\|_p \le \|x\|_p + \|y\|_p.$$

▲ l_p, $1 \le p \le \infty$, is complete. l_p, $1 \le p < \infty$, is spearable, i.e., having a coutable dense subset. l_∞ is not separable. $C[a,b]$ is separable Banach space.

● Bases

● Hamel Basis X = vector space $\{e_s \mid s \in S\}$ is a Hamel Basis iff $(\forall x \in X)$ $\exists ! \{s_1, s_2, \cdots, s_n\}$ $! \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F}$ such that $x = \sum_{i=1}^{n} \alpha_i e_{s_i}$.

(*) Hamel Basis always exists.

● Schauder Basis, $e_1, e_2, e_3, \cdots$ is a Schauder basis if $(\forall x \in X)$ $(\exists ! \alpha_1, \alpha_2, \cdots \in \mathbb{F})$ $x = \sum_{i=1}^{\infty} \alpha_i e_i$.

▲ A Banach space with a Schauder basis is separable. But $\exists$ separable Banach space with no Schauder basis.

§ 2.4 Finite Dimensional Normed Spaces

Y = vector space over $\mathbb{F}$ of dimension $k<\infty$. $\|\cdot\|$ = norm of Y.

Let $v_1, v_2, \cdots, v_k$ be a basis. $\forall y \in Y$, $y = \sum_{i=1}^{k} \alpha_i v_i$ $(\exists !\, \alpha_i)$

Then ($\exists\, C>0$ and $c>0$) such that

$$c\left(\sum_{i=1}^{k} |\alpha_i|\right) \le \left\| \sum_{i=1}^{k} \alpha_i v_i \right\| \le C\left(\sum_{i=1}^{k} |\alpha_i|\right).$$

$\forall\, \alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{F}$, i.e., c and C are independent of α_i.

▲ $(Y, \|\cdot\|)$ finite dimensional $\Rightarrow$ Y is complete.

▲ X = normed linear space, Y = subspace of X.

If $\dim(Y) < \infty \Rightarrow Y$ is a closed subspace of X.

● X = vector space, $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on X.

The norms are equivalent if ($\exists\, m, M > 0$) such that

$$m\|x\|_b \le \|x\|_a \le M\|x\|_b \quad (\forall x \in X).$$

▲ Equivalent norms induce the same topology.

▲ Y = finite dimensional space. any norms are equivalent.

§ 2.5 Compactness and Finite Dimension

▲ M = metric space, $K \subseteq M$, K is compact $\Rightarrow$ K is closed and bounded.

▲ Let X = normed space and $\dim(X) < \infty$, $K \subseteq X$

K is compact $\Leftrightarrow$ K is closed and bounded.

- $\mathcal{B} = \{x \in X \mid \|x\| \le 1\}$

▲ <u>Riesz Lemma</u> $X =$ normed linear space. $Y, Z =$ subspaces of X. Y is closed and is a proper subset of Z. Let $0 < \theta < 1$. Then $(\exists\, z \in Z)$ $\|z\| = 1$ and $d(z, Y) \ge \theta$.

▲ X finite dimensional $\Longleftrightarrow$ $\mathcal{B}$ is compact.

▲ X finite dimensional $\Longleftrightarrow$ X is locally compact.

<u>§ 2.6 Linear Oprators</u>

- Let X, Y be vector spaces over the same field $\mathbb{F}$. A <u>linear operator</u>

$$T : X \longrightarrow Y$$

is a map whose domain $\mathcal{D}(T)$ is a <u>subspace</u> of X and whose range $\mathcal{R}(T)$ is a <u>subset</u> of Y. such that

$$T(x_1 + x_2) = Tx_1 + Tx_2, \quad \forall\, x_1, x_2 \in \mathcal{D}(T)$$
$$T(\alpha x) = \alpha T x, \quad \forall\, x \in \mathcal{D}(T), \ \forall \alpha \in \mathbb{F}.$$

▲ $\mathcal{R}(T)$ is a <u>subspace</u> of Y.

- <u>Null space</u> $\quad n(T) = \{x \in \mathcal{D}(T) \mid Tx = 0\}$.

▲ $n(T)$ is a <u>subspace</u> of X.

§ 2.7. <u>Bounded Linear Operator</u>

- X and Y = normed spaces over $\mathbb{F}$.

$T : X \longrightarrow Y$ is a linear operator.

T is <u>bounded</u> if $(\exists\, c)$

$$\frac{\|Tx\|}{\|x\|} \le c \qquad \forall\, x \in \mathcal{D}(T) \setminus \{0\}.$$

- X and Y = normed spaces over $\mathbb{F}$.

  $\mathcal{B}(X,Y)$ = set of <u>bounded linear operators</u>: $T: X \to Y$ such that $\mathcal{D}(T) = X$. Moreover defining

  $$(T_1 + T_2)x = T_1 x + T_2 x \quad \text{and} \quad (\alpha T)x = \alpha(Tx)$$

▲ $\forall\, T_1$ and $T_2 \in \mathcal{B}(X,Y) \Rightarrow T_1 + T_2 \in B(X,Y)$

$T \in B(X,Y)$, $\alpha \in \mathbb{F} \Rightarrow \alpha T \in B(X,Y)$

$\Rightarrow B(X,Y)$ is a vector space over $\mathbb{F}$.

- <u>Operator norm</u> $\quad T: X \to Y$

$$\|T\| = \sup_{x \in \mathcal{D}(T) \setminus \{0\}} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|.$$

▲ The operator norm is a norm on $B(X,Y)$.

▲ If Y is complete then $B(X,Y)$ is complete.

▲ $T: X \to Y$, X is finite dimensional $\Rightarrow T$ is bounded.

▲ T is continuos at one point $\Rightarrow T$ is bounded

▲ <u>The following statements are equivalent</u>

(i) T is bounded

(ii) T is Lipsitz continuous on $\mathcal{D}(T)$.

(iii) T is continuous at one point.

(iv) T maps bounded sets to bounded sets.

(v) $T(B \cap \mathcal{D}(T))$ is bounded, where B is unit ball.

▲ $T \in B(X,Y) \Longrightarrow n(T)$ is a closed subspace of X.

▲ $X, Y =$ normed space, $Y =$ Banach. $T: X \to Y$ bounded linear operator with $\mathcal{D}(T)$ dense in X. Then $\exists!$ bounded extension $\hat{T}$, $\hat{T} \in B(X, Y)$ and $\|\hat{T}\| = \|T\|$.

§2.8 Linear Functionals

- A functional on X is a map $f: X \to \mathbb{F}$ that is a linear operator.
- The algebraic dual of X is the set of all linear functionals $f: X \to \mathbb{F}$ such that $\mathcal{D}(f) = X$. (It's denoted by X^*).
- $X^{**} =$ algebraic dual of X^*

The following note gives details of X^* and X^{**}:

- $f_1, f_2 \in X^*$, sum of f_1 and f_2 is defined as

$$s(x) = (f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \forall x \in X.$$

product αf of a scalar α and a functional f is defined

$$p(x) = (\alpha f)(x) = \alpha f(x) \quad \forall x \in X.$$

- Choosing a fixed $x \in X$, define a linear functional on X^*

$$g_x(f) = f(x) \quad \forall f \in X^{**}$$

Since $g_x(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x)$

$$= \alpha g_x(f_1) + \beta g_x(f_2)$$

$\Rightarrow g_x \in X^{**}$.

- Define a mapping $C: X \mapsto X^{**}$, $x \mapsto g_x$ (canonical)

$$(C(\alpha x + \beta y))(f) = g_{(\alpha x + \beta y)}(f) = f(\alpha x + \beta y)$$

$$= \alpha f(x) + \beta f(y) = \alpha g_x + \beta g_y$$

"C is 1-1 mapping"

$$= (\alpha(Cx) + \beta(Cy))(f). \Rightarrow C \text{ is linear.}$$

▲ C is a vector space isomophism of X onto $R(C) \subseteq X^{**}$.

● X is algebraically reflexive if $R(C) = X^{**}$.

§2.9 Linear Functional on Finite Dimensional Spaces

Let X = vector space of dimension n over $\mathbb{F}$.

Let $v_1, \cdots, v_n$ be a basis of X

Let $f \in X^*$, Let $x = \sum_{i=1}^{n} c_i v_i \in X$

$f(x) = f(\sum_{i=1}^{n} c_i v_i) = \sum_{i=1}^{n} c_i f(v_i)$. Let $\alpha_i = f(v_i) \in \mathbb{F}$

$= \sum_{i=1}^{n} c_i v_i = [\alpha_1, \alpha_2, \cdots, \alpha_n] \cdot [c_1, c_2, \cdots, c_n]'$

Conversely, given any $\alpha \in \mathbb{F}^n$, we can defined a functional f_α by $f_\alpha(x) = f_\alpha(\sum_{i=1}^{n} c_i v_i) = \sum_{i=1}^{n} c_i \alpha_i$

▲ f_α is a linear functional, i.e., $f_\alpha(ax+by) = a f_\alpha(x) + b f_\alpha(y)$.

▲ $f \mapsto \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ $(\alpha_i = f(v_i), i = 1, \cdots, n)$

is linear, 1-1, onto. (E.F.Y.)

$\Rightarrow$ So it's a vector space isomophism between X^* and $\mathbb{F}^n$

$\Rightarrow \dim(X^*) = n$.

● For $i = 1, \cdots, n$. Let f_i be the functional associated with $\alpha = (0, \cdots, 0, 1, 0, \cdots, 0)$

$f_i(\sum_{j=1}^{n} c_j v_j) = \sum_{j=1}^{n} c_j \alpha_j = c_i$ $\quad f_i(v_j) = \delta_{ij}$

$\Rightarrow f_1, \cdots, f_n$ form a basis for X^*, the dual basis of $v_1, \cdots, v_n$.

▲ $\dim(X) < \infty \Rightarrow X$ is algebraically reflexive, i.e., $R(C) = X^{**}$.

§ 2.10 The dual Space (Normed dual Space)

- X = normed space over $\mathbb{F}$.

  $X' = B(X, \mathbb{F})$ normed dual space of X, bounded, cts.

- X, Y = normed spaces over $\mathbb{F}$. An isomorphism is a linear map $T: X \to Y$ such that T is 1–1, $\mathcal{D}(T) = X$, $\mathcal{R}(T) = Y$ and $\|Tx\| = \|x\|$, $\forall x \in X$.

Chapter 3. Inner Product Spaces, Hilbert Spaces

- X = vector space over $\mathbb{F}$. An inner product on X is a map

$$\begin{aligned} X \times X &\longrightarrow \mathbb{F} \\ x, y &\longrightarrow \langle x, y \rangle \end{aligned}$$

satisfying

$$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$$

$$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

$$\langle x, x \rangle > 0 \quad \text{if} \quad x \neq 0$$

"norm" $\|x\| = \sqrt{\langle x, x \rangle}$, inner product $\Rightarrow$ norm $\Rightarrow$ metric.

▲ $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$

$\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$.

▲ $\mathrm{Re} \langle x, y \rangle = \frac{1}{4} [\|x+y\|^2 - \|x-y\|^2]$

$\mathrm{Im} \langle x, y \rangle = \frac{1}{4} [\|x+iy\|^2 - \|x-iy\|^2]$.

▲ (Parallelogram Law) X = normed space. Then X is a inner product space iff $\|x+y\|^2 + \|x-y\|^2 = 2\cdot\|x\|^2 + 2\cdot\|y\|^2$, $\forall x, y \in X$.

- $x, y \in X$ = inner product space, x is <u>orthogonal</u> (perpendicular, $x \perp y$) if $\langle x, y \rangle = 0$.

▲ (Pythagorean): X = inner product space, $x, y \in X$, $x \perp y$

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

§3.2 *Further Properties*

▲ (<u>Schwarz Inequality</u>): X = inner product space, $x, y \in X \Rightarrow$

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Equality holds iff x and y are linearly dependent.

▲ (<u>Triangle inequality</u>) X = inner product space $\Rightarrow$

$$\|x+y\| \leq \|x\| + \|y\|, \qquad \forall x, y \in X.$$

▲ The inner product is a continuous functions.

- Hilbert space is a complete inner product space.

▲ Every inner product space has a completion that is a Hilbert space (i.e., it can be embeded in a Hilbert space densely) The completion is essentially unique.

§3.3 <u>Orthogonal Complements and Direct Sums</u>

- X = vector space over $\mathbb{F}$. $x, y \in X$. The <u>segment</u> from x to y is $\{\alpha x + (1-\alpha) y \mid 0 \leq \alpha \leq 1\} = \{y + \alpha(x-y) \mid 0 \leq \alpha \leq 1\}$.
- $M \subseteq X$ is <u>convex</u> if $(\forall x, y \in M)$. The segment from x to y lies in M. (e.g. M is a subspace of X, M is convex).

▲ X = Hilbert space, M = nonempty, closed convex subset of X, $x \in X$.

Then $(\exists!\, y \in M)$ $\|x-y\| = \inf_{\hat{y}\in M} \|x-\hat{y}\|$. (Best approximation).

▲ (Projection Theorem) X = Hilbert space. Y = closed subspace.

$x \in X \Rightarrow (\exists!\, y \in Y)$ $x - y \perp Y$. y is the best approx. to x from Y. y is called the <u>orthogonal projection</u> of x onto Y.

● X = vector space, Y, Z subspaces of X, we say that X is the direct sum of Y and Z and write

$$X = Y \oplus Z$$

if $(\forall x \in X)(\exists!\, y \in Y, z \in Z)$ $x = y + z$.

Y, Z are called complementary subspaces, of X.

▲ X = Hilbert space, M = anyset $\subseteq X$, $M^{\perp}$ = orthogonal complement

$= \{x \in X \mid \langle x, y\rangle = 0,\ \forall y \in M\}$

$\Rightarrow$ $M^{\perp}$ is a subspace of X and is closed. $M \subseteq M^{\perp\perp}$.

▲ X = Hilbert space, Y = closed subspace. Then

$$X = Y \oplus Y^{\perp}.$$

▲ If Y is a closed subspace of X, then $Y = Y^{\perp\perp}$.

▲ V = subspace of X, which is a Hilbert space $\Rightarrow$ $\bar{V}$ = subspace

(i) $V^{\perp} = \bar{V}^{\perp}$

(ii) V is dense in X iff $V^{\perp} = \{0\}$.

● <u>Projectors</u>: X = vector space over $\mathbb{F}$, $X = S_1 \oplus S_2$. $x = s_1 + s_2$

Define: $P_1 : X \to X$, $P_1 x = s_1$ and $P_2 : X \to X$, $P_2 x = s_2$

P_1 and P_2 are linear.

P_1 (P_2) is called the projection of X onto S_1 (S_2) in the direction of S_2 (S_1).

$\mathcal{R}(P_1) = S_1$, $\mathcal{N}(P_1) = S_2$, $\mathcal{R}(P_2) = S_2$, $\mathcal{N}(P_2) = S_1$.

● $A : X \to X$ is called an <u>idempodent</u> if $A^2 = A$.

▲ Projectors are idempodents. Every idempodent is a projector.

▲ X = Banach space, S_1 & S_2 are closed subspaces. $X = S_1 \oplus S_2$
$\Rightarrow P_1, P_2$ are bounded.

▲ X = Hilbert space, S = closed subspace, $S_1 = S$, $S_2 = S^\perp$, $X = S_1 \oplus S_2$
$\Rightarrow P_1, P_2$ are bounded, $\|P_1\| \le 1$ and $\|P_2\| \le 1$.

§3.4 Orthonormal Families

● <u>Orthonormal family</u> $(x_\alpha)_{\alpha \in I}$, $x_\alpha \in X$ and $\langle x_\alpha, x_\beta \rangle = \begin{cases} 0 & \text{if } \alpha \ne \beta \\ 1 & \text{if } \alpha = \beta. \end{cases}$

▲ $(x_\alpha)_{\alpha \in I}$ orthonormal $\Rightarrow$ $(x_\alpha)_{\alpha \in I}$ is linearly independent.

▲ $e_1, e_2, \cdots, e_n$ orthonormal family, $Y = \text{span}\{e_1, e_2, \cdots, e_n\}$, $y \in Y$
$y = \sum_{k=1}^{n} \alpha_k e_k \Rightarrow \alpha_i = \langle y, e_i \rangle \Rightarrow y = \sum_{k=1}^{n} \langle y, e_k \rangle e_k$.

▲ $x \in X$. Let $y = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$. y is the best approx. to x from Y.

▲ <u>Gram-Schmidt Process</u>

Let (x_k) be a linearly independent sequence. for $n = 0, 1, 2, \cdots$.

$$\begin{cases} V_{n+1} = x_{n+1} - \sum_{k=1}^{n} \langle x, e_k \rangle e_k \\ \text{If } V_{n+1} = 0 \text{ stop. Otherwise } e_{n+1} = V_{n+1} / \|V_{n+1}\|. \end{cases}$$

§3.5 Series Related to Orthonormal Sequences

▲ H = Hilbert space, (e_k) = orthonormal sequence in H. Then

$$\sum_{k=1}^{\infty} \alpha_k e_k \text{ converges iff } \sum_{k=1}^{\infty} |\alpha_k|^2 \text{ converges.}$$

▲ If $\sum_{k=1}^{\infty} \alpha_k e_k$ converges and $x = \sum_{k=1}^{\infty} \alpha_k e_k \Rightarrow \alpha_k = \langle x, e_k \rangle$.

▲ Let $x \in X$, Let $Y = \overline{\text{span}\{e_1, e_2, \cdots\}}$. Let $y = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$. Then y is the best approximation to x from Y.

▲ $x \in X \Rightarrow \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ converges.

▲ The value of $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ is independent of the order of (e_k).

● Uncountable Orthonormal Families

X = inner product space, $(e_\beta)_{\beta \in I}$ is an orthonormal family in X.

▲ There are at most countably many $\langle x, e_\beta \rangle$ are non-zero.

§3.6 Total Orthonormal Families

● $(e_\beta)_{\beta \in I}$ is a total orthonormal family if $\text{span}\{e_\beta \mid \beta \in I\}$ is dense in H, i.e., $\overline{\text{span}\{e_\beta \mid \beta \in I\}} = H$.

▲ $(e_\beta)_{\beta \in I}$ is total orthonormal family iff $(\forall x \in H)$

$$x = \sum_{\beta \in I} \langle x, e_\beta \rangle e_\beta .$$

- Orthonormal Basis = total orthonormal family (Hilbert basis)

Remark: this is a Schauder basis, not a Hamel basis.

▲ $(e_\beta)_{\beta\in I}$ is an orthonormal basis iff $\{e_\beta \mid \beta\in I\}^\perp = \{0\}$.

▲ <u>Parseval Theorem</u> $(e_\beta)_{\beta\in I}$ is an orthonormal basis iff $(\forall x \in H)$

$$\|x\|^2 = \sum_{\beta\in I} |\langle x, e_i\rangle|^2 .$$

▲ Every Hilbert space has an orthonormal basis.

▲ Any two orthonormal bases of H have the same Cardinality.

- H_1, H_2 = Hilbert spaces. An <u>isomorphism</u> of H_1 and H_2 is a linear map $T: H_1 \to H_2$ ($\mathcal{D}(T) = H_1$, $\mathcal{R}(T) = H_2$) such that $\langle Tx, Ty\rangle = \langle x, y\rangle \quad \forall x, y \in H_1$

* Consequences: $\|Tx\| = \|x\| \ \forall x \in H_1$ implies T is 1-1.

▲ H_1, H_2 = Hilbert spaces over $\mathbb{F}$. Then H_1 and H_2 are isomorphic iff they have the same dimensions.

▲ H = Hilbert space. H is separable iff its dimension is at most countably infinite.

<u>§3.7. Orthogonal Polynomials</u>

▲ P = polynormials, viewed as a subspace of $L_2(-1,1)$. P is dense in $L_2(-1,1)$.

§3.8 Riesz Representation Theorem

▲ Let $y \in H$ = Hilbert space. Define $f_y : H \to \mathbb{F}$ by

$$f_y(x) = \langle x, y \rangle$$

Then f_y is bounded linear with $\|f_y\| = \|y\|$.

▲ Riesz Representation Theorem: H = Hilbert space, $f \in H'$.

Then $(\exists!\ y \in H)$ $f(x) = \langle x, y \rangle$ $(\forall x \in H)$. $\|f\| = \|y\|$.

Sesquilinear Forms

X, Y = vector space over $\mathbb{F}$.

A sesquilinear form is a map $h : X \times Y \to \mathbb{F}$ such that

$h(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 h(x_1, y) + \alpha_2 h(x_2, y)$ $(\forall x_1, x_2 \in X)(\forall y \in Y)(\forall \cdots$

$h(x, \alpha_1 y_1 + \alpha_2 y_2) = \bar{\alpha}_1 h(x, y_1) + \bar{\alpha}_2 h(x, y_2)$ $(\forall x \in X)(\forall y_1, y_2 \in Y) \cdots$

$$\|h\| = \sup\left\{ \frac{|h(x,y)|}{\|x\| \cdot \|y\|} \;\middle|\; \begin{array}{ll} x \in X, & x \neq 0 \\ y \in Y, & y \neq 0 \end{array} \right\} \quad (h \text{ is bounded if } < \infty).$$

▲ $T \in B(X, Y)$. $h : X \times Y \to \mathbb{F}$ by $h(x, y) = \langle Tx, y \rangle \Rightarrow \|h\| = \|T\|$.

▲ Riesz Representation Theorem for Sesquilinear Forms

H_1, H_2 = Hilbert spaces over $\mathbb{F}$.

$h : H_1 \times H_2 \to \mathbb{F}$ is a bounded sesquilinear form. Then

(a) $(\exists!\ T \in B(H_1, H_2))$ $h(x, y) = \langle Tx, y \rangle$

$(\forall x \in H_1)$ $(\forall y \in H_2)$ and $\|h\| = \|T\|$.

(b) $(\exists!\ S \in B(H_2, H_1))$ $h(x, y) = \langle x, Sy \rangle$

$(\forall x \in H_1)$ $(\forall y \in H_2)$ and $\|h\| = \|S\|$.

§3.9 The adjoint Operator (Hilbert-adjoint)

H_1, H_2 = Hilbert spaces. Let $T \in B(H_1, H_2)$. $h: H_1 \times H_2 \to \mathbb{F}$ by $h(x,y) = \langle Tx, y\rangle$. By Riesz, $(\exists !\ T^* \in B(H_2, H_1))$

$$h(x,y) = \langle Tx, y\rangle = \langle x, T^*y\rangle.$$

T^* is called the adjoint of T.

▲ $\|T^*\| = \|T\|$, $T^{**} = T$,

▲ $S: H_1 \to H_2$, $T: H_1 \to H_2$, $S, T \in B(H_1, H_2)$

(a) $(S+T)^* = S^* + T^*$, (b) $(\alpha S)^* = \bar{\alpha} S^*$.

▲ $\|T^*T\| = \|T\|^2 = \|TT^*\|$,

▲ $T^*T = 0$ iff $T = 0$.

▲ $S \in B(H_1, H_2)$, $T \in B(H_2, H_3) \Rightarrow TS \in B(H_1, H_3)$ & $(TS)^* = S^*T^*$.

§3.10 Normal Operator

H = Hilbert space, $T \in B(H,H) \Rightarrow T^* \in B(H,H)$.

① T is Hermitian if $T = T^*$ (self-adjoint)

② T is skew-Hermitian if $T^* = -T$.

③ T is unitary if $T^*T = I = TT^*$.

④ T is normal if $TT^* = T^*T$. (① – ③ are normal).

$S, T \in B(H,H)$ are unitary similar if ($\exists$ an unitary $U \in B(H,H)$)

$$S = U^{-1}TU = U^*TU.$$

▲ $\Rightarrow$ (a) T is Hermitian $\Leftrightarrow$ S is Hermitian

(b) T is skew .. $\Leftrightarrow$ S is skew ..

(c) T is unitary $\Leftrightarrow$ S is unitary.

(d) T is normal $\Leftrightarrow$ S is normal.

Hermitian Operators

▲ $T = T^* \Rightarrow \langle Tx, x\rangle \in \mathbb{R}\ \forall x \in H$ (If $\mathbb{F} = \mathbb{C}$, $\Leftarrow$ holds).

● $T \in B(H,H)$, $T = T^*$, T is called positive definite if

$\langle Tx, x\rangle > 0 \quad (\forall x \in H,\ x \neq 0)$.

▲ S, T are Hermitian $(S, T \in B(H,H))$. ST is Hermitian iff $ST = TS$.

▲ $(T_n) \subseteq B(H,H)$, $T \in B(H,H)$

$T_n \to T \quad (\lim_{n\to\infty} \|T_n - T\| = 0)$

If $T_n^* = T_n\ \forall n$, then $T^* = T$.

Skew-Hermitian Operators

▲ If $\mathbb{F} = \mathbb{C}$, T is skew Hermition $\Leftrightarrow \pm iT$ is Hermitian.

▲ T is skew Hermitian $\Rightarrow \langle Tx, x\rangle$ is purely imaginary $\forall x \in H$.
(If $\mathbb{F} = \mathbb{C}$, the converse holds).

▲ $T_n \to T$. If $T_n^* = -T_n\ \forall n$, then $T^* = -T$.

Unitary Operators

($U^*U = I = UU^*$, U is 1-1, onto and $U^* = U^{-1}$).

▲ U is unitary $\Rightarrow \langle Ux, Uy\rangle = \langle x, y\rangle\ \forall x, y \in H$. $\|Ux\| = \|x\|$.

▲ U is an isometry iff $U^*U = I$.

▲ U is unitary iff U is isometric and surjective.

▲ U is unitary iff U is isometric and normal.

▲ $\dim(H) < \infty$, U is unitary iff U is isometric.

▲ U, V unitary $\Rightarrow UV$ unitary

▲ U, V isometric $\Rightarrow UV$ isometric.

Multiplication Operators

- $H = L_2(a,b)$ Let $\phi \in L_\infty(a,b)$

  $M_\phi : H \longrightarrow H$ (multiplication by ϕ) $M_\phi x = \phi x$ ($\phi(t)x(t)$)

▲ M_ϕ is linear, bounded, $\|M_\phi\| = \|\phi\|_\infty$.

$M_{\phi+\psi} = M_\phi + M_\psi$, $M_{\alpha\phi} = \alpha M_\phi$ ($\forall \alpha \in \mathbb{F}$)

$M_{\phi\psi} = M_\phi M_\psi$. $M_\phi M_\psi = M_\psi M_\phi$, $(M_\phi)^* = M_{\bar{\phi}}$.

▲ M_ϕ is normal

M_ϕ is Hermitian iff ϕ is real a.e.

M_ϕ is positive definite iff $\phi > 0$ a.e.

M_ϕ is skew-Hermitian iff ϕ is purely imaginary. a.e.

M_ϕ is unitary iff $|\phi(t)| = 1$ a.e.

<u>when M_ϕ is a projection ?</u>

- $H = \ell_2$, Let $s = (s_i) \in \ell_\infty$

  $M_s : H \longrightarrow H$ by $M_s x = sx = (s_i x_i)$

▲ M_s is linear, bounded, $\|M_s\| = \|s\|_\infty$. ......

Spectrum

- The resolvent set, $\rho(T)$, of T is the set of $\lambda \in \mathbb{C}$ such that $(\lambda - T)^{-1} \in B(H,H)$. $\{\lambda - T$ is 1-1, $(\lambda - T)^{-1}$ is bounded, $R(\lambda - T) = H\}$

  The <u>spectrum</u> of T, $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

  $\Rightarrow$ eigenvalues $\subseteq \sigma(T)$.

  If $\dim(H) < \infty$, $\{\text{eigenvalues}\} = \sigma(T)$.

Range and Null Space

$T \in B(H,H)$, $\quad T^* \in B(H,H)$

▲ $\mathcal{R}(T)^\perp = \mathcal{N}(T^*) \qquad \mathcal{R}(T^*)^\perp = \mathcal{N}(T)$.

$\mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)} \qquad \mathcal{N}(T^*)^\perp = \overline{\mathcal{R}(T)}$.

▲ $P \in B(H,H)$, $P^2 = P$ (Projector) $\Rightarrow$ P is an orthoprojector iff $P = P^*$ ※

§ Chapter 4. Fundamental Theorem

- M = set. A partial ordering on M is a binary relation $(\leq)$ such that (1) $a \leq a$ $\forall a \in M$, (2) $a \leq b$, $b \leq a \Rightarrow a = b$
  (3) $a \leq b$, $b \leq c \Rightarrow a \leq c$ $\quad \forall a, b, c \in M$.
  $(M, \leq)$ = partially ordered set.

- A total ordering is a partial ordering such that
  (4) $(\forall a, b \in M)$ $\quad a \leq b$ or $b \leq a$.

- M = partially ordered set, $W \subseteq M$, W is called a chain if W is totally ordered.

- $W \subseteq M$, $a \in M$, a is called an upper bound for W if $b \leq a$ $\quad \forall b \in M$.

- $a \in M$, a is called a maximal element of M if $b \in M$, $a \leq b \Rightarrow a = b$ $\quad \forall b \in M$.

Axiom (Zorn's Lemma)

M = partially ordered set. If every chain in M has an upper bound, then M has a maximal element.

▲ Every vector space has a Hamel basis.

▲ Every inner product space has an orthonormal basis.

§4.2 Hahn-Banach Theorem

● X = real vector space $\qquad Z$ = subspace of X.

$f: Z \to \mathbb{R}$ linear functional $\mathcal{D}(f) = Z$.

p = functional defined on X $\quad p: X \to \mathbb{R}$ is called sublinear if

$$\begin{cases} (1) & p(x+y) \le p(x) + p(y) \quad \forall x, y \in X \\ (2) & p(\alpha x) = \alpha p(x) \qquad (\forall x \in X)(\forall \alpha \ge 0) \end{cases}$$

▲ Hahn-Banach Theorem I (Real vector space)

X = real vector space, $\quad Z$ = subspace of X

p = sublinear functional on X.

f = linear fun. on Z such that $f(z) \le p(z)$ $\forall z \in Z$.

Then $(\exists \tilde{f})$ linear functional defined on X such that

$\tilde{f}(z) = f(z)$ $\forall z \in Z$ and

$\tilde{f}(x) \le p(x)$ $\forall x \in X$ (i.e. $\tilde{f}$ is dominated by p).

▲ Hahn-Banach Theorem II (Real vector space) (§4.3)

X = real vector space, p = sublinear functional defined on X such that $p(\alpha x) = |\alpha| p(x)$ $(\forall \alpha \in \mathbb{R})(\forall x \in X)$

Z = subspace of X. f = linear functional on Z such that

$|f(z)| \le p(z)$ $\forall z \in Z$

Then $(\exists \tilde{f})$ lin. fun. on X such that

$\tilde{f}(z) = f(z)$ $\forall z \in Z$ and

$|\tilde{f}(x)| \le p(x)$ $\forall x \in X$.

Linear Functional on Complex vector spaces

X = complex vector space, $X_r = X$ regarded as a real vector space.

$g : X \to \mathbb{C}$ linear functional

$g(x) = g_1(x) + i\, g_2(x)$

▲ g is linear $\Rightarrow$ g_1 and g_2 are real linear. (over $\mathbb{C}$)

$g_1 : X_r \to \mathbb{R}$, $g_2 : X_r \to \mathbb{R}$.

▲ $g_2(x) = -g_1(ix)$

▲ $g : X \to \mathbb{C}$ is linear $\Rightarrow g(x) = g_1(x) - i\, g_1(ix)$

▲ Let h be a linear functional on X_r ($h \in X_r^*$) and let

$$g(x) = h(x) - i\, h(ix)$$

Then g is a complex linear functional on X ($g \in X^*$)

▲ Hahn–Banach Theorem III (Complex vector space)

X = vector space over $\mathbb{C}$, $p(x)$ = real-valued sublinear functional on X such that $p(\alpha x) = |\alpha|\, p(x)$ ($\forall \alpha \in \mathbb{C}$).

Z = subspace of X, f = lin. fun. on Z such that $|f(z)| \le p(z)$ $\forall z \in Z$

Then $(\exists \tilde{f})$ on X such that $\tilde{f}(z) = f(z)$ $\forall z \in Z$ and $|\tilde{f}(x)| \le p(x)$ $\forall x \in X$.

▲ Hahn–Banach Theorem IV (Normed Space)

X = normed space over $\mathbb{F}$, Z is a subspace of X, $f \in Z'$. Then $(\exists \tilde{f} \in X')$ $\tilde{f}(z) = f(z)$ $\forall z \in Z$ and $\|\tilde{f}\| = \|f\|$.

{Hilbert space case} $X = H$, $Z \subseteq H$, $f : Z \to \mathbb{F}$ bounded.

Z is dense in $\overline{Z}$. f is uniformly cts on Z. $\Rightarrow \exists!\ \hat{f}$ on $\overline{Z}$ such that $\hat{f}$ is cts. $\hat{f}$ extends f. Also, $\|\hat{f}\| = \|f\|$.

▲ X = normed space, $x_0 \in X$, $x_0 \neq 0 \Rightarrow (\exists \tilde{f} \in X')$ $\|\tilde{f}\| = 1$ and $\tilde{f}(x_0) = \|x_0\|$.

▲ X = normed space, $x \in X$, $x \neq 0 \Rightarrow (\exists f \in X')$ $f(x) \neq 0$.

▲ If $(x \in X)$ $(f(x) = 0 \ \forall f \in X') \Rightarrow x = 0$

▲ $x_1, x_2 \in X$, $x_1 \neq x_2 \Rightarrow (\exists f \in X')$ $f(x_1) \neq f(x_2)$

▲ $x_1, x_2 \in X$, $(f(x_1) = f(x_2) \ \forall f \in X') \Rightarrow x_1 = x_2$.

▲ $(\forall x \in X)$ $\|x\| = \sup\limits_{f \in X', f \neq 0} \{ |f(x)| / \|f\| \}$.

§4.5 Adjoint Operator (Normed Space adjoint)

● X = normed space over $\mathbb{F}$.

Duality pairing: $x \in X$, $f \in X'$, $\langle x, f \rangle = f(x)$. Resembles inner product $x, f \longrightarrow \langle x, f \rangle : X \times X' \to \mathbb{F}$

▲ $|\langle x, f \rangle| \leq \|x\| \cdot \|f\| \quad \forall x \in X, \ \forall f \in X'$.

▲ Let $x \in X$. Then $(\exists f \in X')$ $\langle x, f \rangle = \|x\| \cdot \|f\|$.

● Let $S \subseteq X$. The annihilator of S is $S^\perp = \{ f \in X' \mid \langle x, f \rangle = 0 \ \forall x \in S \} \subseteq X'$

$= \{ f \in X' \mid f(S) = 0 \}$.

Let $S' \subseteq X'$. The annihilator of S' is

$${}^\perp S' = \{ x \in X \mid \langle x, f \rangle = f(x) = 0 \ \forall f \in S' \} \subseteq X$$

▲ $S^\perp$ is a closed subspace of X'

${}^\perp S'$ is a closed subspace of X.

▲ $S \subseteq {}^\perp(S^\perp)$, $\quad \overline{S} \subseteq {}^\perp(S^\perp)$

$S' \subseteq ({}^\perp(S'))^\perp$

$\overline{S'} = ({}^\perp(S'))^\perp$.

<u>Lemma</u>: X = normed space, Y = closed subspace of X

$x \in X \setminus Y$. Then $(\exists f \in X')$ $f(x) \neq 0$, $f(Y) = 0$.

▲ S = subspace of $X \Rightarrow \overline{S} = {}^{\perp}(S^{\perp})$

▲ S' = subspace of $X' \Rightarrow \overline{S'} = ({}^{\perp}S')^{\perp}$ (harder)

● X = normed space. Y = normed space. $T \in B(X, Y)$

The adjoint (conjugate) of T is an operator $T' : Y' \to X'$ defined by

$\langle Tx, f \rangle := \langle x, T'f \rangle$ $(\forall x \in X, \forall f \in Y')$

$Y \times Y'$ $\quad X \times X'$

$\Rightarrow f(Tx) = T'f(x)$

▲ T' is linear: $T' : Y' \to X'$

▲ T' is bounded and $\|T'\| = \|T\| \Rightarrow T' \in B(Y', X')$

▲ $\eta(T') = R(T)^{\perp}$

▲ $\eta(T) = {}^{\perp}R(T')$

▲ ${}^{\perp}\eta(T') = \overline{R(T)}$

▲ $\eta(T)^{\perp} = \overline{R(T')}$

▲ $S, T \in B(X, Y)$, $(S + T)' = S' + T'$

▲ $T \in B(X, Y)$, $\alpha \in \mathbb{F} \Rightarrow (\alpha T)' = \alpha T'$

▲ $S \in B(X, Y)$, $T \in B(Y, Z) \Rightarrow (TS)' = S'T'$

▲ $I \in B(X, X)$, $Ix = x \Rightarrow I' \in B(X', X')$. $I'f = f$.

▲ $T \in B(X, Y)$, T is bijective, $T^{-1} \in B(Y, X)$

$\Rightarrow (T^{-1})'$ exists and $(T^{-1})' = (T')^{-1}$.

<u>Remark</u>

$$\boxed{\begin{array}{l} ST = I \Rightarrow T \text{ is (1-1) injective} \\ TS = I \Rightarrow T \text{ is surjective (onto)} \end{array}}$$

Relationship with Hilbert-space Adjoint

$$
\begin{tikzcd}
H_2 \arrow[r, "T^*"] \arrow[d, "C_2"'] & H_1 \arrow[d, "C_1"] \\
H_2' \arrow[r, "T'"] & H_1'
\end{tikzcd}
$$

$$\Rightarrow \quad T' = C_1 T^* C_2^{-1} \quad \text{and} \quad T^* = C_1^{-1} T' C_2 .$$

§4.6 Reflexivity

X = normed space. Given $x \in X$ define $g_x : X' \to \mathbb{F}$ by $g_x(f) = f(x)$.

▲ g_x is linear, bounded and $\|g_x\| = \|x\|$.

● The canonical map of X into X'' is

$$C(x) = g_x \quad : X \longrightarrow X''$$

▲ C is a linear map, norm preserving.

C is an isomorphism of X onto $R(C) \subseteq X''$.

● X is reflexive if $R(C) = X''$. (In that case $X \cong X''$)

▲ X is reflexive $\Rightarrow$ X complete [note $X'' = B(X', \mathbb{F})$]

▲ Every finite dimensional space is reflexive.

▲ Every Hilbert space is reflexive.

▲ X' is separable $\Rightarrow$ X = (normed space) is separable.

§4.7 Uniform Bounded Principle (Banach - Steinhaus Theorem).

Review of Baire Catagory Theorem

- X = metric space, $M \subseteq X$
  a) M is nowhere dense if $(\overline{M})^\circ = \emptyset$. ($\overline{M}$ contains no open sets).
  b) M is of the first category if $\exists (A_n)$, $M = \bigcup_{n=1}^{\infty} A_n$ and each A_n is nowhere dense.
  c) M is of the second category if M is not of first category.

▲ Baire Catagory Theorem: X is of 2nd category in X if X is complete.

▲ Uniform Boundedness Principle

X = Banach space, Y = normed space
$\{T_\alpha\}_{\alpha \in I}$ = family of operator in $B(X, Y)$.
Suppose $(\forall x \in X)$ $\{T_\alpha x \mid \alpha \in I\}$ is bounded. Then $\{T_\alpha \mid \alpha \in I\}$ is a bounded subset of $B(X, Y)$. That is if $(\forall x \in X)$ $(\exists\, c_x)$ $\|T_\alpha x\| \leq c_x$, $\forall \alpha \in I$, then $(\exists\, c)$ $\|T_\alpha\| \leq c$ $\forall \alpha \in I$.

§4.8 Strong and weak convergence

- $(x_n) \subseteq X$, $x \in X$. we say $x_n \longrightarrow x$, strongly, if $\|x_n - x\| \longrightarrow 0$ as $n \to \infty$. x is the strong limit of (x_n).

- (x_n) converges to x weakly if $(\forall f \in X')$ $f(x_n) \longrightarrow f(x)$ $\left[\ x_n \xrightarrow{w} x\ \right]$. x is called the weak limit of (x_n).

▲ (x_n) can have at most one weak limit.

▲ Strong convergence $\Rightarrow$ weak convergence.

▲ X = normed space, $\dim X = k < \infty$. $x_n \longrightarrow x$ iff $x_n \xrightarrow{w} x$.

▲ weakly convergent sequences are bounded.

▲ X = normed space $(x_n) \subseteq X$, $x \in X$. Then $x_n \xrightarrow{w} x$ iff

(a) (x_n) is bounded

(b) $(\exists$ total set $M \subseteq X')$ $f(x_n) \to f(x)$ $\forall f \in M$.

▲ (weak convergence in ℓ_p $1 < p < \infty$)

$1 < p < \infty$, $(x_n) \subseteq \ell_p$, $x \in \ell_p$.

$(x_n) = (\xi_j^{(n)})$, $x = (\xi_j)$.

Then $x_n \xrightarrow{w} x$ iff (a) (x_n) is bounded,

(b) $\xi_j^{(n)} \longrightarrow \xi_j$ as $n \to \infty$, $j = 1, 2, \cdots$.

§ 4.9 Convergence of sequence of operators

● X, Y = normed spaces.

$T_n : X \longrightarrow Y$, $n = 1, 2, 3, \cdots$

We say (T_n) converges uniformly (in operator norm) if

$\|T_n - T\| \to 0$. $(\exists\, T \in B(X, Y))$ $\|T_n - T\| \longrightarrow 0$ as $n \to \infty$.

(T_n) converges strongly if $(\forall x \in X)$ $(T_n x)$

$T_n x \to Tx$. is strongly convergent in Y $(\exists\ Tx \in Y$ $\|T_n x - Tx\| \to 0$ as $n \to \infty)$

(T_n) converges weakly if $(\forall x \in X)$ $(T_n x)$

$T_n x \xrightarrow{w} Tx$. is weakly convergent in Y. $(\exists\ Tx \in Y$, $\forall f \in Y'$

$\langle T_n x, f \rangle \longrightarrow \langle Tx, f \rangle$ as $n \to \infty$.

* Limits of all types of convergences are unique.

▲ $X =$ complete, $(T_n) \subseteq B(X,Y)$, $T_n \longrightarrow T$ strongly $\Rightarrow T \in B(X,Y)$

▲ $X, Y =$ Banach spaces, $(T_n) \subseteq B(X,Y)$, (T_n) is strongly convergent iff

(a) $(\|T_n\|)$ is bounded.

(b) ($\exists$ total subset $M \subseteq X$) such that $(\forall x \in M)$ $(T_n x)$ is a Cauchy sequence.

● <u>Convergence of Sequences of Functionals.</u>

$X =$ normed space, $f \in X'$ (Banach space)

$X=$ normed space, $f \in X'$	(f_n) is linear functionals	(f_n) are members of X'
$\|f_n - f\| \to 0$	uniform operator conv.	strong convergence
$f_n(x) \longrightarrow f(x)\ \forall x \in X$	strong operator conv.	weak* convergence
$f_n(x) \xrightarrow{w} f(x)\ \forall x \in X$	weak operator conv.	
$g(f_n) \longrightarrow g(f)\ \forall g \in X''$		weak convergence

▲ strong convergence $\Longrightarrow$ weak convergence $\Rightarrow$ weak* conv.

▲ X is reflexive $\Rightarrow$ weak $\Leftrightarrow$ weak*

▲ $X =$ Banach space, $(f_n) \subseteq X' = B(X, \mathbb{F})$.

(f_n) converges weak* iff

(a) $(\|f_n\|)$ is bounded.

(b) ($\exists$ total subset M of X) $(\forall x \in M)$ $(f_n(x))$ is Cauchy.

§4.12 Open Mapping Theorem

● X, Y = normed spaces, $T \in B(X,Y)$. T is called an open map if $\forall$ open $G \subseteq X$, $T(G)$ is open.

▲ If T is 1-1, T is open iff T^{-1} is continous.

▲ Open Mapping Theorem: X, Y = Banach spaces.
$T \in B(X,Y)$, $R(T) = Y \Rightarrow T$ is an open map.

corollary: X, Y = Banach, $T \in B(X,Y)$, T is bijective
$\Rightarrow T^{-1} \in B(Y,X)$.

§4.13 Closed Graph Theorem

● X, Y = Banach spaces. $T : X \to Y$ linear $\mathcal{D}(T) \subseteq X$.
T may or may not be bounded.
$\mathcal{G}(T) =: \{(x, Tx) \mid x \in \mathcal{D}(T)\} \subseteq X \times Y$ Graph of T

▲ $\mathcal{G}(T)$ is a subspace of $X \times Y$.

● $T : X \to Y$ is a closed operator iff $\mathcal{G}(T)$ is a closed subset of $X \times Y$.

▲ $T : X \to Y$ is closed iff the following holds:
if $(x_n) \subseteq \mathcal{D}(T)$, $x_n \to x \in X$, $Tx_n \to y \in Y$, then $x \in \mathcal{D}(T)$ and $Tx = y$.

▲ $T : X \to Y$, bounded. T is closed iff $\mathcal{D}(T)$ is closed.

▲ $T \in B(X,Y) \Rightarrow T$ is closed.

▲ <u>Closed Graph Theorem</u>

X, Y = Banach spaces, $T : X \longrightarrow Y$, $\mathcal{D}(T) = X$ and

T is closed $\Rightarrow$ T is bounded.

<u>Chapter 7</u>

- X = Banach space over $\mathbb{C}$, $T : X \to X$ closed operator

  $\mathcal{D}(T) \subseteq X$

  $T_\lambda = T - \lambda I = T - \lambda$, $\lambda \in \mathbb{C}$

▲ T_λ is closed, $\mathcal{D}(T_\lambda) = \mathcal{D}(T)$

- <u>Resolnent set</u>

  $\rho(T) = \{\lambda \in \mathbb{C} \mid T_\lambda \text{ is } 1\text{-}1,\ \text{onto},\ T_\lambda^{-1} \text{ is bounded}\}$

  $T_\lambda^{-1} \in B(X, X)$

  <u>Resolnent of T</u>

  $R_\lambda = T_\lambda^{-1} \in B(X, X)$; $\lambda \mapsto R_\lambda$ $\rho(T) \longrightarrow B(X, X)$.

- <u>Spectrum of T</u>

  $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Reason why λ might be in $\sigma(T)$.

1) T_λ is not $1-1$ $(\eta(T_\lambda) \neq \{0\})$

$$\left.\begin{array}{ll} \text{2)} & R(T) \text{ is not closed} \\ \text{3)} & R(T) \text{ is not dense} \end{array}\right\} T_\lambda \text{ is not onto.}$$

● If T_λ is not 1-1, Let $\hat{T}_\lambda : \underbrace{X/\eta(T_\lambda)}_{\text{Banach.}} \to X$,

▲ $\hat{T}_\lambda^{-1}$ is bounded iff $R(T_\lambda)$ is closed.

● Point spectrum

$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \eta(T_\lambda) \neq \{0\}\}$ (eigenvalues)

● Essential spectrum

$\sigma_e(T) = \{\lambda \in \mathbb{C} \mid R(T_\lambda)$ is not closed$\}$

● Defect spectrum

$\sigma_d(T) = \{\lambda \in \mathbb{C} \mid R(T_\lambda)$ is not dense in $X\}$

▲ $\sigma(T) = \sigma_p(T) \cup \sigma_e(T) \cup \sigma_d(T)$.

▲ $T \in B(X,X)$, $T' \in B(X',X')$

a) $\sigma_p(T) = \sigma_d(T')$

b) $\sigma_e(T) = \sigma_e(T')$

c) $\sigma_d(T) = \sigma_p(T')$

▲ $\sigma(T) = \sigma(T')$

▲ X is a Hilbert space.

a) $\sigma_p(T) = \sigma_d(T^*)$ ●

b) $\sigma_e(T) = \sigma_e(T^*)$ ●

c) $\sigma_d(T) = \sigma_p(T^*)$ ●

$\sigma(T) = \sigma(T^*)$ ●

Results can be generalized to closed operators.