F.A.
Note Book II
Ben M. Chen
EE/ME G33
5-2348

1
8
15
6
7

Functional Analysis

B

CB

CB

April 17 Wed

Nomenclature for Convergence of functionals

$(f_n) \in X'$

strong convergence $\|f_n - f\| \to 0$ (uniform operator conv.) $f_n \to f$

weak convergence

$\forall g \in X''$ $g(f_n) \to g(f)$ $f_n \xrightarrow{w} f$

weak* convergence $\forall x \in X$, $f_n(x) \to f(x)$ (strong operator conv.)

$f_n \xrightarrow{w^*} f$

Thm: Strong $\Rightarrow$ weak $\Rightarrow$ weak* (E.F.Y)

Thm: X reflexive $\Rightarrow$ weak $\Leftrightarrow$ weak*

Thm: X = Banach space, $(f_n) \subseteq X' = B(X, \mathbb{F})$

(f_n) converges weak* iff.

(a) $(\|f_n\|)$ is bounded

(b) ($\exists$ total subset M of X) ($\forall x \in M$) $(f_n(x))$ is Cauchy.

P.f: weak* = strong operator convergence

§4.11 Numerical Integration

$X \in C[a,b]$, $\int_a^b x(t)\,dt = ?$

Define : $I : C[a,b] \to \mathbb{F}$

$$I(x) = \int_a^b x(t)\,dt$$

E.F.Y. $I \in C[a,b]'$ w.r.t. $\|\cdot\|_\infty$

Numerical Intergration Farmulas.

$x \in C[a,b]$

$$Q(x) = \sum_{i=1}^{k} w_i \, x(t_i)$$

Prop : $Q \in C[a,b]'$.

P.f: Q is linear.

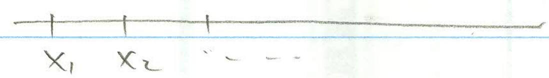
$$|Q(x)| \leq \sum_{i=1}^{k} |w_i| \cdot |x(t_i)| \leq \left(\sum_{i=1}^{k} |w_i| \right) \|x\|$$

$\therefore$ Q is bounded, and

$$\|Q\| = \sup_{x \neq 0} \frac{|Q(x)|}{\|x\|} \leq \sum_{i=1}^{k} |w_i|$$

Prop : $\|Q\| = \sum_{i=1}^{k} |w_i|$

P.f:

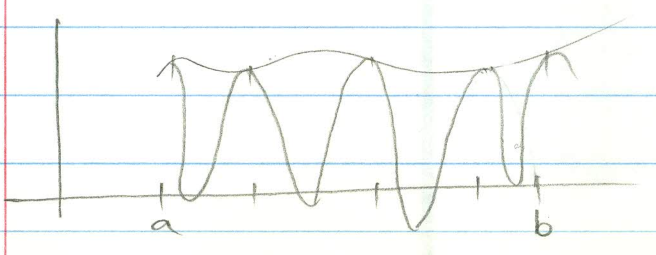


Build cont. x with $\|x\| = 1$

$$x = \begin{cases} 1 & \text{if } w_i > 0 \\ -1 & \text{if } w_i \leq 0 \end{cases}$$

//

Complex case. E.F.Y.



Degree: Q is of degree k if

$Q(x) = I(x)$ for all polynomials x of degree $\le k$ and $Q(x) \neq I(X)$ for some polynomial of degree $k+1$.

Polya's convergence Thm:

Let (Q_n) be a sequence of numerical integration formulas such that $\lim_{n\to\infty} \deg Q_n = \infty$

say

$$Q_n(x) = \underset{\text{(finite sum)}}{\sum_i} w_{ni}\, x(t_{ni})$$

suppose $(\exists\, C)$

$$\sum_i |w_{ni}| \le C \quad \forall n$$

Then $\forall x \in C[a,b]$ $\quad Q_n(x) \to I(x)$ as $n \to \infty$

P.f: We want to show $Q_n \xrightarrow{w^*} I$.

S.T.P. (a) $(\|Q_n\|)$ is bounded.

(b.) ($\exists$ total subset M of $C[a,b]$

$\forall x \in M \quad Q_n(x) \to I(x)$

$\forall n$, $\|Q_n\| = \sum_i |w_i| \le C$ So, (a) holds

Let M = set of polynomial in $C[a,b]$

By Weierstrass, M is dense and hence total

$\lim_{n\to\infty} \deg(Q_n) = \infty$

$(\forall x \in M)$ $\quad Q_n(x) = I(x)$ for sufficiently large n.

Formulas with positive weights

$$Q(x) = \sum_i w_i x(t_i) \qquad w_i > 0, \ \forall i$$

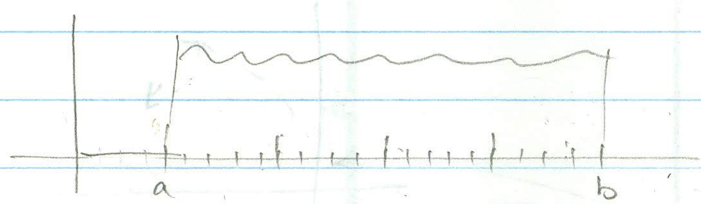
Suppose $\deg(Q) \geq 0$. Let $x(t) = 1$.

$$b - a = \int_a^b 1 \, dt = I(x) = Q(x) = \sum_{i=1} = \sum_{i=1} |w_i|$$

$$\Rightarrow \quad \sum |w_i| = b - a.$$

<u>Steklov's Thm</u>: Let (Q_n) be a sequence of numerical intergration formulas with (+) weights. Such that $\lim_{n \to \infty} \deg(Q_n) = \infty$. Then $(\forall x \in C[a,b])$, $\lim_{n\to\infty} Q_n(x) = I(x)$.

<u>Remark</u>: These results aren't very important.



§4.12 Open Mapping Theorem

X, Y = normed spaces

Def: Let $T \in B(X,Y)$. T is called an open map if $\forall$ open $G \subseteq X$, $T(G)$ is open.

prop. If T is 1-1. T is open iff T^{-1} is cts #

Open Mapping Thm: X, Y = complete spaces (Banach)

$T \in B(X,Y)$, $R(T) = Y$. $\Rightarrow$ T is open map.

cor: X, Y = Banach, $T \in B(X,Y)$, T is bijective $\Rightarrow T^{-1} \in B(Y,X)$ (Bounded Inverse Thm).

Friday 04-19-91

P.f: Phase I: (Reduction).

Let $G \subseteq X$ be open, we must show $T(G)$ is open in Y.

Let $y \in T(G)$, we must show that $(\exists \delta > 0)$ $(B(Y,\delta) \subseteq T(G))$

$(\exists\ x \in G)$ $Tx = y$

$(\exists\ \varepsilon > 0)$ $B(x, \varepsilon) \subseteq G$.

S.T.P. $(\exists\ \delta > 0)$ $B(y, \delta) \subseteq T(B(x,\varepsilon))(\subseteq T(G))$

2

S.T.P. $(\forall \varepsilon > 0)\ (\exists \delta > 0)$

$$T(B(0,\varepsilon)) \supseteq B(0,\delta)$$

[Let $z \in B(y,\delta)$ then $\|z-y\| < \delta$.

$z - y \in B(0,\delta) \subseteq T(B(0,\varepsilon))$, so

$(\exists\, w \in B(0,\varepsilon))\ Tw = z - y = z - Tx$

$T(w+x) = z$

Let $v = w + x$, $v - x = w$

$\|v - x\| = \|w\| < \varepsilon \Rightarrow v \in B(x,\varepsilon)$

$z = Tv \in T(B(x,\varepsilon))$]

S.T.P. $(\exists\, \gamma > 0)\ T(B(0,1)) \supseteq B(0,\gamma)$ (E.F.Y.)

<u>Phase II:</u> (Baire Category thm, Completeness of Y)

$(\forall \delta > 0)$ Let $B_\delta = \{x \in X \mid \|x\| < \delta\} = B(0,\delta)$

$$X = \bigcup_{n=1}^{\infty} B_n, \quad Y = R(T) = T(X) = \bigcup_{n=1}^{\infty} T(B_n)$$

Y is complete $\Rightarrow$ Y is 2nd category. $\Rightarrow$

$(\exists\, n)$ $\overline{T(B_n)}$ contains a nonempty open set.

$$\overline{T(B_n)} \supseteq \underset{\text{open set}}{U} \supseteq B(y,\delta)$$

claim: $\overline{T(B_{2n})} \supseteq B(0,\delta)$

[Let $z \in B(0,\delta)$. Let $w = y + z$ Then
$w \in B(y, \delta) \subseteq \overline{T(B_n)}$

$(\exists (v_k) \subseteq B_n)$ $Tv_k \to w$

$(\exists (u_k) \subseteq B_n)$ $Tu_k \to y$

$T(v_k - u_k) \to w - y = z$

$\| v_k - u_k \| \le \| v_k \| + \| u_k \| < n + n = 2n$

$v_k - u_k \in B_{2n}$ $\forall k$.

$\therefore$ $z \in \overline{T(B_{2n})}$]

Claim : $\overline{T(B_1)} \supseteq B(0,\beta)$, where $\beta = \frac{\delta}{2n}$ (E.F.Y.)

Phase III : (Getting rid of the bar, Completeness of X)

$\overline{T(B_1)} \supseteq B(0,\beta)$

$\overline{T(B_{1/2})} \supseteq B(0,\frac{\beta}{2})$

$\overline{T(B_{\frac{1}{4}})} \supseteq B(0,\frac{\beta}{4})$

$\vdots$

Claim: $T(B_1) \supseteq B(0,\gamma)$, where $\gamma = \frac{\beta}{2}$.

Let $y \in B(0,\gamma)$, want to show $y \in T(B_1)$

$y \in B(0,\frac{\beta}{2}) \subseteq \overline{T(B_{\frac{1}{2}})}$, $(\exists\ x_1 \in B_{\frac{1}{2}})$

$\|y - Tx_1\| < \frac{\beta}{4}$ $\|x_1\| < \frac{1}{2}$

Let $y_1 = y - Tx_1 \in B(0, B\frac{1}{4}) \subseteq \overline{T(B_{1/4})}$

$(\exists\ x_2 \in B_{\frac{1}{4}})$ $\|y_1 - Tx_2\| < \frac{\beta}{8}$ $\|x_2\| < \frac{1}{4}$

Let $y_2 = y_1 - Tx_2 = y - T(x_1 + x_2) \in B(0, \frac{\beta}{8})$
$\subseteq \overline{T(B_{\frac{1}{8}})}$

$(\exists\ x_3 \in B_{\frac{1}{8}})$ $\|y_2 - Tx_3\| < \frac{\beta}{16}$,
$\|x_3\| < \frac{1}{8}$

Let $y_3 = y_2 - Tx_3 = y - T(x_1 + x_2 + x_3)$, $\|y_3\| \le \frac{\beta}{16}$.

By induction ...
$(\exists (x_n))$ such that $\|x_n\| < \frac{1}{2^n}$, $n = 1, 2, \cdots$
and $\|y - T(x_1 + x_2 + \cdots x_n)\| < \frac{\beta}{2^{n+1}} \to 0$ as $n \to \infty$

Let $z_n = x_1 + x_2 + \cdots + x_n$
Then (z_n) is a Cauchy sequence. Let $n > m$

$$\|z_n - z_m\| = \|x_{m+1} + \cdots + x_n\| \le \sum_{k=m+1}^{n} \|x_k\|$$

$$< \sum_{k=m+1}^{n} \frac{1}{2^k} \le \sum_{k=m+1}^{\infty} \frac{1}{2^m} \to 0 \text{ as } m \to \infty$$

X is complete, so $(\exists\ x \in X)$ $x = \lim_{n\to\infty} z_n = \sum_{n=1}^{\infty} x_n$
$z_n \to x$, $\Rightarrow$ $Tz_n \to Tx$
$Tz_n \to y$

$\therefore \quad Tx = y \qquad$ But $\|x\| = \|\sum_{n=1}^{\infty} x_n\| \le \sum_{n=1}^{\infty} \|x_n\|$

$$< \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

$\Rightarrow \quad x \in B(0,1), \quad y = T(x) \in T(B(0,1))$

$\therefore \quad B(0,r) \subseteq T(B(0,1)) \qquad$ #.

Applications: 1. Bounded inverse theorem.

X, Y complete, $T \in B(X,Y)$, T bijective.

$\Rightarrow T^{-1} \in B(Y,X)$.

2. Closed graph thm

Suggested exercises: P290, 1, 2, 5–10.

P295. 1–8, 11–15

Read 9 and 10 for your amusement.

Monday
April 22

§4.13. Closed Graph Theorem

X, Y = Banach Spaces

$T : X \to Y$ linear

$\mathcal{D}(T) \subseteq X$

T may or may not be bounded.

<u>Def:</u> $G(T) =: \{ (x, Tx) \mid x \in \mathcal{D}(T) \} \subseteq X \times Y$

Graph of T

prop: $G(T)$ is a subspace of $X \times Y$.

Norms on $X \times Y$, $\|(x,y)\| = \|x\| + \|y\|$ $\Leftarrow$ we use this one.

$\|(x,y)\|_2 = \sqrt{\|x\|^2 + \|y\|^2}$

$\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}$.

E.F.Y. These are all norms on $X \times Y$ and they are all equivalent.

Also, $(x_n, y_n) \to (x,y)$ iff $x_n \to x$, $y_n \to y$

$\|(x_n, y_n) - (x,y)\| \to 0$ iff $\|x_n - x\| \to 0$, $\|y_n - y\| \to 0$

E.F.Y. X, Y complete $\Rightarrow$ $X \times Y$ is complete (Banach

<u>Def:</u> $T : X \to Y$ is a <u>closed</u> <u>operator</u> iff $G(T)$ is a closed subset of $X \times Y$.

<u>Thm1:</u> $T : X \to Y$ is closed iff the following holds:

~~whenever~~ if $(x_n) \subseteq \mathcal{D}(T)$, $x_n \to x \in X$, $Tx_n \to y \in Y$ then $x \in \mathcal{D}(T)$ and $Tx = y$

Ex: $T : C[0,1] \to C[0,1]$

$Tx = x'$ where $\mathcal{D}(T) = C'[0,1]$

T is unbounded linear operator.

$\Rightarrow$ T is closed.

P.f: Suppose $(x_n) \subseteq \mathcal{D}(T)$ $x_n \xrightarrow{\text{uniform conv}} x \in C[0,1]$

$x_n' = Tx_n \to y \in C[0,1]$

we must show $x \in \mathcal{D}(T) = C'[0,1]$ and $Tx = y$ $(x' = y)$

$x_n \in C'[0,1]$

$x_n(t) = x_n(0) + \int_0^t x_n'(s)\,ds$

as $n \to \infty$ $x_n(t) \to x(t)$

$x_n(0) \to x(0)$

$x_n' \to y$ uniformly $\Rightarrow \int_0^t x_n'(s)\,ds \to \int_0^t y(s)\,ds$ (E.F.Y)

$\Rightarrow x(t) = x(0) + \int_0^t y(s)\,ds$

$\therefore$ $x \in C'[0,1]$ and $x' = y$

Thm: $T : X \to Y$, bounded.

T is closed iff $\mathcal{D}(T)$ is closed.

P.f: $(\Rightarrow)$ T closed. Want to show $\mathcal{D}(T)$ is closed.

Let $(x_n) \subseteq \mathcal{D}(T)$ such that $x_n \to x \in X$.

S.T.P. $x \in \mathcal{D}(T)$, $x_n \to x \Rightarrow (x_n)$ is Cauchy

$\Rightarrow$ (Tx_n) is Cauchy $\Rightarrow Tx_n \to y \in Y$
(boundedness of T)

Y is complete $\Rightarrow$ $x \in \mathcal{D}(T)$ and $Tx = y$
(T is closed).

$(\Leftarrow)$ $\mathcal{D}(T)$ closed. Want T is closed.

Let $(x_n) \subseteq \mathcal{D}(T)$ $\quad x_n \to x \in X$ $\quad$ S.T.P. $x \in \mathcal{D}(T)$

$Tx_n \to y \in Y$ $\quad$ $Tx = y$

$x_n \to x$ $\quad$ $\mathcal{D}(T)$ closed $\Rightarrow x \in \mathcal{D}(T)$

$\Rightarrow Tx_n \to Tx$ $\quad$ (T is bounded)

$Tx_n \to y$

$\therefore Tx = y$.

\#

<u>Cor:</u> $T \in B(X, Y) \Rightarrow T$ is closed.

<u>Closed Graph Thm:</u> X, Y = Banach spaces, $T: X \to Y$

T closed, $\mathcal{D}(T) = X \Rightarrow T$ is bounded.

<u>P.f:</u> Define $S: \underbrace{\mathcal{G}(T)}_{\text{Banach space}} \longrightarrow \underbrace{X}_{\text{Banach}}$

$$S(x, Tx) = x$$

S is linear, 1-1, onto $(R(S) = \mathcal{D}(T) = X)$,

bounded $\quad \|S(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\|$

$= \|(x, Tx)\|$

Finite

$\|S\| \leq 1$

$\therefore$ By open mapping theorem, S^{-1} is bounded:

$(\exists M)$ $\quad \|S^{-1}x\| \leq M\|x\|$ $\quad \forall x \in \mathcal{D}(T) = \mathcal{D}(S^{-1}) = X$

$\|x\| + \|Tx\| = \|(x, Tx)\| \leq M \cdot \|x\|$

$\Rightarrow \|Tx\| \leq (M-1) \cdot \|x\| \quad \forall x \in X$.

$\therefore T$ is bounded.

\#

Application to projectors:

X = Banach space

Y, Z = closed subspaces

s.t. $X = Y \oplus Z$

$(\forall x \in X)\ (\exists!\ y \in Y, z \in Z)\quad x = y + z$

Define $P : X \to X$ by $Px = y$.

P is linear, P is the projector of X onto Y in the ~~director~~ direction of Z.

<u>Then</u> P is bounded.

E.F.Y. Prove this without using the closed graph thm. #

<u>P.f:</u> We'll show that P is closed. hence bounded by C.G.T.

Let $(x_n) \subseteq \mathcal{D}(P) = X$ such that

$x_n \to x \in X$.

$Px_n \to y \in X$

$x_n \in X\ (\exists!\ y_n \in Y, z_n \in Z)\quad x_n = y_n + z_n$

$Px_n = y_n$

$\Rightarrow$ $x_n \to x$

$y_n \to y$ $\quad$ Y is closed $\therefore y \in Y$.

$z_n = x_n - y_n \to x - y$, Z is closed

$(\in Z)$ $\quad\Rightarrow\quad x - y \in Z$.

$$x = \overset{\in Y}{y} + \overset{\in Z}{(x - y)}$$

$\therefore Px = y$. ##

$\boxed{\text{No class on Friday}}$ ###

Wed April 24

Spectral Theory (ch. 7).

X = Banach space over $\mathbb{C}$.

$T: X \to X$ closed operator

$\mathcal{D}(T) \subseteq X$.

<u>Def:</u> $T_\lambda = T - \lambda I \qquad \lambda \in \mathbb{C}$

(E.F.Y.) (T_λ is closed) $\quad \mathcal{D}(T_\lambda) = \mathcal{D}(T)$

<u>Def:</u> <u>resolvent set</u>

$$\rho(T) = \{\lambda \in \mathbb{C} \mid T_\lambda \text{ is 1-1, onto, } T_\lambda^{-1} \text{ is bounded}\}$$

$$(T_\lambda^{-1} \in B(X,X))$$

<u>resolvent</u> of T

$$R_\lambda = T_\lambda^{-1} \in B(X,X)$$

$$\lambda \longmapsto R_\lambda$$

$$\rho(T) \longrightarrow B(X,X)$$

<u>Def:</u> Spectrum of T

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

Finite dimensional case

$$\sigma(T) = \text{set of eigenvalues}$$

Infinite dimensional case.

Reason why λ might be in $\sigma(T)$.

1) T_λ is not 1-1 $(\eta(T_\lambda) \neq \{0\})$

2) $R(T)$ is not closed. } T_λ is not onto.
3) $R(T)$ is not dense

~~4) T_λ^{-1} is unbounded.~~ (which is redundent)

__Note:__ If $T \in B(X,X)$, then $T_\lambda \in B(X,X)$
Then $4 \Rightarrow 2$.

$T_\lambda : X \to R(T_\lambda)$

If 2 is false, then $R(T_\lambda)$ is closed.
is complete.
$T_\lambda : X \to R(T_\lambda)$ is bijective.
By bounded inverse theorem implies
T_λ^{-1} is bounded.

__Thm:__ $T: X \to X$ closed, $\lambda \in \mathbb{C}$. T_λ is 1-1.
Then $4 \Leftrightarrow 2$.

__P.f:__ T_λ is closed, T_λ^{-1} is closed (E.F.Y.)

$S: X \to Y$ closed. $G(S) \subseteq X \times Y$ (x,y)
$G(S^{-1}) \subseteq Y \times X$ (y,x)

suppose $R(T_\lambda)$ is closed. Consider $T_\lambda^{-1} : \underset{\text{Banach}}{R(T_\lambda)} \to \underset{\text{Banach}}{X}$
T_λ^{-1} is closed (regardless of
whether the - - -)

$\therefore$ by the closed graph thm T_λ^{-1} is bounded.

Conversely. T_λ^{-1} is bounded.

T_λ^{-1} is closed.

$\therefore \mathcal{D}(T_\lambda^{-1})$ is closed $\Rightarrow R(T_\lambda)$ is closed.

Def: If T_λ is not 1-1, define

$$\hat{T}_\lambda : \underbrace{X / \eta(T_\lambda)}_{\text{Banach space}} \longrightarrow X$$

$$\hat{T}_\lambda (x + \eta(T_\lambda)) = T_\lambda x.$$

E.F.Y. $\hat{T}_\lambda$ is well-defined

linear

1-1

$R(\hat{T}_\lambda) = R(T_\lambda)$

$\hat{T}_\lambda$ is closed,

is bounded iff T_λ is bounded.

$$\mathcal{D}(\hat{T}_\lambda) = \mathcal{D}(T_\lambda) / \eta(T_\lambda)$$

Thm: $\hat{T}_\lambda^{-1}$ is bounded. iff $R(T_\lambda)$ is closed.

P.f. E.F.Y.

(Nonconventional) subdivision of spectrum:

Point spectrum:

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \eta(T_\lambda) \neq (0)\} \quad \text{(eigenvalues)}$$

Essential spectrum

$$\sigma_e(T) = \{\lambda \in \mathbb{C} \mid R(T_\lambda) \text{ is not closed}\}$$

Defect spectrum

$$\sigma_d(T) = \{\lambda \in \mathbb{C} \mid R(T_\lambda) \text{ is not dense in } X\}$$

$$\sigma(T) = \sigma_p(T) \cup \sigma_e(T) \cup \sigma_d(T)$$

not necessarily disjoint.

From Now on: $T \in B(X, X)$, $T' \in B(X', X')$

$$(T')_\lambda = T' - \lambda I \overset{\text{efy}}{=} (T - \lambda I)' = (T_\lambda)'$$

$$T_\lambda'$$

Thus: a) $\sigma_p(T) = \sigma_d(T')$

b) $\sigma_e(T) = \sigma_e(T')$

c) $\sigma_d(T) = \sigma_p(T')$

Pf: a) $\lambda \in \sigma_p(T)$ iff $\eta(T_\lambda) \neq \{0\}$ iff $\overline{R(T_\lambda')} = X'$

$\eta(T_\lambda)^\perp = \overline{R(T_\lambda')}$ iff $R(T_\lambda')$ is not dense.

iff $\lambda \in \sigma_d(T')$

b) p.f. omitted.

c) similar to (a). by use ${}^{\perp}\eta(T_\lambda') = \overline{R(T_\lambda)}$

<u>Cor:</u> $\sigma(T) = \sigma(T')$

<u>Note:</u> Hilbert space case.
use T^* instead of T'

<u>thus</u>

$$\sigma_p(T) = \sigma_d(T^*)^*$$
$$\sigma_e(T) = \sigma_e(T^*)^*$$
$$\sigma_d(T) = \sigma_p(T^*)^*$$
$$\sigma(T) = \sigma(T^*)^*$$

<u>Remark:</u> All of this generalizes to closed operators for which $\mathcal{D}(T)$ is dense in X.

Final

April 29 Monday

Final Exam, Thurs. May 9, 7-10 a.m.

X = Banach space, $T \in B(X,X)$

Thm: $A \in B(X,X)$, $\|A\|<1 \Rightarrow (\lambda I - A)^{-1} \in B(X,X)$

$$(I-A)^{-1} = \sum_{k=1}^{\infty} A^k \qquad \text{(operator norm)}$$

Remark: Let $x \in \mathbb{C}$, $|x|<1 \Rightarrow$

$\sum_{k=1}^{\infty} x^k$ converges.

$$\sum_{k=1}^{\infty} x^k = \frac{1}{1-x} = (1-x)^{-1}$$

Proof: Let $S_n = \sum_{k=0}^{n} A^k$. claim: (S_n) is Cauchy.

w.l.g. Let $n > m$

$$\|S_n - S_m\| = \|\sum_{k=m+1}^{n} A^k\| \le \sum_{k=m+1}^{n} \|A\|^k < \sum_{k=m+1}^{\infty} \|A\|^k$$

$$= \frac{\|A\|^{m+1}}{1-\|A\|} \to 0 \text{ as } m \to \infty.$$

$\therefore (\exists S \in B(X,X))$ ($\because B(X,X)$ is complete)

$S_n \to S$ in operator norm.

$$S = \sum_{k=1}^{\infty} A^k$$

we need only show $S = (I-A)^{-1}$.

$$(I-A)S = \lim_{n\to\infty} (I-A)S_n$$

$$S(I-A) = \lim_{n\to\infty} S_n(I-A)$$

$$(I-A)S_n = (I-A)\left(\sum_{k=0}^{n} A^k\right) = I - A^{n+1}$$

$$S_n(I-A) = \left(\sum_{k=1}^{n} A^k\right)(I-A) = I - A^{n+1}$$

$$(I-A)S = \lim_{n\to\infty} (I - A^{n+1}) = I$$

$$\because \quad \|A^{n+1}\| \le \|A\|^{n+1} \to 0.$$

$$S(I-A) \longrightarrow I \quad - - -$$

$$\Rightarrow \quad S = (I-A)^{-1} \in B(X,X)$$

<u>Thus</u>: $T \in B(X,X)$, $\lambda \in \mathbb{C}$, $|\lambda| > \|T\| \Rightarrow \lambda \in \rho(T)$

<u>cor</u>: $\sigma(T)$ is bounded.

<u>Def</u>: $r_\sigma(T) = \sup_{\lambda\in\sigma(T)} |\lambda|$ spectrual radius of T

<u>cor</u>: $r_\sigma(T) \le \|T\|$.

<u>Pf</u>: $|\lambda| > \|T\|$, $R_\lambda = T_\lambda^{-1} = (T-\lambda I)^{-1}$

we want to show R_λ exists and $R_\lambda \in B(X,X)$.

$$(T-\lambda I) = -\lambda(I - \lambda^{-1}T) = -\lambda(I-A)$$

where $A = \lambda^{-1}T$ $\quad \|A\| = |\lambda|^{-1}\|T\| < 1$.

$\therefore$ $(T-\lambda I)^{-1} = -\lambda^{-1}(I-A)^{-1}$ exists $\in B(X,X)$ //

$\Rightarrow \lambda \in \rho(T)$. ※

Furthermore, $R_\lambda = -\lambda^{-1}\sum_{k=1}^{\infty} T^k \lambda^{-k}$.

Thm: For all λ with $|\lambda| \geq \|T\|$,

$$R_\lambda = -\sum_{k=1}^{\infty} T^k \lambda^{-(k+1)}$$

Laurent series with operator coff.

Thm Let $\mu \in \rho(T)$, $\lambda \in \mathbb{C}$, $|\lambda - \mu| < \frac{1}{\|R_\mu\|}$

$\Rightarrow \lambda \in \rho(T)$.

Cor: $\rho(T)$ is open, $\sigma(T)$ is closed, in fact compact.

Pf: $T_\lambda = T - \lambda = (T-\mu) - (\lambda - \mu) = T_\mu - (\lambda - \mu)$

$= I\,T_\mu - (\lambda - \mu)\, T_\mu^{-1} T_\mu$

$= (I - (\lambda-\mu) R_\mu)\, T_\mu$

$= (I - A)\, T_\mu$ wher $A = (\lambda - \mu) R_\mu$

$\|A\| = |\lambda - \mu|\,\|R_\mu\| < 1 \Rightarrow (I-A)^{-1} \in B(X,X)$.

$R_\lambda = T_\lambda^{-1} = T_\mu^{-1} (I - A)^{-1} \in B(X,X)$

$\therefore \lambda \in \rho(T)$. #

Furthermore,

$$R_\lambda = R_\mu \sum_{k=0}^{\infty} R_\mu^k (\lambda - \mu)^k$$

Thm: If $\mu \in \rho(T)$, $|\lambda - \mu| < \frac{1}{\|R_\mu\|}$, then

$$R_\lambda = \sum_{k=1}^{\infty} R_\mu^{k+1} (\lambda - \mu)^k$$

which is Taylor series centered on μ. $\therefore$
R_λ is an analytic function of λ.

$f: \mathbb{C} \longrightarrow \mathbb{C}$

$f: \Omega \rightarrow \mathbb{C}$

$f: \Omega \rightarrow Z$ $= B(X,X)$ (Banach space), $\Omega \subseteq \mathbb{C}$

Thm: $\mu, \lambda \in \rho(T) \Rightarrow R_\mu - R_\lambda = (\mu - \lambda) R_\lambda R_\mu$

$$T_\lambda - T_\mu = (\mu - \lambda) I$$

$$R_\lambda (T_\lambda - T_\mu) R_\mu = (\mu - \lambda) R_\lambda R_\mu$$

$$R_\mu - R_\lambda = (\mu - \lambda) R_\lambda R_\mu$$

Cor: R_λ is analytic, and $R_\lambda' = R_\lambda^2$

Pf:

$$R_\lambda' = \lim_{\mu \to \lambda} \frac{R_\mu - R_\lambda}{\mu - \lambda} = \lim_{\mu \to \lambda} R_\lambda R_\mu = R_\lambda^2$$

$\left[\lim_{\mu \to \lambda} R_\mu = R_\lambda \right]$
in operator norm
(optional E.F.Y.).

Thm: $T \in B(X,X) \Rightarrow \sigma(T) \neq \emptyset$.

Pf: Assume $\sigma(T) = \emptyset$, $\rho(T) = \mathbb{C}$.

Then $\lambda \longmapsto R_\lambda$ is an entire function.

Claim: R_λ is bounded on $\mathbb{C}$.

Let $D = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 2\|T\|\}$

$D' = \mathbb{C} \setminus D$.

$R_\lambda|_D$ is bounded because a cts function on a compact set is bounded.

Now, consider $R_\lambda|_{D'}$, $\lambda \in D' \Rightarrow |\lambda| > 2\|T\|$.

$$R_\lambda = -\lambda^{-1} \sum_{k=0}^{\infty} T^k \lambda^{-k}.$$

$$\|R_\lambda\| \leq \frac{1}{|\lambda|} \sum_{k=0}^{\infty} \frac{\|T\|^k}{|\lambda|^k} \leq \frac{1}{|\lambda|} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{2}{|\lambda|} \leq \frac{1}{\|T\|}$$

$\therefore$ $R_\lambda|_{D'}$ is bounded.

$\therefore$ R_λ is bounded.

$\therefore$ By Liouville's thm.

$(\exists\ A \in B(X,X))$ $R_\lambda = A$ $\forall\ \lambda \in \mathbb{C}$

$T_\lambda = A^{-1}$

$T - \lambda I = A^{-1}$ $\quad \otimes \quad \Rightarrow\Leftarrow$

May 1, 91
Wed.

$X =$ Banach space over $\mathbb{C}$

$T \in B(X, X)$

$p(\lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots + \alpha_n \lambda^n$

$p(T) = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_n T^n \in B(X,X)$

E.F.Y.: $Tx = \lambda x$ $\qquad p(T)\, x = p(\lambda)\, x$

Spectrum mapping Thm: $\sigma(p(T)) = p(\sigma(T))$

<u>P.f:</u> $\mu \in \mathbb{C}$

$p(\lambda) - \mu = \alpha_n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$

$p(\lambda) = \mu$ iff $\lambda = \lambda_i$ for some i

Suppose $\mu \in \sigma(p(T))$, Then $p(T) - \mu$ is not 1-1 or not onto

(~~boundedness~~ inverse theorem)

$$p(T) - \mu = \alpha_n \underbrace{(T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_n)}_{\text{these factors commute}}$$

$p(T) - \mu$ not onto $\Rightarrow$ $T - \lambda_i$ not onto, for some i

$p(T) - \mu$ not 1-1 $\Rightarrow$ $T - \lambda_i$ not 1-1 for some i.

$\Bigg\} \Rightarrow \lambda_i \in \sigma(T)$

$\Rightarrow \mu = p(\lambda_i) \in p(\sigma(T))$

Conversely, if $\mu = p(\sigma(T))$, then $(\exists \lambda_i)$

$\lambda_i \in \sigma(T)$ and $\mu = p(\lambda_i)$

$\lambda_i \in \sigma(T) \Rightarrow T - \lambda_i$ is not 1-1
or not onto.

$T - \lambda_i$ not 1-1

$$p(T) = \alpha_n (T-\lambda_1) \cdots (T-\lambda_n)(T-\lambda_i)$$

$$\Rightarrow p(T) - \mu \text{ is not 1-1.}$$

$T - \lambda_i$ not onto.

$$p(T) = \alpha_n (T-\lambda_i)(T-\lambda_1) \cdots (T-\lambda_n)$$

$$\Rightarrow p(T) - \mu \text{ is not onto.}$$

$$\Big\} \Rightarrow \mu \in \sigma(p(T)).$$

\#

$$r_\sigma(T) = \sup_{\lambda \in \sigma(T)} |\lambda|.$$

$$r_\sigma(T) \le \|T\|.$$

Thus: Spectral Radius Formula:

$$r_\sigma(T) = \lim_{k \to \infty} \sqrt[k]{\|T^k\|}.$$

$$R_\lambda = T_\lambda^{-1} = -\lambda^{-1} \sum_{k=0}^{\infty} T^k \lambda^{-k} \qquad \text{if } |\lambda| > \|T\|.$$

Let $\mu = \frac{1}{\lambda}$. $\quad f(\mu) = R_{1/\mu}$ analytic for $|\mu| < \frac{1}{r_\sigma(T)}$

and

$$f(\mu) = -\mu \sum_{k=0}^{\infty} T^k \mu^k$$

Let $\rho =$ rdius of convergence of the power series.

Then $\rho = \frac{1}{r_\sigma(T)}$

By root test, if

$$\limsup_{k\to\infty} \sqrt[k]{\|T^k \mu^k\|} > 1, \quad \text{series divergence.}$$

$$< 1, \quad \text{series converges.}$$

$$= |\mu| \cdot \limsup_{k\to\infty} \sqrt[k]{\|T^k\|}$$

$$|\mu| < \frac{1}{\limsup_{k\to\infty} \sqrt[k]{\|T^k\|}}$$

$$\Rightarrow \rho = \frac{1}{r_\sigma(T)} = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{\|T^k\|}}$$

$$\therefore \quad r_\sigma(T) = \limsup_{k\to\infty} \sqrt[k]{\|T^k\|}.$$

$\sigma(T^k) = [\sigma(T)]^k$ spectral mapping

$$r_\sigma(T^k) = [r_\sigma(T)]^k$$
$$= r_k(T^k) \leq \|T^k\|$$
$$\therefore r_\sigma(T) \leq \sqrt[k]{\|T^k\|} \quad \forall k$$

$$r_\sigma(T) \leq \liminf_{k\to\infty} \sqrt[k]{\|T^k\|} \leq \limsup_{k\to\infty} \sqrt[k]{\|T^k\|} = r_\sigma(T)$$

$$\Longrightarrow \quad r_\sigma(T) = \lim \text{ -- } \text{ -- } \#$$

E.F.Y: $X = \mathbb{C}^n$, $T \in B(X,X)$

(a)
$$T = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$$

Consider $r_\sigma(T)$, $\lim_{k\to\infty} \sqrt[k]{\|T^k\|}$

(b)
$$Te_1 = \lambda_1 e_1$$
$$Te_2 = \lambda_2 e_2$$
$$\vdots$$
$$Te_n = \lambda_n e_n$$

$$T = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

Functional Calculus.

$$p(T), \quad T^{1/2}, \quad \exp(T), \quad T^{\alpha}, \quad \chi_E(T)$$

$$R_\lambda = - \sum_{k=0}^{\infty} T^k \lambda^{-(k+1)} \qquad |\lambda| > r_\sigma(T)$$

$$\lambda^m R_\lambda = - \sum_{k=0}^{\infty} T^k \lambda^{-(k+1-m)}$$

$$\oint_C \lambda^m R_\lambda \, d\lambda = - \sum_{k=0}^{\infty} T^k \oint_C \lambda^{(k+1-m)} d\lambda$$

$$= \begin{cases} 0 & k \neq m \\ 2\pi i & k = m \end{cases}$$

$$= -2\pi i \, T^m$$

$$T^m = - \frac{1}{2\pi i} \oint_C \lambda^m R_\lambda \, d\lambda$$

$$m = 0, 1, 2, \ldots$$

$$p(T) = \alpha_0 I^0 + \alpha_1 T + \cdots + \alpha_n T^n$$

$$= - \frac{1}{2\pi i} \oint_C p(\lambda) R_\lambda \, d\lambda.$$

Let f be locally analytic on an open set containing the spectrum

Define

$$f(T) = -\frac{1}{2\pi i} \oint_C f(\lambda) R_\lambda \, d\lambda$$

—— . — E . — . N . — . D . ——