F. A.

Note Book II

Ben M. Chen

EE/ME G33

5-2348

Feb. 25, 91
Monday

High 88 %   median Score 55%

Problem 1: $x = (x_1, x_2, x_3, \cdots)$

$$\sum_{i=1}^{\infty} |x_i| < \infty$$

$$\Rightarrow (\exists N) \sum_{i=N+1}^{\infty} |x_i| < \frac{\varepsilon}{2}$$

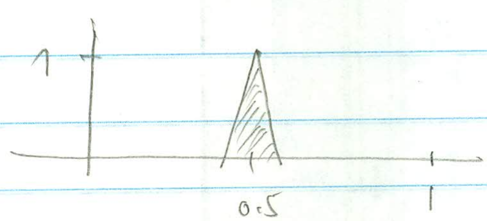
Let $s = (s_1, s_2, \cdots, s_N, 0, 0, \cdots)$

where $|x_i - s_i| < \frac{\varepsilon}{2N}$ $(s_i \in \mathbb{Q})$

$$\|x - s\|_1 < \varepsilon.$$

Problem 3.

$$\frac{|x(0.5)|}{\int_0^1 |x(t)|dt}.$$



Problem 4 Let $(x^{(n)})$ be Cauchy

$$(\forall \varepsilon > 0)(\exists N) \sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^3 < \varepsilon^3$$

$(x_i^{(n)})$ Cauchy $\forall i$

$(\exists\, x_i \in \mathbb{R})$ $x_i^{(n)} \to x_i$ as $n \to \infty$.

Let $x = (x_1, x_2, x_3, \dots)$

Show (i) $x \in \ell_3$

(ii) $\|x^{(n)} - x\|_3 \to 0$ as $n \to \infty$

Do (ii) first. $\Rightarrow$ (i) follows easily

$$\|x^{(n)} - x\|_3^3 = \sum_{i=1}^{\infty} |x_i^{(n)} - x_i|^3$$

Pick k Let $n \geq N$

$$\sum_{i=1}^{k} |x_i^{(n)} - x_i|^3 = \sum_{i=1}^{k} |x_i^{(n)} - \lim_{m\to\infty} x_i^{(m)}|^3$$

$$= \lim_{m\to\infty} \sum_{i=1}^{k} |x_i^{(n)} - x_i^{(m)}|^3$$

$$\leq \limsup_{m\to\infty} \sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^3$$

$$\leq \limsup_{m\to\infty} \varepsilon^3 = \varepsilon^3$$

$$\Rightarrow \sum_{i=1}^{k} |x_i^{(n)} - x_i|^3 \leq \varepsilon^3 \quad \forall k$$

$$\Rightarrow \sum_{i=1}^{\infty} |x_i^{(n)} - x_i|^3 \leq \varepsilon^3$$

$\|x^{(n)} - x\|_3 \leq \varepsilon \qquad \forall\, n \geq N$

Suggested Exercise

p. 167, #6

p. 175, #2-4, 5, 9, 10

§3.7 Orthogonal Polynomials

Thm: P = polynomials, viewed as a subspace of $L_2(-1,1)$. P is dense in $L_2(-1,1)$.

Pf: $C[-1,1]$ is a subspace of $L_2(-1,1)$.

From Math 501, $C[-1,1]$ is dense in $L_2(-1,1)$.

P is dense in $C[-1,1]$ w.r.t. sup norm by Weierstrass

Let $\varepsilon > 0$. Let $x \in L_2(-1,1)$ $(\exists\, y \in C[-1,1])$

$\|x - y\|_2 < \frac{\varepsilon}{2}$

$(\exists\, z \in P) \qquad \|y - z\|_\infty < \frac{\varepsilon}{2\sqrt{2}}$

$$\|y - z\|_2^2 = \int_{-1}^{1} |y(t) - z(t)|^2\,dt \leq \int_{-1}^{1} \|y - z\|_\infty^2\,dt$$

$$= 2 \cdot \|y - z\|_\infty^2$$

$$\|y - z\|_2 \leq \sqrt{2}\, \|y - z\|_\infty$$

$$\Rightarrow \|x - z\|_2 < \varepsilon$$

Let $x_0(t) = 1$, $x_1(t) = t$, $x_2(t) = t^2$, ... (linear...

span$\{x_0, x_1, \ldots\} = P$ which is dense in $L_2(-1,1)$

Let e_0, e_1, e_2 ... be obtained from

$x_0, x_1, x_2, \ldots$ by Gram Schmidt

$\Rightarrow$ $e_0, e_1, e_2, \ldots$ is an o.n. basis of $L_2(-1,1)$

(E.F.Y.)

① $e_n(t)$ is a polynormial of degree exactly n.

② Calculate the first few e_n's.

Def: $P_n(t) = C_n e_n(t)$ such that $P(1) = 1$.

$\Rightarrow$ Legendre polynomials

$L_2(a,b)$ $\quad -\infty < a < b < \infty$

Transform Legendre polynomials to $[a,b]$

$L_2(-\infty,\infty)$ $\quad \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$

$w(t) = e^{-\frac{t^2}{2}} \in L_2(-\infty,\infty)$

Let $x_0(t) = w(t)$, $x_1(t) = w(t)\,t$, $x_2(t) = w(t)\,t^2$, ...

span$\{x_0, x_1, x_2, \ldots\} = \{w(t)p(t) \mid p \in P\} = Q$

Thm: Q is dense in $L_2(-\infty,\infty)$.

Orthonormalize by Gram-Schmidt

E.F.Y. $e_n(t) = C_n\, W(t)\, H_n(t)$
where $H_n(t)$ is a polynomial of degree exactly n.

Hermite polynomial

$L_2(0,\infty)$, $W(t) = e^{-t/2}$
$\Rightarrow$ Laguerre polynomials.

Feb. 28
Wed

§3.8 Riesz Representation Theorem.

H = Hilbert space.
what do bounded linear functionals on H look like? (H')

Let $y \in H$ define $f_y : H \longrightarrow \mathbb{F}$

$$f_y(x) = \langle x, y\rangle \quad \text{linear.}$$

proposition: f_y is bounded, $\|f_y\| = \|y\|$

P.f.
$$|f_y(x)| = |\langle x, y\rangle| \leq \|x\| \cdot \|y\|.$$

$$\forall\ x \neq 0 \quad \frac{|f_y(x)|}{\|x\|} \leq \|y\|$$

$\Rightarrow \|f\| \leq \|y\| \Rightarrow f_y$ is bounded.

take $x = y \Rightarrow =$

Riesz Representation: H = Hilbert space, $f \in H'$

Then $(\exists!\, y \in H)$ $f(x) = \langle x, y \rangle$ $(\forall x \in H)$

$\|f\| = \|y\|$

P.f: If $f = 0$, take $y = 0$

Now assume that $f \neq 0$ $f: H \to \mathbb{F}$

$\eta(f)$ is a closed subspace of H

H complete $\Rightarrow$ $H = \eta(f) \oplus \eta(f)^{\perp}$

Let $g = f|_{\eta(f)^{\perp}}$ $g: \eta(f)^{\perp} \longrightarrow \mathbb{F}$.

E.F.Y: g is 1-1 and also onto

$\therefore$ $\dim(\eta(f)^{\perp}) = \dim(\mathbb{F}) = 1$

Let $z \in \eta(f)^{\perp}$ such that $\|z\| = 1$

Then $\{z\}$ is a basis for $\eta(f)^{\perp}$.

$\Rightarrow$ Every $x \in H$ has ! representation

$x = \alpha z + y$, where $\alpha \in \mathbb{F}$, $y \in \eta(f)$

Let $\beta = f(z)$ Noto that

$f_z(x) = \langle x, z \rangle$ have $f_z(y) = 0$

$\forall\, y \in \eta(f)$

$f_z(z) = \langle z, z \rangle = 1$.

Let $y = \bar{\beta} z$

<u>claim</u> $f = f_y$

Note that

$$f_y(z) = \langle z, y \rangle = \langle z, \bar{\beta} z \rangle = \beta \langle z, z \rangle = \beta$$

$\Rightarrow$

Let $x \in H$,

$$x = \alpha z + u, \quad \alpha \in \mathbb{F}, \quad u \in \eta(f)$$

$$f(x) = \alpha f(z) + f(u)$$

$$= \alpha \beta + 0$$

$$= \alpha f_y(z) + f_y(u) = f_y(x)$$

$\therefore$ $f = f_y$.

$\eta(f) \oplus \eta(f)^\perp$

<u>Uniqueness</u>: Suppose $\exists$ $y_1, y_2 \in H$

$$f(x) = \langle x, y_1 \rangle = \langle x, y_2 \rangle \quad \forall x \in H$$

$\Rightarrow$ $\langle x, y_1 - y_2 \rangle = 0 \quad \forall x \in H$

$\Rightarrow$ $\langle x = y_1 - y_2, y_1 - y_2 \rangle = 0 \Rightarrow y_1 = y_2$

Prop: Let $u_1, u_2 \in H$, If (inner product space)

$$\langle u_1, v\rangle = \langle u_2, v\rangle \quad \forall\, v \in H$$

$$\Rightarrow u_1 = u_2$$

Sesquilinear Forms (Functionals).

X, Y = vector space over $\mathbb{F}$.

Def: A sesquilinear form is a map

$h: X \times Y \to \mathbb{F}$ such that

$$h(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 h(x_1, y) + \alpha_2 h(x_2, y)$$
$$(\forall x_1, x_2 \in X)(\forall y \in Y)(\alpha_1, \alpha_2 \in \mathbb{F})$$

$$h(x, \alpha_1 y_1 + \alpha_2 y_2) = \bar{\alpha}_1 h(x, y_1) + \bar{\alpha}_2 h(x, y_2)$$
$$(\forall x \in X)(\forall y_1, y_2 \in Y)(\forall \alpha_1, \alpha_2 \in \mathbb{F})$$

X, Y = normed spaces.

Df: h is bounded if $(\exists M)$

$$\frac{|h(x,y)|}{\|x\|\,\|y\|} \leq M \quad \forall\, x \in X \ (x \neq 0), \ \forall\, y \in Y \ (y \neq 0)$$

If h is bounded, define

$$\|h\| = \sup\left\{ \frac{|h(x,y)|}{\|x\|\,\|y\|} \;\middle|\; \begin{matrix} x \neq 0, & y \neq 0 \\ x \in X, & y \in Y \end{matrix} \right\}$$

Ex: X, Y = inner product space

Ex: Take $X = Y$. Define $h : X \times X \to \mathbb{F}$ by

$h(x,y) = \langle x, y \rangle$ sesquilinear.

E.F.Y. h is bounded and $\|h\| = 1$

Ex: Let $T \in B(X,Y)$.

define: $h : X \times Y \to \mathbb{F}$ by

$h(x,y) = \langle Tx, y \rangle$ (inner product in Y)

h is a sesquilinear.

Prop. h is bounded and $\|h\| = \|T\|$

Pf: $T \neq 0$. $|h(x,y)| = |\langle Tx, y\rangle| \leq \|Tx\| \cdot \|y\|$

$$\leq \|T\| \cdot \|x\| \cdot \|y\|$$

$\Rightarrow \|h\| \leq \|T\|$. $\forall\, x \neq 0,\ y \neq 0$

$\Rightarrow$ h is bounded.

On the other hand

$$\|h\| = \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|h(x,y)|}{\|x\| \cdot \|y\|}$$

$$= \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle Tx, y\rangle|}{\|x\| \cdot \|y\|} \geq \sup_{\substack{x \neq 0 \\ y \neq 0 \\ y = Tx}} \frac{|\langle Tx, y\rangle|}{\|x\| \cdot \|y\|}$$

$$= \sup_{\substack{x \neq 0 \\ Tx \neq 0}} \frac{\|Tx\|}{\|x\|} = \|T\| \Rightarrow \|h\| = \|T\|$$

Ex: Let $S \in B(Y, X)$

Def: $h: X * Y \to \mathbb{F}$ by

$$h(x,y) = \langle x, Sy \rangle$$

E.F.Y. h is a bounded sesquilinear form with $\|h\| = \|S\|$.

Riesz Rep. Thm. for Sesquilinear Forms

H_1, H_2 = Hilbert spaces over $\mathbb{F}$

$h: H_1 \times H_2 \to \mathbb{F}$ = bounded sesquilinear form. Then

(a) $(\exists!\ T \in B(H_1, H_2))$ $h(x,y) = \langle Tx, y \rangle$ $(\forall x \in H_1)(\forall y \in H_2)$. $\|h\| = \|T\|$.

(b) $(\exists!\ S \in B(H_2, H_1))$ $h(x,y) = \langle x, Sy \rangle$ $(\forall x \in H_1)(\forall y \in H_2)$. $\|h\| = \|S\|$.

Then $y \longmapsto f_y$ $\qquad f_{y_1+y_2} = f_{y_1} + f_{y_2}$

$H \longmapsto H'$ $\qquad f_{\alpha y} = \bar{\alpha} f_y$

is 1-1, onto, norm presuri[ng] and conjugate linear.

March 1, 1991
Friday

Riesz Rep. Thm: H = Hilbert space

$f \in H'$. Then $(\exists!\ y \in H)$.

$f(x) = \langle x, y \rangle \quad \forall x \in H$ and $\|f\| = \|y\|$

Riesz Rep. Thm for Sesquilinear Forms

See previous page:

Pf: (b) Pick $y \in H_2$ Let

$f(x) = h(x, y)$

f is linear.

f is bounded

$|f(x)| = |h(x,y)| \le \|h\| \cdot \|x\| \cdot \|y\|$

$$\frac{|f(x)|}{\|x\|} \le \|h\| \cdot \|y\| \qquad \forall x \ne 0$$

$\Rightarrow f$ is bounded.

$\therefore$ by Riesz $(\exists!\ z \in H_1)$ $f(x) = \langle x, z \rangle \quad \forall x \in H_1$

$h(x,y) = \langle x, z \rangle \quad \forall x \in H$.

z depends on y, so call it Sy

This define $S: H_2 \to H_1$

and

$h(x,y) = \langle x, Sy \rangle \qquad \forall x \in H_1$
$\qquad\qquad\qquad\qquad\qquad \forall y \in H_2$

We must show S is linear, bounded.

Claim: $S(\alpha y) = \alpha S y \quad (\forall \alpha \in \mathbb{F})(\forall y \in H_2)$

$$\langle x, S(\alpha y)\rangle = h(x, \alpha y) = \bar{\alpha}\, h(x, y) = \bar{\alpha}\langle x, Sy\rangle$$
$$= \langle x, \alpha S y\rangle$$

$$\therefore \quad S(\alpha y) = \alpha S y$$

Claim:

$$S(y_1 + y_2) = Sy_1 + Sy_2 \quad \forall\, y_1, y_2 \in H_2$$

E.F.Y. $\Rightarrow$ S is linear.

Claim: S is bounded. Given y, take $x = Sy$

$$\langle x, Sy\rangle = h(x, y)$$

$$\langle x, Sy\rangle = h(x, y)$$

$$\|Sy\|^2 = \langle Sy, Sy\rangle = |h(Sy, y)| \le \|h\| \cdot \|Sy\| \cdot \|y\|$$

$$\Rightarrow \|Sy\| \le \|h\| \cdot \|y\|.$$

$$\frac{\|Sy\|}{\|y\|} \le \|h\| \quad \forall y$$

$\therefore$ S is bounded, and $\|S\| \le \|h\|$

Claim: $\|h\| \le \|S\|$. (redundent)

$$\|h\| = \sup_{\substack{x \ne 0 \\ y = 0}} \frac{|h(x, y)|}{\|x\| \cdot \|y\|}$$

$$|h(x, y)| = |\langle x, Sy\rangle| \le \|x\| \cdot \|Sy\| \le \|S\| \cdot \|x\| \cdot \|y\|$$

$$\Rightarrow \|S\| = \|h\|$$

§3.9 The Adjoint Operator (Hilbert-adjoint)

H_1, H_2 = Hilbert spaces. Let $T \in B(H_1, H_2)$

Def: $h: H_1 \times H_2 \to \mathbb{F}$ by

$h(x,y) = \langle Tx, y \rangle$ bounded sesquilinear form

By Riesz Rep. thm for sesquilinear forms (part (b)) $(\exists!\ T^* \in B(H_2, H_1))$

$$h(x,y) = \langle x, T^*y \rangle \quad \forall x \in H_1, \forall y \in H_2$$

T^* is called the adjoint of T.

Cor: $\|T^*\| = \|T\|$

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \begin{matrix}\forall x \in H_1 \\ \forall y \in H_2\end{matrix}$$

$$\langle T^*y, x \rangle = \langle y, Tx \rangle \quad \begin{matrix}\forall y \in H_2 \\ \forall x \in H_1\end{matrix}$$

$$T^{**}: H_1 \to H_2$$

Prop. $\Rightarrow$ $T^{**} = T$

Thm: $S: H_1 \to H_2$, $T: H_1 \to H_2$, $S, T \in B(H_1, H_2)$

(a) $(S+T)^* = S^* + T^*$

(b) $(\alpha S)^* = \bar{\alpha} S^*$

Pf: $\forall x \in H_1, \forall y \in H_2$

$<x, (S+T)^* y> = <(S+T)x, y>$

$= <Sx, y> + <Tx, y>$

$= <x, S^* y> + <x, T^* y>$

$= <x, (S^* + T^*) y> \quad \forall x, \forall y$

$\Rightarrow (S+T)^* = S^* + T^*$

(b) E.F.Y. //

Lemma X, Y, Z = normed spaces

$S \in B(X, Y)$, $T \in B(Y, Z)$

Then

$TS \in B(X, Z)$ and $\|TS\| \leq \|T\| \cdot \|S\|$

P.f: E.F.Y.

$T : H_1 \to H_2$, $T^* : H_2 \to H_1$

$T^*T : H_1 \to H_1$, $TT^* : H_2 \to H_2$

Thm: $\|T^*T\| = \|T\|^2 = \|TT^*\|$

P.f: $\|T^*T\| \leq \|T^*\| \cdot \|T\| \leq \|T\|^2$

$\|TT^*\| \leq \|T\|^2$

$\|Tx\|^2 = <Tx, Tx> = <x, T^*Tx>$

$\leq \|x\| \cdot \|T^*Tx\|$

$\leq \|x\|^2 \cdot \|T^*T\| \Rightarrow \dfrac{\|Tx\|^2}{\|x\|^2} \leq \|T^*T\| \quad \forall x \neq 0$

$\Rightarrow \|T\|^2 \leq \|T^*T\|$.

$\|T\|^2 = \|T^*\|^2 \le \|TT^*\|$ ※

cor $T^*T = 0$ iff $T = 0$

Thm: $S \in B(H_1, H_2)$, $T \in B(H_2, H_3)$
$(TS \in B(H_1, H_3))$ and $(TS)^* = S^* T^*$

pf E.F.Y.

Examples: Let $A \in \mathbb{C}^{n \times m}$

Define: $T : \mathbb{C}^m \to \mathbb{C}^n$ by

$$Tx = Ax$$

$$T^* : \mathbb{C}^n \to \mathbb{C}^m$$

$\therefore \exists B \in \mathbb{C}^{m \times n}$, such that $T^* y = By \ \forall y \in \mathbb{C}$

Claim $b_{ij} = \overline{a_{ji}} \quad \forall i, j$

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \forall x, y.$$

$$\langle Ax, y \rangle = \langle x, By \rangle$$

$$\Rightarrow \sum_{i=1}^{n} (Ax_i) \overline{y_i} = \sum_{j=1}^{m} x_j \overline{(By)_j}$$

$$\Rightarrow \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} x_j \right) \overline{y_i} = \sum_{j=1}^{m} x_j \overline{\left(\sum_{i=1}^{n} b_{ji} y_i \right)}.$$

$$= \sum_{i=1}^{n} \left[\sum_{j=1}^{n} x_j \overline{b_{ji}} \right] \overline{y_i}$$

$$\Rightarrow$$

$$a_{ij} = \overline{b_{ji}}$$ ※

March 4, Monday

Ex: $H = L_2(0,1)$ $\quad \langle x, y\rangle = \int_0^1 x(t)\,\overline{y(t)}\,dt$

Def: $T \in B(H,H)$ by $\quad Tx(t) = \int_0^t x(s)\,ds$

Let T^* $\quad (\in B(H,H))$

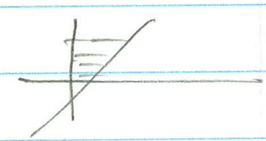
Let $y \in H$. Let $z = T^* y$, $\quad z = ?$

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, z\rangle \quad \forall x \in H$$

$$\int_0^1 Tx(t)\overline{y(t)}\,dt = \int_0^1 x(t)\,\overline{z(t)}\,dt = \int_0^1 x(s)\overline{z(s)}\,ds$$

$$\int_0^1 \int_0^t x(s)\,ds \cdot \overline{y(t)}\,dt = \int_0^1 x(s)\,\overline{z(s)}\,ds$$

$$\int_0^1 \int_0^t x(s)\,\overline{y(t)}\,ds\,dt =$$



F.F.Y. Check that the Hypotheses of Fubini's thm are satisfied.

$$\int_0^1 \int_s^1 x(s)\overline{y(t)}\,dt\,ds = \int_0^1 x(s)\left[\overline{\int_s^1 y(t)\,dt}\right] ds$$

$$\int_0^1 x(s)\left[\overline{\int_0^1 y(t)\,dt}\right] ds = \int_0^1 x(s)\,\overline{z(s)}\,ds$$

$\forall x \in L_2(0,1)$

$$\therefore \quad z(s) = T^* y(s) = \int_s^1 y(t)\,dt$$

§3.10 Normal Operators

H = Hilbert space. $T \in B(H,H) \Rightarrow T^* \in B(H,H)$

Def: T is Hermitian if $T = T^*$ (self-adjoint)

T is skew-Hermitian if $T^* = -T$

T is unitary if $T^*T = I = TT^*$

[This implies is 1-1, onto, and $T^* = T^{-1}$ (E.F.Y.)]

T is normal if $TT^* = T^*T$.

Rmk: Hermitian, skew-Hermitian and Unitary operators are all normal.

Def: $S, T \in B(H,H)$ are unitarily similar if $(\exists$ Unitary $U \in B(H,H))$

$$S = U^{-1}TU = U^*TU$$

prop: T, S unitarily similar.

(a) T is Hermitian $\Rightarrow$ S Hermitian

(b) T is skew ... $\Rightarrow$ S is ...

(c) T is unitary $\Rightarrow$ S is ...

(d) T is normal $\Rightarrow$ S ...

pf. E.F.Y.

Matrices: H = finite dimensional Hilbert.

o.n. basis, $e_1, e_2, \cdots, e_n$

Let $T \in B(H,H)$

Let A = matrix of T w.r.t. basis $e_1, \cdots, e_n$

$A = (a_{ij})$ $\quad Te_j = \sum_{i=1}^{n} e_i a_{ij}, \quad j = 1, \cdots, n.$

$T^* \in B(H,H)$.

Let $B = (b_{ij})$ be the matrix of T^* w.r.t. $e_1, \cdots, e_n$.

E.F.Y. $b_{ij} = \overline{a_{ji}}$. $\quad i,j = 1,2,\cdots,n$.

Def: $A \in \mathbb{C}^{n\times n}$, Define $A^* \in \mathbb{C}^{n\times n}$ by $a^*_{ij} = \overline{a_{ji}}$.

Thm: $T = T^* \iff A = A^*$ (Hermitian matrix)

$T^* = -T \iff A^* = -A$ (skew Hermitian..

$\iff$

$\iff$

Remark: It is crucial that $e_1, \cdots, e_n$ is o.n.

Ex

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not normal.

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is normal. but not $\begin{cases} \text{Hermitian} \\ \text{skew-H} \\ \text{unitary} \end{cases}$

Spectral Theorem (Finite dimensional matrix case)

Let $A \in \mathbb{C}^{n\times n}$ be normal. Then $\exists$ unitary $U \in \mathbb{C}^{n\times n}$ and diagonal $D \in \mathbb{C}^{n\times n}$ such that

$$D = U^* A U = U^{-1} A U.$$

"A is unitary similar to a diagonal matrix". (and conversely)

Hermitian Operator

Thm: $T = T^* \Rightarrow \langle Tx, x\rangle \in \mathbb{R} \quad \forall x \in H$

(If $\mathbb{F} = \mathbb{C}$, then the converse holds)

P.f: $\langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$

$\Rightarrow \langle Tx, x\rangle$ is real.

Conversely, Suppose $\mathbb{F} = \mathbb{C}$, $\langle Tx, x\rangle \in \mathbb{R} \quad \forall x$

$$\langle Tx, x\rangle \overset{\text{real}}{=} \langle x, Tx\rangle = \langle T^*x, x\rangle$$

$$\langle (T - T^*)x, x\rangle = 0 \quad \forall x \in H$$

$\therefore \quad T - T^* = 0.$ (Homework problem)

$\therefore \quad T = T^*.$

$\divideontimes$

Def: $T \in B(H, H)$ $T = T^*$ T is called positive definite if $\langle Tx, x\rangle > 0 \quad (\forall x \in H, x \neq 0)$

$A \in \mathbb{C}^{n\times n}$, $A = A^*$ is p.d. if $x^* A x > 0$ ..

$\langle Ax, x\rangle > 0$

March 6 Wed

Assignment due 25, p.175 #4, P.200. #2, 8
P. 194 #8   P. 207. #6

Recommended. Exercises.

P. 194 4, 6–15, P. 200, 1–6, 8–10, P. 207. 1–10, 12–15

Thm: S, T Hermitian (H = Hilbert space, $S, T \in B(H,H)$)

ST is Hermitian iff $ST = TS$

Thm: $(T_n) \subseteq B(H,H)$, $T \in B(H,H)$

$T_i \to T$ $\left(\lim_{n\to\infty} \|T_n - T\| = 0\right)$.

If $T_n^* = T_n$ $\forall n$, then $T^* = T$.

Pf: we'll show $T_n \to T^*$ Then $T = T^*$ by uniquess of limits.

$$\|T_n - T^*\| = \|T_n^* - T^*\| = \|(T_n - T)^*\|$$

$$= \|T_n - T\| \to 0.$$

Skew-Hermitian Operators

Thm: If $\mathbb{F} = \mathbb{C}$, T skew Hermition $\Leftrightarrow$

$\pm iT$ Hermition.

Pf: E.F.Y.

Thm: T is skew $\Longrightarrow$ $\langle Tx, x\rangle$ is purely imaginary $\forall x \in H$. (If $\mathbb{F} = \mathbb{C}$, the converse holds)

Pf: E.F.Y.

Thm: $T_n \to T$. If $T_n^* = -T_n$ $\forall n$ then $T^* = -T$

E.F.Y.

Unitary Operator $\quad U^*U = I = UU^*$
(U is 1-1, onto and $U^* = U^{-1}$).

Thm: ~~Thm~~ U is unitary $\Rightarrow$ $\langle Ux, Uy\rangle = \langle x, y\rangle$ $\forall x, y \in H$
$\|Ux\| = \|x\|$ $\forall x \in H$. (U is ~~isometric~~ an isometry).

Pf: $\langle Ux, Uy\rangle = \langle x, U^*Uy\rangle = \langle x, y\rangle$ $\forall x, y \in H$

Thm: U is an isometry iff $U^*U = I$

Pf: E.F.Y.

Thm: U is unitary iff U is isometric and surjective.

Pf. E.F.Y.

Thm: U is unitary iff U is isometric and normal.

Pf: E.F.Y.

Ex: Let $H = \ell_2$. $T : H \to H$

$T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$

$T \in B(H, H)$

T is an isometric.

E.F.Y. T is an isometry

T is not unitary

$T^*T \qquad TT^*$

Thm: $\dim(H) < \infty$, U is unitary iff U is isometric

Pf: E.F.Y.

Thm: U, V unitary $\Rightarrow$ UV unitary.

Thm: U, V isometric $\Rightarrow$ UV isometric

Multiplication Operators

$H = L_2(a, b)$ Let $\phi \in L_\infty(a, b)$

Def: $M_\phi : H \to H$ multiplication by ϕ

$M_\phi x = \phi x \qquad \phi(t)x(t)$

E.F.Y. M_ϕ is linear, bounded; $\|M_\phi\| \le \|\phi\|_\infty$.

In fact $\|M_\phi\| = \|\phi\|_\infty$

E.F.Y. $M_{\phi+\psi} = M_\phi + M_\psi$, $M_{\alpha\phi}\,(\alpha \in \mathbb{F}) = \alpha M_\phi$

$M_{\phi\psi} = M_\phi M_\psi$

The map $\phi \mapsto M_\phi$

$L_\infty(a,b) \longrightarrow B(H,H)$

Banach ~~space~~ algebra of isomorphism of $L_\infty(a,b)$ with a ~~subspace~~ algebra of $B(H,H)$.

E.F.Y. $M_\phi M_\psi = M_\psi M_\phi$

E.F.Y. $(M_\phi)^* = M_{\bar{\phi}}$

Cor: M_ϕ is normal

E.F.Y. M_ϕ is Hermitian iff ϕ is real a.e.

M_ϕ is skew-Hermitian iff ϕ is purely imagary a.e.

M_ϕ is positive definite iff $\phi > 0$ a.e.

M_ϕ is unitary iff $|\phi(t)| = 1$ a.e.

E.F.Y. Determine necessary and sufficient condition on ϕ under which M_ϕ is a projector.

$H = l_2$ $\quad s = (s_i) \in l_\infty$

Def: $M_s : H \longrightarrow H$ by

$M_s x = sx \qquad sx = (s_i x_i)$

E.F.Y. M_s is linear, bounded.

$\|M_s\| \le \|s\|_\infty$ $\quad$ $\|M_s\| = \|s\|_\infty$

$s \longmapsto M_s$

$l_\infty \longmapsto B(H,H)$

Banach algebra, isomorphism, etc.

March 8, 90 Friday

Multiplication Operators

on $L_2(a,b) = H$ $\quad$ $\phi \in L_\infty(a,b)$

$M_\phi \in B(H,H)$

$M_\phi x = \phi x$

On $l_2 = H$ $\quad$ $s \in l_\infty$.

$M_s \in B(H,H)$

$M_s x = s x$ $\quad$ $sx = (s_i x_i)$

On $\mathbb{C}^n = H$, $\quad$ $s \in \mathbb{C}^n$

$M_s \in B(H,H)$ by

$M_s x = s x$ $\quad$ $s = (s_1, \cdots, s_n)$

$x = (x_1, \cdots, x_n)$

$sx = (s_1 x_1, \cdots, s_n x_n)$

M_s is linear, $\|M_s\| = \|s\|_\infty$

This is an operator on a finite dimentional space.

$\therefore$ It can be represented by a matrix.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \xrightarrow{M_s} \begin{bmatrix} s_1 x_1 \\ s_2 x_2 \\ \vdots \\ s_n x_n \end{bmatrix} = \begin{bmatrix} s_1 & & 0 \\ & s_2 & \\ & \ddots & \\ 0 & & s_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Eigenvalues + Eigenvectors.

$T \in B(H, H)$

Def: $v \in H$, $v \neq 0$, v is an E.V. of T (if $\exists\ \lambda \in \mathbb{F}$) $Tv = \lambda v$

λ is called the eigenvalue of T associated with v.

$\Rightarrow$ v is E.V. $\Rightarrow$ αv is E.V. $\forall \alpha \neq 0$.

so w.l.g $\Rightarrow$ $\|v\| = 1$.

then

$$\langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle = \lambda$$

Thm: T Hermitian $\Rightarrow$ all e.v. are real.

pos. def $\Rightarrow$ all e.v. are positive.

skew $\Rightarrow$ all e.v. are purely imagary.

Unitary $\Rightarrow$ unit circle. (E.F.Y.)

Ex: $H = L_2(0,1)$ $\quad \phi(t) = t \in L_\infty(0,1)$

$M_\phi : H \to H$.

$M_\phi x = \phi x(t) = t\, x(t)$.

v is E.V. iff $\quad v \neq 0$

$M_\phi v = \lambda v$

$t\, v(t) = \lambda v(t)$.

$(t - \lambda) v(t) = 0$ a.e.

$t - \lambda$ is zero on a set of measured zero.

$\Rightarrow v(t) = 0$ a.e. ⊗.

M_ϕ has no eigenvalue or eigenvectors

There is more to "spectrum" than "eigenvalues".

E.F.Y. Under what conditions on ϕ does M_ϕ have eigenvalues?

Def: The resolvent set $\rho(T)$ of T is the set of $\lambda \in \mathbb{C}$ such that $(\lambda - T)^{-1} \in B(H,H)$

$[\lambda - T$ is 1-1, $(\lambda - T)^{-1}$ is bounded $\mathcal{R}(\lambda - T) = H]$

The spectrum of T, $\sigma(T) = \mathbb{C} \setminus \rho(T)$

eigenvalues $\subseteq \sigma(T)$

If $\dim(H) < \infty$, $\{\text{eigenvalues}\} = \sigma(T)$

E.F.Y. $x(t) \xrightarrow{M_\phi} t\,x(t)$ on $[0, 1]$

$(\lambda - M_\phi)^{-1}$ is unbounded iff $\lambda \in [0, 1]$

in fact $\sigma(M_\phi) = [0, 1]$.

Range and Null Space

$T \in B(H, H)$, $T^* \in B(H, H)$

Thm: $R(T)^\perp = \eta(T^*)$

$R(T^*)^\perp = \eta(T)$

$\eta(T)^\perp = \overline{R(T^*)}$

$\eta(T^*)^\perp = \overline{R(T)}$

Pf: $z \in R(T)^\perp \Leftrightarrow \langle y, z \rangle = 0 \quad \forall y \in R(T)$

$\Leftrightarrow \langle Tx, z \rangle = 0 \quad \forall x \in H$

$\Leftrightarrow \langle x, T^* z \rangle = 0 \quad \forall x \in H$

$\Leftrightarrow T^* z = 0 \Leftrightarrow z \in \eta(T^*)$

Execise: λ = eigenvalue of T.

$$\eta(\lambda - T) \neq \{0\}$$

$$H \neq \eta(\lambda - T)^{\perp} = \overline{R\{(\lambda - T)^*\}} = \overline{R(\bar{\lambda} - T^*)}$$

$$R(\bar{\lambda} - T^*) \neq H \quad \therefore \quad \bar{\lambda} \in \sigma(T^*)$$

(not necessarilly eigenvalues)

E.F.Y. $P \in B(H, H)$, $P^2 = P$ (projector).

Prove that: P is an orthoprojector iff $P = P^*$ ※

Monday March 11

E.F.Y. from the above $\Leftarrow$ easy

$\Rightarrow$ (E.F.Y P^* is a projector)

Chapter 4. Fundamental Theorems

Hahn-Banach theorem	Consequences of Baire Category thm
Zorn's lemma	completeness needed
completeness not needed	Uniform boundedness principle.
	Open mapping theorem.
	closed graph theorem

§ 4.1 Zorn's Lemma

Def: M = set. A partial ordering on M is a binary relation ($\leq$) such that

(1) $a \leq a \quad \forall a \in M$

(2) $a \leq b,\ b \leq a \Rightarrow a = b \quad \forall a, b \in M$

(3) $a \leq b,\ b \leq c \Rightarrow a \leq c, \quad \forall a, b, c \in M$

$(M, \leq)$ = partially ordered set.

Def: A <u>total</u> <u>ordering</u> is a partial ordering such that

(4) $(\forall a, b \in M) \quad a \leq b$ or $b \leq a$

Def: M = partially order set. $W \subseteq M$

W is called a <u>chain</u> if W is totally ordered.

Def: $W \subseteq M$, $a \in M$ a is called an <u>upper</u> <u>bound</u> for W if $b \leq a \quad \forall b \in W$.

Def: $a \in M$, a is called a <u>maximal</u> <u>element</u> of M if $b \in M$, $a \leq b \Rightarrow a = b$, $\forall b \in M$

$M \neq \emptyset$

Ex: $(\mathbb{R}, \leq)$ usual ordering $\Rightarrow$ totally ordered set

Ex: $(\mathbb{C}, \leq)$ ordering defined by $a \leq b$ iff
Re $a \leq$ Re b, Im $a \leq$ Im b, partially ordering

Ex: X = set. M = set of all subsets of X.
$A, B \in M$, $A \leq B$ iff $A \subseteq B$
X is an upper bound for the set
X is (the unique) maximal element of M.

Axiom (Zorn's Lemma):

M = partially ordered set. If every chain in M has an upper bound, then M has a maximal element.

Thm: Every vector space has a Hamel basis.

Pf: X = vector space $(X \neq \{0\})$
Let M = set of all linear independent subsets of X. $S_1, S_2 \in M$, say $S_1 \leq S_2$ iff $S_1 \subseteq S_2$.
$(M, \leq)$ is a partially ordered set.

Show: Every chain in M has an upper bound.

Let $W \subseteq M$ be a chain (W is totally ordered)

Let $S = \bigcup W$. Certainly $(\forall \hat{S} \in W)\ \hat{S} \subseteq S$

Is S linealy independent. Let $x_1, x_2, \cdots x_n \in S$
We want to show $x_1, x_2, \cdots, x_n$ are independent.

$\forall i \quad x_i \in S \Rightarrow (\exists S_i \in W)\ x_i \in S_i$

$S_1, S_2, \cdots, S_n \in W \quad W$ is a chain

By an easy induction argument,

$\exists j \quad S_i \subseteq S_j \quad , i = 1, 2, \cdots n$

$x_1, \cdots, x_n \in S_j$, S_j is linearly indep..

$\Rightarrow x_1, x_2, \cdots, x_n$ is l. i.

$\Rightarrow S$ is an upper banded for W.

$\therefore$ Every chain has an upper bounded

$\therefore$ by Zorn's lemma M has a maximal element B.

Claim: B is a Hamel basis for X.

Proof by contradiction. Assume not a basis

Then $(\exists x \in X)$ x is not a (finite) linear comb. of B $\therefore \hat{B} = B \cup \{x\}$ is l. ind.

(E.F.Y). $B \subseteq \hat{B}$, $B \neq \hat{B}$ $\therefore B$ is not maximal

X

Thm: Every inner product space has an o.n. basis.

Pf: E.F.Y.

§4.2 Hahn–Banach theorem:

Extension theorem

X = real vector space

Z = subspace of X

$f : Z \longrightarrow \mathbb{R}$ linear functional $\mathcal{D}(f) = Z$.

Def: X = real vector space

p = functional defined on X $p : X \longrightarrow \mathbb{R}$

p is called <u>sublinear</u> if

(1) $p(x+y) \le p(x) + p(y)$ $\forall x, y \in X$ (subadd...

(2) $p(\alpha x) = \alpha p(x)$ $(\forall x \in X)(\forall \alpha \ge 0)$ (p...

Ex: linear $\Rightarrow$ sublinear

Ex: let $\|\cdot\|$ be a norm on X, $\beta > 0$

$$p(x) = \beta \|x\|$$

p is a sublinear functional.

Recommended Exercises

P. 212 1–5

P. 218 all

P. 224 all

Mid term Wed. April 3 part I
Fri. April 5 Part II

Hahn-Banach Theorem I: (Real vector space)

X = real vector space.

p = sublinear functional on X

Z = subspace of X

f = linear functional defined on Z such that

$f(z) \le p(z) \quad \forall\, z \in Z$ (f is dominated by z)

Then $(\exists\, \hat{f})$ linear functional defined on X such that

$$\hat{f}(z) = f(z) \quad \forall\, z \in Z.$$

and $\hat{f}(x) \le p(x) \quad \forall\, x \in X$ ($\hat{f}$ is dominated by p

P.f: Let M be the set of all extensions g of f such that $g(x) \le p(x) \quad \forall\, x \in \mathcal{D}(g)$.

Partially order M by $f_1 \le f_2$ if f_2 is an extension of f_1

Must show every chain in M has an upper bound

Let $W \subseteq M$ be a chain. Define linear functional h as follows

$\mathcal{D}(h) = \bigcup_{g \in W} \mathcal{D}(g)$. Given $x \in \mathcal{D}(h)$

$x \in \mathcal{D}(g)$ for some $g \in W$. Define $h(x) = g(x)$

1) h is well definied (W is a chain) E.F.Y.

2) $\mathcal{D}(h)$ is a vector space (subspace of X)

$x, y \in \mathcal{D}(h) \Rightarrow x+y \in \mathcal{D}(h)$ E.F.Y.

$\alpha \in \mathbb{R}$, $x \in \mathcal{D}(h) \Rightarrow \alpha x \in \mathcal{D}(h)$

3) h is linear

4) $h(x) \le p(x)$

$\therefore h \in M$.

$g \le h \quad \forall g \in W$

$\therefore h$ is an upper bound for W.

By Zorn, M has a maximal element $\tilde{f}$.

<u>claim:</u> $\mathcal{D}(\tilde{f}) = X$, If so, we're done.

pf by contradiction. Let $S = \mathcal{D}(\tilde{f})$

Assume $S \ne X$, Then $\exists\ v \in X \setminus S$

Let $\hat{S} = S \oplus \operatorname{span}\{v\} = \{s + \alpha v \mid s \in S, \alpha \in \mathbb{R}\}$

We will show that $\exists\ \hat{f}$ defined on $\hat{S}$, $\hat{f}$ linear extension of $\tilde{f}$, $\hat{f}(x) \le p(x) \quad \forall x \in \hat{S}$.

This will contradict maximality of $\tilde{f}$.

What propeties would $\hat{f}$ have?

$$\hat{f}(s + \alpha v) = \hat{f}(s) + \alpha \hat{f}(v)$$
$$= \tilde{f}(s) + \alpha \hat{f}(v)$$

$\hat{f}(v)$ must be chosen so that

$$(*) \qquad \hat{f}(s+\alpha v) \le p(s+\alpha v) \qquad \forall s \in S,\ \forall \alpha \in R$$

$$\hat{f}(s) + \alpha \hat{f}(v) \le p(s+\alpha v)$$

$$(1) \qquad \hat{f}(v) \le \frac{p(s_1+\alpha_1 v) - \hat{f}(s_1)}{\alpha_1} \qquad \forall s \in S,\ \forall \alpha >$$

For $\alpha = 0$, $(*)$ is already satisfied. ~~Let~~

For $\alpha < 0$, Let $\alpha_2 = -\alpha$

$$\tilde{f}(s_2) - \alpha_2 \hat{f}(v) \le p(s_2 - \alpha_2 v) \qquad \forall s_2 \in S$$

$$\forall \alpha_2 > 0$$

$$(2) \qquad \hat{f}(v) \ge \frac{\tilde{f}(s_2) - p(s_2 - \alpha_2 v)}{\alpha_1}$$

Let $\displaystyle b = \sup_{\substack{\alpha_2 > 0 \\ s_2 \in S}} \frac{\tilde{f}(s_2) - p(s_2 - \alpha_2 v)}{\alpha_2}$

Let $\displaystyle B = \inf_{\substack{\alpha_1 > 0 \\ s_1 \in S}} \frac{p(s_1 + \alpha_1 v) - \hat{f}(s_1)}{\alpha_1}$

we need $\quad b \le \hat{f}(v) \le B$

S.T.P. $\quad b \le \hat{f}(v) \le B$

$b \le B$ iff

$$\frac{\tilde{f}(s_2) - p(s_2 - \alpha_2 v)}{\alpha_2} \le \frac{p(s_1 + \alpha_1 v) - \tilde{f}(s_1)}{\alpha_1}$$

$\forall \alpha_1, \alpha_2 > 0 \quad \forall s_1, s_2 \in S$

iff $\alpha_1 \tilde{f}(s_2) - \alpha_1 p(s_1 - \alpha_2 v) \leq \alpha_2 p(s_1 + \alpha_1 v) - \alpha_2 \tilde{f}(s_1)$ $\forall$

iff $\alpha_1 \tilde{f}(s_2) + \alpha_2 \tilde{f}(s_1) \leq \alpha_1 p(s_2 - \alpha_2 v) + \alpha_2 p(s_1 + \alpha_1 v)$

$\forall$

iff (3) $\tilde{f}(\alpha_1 s_2 + \alpha_2 s_1) \leq p(\alpha_1 s_2 - \alpha_1\alpha_2 v) + p(\alpha_2 s_1 + \alpha_1\alpha_2 v)$

But (3) is true, because

$$\tilde{f}(\alpha_1 s_2 + \alpha_2 s_1) \leq p(\alpha_1 s_2 + \alpha_2 s_1)$$

$$= p(\alpha_1 s_2 - \alpha_1\alpha_2 v + \alpha_2 s_1 + \alpha_2\alpha_1 v)$$

$$\leq p(\alpha_1 s_2 - \alpha_1\alpha_2 v) + p(\alpha_2 s_1 + \alpha_2\alpha_1 v)$$ ///

Can we get by without Zorn's Lemma?

Yes if X/Z is finite dimensional or countable dimensional. //

§4.3 Hahn–Banach Theorem II (Real vector space)

X = real vector space, p = sublinear functional defined on X such that $p(\alpha x) = |\alpha| p(x)$ $(\forall \alpha \in \mathbb{R})(\forall x \in X)$ Z = subspace of X,

f = linear functional Z such that

$$|f(z)| \leq p(z) \quad \forall z\ Z$$

Then $\exists$ linear $\tilde{f}$ defined on X such that

$\tilde{f}(z) = f(z) \quad \forall z \in Z$ and

$$|\tilde{f}(x)| \le p(x) \qquad \forall x \in X.$$

Pf.: From Hahn-Banach I, $\exists \tilde{f}$ defined on X such that $\tilde{f}(x) \le p(x), \ \forall x \in X$

$$\tilde{f}(-x) \le p(-x) \qquad \forall x \in X$$

$$\Rightarrow \quad -\tilde{f}(x) \le |-1|\, p(x) = p(x)$$

$$\tilde{f}(x) \ge -p(x)$$

$$-p(x) \le \tilde{f}(x) \le p(x) \qquad \forall x$$

$$|\tilde{f}(x)| \le p(x) \qquad \forall x \in X. \quad \#$$

Linear functional on Complex vector spaces.

X — complex vector space.

$X_r = X$, regarded as a real vector space

$g: X \to \mathbb{C}$ linear functional

$g(x) = g_1(x) + i g_2(x) \qquad g_1 = Re(g(x))$
$\qquad\qquad\qquad\qquad\qquad g_2 = Im(g(x))$

Thm: g linear $\Rightarrow g_1, g_2$ are real linear

$g_1 : X_r \longrightarrow \mathbb{R}$, $g_2 : X_r \longrightarrow \mathbb{R}$.

P.f.: E.F.Y.

$g(x) = g_1(x) + i g_2(x)$

$g(ix) = g_1(ix) + i g_2(ix) = i\, g(x)$

$g(x) = g_2(ix) - i g_1(ix)$

Thus $g_2(x) = -g_1(ix)$

Cor: $g : X \longrightarrow \mathbb{C}$ linear $\Rightarrow g(x) = g_1(x) - i g_1(ix)$.

March 15
Friday

Springbreak

Linear Functionals on Complex Vector Spaces

X = complex vector space.

$X_r = X$, viewed as real space.

<u>Thm:</u> $g \in X^*$ $\quad g = g_1 + i g_2 \Rightarrow g_1, g_2 \in X_r^*$

$$\Rightarrow g_2(x) = -g_1(ix)$$

$$\Rightarrow g(x) = g_1(x) - i g_1(ix) \qquad \forall\, x \in X.$$

<u>Thm:</u> Let h be a linear functional on X_r $(h \in X_r^*)$ and let

$$g(x) = h(x) - i h(ix)$$

Then g is a complex linear functional on X $(g \in X^*)$

<u>Pf:</u> $g(x+y) = g(x) + g(y)$ ✓

$g(\alpha x) = \alpha g(x) \quad \forall\, \alpha \in \mathbb{C} \ \forall\, x \in X.$

$\alpha = \beta + \gamma i, \quad \beta, \gamma \in \mathbb{R}$

$$\begin{aligned} g(\alpha x) &= h(\alpha x) + i h(i \alpha x) \\ &= h(\beta x + \gamma i x) - i h(-\gamma x + \beta i x) \\ &= \beta h(x) + \gamma h(ix) + i \gamma h(x) - i \beta h(ix) \\ &= \cdots = \alpha g(x) \end{aligned}$$

☆

Hahn – Banach Thm III (complex vectorspace)

X = vector space over $\mathbb{C}$, $p(x)$ = real-valued valued sublinear functional on X such that $p(\alpha x) = |\alpha| \, p(x)$ $(\forall \alpha \in \mathbb{C})$, Z = subspace of X, f = linear functional on Z such that $|f(z)| \leq p(z)$, $\forall z \in Z$.

Then $\exists$ linear $\tilde{f}$ on X such that $\tilde{f}(z) = f(z)$ $\forall z \in Z$ and $|\tilde{f}(x)| \leq p(x)$ $\forall x \in X$.

Pf: $f : Z \to \mathbb{C}$, $f(x) = f_1(x) - i f_1(ix)$

$f_1 : Z_r \to \mathbb{R}$

$$|f_1(z)| \leq |f(z)| \leq p(z) \quad (\forall z \in Z)$$

$\therefore$ by Real Hahn-Banach II, f_1 has an extention $\tilde{f}_1 : X_r \to \mathbb{R}$ satisfying

$$|\tilde{f}_1(x)| \leq p(x) \quad \forall x \in X.$$

Define $\tilde{f}(x)$ by $\tilde{f}(x) = \tilde{f}_1(x) - i\tilde{f}_1(ix)$

$\Rightarrow \tilde{f}$ is complex linear. It's an extension.

We need only to show

$$|\tilde{f}(x)| \leq p(x) \quad \forall x \in X$$

Let $x \in X$ $\quad \tilde{f}(x) = r e^{i\theta}$

where $r = |\tilde{f}(x)|$, $|e^{i\theta}| = 1$.

$$|\tilde{f}(x)| = r = e^{-i\theta} \tilde{f}(x) = \tilde{f}(e^{-i\theta} x)$$
$$= \tilde{f}_1(e^{-i\theta} x) - i \underbrace{\tilde{f}_1(i e^{-i\theta} x)}_{0}$$
$$= \tilde{f}_1(e^{-i\theta} x) = |\tilde{f}_1(e^{-i\theta} x)| \leq p(e^{-i\theta} x)$$
$$= |e^{-i\theta}| \cdot p(x)$$

※

Hahn-Banach Thm (Normed Space)

X = normed space over $\mathbb{F}$, Z is a subspace of X, $f \in Z'$. Then $(\exists\, \tilde{f} \in X')$ $\tilde{f}(z) = f(z)$ $\forall z \in Z$ and $\|\tilde{f}\| = \|f\|$.

<u>Pf:</u> Let $p(x) = \|f\| \cdot \|x\|$. p satisfies the hypothesis of H-B II.

$$|f(z)| \leq \|f\| \cdot \|z\| = p(z) \quad \forall z \in Z.$$

So by H-B II ($\exists$ linear $\tilde{f}$, defined on X)

such that $\quad |\tilde{f}(x)| \leq p(x)$, $\quad \forall x \in X$

$$\therefore \; x \neq 0 \quad \frac{|\tilde{f}(x)|}{\|x\|} \leq \|f\| \quad \forall x \in X.$$

$$\|\tilde{f}\| \leq \|f\| \Rightarrow \text{obviously } \|\tilde{f}\| \geq \|f\|$$

$$\therefore \|\tilde{f}\| = \|f\|.$$

※

Hilbert Space Case :

$X = H =$ Hilbert space.

$Z \subseteq H$.

$f : Z \to \mathbb{F}$ bounded.

Z is dense in $\overline{Z}$,

f is uniformly cts. on Z

$\exists !$ $\hat{f}$ defined on $\overline{Z}$ such that $\hat{f}$ is cts. $\hat{f}$ extends f. Also, $\|\hat{f}\| = \|f\|$

$$\hat{f} : \overline{Z} \to \mathbb{F}$$

$\overline{Z}$ is a Hilbert space, so by Riesz Rep thm

$\exists !$ $y \in \overline{Z}$. $\hat{f}(z) = \langle z, y \rangle$ $\forall z \in \overline{Z}$

$$\|\hat{f}\| = \|y\|$$

$$y \in \overline{Z} \Rightarrow y \in H$$

Defined $\tilde{f}$ on H by $\tilde{f}(x) = \langle x, y \rangle$ $\forall x \in X$

$\tilde{f}$ extends $\hat{f}$

$$\|\tilde{f}\| = \|y\| = \|\hat{f}\| = \|f\| \quad \divideontimes$$

E.F.Y: Show that $\hat{f}$ has other bounded extensions, but there is only one that satisfies $\|\tilde{f}\| = \|\hat{f}\|$.

Thm: X = normed space, $x_0 \in X$, $x_0 \neq 0$

Then $(\exists \tilde{f} \in X')$ $\|\tilde{f}\| = 1$ and $\tilde{f}(x_0) = \|x_0\|$

Pf: Let $Z = \text{span}\{x_0\} = \{\alpha x_0 \mid \alpha \in \mathbb{F}\}$

Define $f : Z \to \mathbb{F}$, $f(\alpha x_0) = \alpha \|x_0\|$

f is linear, f is bounded

$$\|f\| = \sup_{\substack{\alpha \in \mathbb{F} \\ \alpha \neq 0}} \frac{|f(\alpha x_0)|}{\|\alpha x_0\|} = 1 . \qquad f(x_0) = \|x_0\|$$

By H-B, f has an extension $\tilde{f}$ defined on X

such that $\|\tilde{f}\| = \|f\| = 1$ ※

Hilbert Space Case: $X = H$ = Hilbert Space

$x_0 \neq 0$, Let $\hat{x} = \frac{1}{\|x_0\|} x_0$.

Let $\tilde{f}(x) = \langle x, \hat{x} \rangle$ $\forall x \in X$. ※

Cor: X = normed space. $x \in X$, $x \neq 0 \Rightarrow (\exists f \in X')$

$f(x) \neq 0$.

Cor: If $(x \in X)$ $(f(x) = 0 \ \forall f \in X') \Rightarrow x = 0$.

Cor: $x_1, x_2 \in X$, $x_1 \neq x_2 \Rightarrow (\exists f \in X')$ $f(x_1) \neq f(x_2)$

Cor: $x_1, x_2 \in X$, $(f(x_1) = f(x_2) \ \forall f \in X') \Rightarrow x_1 = x_2$

"The dual space separates the points of X'.

Cor: $(\forall x \in X)$ $\quad \|x\| = \sup\limits_{\substack{f \in X' \\ f \neq 0}} \dfrac{|f(x)|}{\|f\|}$

If $x=0 \Rightarrow$ trivial.

p.f: $|f(x)| \leq \|f\| \cdot \|x\| \Rightarrow \dfrac{|f(x)|}{\|f\|} \leq \|x\|$

$\forall f \in X', \quad f \neq 0$

So $\sup\limits_{\substack{f \neq 0 \\ f \in X'}} \dfrac{|f(x)|}{\|f\|} \leq \|x\|$

But by previous theorem, $\exists \hat{f} \in X'$

$\|\hat{f}\| = 1, \quad \hat{f}(x) = \|x\|$.

$\dfrac{|\hat{f}(x)|}{\|\hat{f}\|} = \hat{f}(x) = \|x\| \therefore \sup\limits_{\substack{f \in X' \\ f \neq 0}} \dfrac{|f(x)|}{\|f\|} = \|x\|$.

End § 4.3. Skip § 4.4.

Mar. 25 Monday

1st day after Break.

Exam Next week. §3.5 → 4.1 – 4.3, 4.5 + 4.6.

Hahn – Banach Thm:

§4.5 Adjoint Operator (Conjugate operator) (Normed – space adjoint)

X = normed space over $\mathbb{F}$.

Duality pairing: $x \in X$, $f \in X'$

$\langle x, f \rangle = f(x)$, Resembles inner product.

$x, f \longrightarrow \langle x, f \rangle$

$X \times X' \longrightarrow \mathbb{F}$

bilinear map

Thm: $|\langle x, f \rangle| \le \|x\| \cdot \|f\| \quad \forall x \in X, \ \forall f \in X'$

P.f: E.F.Y. (Resembles Schwarz inequality).

Thm: Let $x \in X$. Then $(\exists f \in X')$

$\langle x, f \rangle = \|x\| \cdot \|f\|$.

P.f: Recall this consequence of Hahn – Banach:

$(\forall x \neq 0)\ (\exists f \in X')\ f(x) = \|x\|$ and $\|f\| = 1$

$\langle x, f \rangle = f(x) = \|x\| = \|x\| \cdot \|f\|$.

Def: Let $S \subseteq X$. The <u>annihilator</u> of S is

$$S^\perp = \{ f \in X' \mid \langle x, f \rangle = 0 \ \forall x \in S \} \subseteq X'$$
$$= \{ f \in X' \mid f(S) = 0 \}$$

Let $S' \subseteq X'$. The <u>annihilator</u> of S' is

$$^\perp S' = \{ x \in X \mid \langle x, f \rangle = 0 \ \forall f \in S' \} \subseteq X$$

Thm: $S^\perp$ is a closed subspace of X'

$^\perp S'$ is a closed subspace of X.

P.f: E.F.Y.

Thm: $S \subseteq {}^\perp(S^\perp)$, $\overline{S} \subseteq {}^\perp(S^\perp)$

$S' \subseteq ({}^\perp(S'))^\perp$

$\overline{S}' = ({}^\perp(S'))^\perp$.

P.f: E.F.Y.

Lemma: X = normed space, Y = closed subspace of X, $x \in X \setminus Y$. Then $(\exists f \in X')$ $f(x) \neq 0$, $f(Y) = 0$.

P.f: Let $\hat{Y} = Y \oplus \text{span}\{x\} = \{ y + \alpha x \mid y \in Y, \alpha \in \mathbb{F} \}$ unique.

Define $\hat{f} : \hat{Y} \to \mathbb{F}$ by $\hat{f}(y + \alpha x) = \alpha$

$\hat{f}$ is well-defined, $\hat{f}$ is linear

$\hat{f}(Y) = 0$ $\quad \hat{f}(x) = 1 \neq 0$.

We just need to show $\hat{f}$ is bounded.

Y is closed, so $\inf\limits_{y \in Y} \|x - y\| := \beta > 0$

$$\frac{|\hat{f}(y + \alpha x)|}{\|y + \alpha x\|} = \frac{|\alpha|}{\|y + \alpha x\|} \quad \text{if } \alpha = 0, \; = 0$$

$\alpha \neq 0$

$$\|y + \alpha x\| = |\alpha| \left\| \frac{1}{\alpha} y + x \right\| = |\alpha| \cdot \left\| x - \left(-\frac{1}{\alpha} y\right) \right\| \geq |\alpha| \cdot \beta > 0.$$

$$\frac{|\hat{f}(y + \alpha x)|}{|y + \alpha x|} \leq \frac{|\alpha|}{|\alpha| \beta} = \frac{1}{\beta}$$

$\therefore$ $\hat{f}$ is bounded and $\|\hat{f}\| \leq \frac{1}{\beta}$.

$\therefore$ By Hahn-Banach $\hat{f}$ has an extension

$f \in X$ such that $\|f\| = \frac{1}{\beta}$, $f(x) = 1 \neq 0$, $f(Y) = 0$

Remark: $\quad f(x) = 1$, $\|f\| \leq \dfrac{1}{\inf\limits_{y \in Y} \|x - y\|}$.

Thm: S = subspace of $X \Rightarrow \bar{S} = {}^{\perp}(S^{\perp})$

Pf: (1) $\bar{S} \subseteq {}^{\perp}(S^{\perp})$ from above

we will show that if $x \notin \bar{S}$, then $x \notin {}^{\perp}(S^{\perp})$

Applying lema $(\exists f \in X')$ $\langle x, f \rangle \neq 0$, but $f(\bar{S}) = 0$.

$f(\bar{S}) = 0 \Rightarrow f(S) = 0 \Rightarrow f \in S^{\perp}$

$\langle x, f \rangle \neq 0 \Rightarrow x \notin {}^{\perp}(S^{\perp})$. ✡

Thus S' = subspace of $X' \Rightarrow \overline{S'} = ({}^{\perp}S')^{\perp}$.

P.f: Harder.

Def: X = normed space. Y = normed space

$T \in B(X, Y)$, $T : X \to Y$.

the addjoint (conjugate) of T is an operator $T' : Y' \to X'$ defined by

$$\underset{Y \times Y'}{\langle Tx, f \rangle} := \underset{X \times X'}{\langle x, T'f \rangle} \quad \forall x \in X, \forall f \in Y'$$

$$\Rightarrow f(Tx) = T'f(x) \quad \forall x \in X, \forall f \in Y'$$

$$f \circ T(x) = T'f(x) \quad \forall x \in X.$$

$$T'f = f \circ T. \quad \forall f \in Y'$$

Prop: T' is linear : $T' : Y' \to X'$

Prop: T' is bounded and $\|T'\| \le \|T\|$

$$\|T'f\| = \|f \circ T\| \le \|f\| \cdot \|T\|$$

$$\Rightarrow \frac{\|T'f\|}{\|f\|} \le \|T\| \quad \forall f \neq 0.$$

$\therefore$ T' is bounded and $\|T'\| \le \|T\|$

$T' \in B(Y', X')$

Thm: $\|T'\| = \|T\|$

p.f: ~~$\langle x, T'f \rangle = \langle Tx, f \rangle$~~

$\langle Tx, f \rangle = \langle x, T'f \rangle \le \|x\| \cdot \|T'f\|$

$\le \|x\| \cdot \|T'\| \cdot \|f\|.$

$$\|Tx\| = \sup_{\substack{f \in Y' \\ f \neq 0}} \frac{|f(Tx)|}{\|f\|} = \sup_{\substack{f \in Y \\ f \neq 0}} \frac{\langle Tx, f \rangle}{\|f\|}$$

$\le \|x\| \cdot \|T'\| \qquad \forall\, x \in X$

$\therefore \quad \|T\| \le \|T'\|.$ #

Wed. March 27

$$\|x\|^2 = \sum_k |\langle x, e_k\rangle|^2$$

$$\langle x, y\rangle = \sum_k \text{---}$$ Polarization.

X, Y = normed spaces.

$T: X \to Y \qquad T \in B(X, Y)$

$T': Y' \to X'$

$$\langle Tx, f\rangle = \langle x, T'f\rangle \quad \forall x \in X, \forall f \in Y'$$

$$T'f = f \circ T = fT$$

Thm: $T' \in B(Y', X')$, $\|T'\| = \|T\|$

Thm: $\eta(T') = R(T)^{\perp}$

Pf: $f \in \eta(T')$ iff $T'f = 0$ iff

$\langle x, T'f\rangle = 0 \quad \forall x \in X$

iff $\langle Tx, f\rangle = 0 \quad \forall x \in X$.

iff $\langle y, f\rangle = 0 \quad \forall y \in R(T)$ iff $f \in R(T)^{\perp}$.

Thm: $\eta(T) = {}^{\perp}R(T')$ Pf: E.F.Y.

~~Thm~~ cor: ${}^{\perp}\eta(T') = \overline{R(T)}$

cor: $\eta(T)^{\perp} = \overline{R(T')}$

Thm: $S, T \in B(X,Y)$, $(S+T)' = S' + T'$.

Thm: $T \in B(X,Y)$, $\alpha \in \mathbb{F} \Rightarrow (\alpha T)' = \alpha T'$

Thm: $S \in B(X,Y)$, $T \in B(Y,Z) \Rightarrow (TS)' = S'T'$

Thm: $I \in B(X,X)$, $Ix = x \Rightarrow I' \in B(X', X')$, $I'f = f$

Thm: $T \in B(X,Y)$, T bijective, $T^{-1} \in B(Y,X)$

Then $(T^{-1})'$ exists and $(T^{-1})' = (T')^{-1}$.

Remarks: Say we want to show A^{-1} exists and $A^{-1} = B$.

S.T.P. $BA = I$, $\longrightarrow$ A is 1–1, injective

$AB = I$ $\longrightarrow$ A is surjective

P.f: S.T.P. $(T^{-1})' T' = I'$

$(T')(T^{-1})' = I'$

$$(T^{-1})' T' = (T T^{-1})' = I'$$

$$(T')(T^{-1})' = (T^{-1} T)' = I'$$

//

Direct proof that T' is 1–1.

This is equivalent to show $\eta(T') = \{0\}$.

Note that $\eta(T') = R(T)^{\perp} = Y^{\perp} = \{0\}$.

Finite Dimensional Case.

$$T : X \to Y. \qquad \dim(X) = m, \ \dim(Y) = n$$

$u_1, \cdots, u_m$ basis for X, $\quad v_1, \cdots, v_n$ basis for Y.

Let A be the matrix of T w.r.t. the bases.

$$\underset{(n\times m)}{A} = (a_{ij}), \text{ where } Tu_j = \sum_{i=1}^{n} v_i a_{ij}, \ j = 1, \cdots, m,$$

$$T' : Y' \to X'$$

$v_1', \cdots, v_n'$ dual basis of $v_1, \cdots, v_n$ $\qquad$ $u_1', \cdots, u_n'$ dual basis of $u_1, \cdots, u_n$.

$\langle v_j, v_i' \rangle = \delta_{ij}$ $\qquad$ $\langle u_k, u_l' \rangle = \delta_{kl}$.

Let B be the matrix of T' w.r.t. dual bases.

$$\underset{m\times n}{B} = (b_{kl}) \qquad T'_{v_l'} = \sum_{k=1}^{n} u_k' b_{kl}, \ l = 1, \cdots, n$$

$$\langle Tx, f \rangle = \langle x, T'f \rangle$$

$$\langle Tu_j, v_l' \rangle = \langle u_j, T' v_l' \rangle$$

$$\Rightarrow \sum_{i=1}^{n} a_{ij} \underbrace{\langle v_i, v_l' \rangle}_{\delta_{il}} = \sum_{k=1}^{m} b_{kl} \underbrace{\langle u_j, u_k' \rangle}_{\delta_{jk}}$$

$$\Longrightarrow a_{lj} = b_{jl} \Longrightarrow \underline{A^T = B}$$

Relationship with Hilbert-space Adjoint.

$X = H_1$, $Y = H_2$, Hilbert spaces

$T \in B(H_1, H_2)$

$T^* \in B(H_2, H_1)$

$T' \in B(H_2', H_1')$

Given $z \in H_1$, define $f_z \in H'$ by $f_z(x) = \langle x, z \rangle$
inner product

<u>Def:</u> $C_1 : H_1 \to H_1'$ by $C_1(z) = f_z$

By Riesz's, C_1 is a norm-preserving bijection.

conjugate linear

$C_1(z_1 + z_2) = C_1(z_1) + C_1(z_2)$ $\quad C_1(\alpha z) = \bar{\alpha} C_1(z)$

Def C_2: similarly $C_2 : H_2 \to H_2'$

$$\begin{tikzcd} H_2 \arrow[r, "T^*"] \arrow[d, "C_2"'] & H_1 \arrow[d, "C_1"] \\ H_2' \arrow[r, "T'"] & H_1' \end{tikzcd}$$

$\langle Tx, y \rangle_i = \langle x, T^* y \rangle_i \quad \forall x \in X, \ y \in Y.$

$\langle Tx, f \rangle_d = \langle x, T' f \rangle_d. \quad \forall x \in X, \ f \in Y'$

$\langle Tx, f_z \rangle_d = \langle x, T' f_z \rangle_d \quad \forall x \in X, \ \forall z \in Y$

$f_z(Tx) = \langle Tx, z \rangle_i = \langle x, T^* z \rangle_i$

$= f_{T^* z}(x) = \langle x, f_{T^* z} \rangle_d.$

$$\langle x, T' f_z \rangle_d = \langle x, f_{T^* z} \rangle_d .$$

$$T' f_z = f_{T^* z}, \quad \forall z \in H_2$$

$$z \in Y, \quad f_z = \langle x, z \rangle$$

$$C_2(z) = f_z$$

$$T' C_2 z = C_1 T^* z \quad \forall z \in H_2$$

$$\Rightarrow \quad T' C_2 = C_1 T^*$$

or $\quad T' = C_1 T^* C_2^{-1} \quad$ or $\quad T^* = C_1^{-1} T' C_2 \quad$ ##

Mar. 29
Friday

§4.6 Reflexivity

X = normed space

Given $x \in X$, define: $g_x : X' \to \mathbb{F}$ by
$g_x(f) = f(x)$.

<u>Thm</u>: g_x is linear, bounded and $\|g_x\| = \|x\|$.

<u>Pf</u>: $\|g_x\| = \sup\limits_{f \neq 0} \dfrac{|g_x(f)|}{|f|} = \sup\limits_{f \neq 0} \dfrac{|f(x)|}{\|f\|} \underset{\uparrow}{=} \|x\|$
Hahn–Banach thm

<u>Def</u>: The <u>canonical map</u> of X into X'' is

$$C(x) = g_x$$

$$C : X \to X''$$

Thm: C is a linear map, norm preseving

$\Rightarrow$ C is an isomorphism of X onto $R(C)$

$R(C) \subseteq X''$

Def: X is reflexive if $R(C) = X''$. In that case

$X \cong X''$.

$g_x(f) = f(x) \quad \Rightarrow \quad \langle f, g_x \rangle = \langle x, f \rangle$

$\qquad\qquad X' \times X'' \qquad\qquad X \times X'$

Think of $g_x = x \quad \Rightarrow \langle f, x \rangle = \langle x, f \rangle$.

in the duality space.

Thm: X reflexive $\Rightarrow$ X complete.

P.f: $X'' = B(X', \mathbb{F})$ [$B(Y, Z)$ is complete if Z is complete

Thm: Every Finite dimensional space is reflexive.

P.f: $\dim(X) = n < \infty$. All linear functionals

are bounded. $X' = X^*$, $\dim(X') = \dim(X^*) = n$

$X'' = X^{**}$ $\dim(X'') = n$.

$\therefore$ C is a surjection $C: X \to X''$ #

Ex. $1<p<\infty$, l_p is reflexive

$\frac{1}{p}+\frac{1}{q}=1$

Naive proof: $l_p' \cong l_q$, $l_q' \cong l_p$. $l_p'' \cong l_q' \cong l_p$.

we have ~~sh~~ to show the canonical map is surjective.

Given $y=(y_i)\in l_q$, define

$$f_y : l_p \to \mathbb{F}$$

by $f_y(x) = \sum_{i=1}^{\infty} x_i y_i$, absolutely convergent (Hölder).

f_y is linear, bounded. $\|f_y\| \le \|y\|_q$ (Hölder)

$\therefore f_y \in l_p'$

In fact we showed $\|f_y\| = \|y\|_q$ and the linear map

$y \longmapsto f_y$ is a surjection

It is an isomorphism, $l_q \cong l_p'$

$l_p \cong l_q'$ $\quad x \longmapsto f_x \quad$ by $f_x(y) = \sum_{i=1}^{n} y_i x_i$

<u>Canonical map</u> $C : l_p \to l_p''$: $C(x) = g_x$

$\underset{\substack{\uparrow\\ l_p}}{g_x}\,(\underset{\substack{\uparrow\\ l_p' = l_q}}{f}) = f(x)$ $\qquad g_x(f_y) = f_y(x) = \sum_{i=1}^{n} x_i y_i$

Is the cannonical map a surjection?

$f = f_y \ \exists y \in l_q$

Let $g \in l_p''$. We want to show that $(\exists x \in l_p)$

$$g = g_x, \quad g \in l_p'', \quad g : l_p' \longrightarrow \mathbb{F}$$

~~$f \in l_p'$, $f = f_y$, $\exists y \in l_p$~~

$$g : l_q \rightarrow \mathbb{F} \Rightarrow g \in l_q'$$

$\Rightarrow$ $(\exists x \in l_p)$ such that

$$g(f_y) = g(y) = \sum_{i=1}^{\infty} x_i y_i \quad \forall y \in l_q \quad (f_y \in l_p')$$

$$g(f_y) = f_y(x) \quad \Rightarrow \quad g = g_x.$$

__Duality Pairing__

$$f_y(x) = \sum_{i=1}^{\infty} x_i y_i$$

$$\underset{l_p \quad l_p'}{\langle x, f_y \rangle} = \sum_{i=1}^{\infty} x_i y_i, \quad \underset{l_p, \, l_q}{\langle x, y \rangle} = \sum_{i=1}^{\infty} x_i y_i$$

$$\underset{l_p, \quad l_q'}{\langle y, f_x \rangle} = \sum_{i=1}^{\infty} x_i y_i, \quad \underset{l_q \quad l_p}{\langle y, x \rangle} = \sum_{i=1}^{\infty} x_i y_i.$$

__Thm:__ Every Hilbert space is Reflexive.

__P.f:__ H = Hilbert space.

Show H' is a Hilbert space with inner product

$$\langle f_u, f_v \rangle_1 = \langle v, u \rangle \qquad f_u(x) = \langle x, u \rangle$$

Indeed this is an inner product. Furthermore,

$$\sqrt{\langle f_u, f_u \rangle_1} = \sqrt{\langle u, u \rangle} = \|u\|^2 = \|f_u\|^2$$

$\Rightarrow$ H' is Hilbert space.

Define $\quad C_1 : H \to H'$ $\quad$ norm-preserving bijection.

$\qquad\qquad v \to f_v$ $\quad$ (conjugate linear)

$C_2 : H' \to H''$ $\quad$ norm-preserving bijection

$\qquad f \to g_f$ $\quad g_f(h) = \langle h, f \rangle_1$.

Let Isomorphism.

$C_3 : H \to H''$ be the composite map.

$C_3 = C_2 C_1$ $\quad$ norm-preserving bijection

linear !

In fact C_3 is C,

$g_{f_v} : H' \to \mathbb{F}$

$C_3(v) = g_{f_v}$

$g_{f_v}(f_u) = \langle f_u, f_v \rangle_1 = \langle v, u \rangle = f_u(v)$.

This shows that $g_{f_v} = g_v$. $\left(g_v(f) = f(v) \right)$

$\Rightarrow \quad C_3(v) = C(v)$.

$\therefore \quad H$ is reflexive. $\quad$ - - - - $\#$

April 1
Monday

Ex: $c_0 = \{x = (x_n) \mid \lim_{n\to\infty} x_n = 0\}$

$\|x\|_\infty = \sup_i |x_i|$ $\quad c_0 \subseteq l_\infty$

$(c_0, \|\cdot\|_\infty)$ is a Banach space.

we know $c_0' \cong l_1$, $\quad l_1' \cong l_\infty$

Canonical map.

$C : c_0 \to c_0'' \ (\cong l_\infty)$

$C : c_0 \to l_\infty$

E.F.Y. $\quad R(C) = c_0$

So. c_0 is not reflexive.

Remark:

c_0 is sepraable, $\quad l_\infty$ is not separable.

So $c_0 \ncong l_\infty \cong c_0''$

$\therefore$ c_0 is not reflexive.

Thm: X' is separable $\Rightarrow$ $X = $ (normed space) is separable

P.f: Suppose X' is separable ($\exists$ contable dense subset)

Let $U' = \{f \in X' \mid \|f\| = 1\}$

U' is separable (E.F.Y.)

$\exists$ $(f_n) \subseteq U'$ such that $\{f_n \mid n = 1, 2, 3, \cdots\}$ is dense in U', $\|f_n\| = 1$ $\forall n$.

$1 = \|f_n\| = \sup_{\|x\|=1} |f_n(x)|$

$(\forall n)(\exists x_n \in X)$ $|f_n(x_n)| \geq \frac{1}{2}$

Let $Y = \overline{\text{span}\{x_1, x_2, x_3, \cdots\}} \subseteq X$ is closed subspace. Y is separable (E.F.Y. Fi-----)

If $Y \neq X$, $\exists\, x \in X \setminus Y$

$\Rightarrow (\exists f \in X') \quad f(x) \neq 0, \quad f(Y) = 0$

w.L.G. $\|f\| = 1$. $f \in U'$

$(\exists f_{n_k}) \quad f_{n_k} \to f$ (in X' norm)

$\underline{|f_{n_k}(x_{n_k})| \geq \frac{1}{2}}$

$$|f(x_{n_k}) - f_{n_k}(x_{n_k})| \leq \|f - f_{n_k}\| \cdot \underbrace{\|x_{n_k}\|}_{1} \to 0 \text{ as } k \to \infty$$

but $f(x_{n_k}) = 0 \Rightarrow |f_{n_k}(x_{n_k})| \to 0$ as $k \to \infty$.

$\times\times$

Cor: l_1 is not reflexive

p.f: $l_1' \cong l_\infty$

$l_1'' \cong l_\infty'$

l_∞ is not separable, so $l_\infty' \cong l_1''$ is not separable. $\Rightarrow l_1$ is not separable

so $l_1 \ncong l_1''$ //

#

Riesz

3.7

Exam. weds. Fri.

Bring Paper. → 3.5 – 3.10, → 4.1 – 4.6 (skiping 4.4)

sesquilinear T^*.

normal operator – OPERATOR.

multiplication operator

eigenvalue

Zorn's Lemma.

Hahn-Banach. I, II, III, IV.

Complex linear functional.

important corollaries

duality pairing. normed space adjoint.

reflexivity. canoical map.

Examination?

April 19 Monday

mid. 81% . high 94%

Chapter 4, Part I

§4.7. uniform bounded Principle (Banach–Steinhaus Thm)

Review. of Baire Category Thm.

X = metric space.

Def: $M \subseteq X$

a) M is nowhere dense if $(\bar{M})^\circ = \phi$.

(This means $\bar{M}$ contains no open sets)

b) M is of first category (meagre) if $\exists\ (A_n)$, $M = \bigcup_{n=1}^{\infty} A_n$ and each A_n is nowhere dense

c) M is of second category if M is not of first category (non meagre)

Baire Category Thm. X = a complete metric space.

$\Rightarrow$ X is of the second category (in X)

Pf: Let (A_n) be a sequence of nowhere dense sets in X

Let $S = \bigcup_{n=1}^{\infty} A_n$, We'll show that $S \neq X$

Let V be an open set in X. $V \not\subseteq \bar{A}_1$, so,

$x_1 \in V \setminus \bar{A}_1$ ---- $x_n \rightarrow x \notin S$

$\#$

Uniform Boundedness Principle:

$X =$ Banach space, $Y =$ normed space.

$\{T_\alpha\}_{\alpha \in I}$ = family of operator in $B(X, Y)$.

Suppose $(\forall x \in X)$ $\{T_\alpha x \mid \alpha \in I\}$ is bounded.

Then $\{T_\alpha \mid \alpha \in I\}$ is a bounded subset of $B(X, Y)$.

That is if $(\forall x \in X)(\exists\, c_x)$ $\|T_\alpha x\| \le c_x$, $\forall \alpha \in I$

then $\exists$ $(\exists C)$ $\|T_\alpha\| \le C$ $\forall \alpha \in I$

P.f: For $k = 1, 2, 3, \ldots$

$$A_k = \{x \in X \mid \|T_\alpha x\| \le k \quad \forall \alpha \in I\}$$

A_k is closed. (E.F.Y.)

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

$$\bigcup_{k=1}^{\infty} A_k \subseteq X$$

Conversely, $\forall x \in X$ $(\exists k)$ $x \in A_k$

take $k \geq c_x$ $\Rightarrow$ $x \in \bigcup_{k=1}^{\infty} A_k$.

$\Rightarrow X = \bigcup_{k=1}^{\infty} A_k$.

X complete $\Rightarrow$ X of second category.

$\Rightarrow (\exists k)$ $A_k = \overline{A_k}$ has nonempty interior.

A_k contains an open ball $B(w, r) = \{x \in X \mid \|x - w\| < r\}$

$\therefore \forall y \in B(w, r)$, $\|T_\alpha y\| \le k$, $\forall \alpha \in I\}$

Let $x \in B(0, r)$. $y = w + x$, Then $y \in B(w, r)$

and $x = y - w$

$T_\alpha x = T_\alpha y - T_\alpha w$

$\|T_\alpha x\| \le \|T_\alpha y\| + \|T_\alpha w\|$

$\le k + k = 2k \qquad \forall \alpha \in I.$

$\forall x \in B(0,r), \quad \|T_\alpha x\| \le 2k, \qquad \forall \alpha \in I.$

$\|T_\alpha\| = \sup_{\|x\|=1} \|T_\alpha x\|,$

Let $x \in X$, $\|x\| = 1$, $\hat{x} = \frac{r}{2} x$

$\Rightarrow \|\hat{x}\| = \frac{r}{2}$, so $\hat{x} \in B(0, r)$

$\therefore \|T_\alpha \hat{x}\| \le 2k \qquad \forall \alpha \in I$

$x = \frac{2}{r} \hat{x}$

$\|T_\alpha x\| = \frac{2}{r} \|T_\alpha \hat{x}\| \le \frac{4k}{r} \qquad \forall \alpha \in I.$

$\|T_\alpha\| = \sup_{\|x\|=1} \|T_\alpha x\| \le \frac{4k}{r} \qquad \forall \alpha \in I.$ ※

Example: X = vector space of polynomials.

$x \in X \Rightarrow x(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$

$\|x\| = \max |a_i|$, This is a norm on X.

use uniform boundedness principle to show that

X is not complete. ....

Alternate proof: show directly that X is of first cat.

X ~~has~~ a countable Hamel basis

$x_0 = 1, \quad x_1(t) = t, \quad \dots$

Let $A_n = \text{span}\{x_0, \dots, x_n\}$.

A_n is closed, nowhere dense.

$X = \bigcup_{n=1}^{\infty} A_n$ . . .

※

Example: $L_2(0,2\pi)$, $\varphi_n(t) = e^{int} \in L_2(0,2\pi)$, $n \in \mathbb{Z}$

$\{\varphi_n\}_{n=-\infty}^{\infty}$ is an orthogonal of $L_2(0,2\pi)$

$\|\varphi_n\|_2^2 = 2\pi$.

Given $x \in L_2(0,2\pi)$ $\quad x = \sum_{n=-\infty}^{\infty} c_n \varphi_n$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} \quad \text{where} \quad c_n = \frac{\langle x, \varphi_n\rangle}{\langle \varphi_n, \varphi_n\rangle}$$

$$= \underbrace{\frac{1}{2\pi}\int_0^{2\pi} x(t) e^{-int}\,dt}_{\text{coff.}}$$

convergent in L_2-norm

~~n. pointwise converge (a.e.)~~

is called Fourier series of x

March 10 Wed

Fourier Series

$$x \in L_2(0,2\pi) \Rightarrow x = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

convergence in $L_2(0,2\pi)$ (not pointwise)

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} x(t) e^{-int}\,dt$$

We'll show existence of a cont. x whose Fourier series divergences at a point

Let

$$X = \{x \in [0,2\pi] \mid x(0) = x(2\pi)\}$$

$$\|x\| = \sup_{t \in [0,2\pi]} |x(t)|$$

E.F.Y. X is a closed subspace of $C[0,2\pi]$.

$\therefore$ X is a Banach space.

for $k=1,2,\cdots$ Define

$$f_k : X \longrightarrow \mathbb{C}$$

by $\quad f_k(x) = \sum_{n=-k}^{k} c_n$ where c_n ---

f_k is linear, which is also kth partial sum of Fourier series of x ~~exauted~~ evaluated at $t=0$.

We'll use $\{f_k\}$ to show that $(\exists x \in X)$ whose Fourier series diverges at $t=0$

$$f_k(x) = \sum_{n=-k}^{k} c_n = \frac{1}{2\pi} \sum_{n=-k}^{k} \int_0^{2\pi} x(t) e^{-int} dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x(t) \left[\sum_{n=-k}^{k} e^{-int} \right] dt$$

Let $\quad g_k(t) = \sum_{n=-k}^{k} e^{-int} \in X$

$$f_k(x) = \frac{1}{2\pi} \int_0^{2\pi} x(t)\, g_k(t)\, dt$$

$$|f_k(x)| \leq \frac{1}{2\pi} \int_0^{2\pi} |x(t)| \cdot |g_k(t)| dt$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |g_k(t)| dt \cdot \|x\|$$

$\therefore f_k$ is bounded and $\quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$\|f_k\| \leq \frac{1}{2\pi} \int_0^{2\pi} |g_k(t)| dt$$

optional E.F.Y. $\quad \|f_k\| = \frac{1}{2\pi} \int_0^{2\pi} |g_k(t)| dt$

optional

E.F.Y. $f_k(t) = \sum_{n=-k}^{k} e^{-int}$ Geometric progression

$$= \frac{\sin(k+\frac{1}{2})t}{\sin(\frac{1}{2}t)}$$

$$\|f_k\| = \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{\sin(k+\frac{1}{2})t}{\sin\frac{1}{2}t} \right| dt$$

$$\geqslant \frac{1}{\pi}\int_0^{2\pi} \frac{|\sin(k+\frac{1}{2})t|}{t} dt$$

$$= \cdots$$

$$\Rightarrow \|f_k\| \geqslant \frac{2}{\pi^2}\sum_{n=1}^{2k} \frac{1}{n} \longrightarrow \infty \text{ as } k\to\infty$$

$\therefore \{\|f_k\|\}$ is unbounded.

$f_k : X \longrightarrow \mathbb{C}$

$\therefore (\exists\, x \in X)$ $\{f_k(x)\}$ is unbounded.

$$f_k(x) = \sum_{n=-k}^{k} c_n$$

???

$\therefore$ Fourier series of X diverges at 0. #

###

§4.8 Strong and weak convergence

X = normed space.

Def: $(x_n) \subseteq X$, $x \in X$

we say $x_n \to x$ strongly if

$\| x_n - x \| \to 0$ as $n \to \infty$

x is the strong limit of (x_n).

x_n converges to x weakly if

$(\forall f \in X')$ $f(x_n) \to f(x)$

we write $x_n \xrightarrow{w} x$

x is called the weak limit of (x_n).

Thm: (x_n) can have at most one weak limit.

Pf: Suppose $x_n \xrightarrow{w} y$

$x_n \xrightarrow{w} z$.

Then $(\forall f \in X')$.

$f(x_n) \to f(y)$

$f(x_n) \to f(z)$

$\mathbb{F}$ is a Hausdorff space $\Rightarrow f(y) = f(z)$

$\forall f \in X'$.

$\therefore$ $y = z$ (consequence of Hahn-Banach)

Thm: Strong convergence $\Rightarrow$ weak convergence.

P.f: E.F.Y.

Ex: H = Hilbert space of infinite dimension.

$x_n \xrightarrow{w} x$ iff $\langle x_n, z\rangle \to \langle x, z\rangle \ \forall z \in H$

Let (e_n) be an orthonormal sequence in H. (Riesz)

$\therefore$ (e_n) does not have a strong limit.

$e_n \xrightarrow{w} 0$.

By Bessel's ineq. $\forall z \in H$

$$\sum_{n=1}^{\infty} |\langle z, e_n\rangle|^2 \le \|z\|^2 < \infty.$$

$\therefore$ $|\langle z, e_n\rangle|^2 \to 0$ as $n \to \infty$

$\therefore$ $\langle e_n, z\rangle \to 0$ as $n \to \infty$ $\forall z \in H$.

$\therefore$ $e_n \xrightarrow{w} 0$

Thm: X = normed space, $\dim X = k < \infty$.

$x_n \to x$ iff $x_n \xrightarrow{w} x$.

P.f: ($\Leftarrow$) Suppose $x_n \xrightarrow{w} x$

Let $e_1, e_2, \cdots, e_k$ be a basis for X.

Let $f_1, f_2, \cdots, f_k$ be the dual basis in X'

$f_i(e_j) = \delta_{ij}$

$x_n \xrightarrow{w} x \Rightarrow f_i(x_n) \to f_i(x)$ $i=1,\cdots,k$

$$x = \sum_{j=1}^{k} \alpha_j e_j$$

$$x_n = \sum_{j=1}^{k} \alpha_j^{(n)} e_j$$

$f_i(x) = \alpha_i \qquad f_i(x_n) = \alpha_i^{(n)}$

$f_i(x_n) \to f_i(x) \Rightarrow \alpha_i^{(n)} \longrightarrow \alpha_i$ as $n \to \infty$

$i = 1, 2, \dots, n.$

$$\|x - x_n\| = \left\| \sum_{j=1}^{k} (\alpha_j - \alpha_j^{(n)}) e_j \right\|$$

$$\leq \sum_{j=1}^{k} \underbrace{|\alpha_j - \alpha_j^{(n)}|}_{\to 0} \cdot \|e_j\| \to 0 \text{ as } n \to \infty$$

$\therefore x_n \to x$ strongly. ✕

<u>Thm:</u> Weakly convergent sequences are bounded.

<u>p.f:</u> Suppose $x_n \xrightarrow{w} x$. We must show $(\|x_n\|)$ is bounded.

$\forall f \in X' \qquad f(x_n) \to f(x)$

Define $g_n : X' \to \mathbb{F}$ by $g_n(f) = f(x_n)$.

$g_n \in X''$

We know $\|g_n\| = \|x_n\|$ (Consequence of Hahn – Banach.)

So. S.T.P. $(\|g_n\|)$ is bounded.

$\forall f \in X' \qquad |g_n(f)| = |f(x_n)| \to |f(x)|$

So $\{g_n(f) \mid n = 1, 2, \dots\}$ is bounded, $\forall f \in X'$.

$g_n : X' \to \mathbb{F}$

complete.

So by uniform boundedness.

$\{\|g_n\| \mid n = 1, 2, \dots\}$ is bounded. ✕

Thm: X = normed space $(x_n) \subseteq X$, $x \in X$.

Then $x_n \xrightarrow{w} x$ iff

(a) (x_n) is bounded.

(b) $(\exists$ total set $M \subseteq X')$ $f(x_n) \to f(x)$ $\forall f \in M$.

Pf: $(\Rightarrow)$ done.

$(\Leftarrow)$ M total means $\overline{\text{span } M} = X'$. E.F.Y.

Friday April 12

$(\Leftarrow)$ (x_n) bounded $\Rightarrow$ $\exists \hat{c} > 0$ $\|x_n\| \le \hat{c}$ $\forall n$.

Let $c = \max\{\hat{c}, \|x\|\}$.

$$f(x_n) \longrightarrow f(x) \quad \forall f \in M$$

$$\Rightarrow f(x_n) \to f(x) \quad \forall f \in \text{span}(M) \quad \text{(E.F.Y.)}$$

$$\underline{\overline{\text{span}(M)} = X'}.$$

We want to show $f(x_n) \to f(x)$ $\forall f \in X'$.

Let $f \in X'$, Let $\varepsilon > 0$.

$(\exists g \in \text{span}(M))$ $\|f - g\| < \frac{\varepsilon}{3c}$.

$g(x_n) \to g(x)$ as $n \to \infty$, so

$(\exists N)$ $|g(x_n) - g(x)| < \frac{\varepsilon}{3c}$ $\forall n \ge N$.

Let $(n \ge N)$ Then

$$|f(x) - f(x_n)| \le |f(x) - g(x)| + |g(x) - g(x_n)| + |g(x_n) - f_n(x)|$$

$$\le \|f - g\| \cdot \|x\| + |g(x) - g(x_n)| + \|g - f\| \cdot \|x_n\|$$

$$\le \frac{\varepsilon}{3c} \cdot c + \frac{\varepsilon}{3} + \frac{\varepsilon}{3c} \cdot c = \varepsilon.$$

$f_n(x_n) \to f(x)$ $\forall f \in X$. $\therefore x_n \xrightarrow{w} x$ $\blacksquare$

<u>Theorem</u> : (Weak convergence in l_p $1<p<\infty$)

$1<p<\infty$, $(x_n) \subseteq l_p$, $x \in l_p$

$(x_n) = (x_j^{(n)})$, $x = (x_j)$

Then $x_n \xrightarrow{w} x$ iff

(a) (x_n) is bounded.

(b) $x_j^{(n)} \longrightarrow x_j$ as $n \to \infty$, $j = 1, 2, 3, \cdots$

<u>p.f.</u> ($\Rightarrow$) easy. $x \longmapsto x_j$ is linear functional.

($\Leftarrow$) $l_p' = l_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

$$f_y(x) = y(x) = \langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i, \quad x \in l_p,\ y \in l_q.$$

$$e_j = (0, \cdots, 0, \underset{\substack{\uparrow \\ j}}{1}, 0, 0 \cdots 0)$$

$M = \{ e_j \mid j = 1, 2, 3, \cdots \}$ is a total subset of $l_q = l_p'$

$\langle x, e_j \rangle = x_j$

$\langle x_n, e_j \rangle = x_j^{(n)}$

(b) says $x_j^{(n)} \longrightarrow x_j$ $\forall j$

$e_j(x_n) \to e_j(x)$ $\forall j$, $\forall e_j \in M$.

$\therefore$ by the previous theom $\Rightarrow$ $x_n \xrightarrow{w} x$

Weak Topology : $\begin{cases} \text{strong convergence } \forall f \\ \|x_n - x\| \to 0. \end{cases}$

Weak Topology :

Def: X = normed space. The weak topology on X is the weakest topology such that $\forall f \in X'$, $f : X \to \mathbb{F}$ is continuous. i.e. it is the weakest top. that contains

$$f^{-1}(G) = \{x \in X \mid f(x) \in G\} \quad \begin{matrix} \forall f \in X' \\ \forall \text{ open } G \subseteq \mathbb{F} \end{matrix} \Big\}$$

Optinal. E.F.Y: $x_n \xrightarrow{w} x$ iff x_n conveges to x w.r.t. weak topology.

E.F.Y. The weak top is a Hausderff top.

X = metric space.

$S \subseteq X$ $\qquad x \in \overline{S}$ iff $\exists$ sequence $(x_n) \subseteq S$, $x_n \to x$

Topological vector spaces.

$\boxed{\text{skip 4.9}}$

4.9. Convergence of sequence of operators

$X, Y =$ normed spaces

$T_n : X \to Y, \quad n = 1, 2, \cdots$

We say (T_n): converges uniformly (in operator norm) if $\exists\, T \in B(X, Y)$ $\|T_n - T\| \to 0$ as $n \to \infty$.

(T_n) conveges strongly if

$(\forall x \in X)$ $(T_n x)$ is strongly convergent in Y

$(\exists\, Tx \in Y \quad \|T_n x - Tx\| \to 0$ as $n \to \infty)$

(T_n) converges weakly if.

$(\forall x \in X)$ $(T_n x)$ is weakly convergent in Y

$(\exists\, Tx \in Y)$ $\forall f \in Y'$ $\langle T_n x, f\rangle \to \langle Tx, f\rangle$ as $n \to \infty$

Limits of all types are unique.

E.F.Y. uniform $\underset{\not\Leftarrow}{\Rightarrow}$ strong $\underset{\not\Leftarrow}{\Rightarrow}$ weak

Weak and strong coucongence are manigful for unbounded operators.

Ex: ℓ_2 $\quad e_1, e_2, \cdots =$ standard basis.

(T_n), $n = 1, 2, \cdots$

$$T_n e_i = \begin{cases} 0 & \text{if } 1 \le i \le n \\ e_i & \text{if } i > n \end{cases}$$

$$T_n(x_1, x_2, \cdots) = (0, 0, \cdots, 0, x_{n+1}, x_{n+2}, \cdots)$$

$\forall x \in X \quad \|T_n x\| = \sqrt{\sum_{i=n}^{\infty} |x_i|^2} \to 0$ as $n \to \infty$

Let $T = 0$,

$T_n x \to 0x, \quad \forall x \in X.$

$T_n \to 0$ strongly.

But $\|T_n - 0\| = \|T_n\| = 1 \quad \forall n.$

T_n is not uniformly convergent to zero.

Monday April, 15

Assign't #6.

Due. Monday, April 29.

P. 255, # 10, P. 262, # 4, 5–7, 10

P. 269 # 10

Suggested Exercises.

P. 255, 6–8, 10, 11, 14,

P. 262, 1–8, 10,

P. 268, 1–6, 9, 10

Convergence of sequences of operators (T_n)

<u>uniform</u> $\|T_n - T\| \to 0$.

<u>strong</u> $T_n x \to Tx$ (strongly) $\forall x \in X$

<u>weak</u> $T_n x \xrightarrow{w} Tx \quad \forall x \in X.$

<u>Ex:</u> $T_n : l_2 \to l_2$

$T_n x = T_n(x_1, x_2, \dots) = (\underbrace{0, \dots, 0}_{n}, x_1, x_2, x_3, \dots)$

$T_n \in B(l_2, l_2)$, $\|T_n x\| = \|x\| \ \forall x, \forall n$, are isometrics

claim: $T_n \xrightarrow{w} 0$ weakly

S.T.P $(\forall x \in l_2) \ T_n x \xrightarrow{w} 0x = 0$

$\Rightarrow \langle T_n x, y\rangle \to \langle 0, y\rangle = 0 \quad \forall y \in l_2$

$$|\langle T_n x, y\rangle| = |(0,0,\cdots,0,x_1,x_2,\cdots)(\overline{y_1}, \overline{y_2}, \cdots)^T|$$
$$= \left|\sum_{i=1}^{\infty} x_i \overline{y_{n+i}}\right| \le \underbrace{\sqrt{\sum_{i=1}^{\infty} |x_i|^2}}_{\|x\|} \cdot \sqrt{\sum_{i=n}^{\infty} |y_{n+i}|^2} \longrightarrow 0.$$

$\therefore T_n \to 0$ weakly. But

$T_n \not\to 0$ stongly since

$\forall x \in l_2, \quad \|T_n x\| = \|x\| \not\to 0.$

<u>Thm</u> X = complete, $(T_n) \subseteq B(X,Y)$, $T_n \to T$ strongly.

Then $T \in B(X,Y)$.

<u>Pf:</u> $\forall x \in X, \quad T_n x \longrightarrow Tx$

Note: $T: X \to Y$, T is linear (E.F.Y.)

$(T_n x)$ convergent $\Rightarrow$ $(T_n x)$ bounded

By uniform boundedness priciple, (T_n) bounded,

i.e., $\exists\, c$, $\|T_n\| \le c \quad \forall n$

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \le \liminf_{n\to\infty} \|T_n\| \cdot \|x\| \le c \cdot \|x\|$$

$\therefore$ T is bounded and $\|T\| \le c$. #

<u>Thm:</u> X, Y = Barach spaces, $(T_n) \subseteq B(X,Y)$. (T_n) is strongly convergent iff

(a) $(\|T_n\|)$ is bounded.

(b) $\exists$ total subset $M \subseteq X$ such that

$(\forall x \in M)$ $(T_n x)$ is a Cauchy sequence.

Pf: ($\Rightarrow$) (a) (T_n) strongly convergent $\Rightarrow$ $(\|T_n\|)$ bounded.

(b) is trivial.

($\Leftarrow$) $(\|T_n\|)$ bounded, so $(\exists c>0)\ \|T_n\| \le c\ \forall n$

M total $\Rightarrow$ $\overline{\text{span}(M)} = X$

$x \in \text{span}(M) \Rightarrow (T_n x)$ Cauchy (E.F.Y.)

We want to show $(\forall x \in X)$ $(T_n x)$ is Cauchy.

Let $x \in X$, $\varepsilon > 0$

$(\exists\ y \in \text{span}(M))$ $\|x-y\| < \frac{\varepsilon}{3c}$

$(T_n y)$ is Cauchy, so $(\exists N)(\forall m,n \ge N)$

$\|T_m y - T_n y\| < \frac{\varepsilon}{3}$

Now $(\forall m,n \ge N)$ $\|T_m x - T_n x\|$

$\le \|T_m x - T_m y\| + \|T_m y - T_n y\| + \|T_n y - T_n x\|$

$\le \|T_m\| \cdot \|x-y\| + \|T_m y - T_n y\| + \|T_n\| \cdot \|y - x\|$

$< c \cdot \frac{\varepsilon}{3c} + \frac{\varepsilon}{3} + c \cdot \frac{\varepsilon}{3c} = \varepsilon$

$(T_n x)$ is Cauchy $\forall x \in X \Rightarrow (T_n x)$ is conv. $\because Y$ compl

$\Rightarrow T_n$ converges strongly

Convergence of Sequences of Functionals

X = normed space, $f \in X'$ (Banach space)

Dual Nature of Functionals. (Clash ...)

	(f_n) is linear functional	members of X'
$\|f_n - f\| \to 0$	uniform operator convergence	strong convergence.
$f_n(x) \to f(x)\ \forall x \in X$	strong operator convergence $\parallel$	weak-$*$ convergence
$f_n(x) \xrightarrow{w} f(x)\ \forall x \in X$	weak operator convergence	
$g(f_n) \to g(f)\ \forall g \in X''$		weak convergence

Nomenclature for Functionals

strong convergence	$\|f_n - f\| \to 0$
weak convergence	$g(f_n) \to g(f) \ \forall g \in X''$
weak-* convergence	$f_n(x) \to f(x) \ \forall x \in X$