Math 502

Note Book I

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EE/ME G33

5-2348

Funtional Analysis / Spring-91

Jan. 14, 91 Monday | Neill 311

chapters 2 - 4

l_∞ space

Section 1-6.

1.6 Completion of metric spaces.

Ex: $\mathbb{Q}$ = rational numbers

d = usual metric.

$\mathbb{Q}$ is not complete.

$\mathbb{Q} \subseteq \mathbb{R}$, $\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{R}$ is complete.

Ex: $P[a,b]$ polynomials restricted to $[a,b]$

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

$P[a,b]$ not complete.

$C[a,b]$ = continuous functions on $[a,b]$

$C[a,b]$ is complete.

$P[a,b]$ is dense in $C[a,b]$.

Ex: $C[a,b]$

$$d(f,g) = \int_a^b |f(x) - g(x)|\,dx$$

not complete

$L_1[a,b]$ is complete

$C[a,b] \subseteq L_1(a,b)$

$C[a,b]$ is dense in $L_1(a,b)$

Def: (X,d), $(\tilde{X},\tilde{d})$ metric spaces

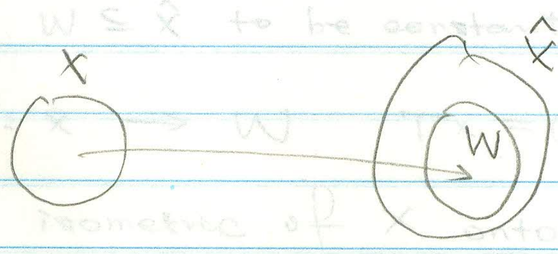
$T: X \to \tilde{X}$ is called an isometry

if $(\forall\, x,y \in X)$ $d(x,y) = \tilde{d}(Tx,Ty)$

E.F.Y. Isometry is 1-1 map.

Def: X, $\tilde{X}$ are isometric if $\exists$ isometry $T: X \to \tilde{X}$
isomorphic
onto $\tilde{X}$.

Thm Let (X,d) = metric space. Then $\left(\exists\, (\hat{X},\hat{d})\right.$ complete metric space$\left.\right)$ and a subspace W, such that W is isometric to X and W is dense in $\hat{X}$.



(Thm cont.) Furthermore, $\hat{X}$ is unique up to isometry.

$(\hat{X}, \hat{d})$ is called the completion of (X, d).

p.f: (X, d)

$\hat{X}$ = space of Cauchy sequences. But we consider $x = (x_n)$, $y = (y_n)$ equivalent if $\lim_{n \to \infty} (x_n, y_n) = 0$.

Let $x, y \in \hat{X}$, $x = (x_n)$, $y = (y_n)$

$(d(x_n, y_n))$ is a Cauchy sequence of real number. ($\mathbb{R}$ is complete)

Define: $\hat{d}(x, y) = \lim_{n \to \infty} d(x_n, y_n)$

must show $\hat{d}(x, y)$ is well defined.

$(\hat{X}, \hat{d})$ is complete.

$X \qquad \hat{X}$

$\cdot \longrightarrow$ seq.

define $W \subseteq \hat{X}$ to be constant sequences

$T: X \to W \qquad Tx = (x, x, x, \dots)$

T is isometric of X onto W.

Show W is dense in $\hat{X}$. E.F.Y.

Suppose we also have $(\tilde{X}, \tilde{d})$ with dense subspace V and an isometry $S : X \to V$.

Consider map

$ST^{-1} : W \to V$ is isometry

[isometry is uniformly continuos]

ST^{-1} has a unique extension $U : \hat{X} \to \tilde{X}$.
is isometry ~~onto~~ $\hat{X}$ onto $\tilde{X}$. #

Jan. 16, 91 wednesday

$\boxed{\text{Phys. Sci. B-14. Meet in}}$

Chapter 2. Normed spaces, Banach spaces.

2.1 vector spaces on ~~the~~ a field

$$\mathbb{F} = \begin{cases} \mathbb{R} & \text{real numbers} \\ \mathbb{C} & \text{complex numbers} \end{cases}$$

2.2.

Def. Let X be a vector space over $\mathbb{F}$.
A norm on X is a function

$$X \longmapsto \mathbb{R}$$

$$x \longmapsto \|x\|$$

satisfying

(i) $\|x\| > 0$ if $x \neq 0$.

(ii) $\|x\| = 0$ if $x = 0$.

(iii) $\|\alpha x\| = |\alpha| \cdot \|x\|$ $(\forall x \in X), (\alpha \in \mathbb{F})$

(iv) $\|x + y\| \leq \|x\| + \|y\|$ $(\forall x, y \in X)$
(triangle inequality)

Think of $\|x\|$ as the "length" of x.

$(X, \|\cdot\|)$ is called a normed space.

Def. $d(x, y) = \|x - y\|$

E.F.Y. $d(x, y)$ is a metric on X.

Def: Banach space is a complete normed space.

Prop: $\big| \|x\| - \|y\| \big| \leq d(x,y)$ $(\forall x, y \in X)$

p.f: E.F.Y. show that $\|x\| - \|y\| \leq \|x-y\|$

$\|y\| - \|x\| \leq \|x-y\|$

coro: $\|\cdot\|$ is a continuous function on X (Lipschitz with $L=1$).

Ex: $\mathbb{F}^n$ ($\mathbb{R}^n$ or $\mathbb{C}^n$)

$\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ Euclidean norm

Barach space.

Ex: $C[a,b]$, $\|f\| = \sup_{x \in [a,b]} |f(x)|$, $C(k)$

E.F.Y. this is a norm.

$d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$

Barach space.

Ex: $C[a,b]$, $\|f\|_1 = \int_a^b |f(x)|\,dx$

E.F.Y. $\|\cdot\|_1$ is a norm

$d_1(f,g) = \int_a^b |f(x) - g(x)|\,dx$

$(C[a,b], \|\cdot\|_1)$ is not a Barach space

not complete.

The completion of this space is $L_1(a,b)$.

Ex: $L_1(a,b)$

$$\|f\|_1 = \int_a^b |f(x)|\,dx$$

$L_1(a,b)$ is complete (Banach space)

Ex: $1 \le p < \infty$

$L_p(a,b) = \cancel{\int_a^b |f(x)|}$ measurable

functions such that $\int_a^b |f(x)|^p\,dx < \infty$.

$$\|f\|_p = \left[\int_a^b |f(x)|^p\,dx\right]^{\frac{1}{p}}$$

This is a Banach space.

Ex: $L_\infty(a,b)$

$\|f\|_\infty$ = esstianl supremum.

Banach space.

Ex: ℓ^∞ = space of bounded sequences (of $\mathbb{R}$ or $\mathbb{C}$)

$$x = (x_1, x_2, x_3, \ldots)$$

ℓ^∞ is a vector space.

$\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$. E.F.Y. $\|\cdot\|_\infty$ is a norm.

Thm. ℓ_∞ is complete

Pf: E.F.Y. Let $(X^{(n)})$ be a cauchy sequence in ℓ_∞. $d(x^{(m)}, x^{(n)}) = \sup |x_i^{(m)} - x_i^{(n)}|$

Cauchy $\Rightarrow$ Cachy in each componet

$x_i^{(n)}$, i = fixed, is chauchy in $\mathbb{F}$.

F is complete, so $(\exists x_i \in \mathbb{F})$

$$\lim_{n\to\infty} x_i^{(n)} = x_i$$

Define

$$x = (x_1, x_2, x_3, \cdots)$$

i) $x \in \ell_\infty$

ii) $d(x^{(n)}, x) \to 0$ as $n \to \infty$.

It is easier show (ii) first

$$d(x^{(n)}, x) = \sup_{i\in\mathbb{N}} |x_i^{(n)} - x_i|$$

$$|x_i^{(n)} - x_i| \le \lim_{n\to\infty} |x_i^{(m)} - x_i^{(n)}|$$

$$\le \lim_{n\to\infty} d(x^{(m)}, x^{(n)}) \le \varepsilon$$

If n is sufficient large.

To prove one

$$X = (X - X^m) + X^m \qquad *$$

Ex: $1 \le p < \infty$

ℓ_p = set of sequence $X = (X_i)$ such that $\sum_{i=1}^{\infty} \|X_i\|^p < \infty$.

Conjugate exponents

$p > 1$. Let $q = \frac{p}{p-1}$ then $\frac{1}{p} + \frac{1}{q} = 1$

$p = \frac{q}{q-1}$, $p = 1$, $q = \infty$ are considered as c. e.

Lemma Let $p, q > 1$ be complex conjugate exponent. Then $\forall \alpha, \beta \geq 0$

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\alpha^q}{q}$$

$$f(\alpha, \beta) = \frac{\alpha^p}{p} + \frac{\beta^q}{q} - \alpha\beta$$

See Geometric interpretation on text on p. 13

Jan. 18, 91
Friday

Suggested Exercises: p.56. 10, 12, 13–15,
p.64, 3, 8, 10–15
p.70, all

l_∞ = bounded sequences Banach space

$$\|x\|_\infty = \sup_i |x_i|$$

l_p = supremum statisfying $\sum_{i=1}^{\infty} |x_i|^p < \infty$ $\quad 1 \le p < \infty$

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p}$$

Congugate exponents $\frac{1}{p} + \frac{1}{q} = 1$

Lemma: $\alpha, \beta \ge 0$, $p, q > 1$, conjugate

$$\Rightarrow \quad \alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

<u>Hölder's Inequality for Sequences:</u>

$x = (x_i)$, $y = (y_i)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Then $\sum_{i=1}^{\infty} |x_i y_i| \le \|x\|_p \cdot \|y\|_q$

<u>Pf</u>: case $p = 1, \infty$, E.F.Y.

Now, assume $1 < p < \infty$

w.l.g. $0 < \|x\|_p < \infty$, $0 < \|y\|_q < \infty$

Special case $\|x\|_p = 1 = \|y\|_q$

Apply lemma

$$\alpha = |x_i|, \quad \beta = |y_i|$$

$$|x_i y_i| \leq \frac{|x_i|^p}{p} + \frac{|y_i|^q}{q}$$

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \frac{1}{p} \underbrace{\sum_{i=1}^{\infty} |x_i|^p}_{\|x_p\|_p^p = 1} + \frac{1}{q} \underbrace{\sum_{i=1}^{\infty} |y_i|^q}_{\|y\|_q^q = 1.}$$

$$\Rightarrow \sum_{i=1}^{\infty} |x_i y_i| \leq 1. \qquad (*)$$

General case: $0 < \|x\|_p < \infty$

$0 < \|x\|_q < \infty$

Take

$$\hat{x} = \frac{x}{\|x\|_p}, \quad \hat{x}_i = \frac{x_i}{\|x\|_p}$$

$$\hat{y} = \frac{y}{\|y\|_q}, \quad \hat{y}_i = \frac{y_i}{\|y\|_q}$$

Then $\|\hat{x}\|_p = 1$, $\|\hat{y}\|_q = 1$.

Apply $(*)$ to $\hat{x}, \hat{y}$... Result falls out #

Remark: Hölder holds for finite sums as well.

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

Minkowski Inequality for Sequences

Let $1 \le p \le \infty$, $x, y \in l_p$. Then $x+y \in l_p$ and $\|x+y\|_p \le \|x\|_p + \|y\|_p$

Case $p=1, \infty$. E.F.Y. Now assume $1<p<\infty$.

P.f. $\|x+y\|_p^p = \sum_{i=1}^{\infty} |x_i + y_i|^p = \lim_{j \to \infty} \sum_{i=1}^{j} |x_i + y_i|^p$

$$\sum_{i=1}^{j} |x_i + y_i|^p = \sum_{i=1}^{j} |x_i + y_i| \cdot |x_i + y_i|^{p-1}$$

$$\le \sum_{i=1}^{j} |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^{j} |y_i| \cdot |x_i + y_i|^{p-1}$$

$$\le \left(\sum_{i=1}^{j} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{j} |x_i + y_i|^p \right)^{\frac{1}{q}} + \text{———}$$

$$\le \|x\|_p + \qquad + \|y\|_p$$

$$\Rightarrow \sum_{i=1}^{j} |x_i + y_i|^p \le (\|x\|_p + \|y\|_p) \underbrace{\left(\sum_{i=1}^{j} |x_i + y_i|^p \right)^{\frac{1}{q}}}_{>0 \text{ if } j \text{ is large enough.}}$$

$$\left(\sum_{i=1}^{j} |x_i + y_i|^p \right)^{1 - \frac{1}{q} = \frac{1}{p}} \le \|x\|_p + \|y\|_p$$

$$\Rightarrow \left(\sum_{i=1}^{j} |x_i + y_i| \right) \le (\|x\|_p + \|y\|_p)^p < \infty$$

$$\therefore \|x+y\|_p^p = \sum_{i=1}^{\infty} |x_i + y_i|^p \le (\|x\|_p + \|y\|_p)^p < \infty$$

$\Rightarrow$ $x+y \in l_p$ and $\|x+y\|_p \le \|x\|_p + \|y\|_p$. ※

Cor: l_p is a vector space.

Cor: $\|\cdot\|_p$ is a norm on l_p.

<u>Thm</u>: l_p is complete.

<u>Pf</u>: Let $(x^{(n)})$ be a Cauchy seq. in l_p

$$\|x^{(n)} - x^{(m)}\|_p \to 0 \quad \text{as} \quad m, n \to \infty$$

$\therefore$ for all i, $(x_i^{(n)})$ is a Cauchy seq. in $\mathbb{F}$.

$$|x_i^{(n)} - x^{(n)}| \leq \|x^{(n)} - x^{(m)}\|_p$$

$\mathbb{F}$ is complete, so $(\exists x_i) \Rightarrow \lim_{n\to\infty} x_i^{(n)} = x_i$

Define $x = (x_i)$

We want to show

(i) $x \in l_p$

(ii) $\lim_{n\to\infty} \|x^{(n)} - x\|_p = 0$

Prove (ii) first, then get (i).

$$\|x^{(n)} - x\|_p^p = \sum_{i=1}^{\infty} |x_i^{(n)} - x_i|^p = \lim_{j\to\infty} \sum_{i=1}^{j} |x_i^{(n)} - x_i|^p$$

$$\sum_{i=1}^{j} |x_i^{(n)} - x_i|^p = \lim_{m\to\infty} \sum_{i=1}^{j} |x_i^{(n)} - x_i^{(m)}|^p$$

$$\leq \lim_{m\to\infty} \sup \underbrace{\|x^{(n)} - x^{(m)}\|_p^p}_{\text{small for } n \text{ large.}}$$

and so on ......

(i) $x = (x - x^{(n)}) + x^{(n)} \in l_p$. #

<u>Thm:</u> separable (coutable dense set)

l_p is separable.

<u>P.f:</u> S = set of rational seq. with only finitely many nonzero terms.

E.F.Y. S is countable, dense.

<u>Thm</u>: l_∞ is not separable.

E.F.Y. Think of an uncountable set S such that

$\forall x \in S$, $\|x\|_\infty = 1$

$\forall x, y \in S$, $x \neq y \Rightarrow \|x - y\|_\infty = 1$

Read §2.3.

$C[a,b]$ is separable Banach space.

#

Jan. 21, 90
Monday

Assignment # 1 due Friday, Feb. 1

p. 70 # 8, 14

p. 76 # 1, 2, 8.

Suggested Exercises. P. 76 1–10, P. 82. 7
(and other if you need work on compactness.

Bases

Hamel Basis

$X =$ vector space $\{e_s \mid s \in \mathcal{S}\}$
is a Hamel Basis
iff $(\forall x \in X)$

$\exists! \{s_1, \dots, s_n\} \subseteq \mathcal{S}$, $! \alpha_1, \dots, \alpha_n \in \mathbb{F}$

$x = \sum_{i=1}^{n} \alpha_i e_{s_i}$

Hamel Basis always exists (Zorn's).
Doesn't depend on topology.

Schauder Basis

$e_1, e_2, e_3, \dots$

is a Schauder basis if $\forall x \in X$

$(\exists! \ \alpha_1, \alpha_2, \cdots \in \mathbb{F})$

$$x = \sum_{i=1}^{\infty} \alpha_i e_i$$

(convergence in mean topology)

<u>Thm:</u> A Banach space with a Schauder basis is separable

<u>P.f:</u> E.F.Y.

<u>Ex:</u> $l_p \quad 1 \le p < \infty$

$e_1 = (1, 0, 0, \cdots)$

$e_2 = (0, 1, 0, \cdots)$

$\vdots$

$e_n = (0, 0, 0, \cdots 1, \cdots)$

$\vdots$

Schauder basis.

$\Rightarrow \exists$ separable Banach space with no Schauder Basis.

2.4 Finite dimensional Normed Spaces

Y = vector space over $\mathbb{F}$ of dimension $k<\infty$.

$\|\cdot\|$ = norm on Y.

Let $v_1, v_2, \dots, v_k$ be a basis

$\forall y \in Y, \quad y = \sum_{i=1}^{k} \alpha_i v_i \qquad (\exists!\ \alpha_i)$

$\|y\| = \|\sum_{i=1}^{k} \alpha_i v_i\| \le \sum_{i=1}^{k} |\alpha_i| \cdot \|v_i\| \le c \sum_{i=1}^{k} |\alpha_i|$

where $c = \max_{1\le i\le k} \|v_i\|$

<u>Thm:</u> $(\exists\, c>0)\ \|\sum_{i=1}^{k} \alpha_i v_i\| \ge c\left(\sum_{i=1}^{k} |\alpha_i|\right)$ (*)

$\forall \alpha_1, \dots, \alpha_k \in \mathbb{F}$.

<u>P.f:</u> Rewrite (*) (w.l.g. $(\alpha_1, \dots, \alpha_k) \ne 0$)

Let $s = \sum_{i=1}^{k} |\alpha_i| > 0, \quad \beta_i = \frac{\alpha_i}{s}$

$\sum_{i=1}^{k} |\beta_i| = 1$

$(*) \iff \|\sum_{i=1}^{k} \beta_i v_i\| \ge c \qquad (**)$

$\forall \beta_1, \dots, \beta_k \quad \sum_{i=1}^{k} |\beta_i| = 1$

Let $K = \{(\beta_1, \dots, \beta_k) \in \mathbb{F}^k \mid \sum_{i=1}^{k} |\beta_i| = 1\}$

K is closed and bounded $\Rightarrow K$ is compact.

Define $f: K \to \mathbb{R}$ by $f(\beta_1, \dots, \beta_k) = \| \sum_{i=1}^{k} \beta_i v_i \|$

f is continous

$$|f(\beta_1, \dots, \beta_k) - f(\gamma_1, \dots, \gamma_k)| \le f(\beta_1 - \gamma_1, \dots, \beta_k - \gamma_k)$$

$$\le C \sum_{i=1}^{k} |\beta_i - \gamma_i|$$

f is continuous on compact set, so

$$(\exists \hat{\beta} \in K) \quad f(\hat{\beta}) = \inf_{\beta \in K} f(\beta)$$

$f(\hat{\beta}) > 0$. Let $c = f(\hat{\beta}) > 0$.

$$\forall \beta \in K \quad \| \sum_{i=1}^{k} \beta_i v_i \| = f(\beta) \ge f(\hat{\beta}) = c > 0.$$

This is $(**)$. $\quad$ ⨳

<u>Thm:</u> $(Y, \|\cdot\|)$ finite dimensionl $\Rightarrow$ Y is complete.

<u>Pf</u>: Let $v_1, \dots, v_k$ be a basis

$$(\exists c > 0) \quad \| \sum_{i=1}^{k} \alpha_i v_i \| \ge c \left(\sum_{i=1}^{k} |\alpha_i| \right) \quad (\forall \alpha \in \mathbb{F}^k)$$

Let (y_n) be a Cauchy seq.

$$\| y_n - y_m \| \to 0 \quad \text{as} \quad m, n \to \infty$$

$$y_n = \sum_{i=1}^{k} \alpha_{n_i} v_i \qquad \forall n$$

Show that $\forall i$ $(a_{n_i})_{n=1}^{\infty}$ is Cauchy in $\mathbb{F}$.

$(\exists \alpha_i)\ \alpha_{n_i} \to \alpha_i$ as $n \to \infty$ $(\forall i)$

Let

$$y = \sum_{i=1}^{n} \alpha_i v_i \in Y.$$

Show $\|y_n - y\| \to 0$ as $n \to \infty$. ✩

<u>cor:</u> X normed linear space.

Y subspace of X

if $\dim(Y) < \infty \Rightarrow Y$ is a closed subspace of X

<u>Pf:</u> E.F.Y.

<u>Def:</u> $X =$ vector space

$\|\cdot\|_a$, $\|\cdot\|_b$ norms on X.

The norms are equivalent if $(\exists, m, M > 0)$

$$m\|x\|_b \le \|x\|_a \le M\|x\|_b \quad (\forall x \in X)$$

<u>Ex:</u> $l_1 \subseteq l_\infty$

$\|\cdot\|_1$, $\|\cdot\|_\infty$ are two norms on l_1

E.F.Y. Show that they are not equivalent.

$$? \ \|x\|_1 \le \|x\|_\infty \le \|x\|_1$$

<u>Thm:</u> Equivalent norms induce the same topology.

<u>Thm:</u> $Y =$ finite dimensional space

$\|\cdot\|_a$, $\|\cdot\|_b$ norms on $Y \Rightarrow \|\cdot\|_a, \|\cdot\|_b$ are equivalent.

p.f: $v_1, \dots, v_k$ basis for Y.

$$y = \sum_{i=1}^{k} \alpha_i v_i$$

$(\exists\, C_a)$, $\|\sum_{i=1}^{k} \alpha_i v_i\|_a \leq C_a\left(\sum_{i=1}^{k} |\alpha_i|\right)$

$(\exists\, c_b)$, $\|\sum_{i=1}^{k} \alpha_i v_i\|_b \geq c_b\left(\sum_{i=1}^{k} |\alpha_i|\right)$

$$\|y\|_a \leq C_a\left(\sum_{i=1}^{k} |\alpha_i|\right) \leq \underbrace{\frac{C_a}{c_b}}_{M} \|y\|_b .$$

get the ~~ot~~ other ineq. by reversing roles of norms. ✗

Jan. 24, 90
Wednesday

2.5 Compactness and Finite Dimension.

Thm: M = metric space, $K \subseteq M$, K compact $\Rightarrow$ K is closed and bounded

Remark: The converse is false in general

Thm: Let X = normed space, $\dim(X) < \infty$, $K \subseteq X$. K is compact $\Leftrightarrow$ K is closed and bounded

P.f: $\Leftarrow$ K is closed and bounded.

Let $v_1, v_2, \cdots, v_k$ be a basis for X

Let $\hat{K} = \{\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{F}^k \mid \sum_{i=1}^{k} \alpha_i v_i \in K\} \subseteq \mathbb{F}^k$

E.F.Y. $\hat{K}$ is closed and bounded in $\mathbb{F}^k$.

$\therefore \hat{K}$ is compact $\Rightarrow \therefore K$ is compact.

Let (x_n) be a sequence in K. $x_n = \sum_{i=1}^{k} \alpha_{n_i} v_i$

Let $\alpha^{(n)} = (\alpha_{n_1}, \cdots, \alpha_{n_k})$

$(\alpha^{(n)}) \subseteq \hat{K}$

$\hat{K}$ is compact so $\exists$ convergent subsequence

$(\alpha^{(n_j)})$

$(x^{(n_j)})$ is convergent to a point in K. $\#$

$B = \{ x \in X \mid \|x\| \leq 1 \}$ closed and bounded.

X = finite dimensional $\Rightarrow B$ is compact

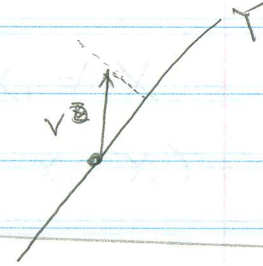
and in fact every closed ball is compact

Riesz Lemma: X = normed linear space

x, z = subspaces of X, $Y \subseteq Z$, $Y \neq Z$, Y is cl

Let $0 < \theta < 1$. Then $(\exists\, z \in Z)$ $\|z\| = 1$

and $d(z, Y) \geq \theta$



P.f: Let $v \in Z \setminus Y$ Let $a = d(v, Y) > 0$ $\because$ Y is closed

$0 < \theta < 1$, so $a < \frac{a}{\theta}$

So $(\exists\, \hat{y} \in Y)$ $\|v - \hat{y}\| < \frac{a}{\theta}$

$v - \hat{y} \in Z$ Let $z = \frac{v - \hat{y}}{\|v - \hat{y}\|}$

$\|z\| = 1$, we must show that

$\|z - y\| \geq \theta \quad \forall\, y \in Y$

Let $y \in Y$

$$\|z - y\| = \left\| \frac{v - \hat{y}}{\|v - \hat{y}\|} - y \right\| = \frac{1}{\|v - \hat{y}\|} \{ \| v - \underbrace{\hat{y} - \|v - \hat{y}\| y}_{\in Y} \|$$

$$\geq \frac{1}{\|v - \hat{y}\|} a \geq \frac{\theta}{a} \cdot a = \theta$$

Thm: X finite dimensional $\Longleftrightarrow$ $\mathbb{B}$ is compact

P.f: $(\Leftarrow)$ X infinite dimensional $\Rightarrow$ $\mathbb{B}$ is not compact.

Pick $x_1 \in \mathbb{B}$ with $\|x_1\| = 1$

Let $X_1 = \text{span}\{x_1\}$ $\quad X_1 \neq X$

So by Riesz Lemma $(\exists\, x_2 \in X)$

$d(x_2, X_1) \geq \frac{1}{2}$, $\|x_2 - x_1\| \geq \frac{1}{2}$

Let $X_2 = \text{span}\{x_1, x_2\}$ $\quad X_2 \neq X$, so, $\exists\, x_3 \in X$

$d(x_3, X_2) \geq \frac{1}{2}$

$\Rightarrow \|x_3 - x_2\| \geq \frac{1}{2}$

$\|x_3 - x_1\| \geq \frac{1}{2}$

Suppose we have $x_1, \dots, x_k$

such that $\|x_i - x_j\| \geq \frac{1}{2}$ $\forall\, i, j$

Let $X_k = \text{span}\{x_1, \dots, x_k\}$ $\quad X_k \neq X$.

So by Riesz, $\exists\, x_{k+1} \in X$ such that

$\|x_{k+1} - x_j\| \geq \frac{1}{2}$, $\quad j = 1, \dots, k$.

$\therefore$ by induction

$\exists$ seq. $(x_k) \subseteq \mathbb{B} \ni \|x_j - x_i\| \geq \frac{1}{2}$

$\forall i, j, i \neq j$. (x_k) has no convergent subsequence

$\therefore \mathcal{B}$ is not compact. #

$d(x, y) = \|x - y\| = \|x - a - (y - a)\| = d(x+a, y+a)$.

$d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = |\alpha| \cdot d(x, y)$

Def: X is locally compact iff every point in X has a neighborhood whose closure is compact.

Thm: X = normed space.

X finite dimensional $\Longleftrightarrow$ X is locally compact

Jan. 25, 91 Friday

§2.6 Linear Oprators

Definition: Let X, Y be vector spaces over the same field $\mathbb{F}$. A linear operator

$$T : X \to Y.$$

is a map whose domain

$$\mathcal{D}(T) \text{ is a subspace of } X$$

whose range

$$R(T) \text{ is a subset of } Y$$

such that

$$T(x_1+x_2) = Tx_1 + Tx_2 \quad \forall\, x_1, x_2 \in \mathcal{D}(T)$$
$$T(\alpha x) = \alpha\, Tx \quad \forall\, x \in \mathcal{D}(T),\ \forall \alpha \in \mathbb{F}$$

Watkin's convention	Kreyszig's convention
$T: X \to Y$	$T: X \to Y$
$\mathcal{D}(T) \subseteq X$	$\mathcal{D}(T) = X$
	$T: \mathcal{D}(T) \to Y$

Prop: $\mathcal{R}(T)$ is a subspace of Y. E.F.Y.

Def: Null space

$$\mathcal{N}(T) = \{x \in \mathcal{D}(T) \mid Tx = 0\}$$

prop: $\mathcal{N}(T)$ is a subspace of X

Read §2.6 optional problems P. 90.

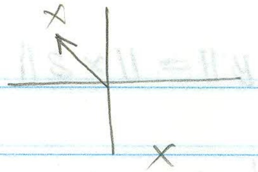
§2.7. Bounded Linear Operators

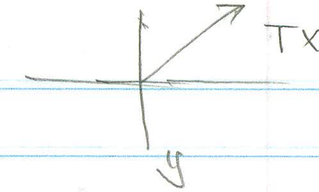
Def: X, Y = normed spaces over $\mathbb{F}$

$T: X \to Y$ linear operator

T is bounded if $(\exists C)$

$$\frac{\|Tx\|}{\|x\|} \le C \quad \forall\, x \in \mathcal{D}(T) \setminus \{0\}.$$





Boundedness means $\exists$ upper bound on magnification that T can cause.

<u>Ex:</u> $T : C[0,1] \longrightarrow C[0,1]$

$\mathcal{D}(T) = C^1[0,1] =$ cont. differentiable func.

for $x(t) \in \mathcal{D}(T)$ $\quad Tx = x'$.

Let $m \in \mathbb{N}$

$$x_m(t) = e^{mt}$$

$$Tx_m = me^{mt}$$

$$\frac{\|Tx_m\|}{\|x_m\|} = m \quad \forall\, m$$

$\Rightarrow$ <u>T is unbounded</u>

<u>Ex:</u> $S : C[0,1] \rightarrow C[0,1]$.

$\mathcal{D}(S) = C[0,1]$.

$Sx = y$ where $y(t) = \int_0^t x(s)\,ds$

$$\|Sx\| = \|y\| = \sup_{t\in[0,1]} |y(t)|$$

$$|y(t)| = \left|\int_0^t x(s)\, d(s)\right| \leq \int_0^t |x(s)|\, ds$$

$$\leq \int_0^t \|x\|\, ds = t \cdot \|x\| \quad \forall\, t \in [0,1]$$

$$\Rightarrow |y(t)| \leq \|x\| \Rightarrow \|Sx\| = \|y\| \leq \|x\|$$

$$\frac{\|Sx\|}{\|x\|} \leq 1 \qquad \forall\, x \in \mathcal{D}(S)\setminus\{0\}$$

$\Rightarrow S$ is <u>bounded</u>.

<u>Ex:</u> $V : L_2(0,1) \to L_2(0,1)$.

$$Vx = y \qquad y(t) = \int_0^t x(s)\, ds$$

E.F.Y. prove that V is bounded.

<u>Ex:</u> $A \in \mathbb{F}^{n\times m}$ = $n\times m$ matrices over $\mathbb{F}$.

<u>Def:</u> $T : \mathbb{F}^m \longrightarrow \mathbb{F}^n$ by

$$\mathcal{D}(T) = \mathbb{F}^m, \qquad Tx = Ax$$

Let $Tx = y$

$$\|y\| = \sqrt{\sum_{i=1}^n |y_i|^2}$$

$$\sum_{i=1}^n$$

$$\|Tx\|^2 = \|y\|^2 = \sum_{i=1}^n |y_i|^2 = \left|\sum_{j=1}^m \alpha_{ij} x_j\right|^2$$

$$\Rightarrow \quad \|Tx\|^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{m}|a_{ij}|^2 \sum_{j=1}^{m}|x_j|^2\right)$$

$$= \left(\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|^2\right)\underbrace{\left(\sum_{j=1}^{m}|x_j|^2\right)}_{\|x\|^2}$$

$$\Rightarrow \frac{\|Tx\|^2}{\|x\|^2} \le \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|^2} \quad \forall\, x \in \mathcal{D}(T)\setminus\{0\}.$$

<u>Def:</u> X, Y normed spaces over $\mathbb{F}$.

$B(X,Y)$ = set of bounded linear operators: $T : X \to Y$ such that $\mathcal{D}(T) = X$

<u>Def:</u> $(T_1 + T_2)x = T_1x + T_2x$

$(\alpha T)\,x = \alpha(Tx)$

<u>Thm:</u> (a) T_1 and $T_2 \in B(X,Y) \Rightarrow T_1 + T_2 \in B(X,Y)$.

(b) $T \in B(X,Y)$, $\alpha \in \mathbb{F} \Rightarrow \alpha T \in B(X,Y)$.

$B(X,Y)$ is a vector space over $\mathbb{F}$.

P.f. E.F.Y

<u>Def:</u> operator norm $T : X \to Y$

$$\|T\| = \sup_{x \in \mathcal{D}(T)\setminus\{0\}} \frac{\|Tx\|}{\|x\|}$$

Lemma : ~~Thm~~ $\|T\| = \sup_{\|x\|=1} \|Tx\|$

P.f: E.F.Y.

Thm: The operator norm is a norm on $B(X,Y)$

p.f: We must show

(i) $\|T\| > 0$ if $T \neq 0$.

(ii) $\|0\| = 0$

(iii) $\|\alpha T\| = |\alpha| \cdot \|T\| \quad \forall T \in B(X,Y), \forall \alpha \in \mathbb{F}$

(iv) $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$

p.f. (iv)

$$\|T_1 + T_2\| = \sup_{\|x\|=1} \|(T_1+T_2)x\|$$

$$= \sup_{\|x\|=1} \|T_1 x + T_2 x\|$$

$$\leq \sup_{\|x\|=1} \left(\|T_1 x\| + \|T_2 x\|\right)$$

$$\leq \sup_{\|x\|=1} \|T_1 x\| + \sup_{\|x\|=1} \|T_2 x\| = \|T_1\| + \|T_2\|$$

#

(i), (ii), (iii) E.F.Y.

Thm If Y is complete then $B(X,Y)$ is complete.

P.f: Routine.

Let (T_n) be a Cauchy seq. in $B(X,Y)$

$$\lim_{m,n\to\infty} \|T_m - T_n\| = 0$$

$$\|T_m - T_n\| = \sup_{\|x\|\neq 0} \frac{\|T_m x - T_n x\|}{\|x\|}$$

$(T_n x)$ cauchy in Y $\cancel{\forall x \ \|x\|=1}$

$\exists$ limit. Call it Tx.

show T is linear bounded

$\|T_n - T\| \to 0$ as $n \to \infty$. #

Jan, 28, 91
Monday

Bounded Linear Operators

Thm: $T : X \to Y$, X is finite dimensional

$\Rightarrow T$ is bounded

P.f: We must find M such that

$$\frac{\|Tv\|}{\|v\|} = M \quad \forall \ v \in \mathcal{D}(T) \setminus \{0\}$$

$\mathcal{D}(T)$ is finite dimensional.

Let $v_1, \cdots, v_k$ be a basis for $\mathcal{D}(T) \setminus \{0\}$

$$(\exists c) \quad \left\| \sum_{i=1}^{k} \alpha_i v_i \right\| \geq c \left(\sum_{i=1}^{k} |\alpha_i| \right) \quad \forall \ \alpha \in \mathbb{F}^k$$

Let $v \in \mathcal{D}(T)$ Then $\dfrac{\|Tv\|}{\|v\|} = \dfrac{\left\| \sum_{i=1}^{k} \alpha_i T v_i \right\|}{\left\| \sum_{i=1}^{k} \alpha_i v_i \right\|}$

$$\leq \frac{\sum_{i=1}^{k} |\alpha_i| \cdot \|Tv_i\|}{c \sum_{i=1}^{k} |\alpha_i|} \leq \frac{\max_{i=1,\cdots,k} \|Tv_i\| \cdot \sum_{i=1}^{k} |\alpha_i|}{c \sum_{i=1}^{k} |\alpha_i|}$$

where $M = \left(\max_{i=1,\cdots,k} \|Tv_i\| / c \right)$ ##

$T: X \to Y$

Suppose T is bounded.

$$\|T\| = \sup_{x \in \mathcal{D}(T)\setminus\{0\}} \frac{\|Tx\|}{\|x\|}$$

Let $z \in \mathcal{D}(T)\setminus\{0\}$

$$\frac{\|Tz\|}{\|z\|} \le \|T\|$$

$$\|Tz\| \le \|T\| \cdot \|z\| \qquad \forall\ z \in \mathcal{D}(T) \text{ (even } 0)$$

Let $x, y \in \mathcal{D}(T)$ Let $z = x - y$.

Then $Tz = Tx - Ty$

$$\Rightarrow \quad \|Tx - Ty\| \le \|T\| \cdot \|x-y\|$$

$\therefore$ T is Lipschitz cont. throughout $\mathcal{D}(T)$

with Lipschitz continuous $\|T\|$

Thm:

T cts at one point $\Rightarrow$ T bounded

P.f.: T cts at x_0. Then $\forall \varepsilon > 0, (\exists \delta > 0)(\forall x \in \mathcal{D}(T))$
$(\|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < \varepsilon)$

Take $\varepsilon = 1$ $(\exists \delta) \Rightarrow (\|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < 1)$

Want to show $(\exists M)$

$$\frac{\|Tx\|}{\|x\|} \le M \quad \forall\ x \in \mathcal{D}(T)\setminus\{0\}$$

Let $v \in \mathcal{D}(T) \setminus \{0\}$. Let $\hat{v} = \alpha v$, where $\alpha = \frac{\delta}{2\|v\|}$

Then $\|\hat{v}\| = |\alpha| \cdot \|v\| = \frac{\delta}{2} < \delta$.

Let $x = x_0 + \hat{v} \in \mathcal{D}(T)$.

Then $\hat{v} = x - x_0$

$\|\hat{v}\| = \|x - x_0\| < \delta$

So $\|Tx - Tx_0\| < 1$

$\Rightarrow \|T\hat{v}\| < 1$

$\Rightarrow v = \frac{1}{\alpha}\hat{v} \Rightarrow \|Tv\| = \frac{1}{\alpha}\|T\hat{v}\| = \frac{2\|v\|}{\delta}$.

$\Rightarrow \frac{\|Tv\|}{\|v\|} < \frac{2}{\delta} \quad \forall v \in \mathcal{D}(T) \setminus \{0\}$

$\therefore$ T is bounded.

<u>Theorem:</u> The following are equivalent

(i) T is bounded.

(ii) T is Lip. continous on $\mathcal{D}(T)$.

(iii) T is cts at one point.

(iv) T maps bounded sets to bounded sets

(v) $T(\mathcal{B} \cap \mathcal{D}(T))$ is bounded, where

$$\mathcal{B} = \{x \in X \mid \|x\| < 1\}$$

<u>P.f.</u> E.F.Y (iv) and (v).

Thm: Let $T \in B(X,Y)$. Then $\eta(T)$ is a closed subspace of X.

Pf. E.F.Y.

EX: Define $T_1 : C[0,1] \to C[0,1]$, where $\mathcal{D}(T_1) = C^1[0,1]$

$T_1 x = x'$.

$T_2 : C[0,1] \to C[0,1]$, $\mathcal{D}(T_2) = P$ = polynomial

$T_2 x = x'$.

Then $T_1 \neq T_2$, But $T_2 = T_1|_P$

$\Rightarrow$ T_2 is called restriction of T_1

T_1 is called extension of T_2.

Thm: Let X, Y = normed space, Y Banach.

$T: X \to Y$ bounded linear operator with $\mathcal{D}(T)$ dense in X. Then $\exists!$ bounded extension $\hat{T}$

$\hat{T} \in B(X,Y)$ $(\mathcal{D}(\hat{T}) = X)$. Furthermore $\|\hat{T}\| = \|T\|$

P.f Let $x \in X$. $\left(\exists (x_n) \subseteq \mathcal{D}(T)\right)$ $\lim_{n\to\infty} x_n = x$

Define

$$\hat{T}x = \lim_{n\to\infty} Tx_n$$

The issues: Convergence of (Tx_n).

$\hat{T}$ is well defined.

$\hat{T}$ is linear. and unique.

$\hat{T}$ is an extension of T, $\hat{T}$ is bounded.

§2.8 Linear Functionals

Let X = vector space over $\mathbb{F}$

Def: A functional on X is a map.

$f: X \to \mathbb{F}$ $\quad \mathcal{D}(f) \subseteq X$

Def: A linear functional is a functional

$f: X \to \mathbb{F}$ that is a linear operator.

Bounded linear funtional (normed space)

Bounded iff cts

See examples in text.

X = vector space over $\mathbb{F}$.

Def: The algebraic dual of X is the set of all linear functionals $f: X \to \mathbb{F}$ such that $\mathcal{D}(f) = X$. It's denoted by X^*

X^* has a vector space structure

X^{**} = algebraic dual of X^*.

What is in X^{**}?

Given $x \in X$, define $g_x : X^* \to \mathbb{F}$

by $g_x(f) = f(x)$

E.F.Y. g_x is a linear map, so $g_x \in X^{**}$.

Jan. 30, 91
Wednesday

X = vector space over $\mathbb{F}$

X^* = set of linear functionals on X (with $\mathcal{D}(f) = X$)

X^{**}

Canonical map from X into X^{**}

$x \longmapsto g_x$

$g_x : X^* \to \mathbb{F}$

$g_x(f) = f(x)$

$\mathcal{D}(g_x) = X^*$, g_x is linear

$g_x \in X^{**}$.

$C : X \to X^{**}$ $\quad C(x) = g_x$, $\mathcal{D}(C) = X$.

E.F.Y. C is linear:

Thm: C is 1-to-1.

p.f: Suppose $C(x) = 0$, want to show $x = 0$.

$g_x = 0 \quad g_x(f) = 0 \quad \forall f \in X^*$

$f(x) = 0 \quad \forall f \in X^*$

E.F.Y. Show that if $x \neq 0$, then $(\exists f \in X^*)$ $f(x) \neq 0$

Hint: Use Zorn's lemma to show that

X has a Hamel basis containing x.
use the basis represatation to build $f \in X^*$.
such that $f(x) \neq 0$.

C is a vector space isomophism of X onto
$R(C) \subseteq X^{**}$.

Def: X is algebraically reflexive if $R(C) = X^{**}$.

§ 2.9. Linear Functionals on Finite dimensional spaces

Let X = vector space of dimension n over $\mathbb{F}$.
Let $v_1, \cdots, v_n$ be a basis of X.
Let $f \in X^*$, Let $x = \sum_{i=1}^n c_i v_i \in X$

$f(x) = f\left(\sum_{i=1}^n c_i v_i\right) = \sum_{i=1}^n c_i f(v_i)$. Let $\alpha_i = f(v_i) \in \mathbb{F}$

$f\left(\sum_{i=1}^n c_i v_i\right) = \sum_{i=1}^n c_i \alpha_i$ f is represented by

$\alpha = (\alpha_1, \cdots, \alpha_n)$

$$= [\alpha_1 \cdots, \alpha_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Conversely, given any $\alpha \in \mathbb{F}^n$, we can defined a functional f_α by

$$f_\alpha(x) = f_\alpha\left(\sum c_i v_i\right) = \sum_{i=1}^n c_i \alpha_i$$

E.F.Y. Check that f_α is a linear functional.

The map $f \mapsto \alpha = (\alpha_1, \cdots, \alpha_n)$ $(\alpha_i = f(v_i), i=1,\cdots,n)$ is linear, 1-1, onto (E.F.Y.).

So it's a vector space isomophism between X^* and $\mathbb{F}^n$. $\therefore$ $\dim X^* = n$

For $i=1,\cdots,n$, Let f_i be the functional associated with $\alpha = (0,0,\cdots,0,1,0,\cdots,0)$

$$f\left(\sum_{j=1}^n c_j v_j\right) = \sum_{j=1}^n c_j \alpha_j = c_i$$

$$f(v_j) = \delta_{ij}$$

E.F.Y. $f_1, \cdots, f_n$ form a basis for X^*, the dual basis of $v_1, \cdots, v_n$.

Cor: $\dim(X) < \infty \Rightarrow X$ is algebraically reflexive.

p.f: $\dim(X) = \dim(X^*) = \dim(X^{**})$
$\Rightarrow \therefore R(C) = X^{**}$.

2.10 The dual Space (Normed dual space).

Reminder: X, Y = normed spaces (over $\mathbb{F}$)
$B(X,Y)$ = bounded linear operators $T: X \to Y$ with $\mathcal{D}(T) = X$.

$B(X, Y)$ is complete if Y is complete.

<u>Def:</u> X = normed space over $\mathbb{F}$.

$X' = B(X, \mathbb{F})$ dual space of X
or normed dual.

<u>bounded</u> linear functional
<u>continuous</u>

Note: X is finite dimensional $\Rightarrow X' = X^*$

X' is a Banach space.

<u>Def:</u> X, Y = normed spaces over $\mathbb{F}$.

An <u>isomorphism</u> is a linear map

$T: X \to Y$

such that T is 1-1, $\mathcal{D}(T) = X$, $R(T) = Y$.

and $\|Tx\| = \|x\| \quad \forall x \in X$.

<u>Ex:</u> The dual space of $\mathbb{F}^n$ $(F_n' = F_n^*)$.

$x = (x_1, \dots, x_n)$

$f \in F_n'$. Then $(\exists !\ \alpha = (\alpha_1, \dots, \alpha_n))$

$f(x) = \sum_{i=1}^{n} x_i \alpha_i \quad \forall x \in \mathbb{F}$. (E.F.Y.)

and conversely

$\alpha \xmapsto{T} f$

$\mathbb{F}^n \qquad\qquad \mathbb{F}^{n'}$

$(\alpha_1, \cdots, \alpha_n) \longrightarrow f(x) = \sum x_i \alpha_i$

E.F.Y. T is linear.

T is 1-1

T is onto. $\qquad \|T\alpha\| = \|\alpha\|$

Let $T\alpha = f$

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)|$$

Let $\|x\| = 1$

$$|f(x)| = \left|\sum_{i=1}^n x_i \alpha_i\right| \overset{\text{Cauchy}\cdots}{\leq} \sqrt{\sum_{i=1}^n |x_i|^2} \cdot \sqrt{\sum_{i=1}^n |\alpha_i|^2}$$

$$= \|x\| \cdot \|\alpha\| = \|\alpha\| \qquad \forall \|x\| = 1.$$

$$\Rightarrow \|f\| = \sup_{\|x\|=1} |f(x)| \leq \|\alpha\|.$$

Now take $x = (\overline{\alpha_1}, \overline{\alpha_2}, \cdots, \overline{\alpha_n})$

$$f(x) = \sum_{i=1}^n x_i \alpha_i = \sum_{i=1}^n |\alpha_i|^2 = \|\alpha\|^2 = \|x\| \cdot \|\alpha\|.$$

$$\frac{|f(x)|}{\|x\|} = \|\alpha\|. \Rightarrow \|f\| = \|\alpha\|. \quad \#$$

$\therefore \mathbb{F}^{n'} \approx \mathbb{F}^n.$ $\qquad$ #

Friday
Feb. 1

Assign #2. due Friday 2/15

P. 101 6, 10

P. 110 11, 14

P. 126 8

Recommend Exercise

P. 101 all, P 110, all

P. 116 4, 5, 8, 10, 12, 13,

P. 126 8 – 12

Example : The dual of l_1 is l_∞.

$$l_1 = \{x = (x_i) \mid \sum_{i=1}^{\infty} |x_i| < \infty\}$$

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$$

Schauder basis

$e_1 = (1, 0, 0 \cdots)$
$e_2 = (0, 1, 0 \cdots)$
$e_3 = (0, 0, 1, 0, \cdots)$

$$x = \sum_{i=1}^{\infty} (x_i \cdot e_i)$$

Check that this series converges to x

in l_1 norm.

Let $f \in l_1'$

$$f(x) = f\left(\sum_{i=1}^{\infty} x_i e_i\right) \overset{\text{cont. } f}{=} \sum_{i=1}^{\infty} x_i f(e_i)$$

Let $\alpha_i = f(e_i)$, $\alpha = (\alpha_i)$

$$f(x) = \sum_{i=1}^{\infty} x_i \alpha_i$$

$$|\alpha_i| = |f(e_i)| \leq \|f\| \, \|e_i\|_1 = \|f\| \quad \forall i.$$

$$\Rightarrow \alpha \in l_\infty$$

$$\|\alpha\|_\infty \leq \|f\|$$

$$|f(x)| = \left|\sum_{i=1}^{\infty} x_i \alpha_i\right| \underset{\text{Hölder}}{\leq} \|x\|_1 \cdot \|\alpha\|_\infty \quad \forall x \in l_1 \setminus \{0\}$$

$$\frac{|f(x)|}{\|x\|_1} \leq \|\alpha\|_\infty \quad \Rightarrow \quad \|\alpha\|_\infty \geq \|f\|$$

$$\Rightarrow \|f\| = \|\alpha\|_\infty$$

Define:

$$c : l_1' \to l_\infty$$

by $c(f) = \alpha = (\alpha_1, \cdots)$ $\quad (\alpha_i = f(e_i))$

$$= (f(e_1), f(e_2), f(e_3), \cdots)$$

E.F.Y. c is linear. $\mathcal{D}(c) = l_1'$

$$\|cf\|_\infty = \|f\| \quad \forall f \in l_1'$$

We need only check that c is surjective.

Given $\alpha \in l_\infty$. define

$$f_\alpha \text{ by } f_\alpha(x) = \sum_{i=1}^{\infty} x_i \alpha_i$$

E.F.Y. f_α is defined on all of l_1,

f_α is bounded

f_α is linear $\quad f_\alpha \in l_1'$

$$C(f_\alpha) = \alpha$$

Ex: $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

The dual of l_p is l_q

$$l_p = \{ x = (x_i) \mid \sum_{i=1}^{\infty} |x_i|^p < \infty \}$$

$$\|x\| = \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p}$$

$e_1, e_2, e_3, \cdots$ is a Schauder basis for l_p.

$x = \sum_{i=1}^{\infty} x_i e_i$. E.F.Y. Confirm that this ~~series~~ sires converges to x in l_p norm

Let $f \in l_p'$ $\quad f(x) = f\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{\infty} x_i f(e_i)$

Let $\alpha = (\alpha_i)$, where $\alpha_i = f(e_i)$

$$|f(x)| = \left| \sum_{i=1}^{\infty} x_i \alpha_i \right| \underset{\text{Hölder}}{\leq} \|x\|_p \cdot \|\alpha\|_q$$

$$\frac{|f(x)|}{\|x\|_p} \leq \|\alpha\|_q \overset{?}{<} \infty.$$

$$\Longrightarrow \|f\| \leq \|\alpha\|_q \overset{?}{<} \infty$$

We'd like to show that $\alpha \in l_q$ and $\|f\| = \|\alpha\|_q$

Define $X^{(n)} = (X^{(n)}_i)$, $n = 1, 2, 3, 4, 5, \cdots$ by

$$x_i^{(n)} = \begin{cases} \frac{|\alpha_i|^q}{\alpha_i} & \text{if } \alpha_i \neq 0 \text{ and } i \leq n \\ 0 & \text{if } \alpha_i = 0 \text{ or } i > n \end{cases}$$

$$f(x^{(n)}) = \sum_{i=1}^{\infty} x_i^{(n)} \alpha_i = \sum_{i=1}^{n} |\alpha_i|^q$$

$$\|x^{(n)}\|_p^p = \sum_{i=1}^{n} |x_i^{(n)}|^p = \sum_{i=1}^{n} |\alpha_i|^{(q-1)p} = \sum_{i=1}^{n} |\alpha_i|^q$$

$$|x_i^{(n)}| = \left| \frac{|\alpha_i|^q}{|\alpha_i|} \right| = |\alpha_i|^{q-1}$$

$$\|f\| \geq \frac{|f(x^{(n)})|}{\|x^{(n)}\|_p} = \left(\sum_{i=1}^{n} |\alpha_i|^q \right)^{1 - \frac{1}{p} = \frac{1}{q}}$$

$$\Rightarrow \quad \|f\|^q \geq \sum_{i=1}^{n} |\alpha_i|^q \quad \forall n$$

$$\therefore \sum_{i=1}^{\infty} |\alpha_i|^q \leq \|f\|^q.$$

$\therefore \quad \alpha \in l_q$

and $\|\alpha\|_q \leq \|f\|$

We already knew $\|f\| \leq \|\alpha\|_q$

$$\Rightarrow \quad \|f\| = \|\alpha\|_q$$

Define: $C : l_p' \longrightarrow l_q$ by

$$C(f) = (f(e_1), f(e_2), \cdots)$$

$$\mathcal{D}(C) = l_p'$$

C is linear.

$\|Cf\|_q = \|f\| \quad \forall f$

It remains only to show that C is surjective.
Given $\alpha \in l_q$, Define

$$f_\alpha : l_p \longrightarrow \mathbb{F}$$

by $f_\alpha(x) = \sum_{i=1}^{\infty} x_i \alpha_i$ Hölder

$\mathcal{D}(f_\alpha) = l_p$. f_α is bounded.

f_α is linear $\therefore f_\alpha \in l_p'$

$C(f_\alpha) = \alpha$. $\#$

Ex: $L_1(E)' = L_\infty(E)$.

Ex: $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

$L_p(E)' = L_q(E)$

Let $y \in L_q$. Define a linear functional

$$f_y : L_p \to \mathbb{F} \text{ by}$$

$$f_y(x) = \int_E x(t)\,y(t)\,dt$$

$$|f_y(x)| = \left|\int_E x(t)\,y(t)\,dt\right| \le \sqrt[p]{\int_E |x(t)|^p dt} \cdot \sqrt[q]{\int_E |y(t)|^q dt}$$

$$= \|x\|_p \cdot \|y\|_q < \infty \quad \forall x \in L_p$$

So $\mathcal{D}(f_y) = L_p(E)$

and
$$\|f_y\| = \sup_{x \neq 0} \frac{|f_y(x)|}{\|x\|_p} \leq \|y\|_q$$

We can show that
$$\|f_y\| = \|y\|_q \quad \forall\, y \in L_q(E).$$

by taking $x(t) = \dfrac{|y(t)|^q}{y(t)}$ optional E.F.Y.

$$y \longmapsto f_y \qquad L_q(E) \to L_p(E)'$$

is linear norm preserving surjection ? ......

Monday Feb. 4, 91

Midterm Exam #1. Weds - Fri. Feb 20, 22.

Chapter 3 Inner Product Spaces, Hilbert Spaces.

X = vector space over $\mathbb{F}$. An inner product on X is a map

$$X \times X \longrightarrow \mathbb{F}$$
$$x, y \longrightarrow \langle x, y \rangle$$

Inner Product Space

Satisfying

$$\left.\begin{aligned} \langle x_1 + x_2, y \rangle &= \langle x_1, y \rangle + \langle x_2, y \rangle \\ \langle \alpha x, y \rangle &= \alpha \langle x, y \rangle \end{aligned}\right\} \text{linearity in the first argument}$$

$\langle x, y \rangle = \overline{\langle y, x \rangle}$ (symmetry)

$\langle x, x \rangle > 0$ if $x \neq 0$

In an inner product space, define the "norm" by

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

inner product $\Rightarrow$ norm $\Rightarrow$ metric

Def: A Hilber Space is a complete inner product space.

Rmks: If $\mathbb{F} = \mathbb{R}$, $\langle x, y\rangle = \langle y, x\rangle \quad \forall x, y$

E.F.Y.
$$\left.\begin{aligned} &\langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle \\ &\langle x, \alpha y\rangle = \bar{\alpha}\langle x, y\rangle \end{aligned}\right\} \text{conjugate linearity in 2nd argument.}$$

If $\mathbb{F} = \mathbb{R}$, we have actual linearity in 2nd argument.

Recovering the inner product from the norm.

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x+y\rangle + \langle y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \end{aligned}$$

$$\|x-y\|^2 = \langle x, x\rangle - \langle y, x\rangle - \langle x, y\rangle + \langle y, y\rangle$$

(1) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle y, x\rangle + \langle x, y\rangle$

(2) $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - \langle y, x\rangle - \langle x, y\rangle$

$$\Rightarrow \quad \|x+y\|^2 - \|x-y\|^2 = 2\langle x, y\rangle + 2\langle y, x\rangle$$

$$= 2\left[\langle x, y\rangle + \overline{\langle x, y\rangle}\right] = 4\,\mathrm{Re}\langle x, y\rangle$$

$$\mathrm{Re}\langle x, y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right]$$

In real case, this is all that's we need.

E.F.Y. $\mathrm{Im}\langle x, y\rangle = \frac{1}{4}\left[\|x+iy\|^2 - \|x-iy\|^2\right]$

Polarization identity $\Rightarrow$

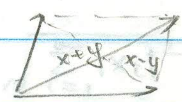
1) The inner product can be recovered by the norm.

2) The inner prod. is uniquely determined by norm

Thm: Let X be an inner product space. Then

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X$$

add (1) and (2), Parallelogram Law



Remark: The converse is true as well.

If X is a normed space for which the parallelogram law holds, then X is an inner product space.

P.f: P. Jardan and J. von Neumann

Annal of Math, 1935.

Def: $x, y \in X$ inner product space.

x is orthogonal to y if $\langle x, y\rangle = 0$
perpendicular

$x \perp y$

Thm (Pythagoras) : X = inner product space, $x, y \in X$

$x \perp y \implies \|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Pf: E.F.Y.

Ex: $\mathbb{R}^n$, $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$

$\mathbb{C}^n$, $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$. $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$

Ex: $E \subseteq \mathbb{R}^n$, Lebesgue measure

$L_2(E)$, $\langle x, y \rangle = \int_E x(t)\overline{y(t)}\, dt$

$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\int_E |x(t)|^2\, dt}$.

Ex: l_2 $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$

$\|x\| = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$.

Ex: l_p ; $p \neq 2$, $x = (1, 0, 0, \dots)$

$y = (0, 1, 0, \dots)$

Show that parallelogram law is violated if $p \neq 2$.

l_p is not an inner product space. $p \neq 2$.

Ex: $C[0,1]$, $\|x\| = \sup_{t \in [0,1]} |x(t)|$

E.F.Y: $C[0,1]$ ~~that~~ is not an inner product space.

§3.2 Further Properties:

Norm properties: $\|x\| = \sqrt{<x,x>}$

1) if $x \neq 0$ then $\|x\| > 0$

2) $\|\alpha x\| = |\alpha| \cdot \|x\|$

3) $\|x+y\| \leq \|x\| + \|y\|$

Schwarz Inequality: X = inner product space

$x, y \in X \Rightarrow |<x,y>| \leq \|x\| \cdot \|y\|$

Equality holds iff x and y are linear dependent.

P.f: See proof in text (short, hocus-pocus). #

Wed 02-06 P.f: If $x = 0$ or $y = 0$ equality holds (and x and y are linearly independent)

Assume $x \neq 0$, $y \neq 0$.

Case I: $<x,y>$ is real, consider the function

$$f(t) = \|x + ty\|^2 \geq 0 \cdot t \in \mathbb{R}$$

$$= <x+ty, x+ty>$$

$$= <x,x> + t<y,x> + t<x,y> + t^2 <y,y>$$

$$= \|x\|^2 + 2<x,y> + t^2\|y\|^2$$

$$= C + Bt + At^2$$

disicrimanl $\quad B^2 - 4AC \le 0$

$$\left(\frac{B}{2}\right) \le AC$$

$$(\langle x, y\rangle)^2 \le \|x\|^2 \cdot \|y\|^2$$

$$\Rightarrow |\langle x, y\rangle| \le \|x\| \cdot \|y\|$$

Case II. $\langle x, y\rangle \in \mathbb{C}$, $\quad \langle x, y\rangle = re^{i\theta}$

$$|\langle x, y\rangle| = r$$

Let $\hat{x} = e^{-i\theta} x \Rightarrow \|\hat{x}\| = \|x\|$

$\langle \hat{x}, y\rangle = e^{-i\theta} \langle x, y\rangle = r$ is real

$$|\langle x, y\rangle| = r = |\langle \hat{x}, y\rangle| \overset{\text{Case 1}}{\le} \|\hat{x}\| \cdot \|y\| = \|x\| \cdot \|y\|.$$

We have equality iff $B^2 - 4AC = 0$ in $\hat{f}(t) = 0$.

$$\hat{f}(t) = \|\hat{x}\|^2 + 2\langle \hat{x}, y\rangle t + \|y\|^2 t^2$$

iff $\hat{f}(t) = 0$ $(\exists t \in \mathbb{R}) \Leftrightarrow \|\hat{x} + ty\| = 0$ for some t $\Leftrightarrow \hat{x} = -ty \Rightarrow x = -e^{i\theta} t y$ //

<u>Thm</u> (Triangle inequality) X = inner product space $\Rightarrow \|x + y\| \le \|x\| + \|y\|$. $\forall x, y \in X$.

P.f.

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x,x\rangle + |\langle x,y\rangle| + |\langle y,x\rangle| + \langle y,y\rangle$$
$$\leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2 \qquad //$$

Cor: The "Norm" in an inner product space is indeed a norm.

Thm The inner product is a continuous function

$$(x,y) \to \langle x,y\rangle$$
$$X\times X \to \mathbb{F}$$

P.f. Let $(x,y)\in X\times X$. Let $\varepsilon>0$. we must find $(\delta>0)$ such that if

$\|\hat{x}-x\|<\delta$ and $\|\hat{y}-y\|<\delta$, then

$|\langle\hat{x},\hat{y}\rangle - \langle x,y\rangle| < \varepsilon$

Scratch work: $|\langle\hat{x},\hat{y}\rangle - \langle x,y\rangle|$

$$= |\langle\hat{x},\hat{y}\rangle - \langle x,\hat{y}\rangle + \langle x,\hat{y}\rangle - \langle x,y\rangle|$$
$$= |\langle\hat{x}-x,\hat{y}\rangle| + |\langle x,\hat{y}-y\rangle|$$
$$\leq \|\hat{x}-x\|\cdot\|\hat{y}\| + \|x\|\cdot\|\hat{y}-y\|$$

P.f.

Let $M = \max\{\|x\|, \|y\|+1\}$

Let $\delta = \min\{\frac{\varepsilon}{2M}, 1\}$

If $\|\hat{y} - y\| < \delta$, then $\|\hat{y} - y\| < 1$

cont. of norm

$$\|\hat{y}\| \le \|\hat{y} - y\| + \|y\| \le \|y\| + 1 \le M$$

$\vdots$ etc.

<u>Thm:</u> Every inner product space has a completion that is a Hilbert space (i.e., it can be embeded in a Hilbert space densely).

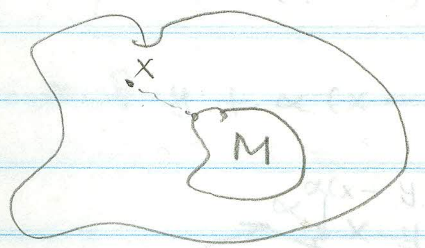
The completion is essentially unique.

The issue: Can we define an inner product on the completion?

Answer: Yes. because the inner product is cts

§3.3 Orthogonal Complements and Direct Sums

Metric space X

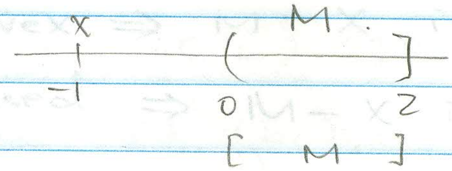


Normed space

$$d(x, M) = \inf_{y \in M} d(x, y) = \inf_{y \in M} \|x - y\|$$

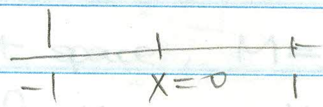
y is called a best approximation to x from M iff $\|x - y\| = d(x, M)$ $\left(\|x-y\| \le \|x - \hat{y}\| \ \forall \hat{y} \in M \right)$

existence, uniqueness and characterization

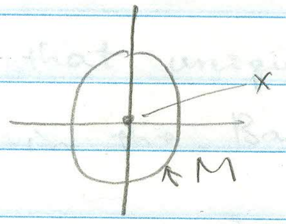


no best approximatio

unique best approximati



$M = \{-1, 1\}$ no uniqueness

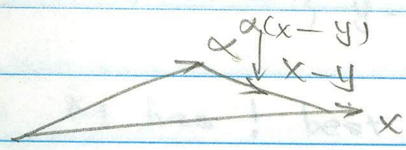


Def: X = vector space over $\mathbb{F}$,

$x, y \in X$ The segment from x to y

is $\{\alpha x + (1-\alpha) y \mid 0 \le \alpha \le 1\}$

$= \{y + \alpha(x-y) \mid 0 \le \alpha \le 1\}$.



Def: $M \subseteq X$ is convex if $(\forall x, y \in M)$ the segment from x to y lies in M.

Ex: M = subspace of X

$\Rightarrow$ M is convex.

EX: $M - x = \{ y - x \mid y \in M \}$

M convex $\Rightarrow$ $M - x$ is convex

closed $\Rightarrow$ $M - x$ is closed. (in normed space)

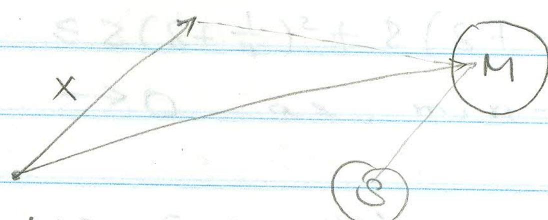
<u>Thm</u>: X = Hilbert space, M = nonempty, closed convex subset of X, $x \in X$. Then $(\exists !\ y \in M)$

$$\|x - y\| = \inf_{\hat{y} \in M} \|x - \hat{y}\| \quad \text{(best approximate)}$$

E.F.Y. Show that uniqueness does not necessarily hold in the Banach space. $(\mathbb{R}^2, \ldots)$

02-08-91
Friday

<u>P.f:</u> Simplifying transformation:



Let $S = M - x = \{ y - x \mid y \in M \}$

E.F.Y. M has ! best approx. to x iff S has ! best approx. to 0.

S.T.P. $(\exists !\ s \in S)$ $\|s\| = \inf_{\hat{s} \in S} \|\hat{s}\|$

Note: S is nonempty, convex and closed.

Existence

Let $\delta = \inf_{\hat{s} \in S} \|\hat{s}\|$

$(\forall n \in \mathbb{N})(\exists s_n \in S) \quad \|s_n\| < \delta + \frac{1}{n}$.

We'll show that (s_n) is Cauchy

Parallelogram law:

$$\|s_n - s_m\|^2 + \|s_n + s_m\|^2 = 2\|s_n\|^2 + 2\|s_m\|^2$$

$$s_n + s_m = 2\left(\tfrac{1}{2}s_m + \tfrac{1}{2}s_n\right)$$

$s_n, s_m \in S \Rightarrow \frac{1}{2}s_n + \frac{1}{2}s_m \in S$ by convex.

$$\|\tfrac{1}{2}s_n + \tfrac{1}{2}s_m\| \geq \delta$$

$$\Rightarrow \|s_n + s_m\| \geq 2\delta$$

$$\Rightarrow \|s_n + s_m\|^2 \geq 4\delta^2$$

$$\Rightarrow \|s_n - s_m\|^2 = 2\|s_n\|^2 + 2\|s_m\|^2 - \|s_n + s_m\|^2$$
$$\leq 2\left(\delta + \tfrac{1}{n}\right)^2 + 2\left(\delta + \tfrac{1}{m}\right)^2 - 4\delta^2$$
$$\to 0 \quad \text{as}, \ m, n \to \infty.$$

X complete $\Rightarrow (\exists s \in X) \ s_n \to s$.

S is closed, so $s \in S$.

Finally $\|s\| = \lim_{n\to\infty} \|s_n\| = \delta$.

Uniqueness

Suppose $s_1, s_2 \in S$

$$\|s_1\| = \|s_2\| = \delta = \inf_{\hat{s} \in S} \|\hat{s}\|.$$

use the ~~pap~~rrallelogram law and convex $\Rightarrow s_1 = s_2$

#

Cor: X = Hilbert space, M = closed subspace, $x \in X$

$$\Rightarrow (\exists!\ y \in M) \quad \|x-y\| = \inf_{\hat{y} \in M} \|x - \hat{y}\|$$

Thm: Projection Theorem: X = Hilbert Space

Y = closed subspace, $x \in X$. $\Rightarrow$ $(\exists!\ y \in Y)$

$x - y \perp Y$. y is the best approximation to x from Y.

Terminology: y is called the orthogonal projection of x onto Y.

P.f: uniquess: Suppose $\exists\ y_1, y_2 \in Y$

$x - y_1 \perp Y$, $x - y_2 \perp Y$

Then $y_2 - y_1 = (x - y_1) - (x - y_2) \perp Y$

but $y_2 - y_1 \in Y \Rightarrow \langle y_2 - y_1, y \rangle = 0 \quad \forall y \in Y$

$\Rightarrow \langle y_2 - y_1, y_2 - y_1 \rangle = 0 \Rightarrow y_1 = y_2$.

Existence: (see text)

Let y be the best approx. to x from Y.

S.T.P $x - y \perp Y$

suppose not, Then $(\exists\ z \in Y)$ $\langle x-y, z \rangle \neq 0$

$\langle x-y, z \rangle = \gamma e^{i\theta}$ $\quad \gamma = |\langle x-y, z\rangle| > 0$

Let $\hat{z} = e^{i\theta} z$, Then $\langle x-y, \hat{z}\rangle = \gamma > 0$

Claim : $\|x - (y + t\hat{z})\| < \|x - y\|$

for small positive values of t.

$y + t\hat{z} \in Y \Rightarrow$ This would be a contradiction

Picture $\Leftarrow$

Let $f(t) = \|x - (y + t\hat{z})\|^2$

$$= \|(x-y) - t\hat{z}\|^2$$

$$= \langle (x-y) - t\hat{z}, (x-y) - t\hat{z}\rangle$$

$$= \langle x-y, x-y\rangle + 2t\langle x-y, \hat{z}\rangle + t^2\langle \hat{z}, \hat{z}\rangle$$

$$= \|x-y\|^2 + 2t\langle x-y, \hat{z}\rangle + t^2\underbrace{\|\hat{z}\|^2}_{>0}$$

f has a uniquess minimal

$\Rightarrow f'(t_0) = 0, \quad t_0 > 0 \Rightarrow f(t_0) < f(t_0) \otimes$

Def: X = vector space, Y, Z subspaces of X, we say that X is the direct sum of Y and Z and write.

$$X = Y \oplus Z$$

If $(\forall x \in X)(\exists!\ y \in Y, z \in Z)\ x = y + z$

Y, Z are called complementary subspaces of X.

X = Hilbert space.

M = anyset $\subseteq X$, $M^\perp$ = orthogonal complement

$= \{x \in X \mid \langle x, y\rangle = 0 \ \forall\ y \in M\}$

E.F.Y. $M^{\perp}$ is a subspace of X and is closed.

$M \subseteq M^{\perp\perp}$

Thm: X = Hilbert space, Y = closed subspace.
Then
$$X = Y \oplus Y^{\perp}$$ (Y and its orth. compl. are complement. subspace)

Pf: Let $x \in X$. we must show that ($\exists ! \, y \in Y$, $y^{\perp} \in Y^{\perp}$) such that $x = y + y^{\perp}$

Let y be the orthogonal projection of x onto Y $\Rightarrow$ $x - y \perp Y$

$x - y \in Y^{\perp}$ $\Rightarrow$ $y^{\perp} \triangleq x - y$

$\Rightarrow$ $x = y + y^{\perp}$, $y \in Y$, $y^{\perp} \in Y^{\perp}$.

uniqueness. E.F.Y.

Monday
Feb. 11

Assignment #3 due Friday March 1.

P. 136	# 15.	Recommended.
P. 140	# 5, 10	P. 135, 1–4, 6, 7, 9–11, 15
P. 150	# 7	P. 140, 4–10
P. 159	# 7	P. 150, 2, 3, 5–10, P. 159 7–10

Midterm Exam Feb 20, 22 up to § 3.4

Projection Theorem:

X = Hilbert space, Y = closed subspace, $x \in X$

Then $(\exists!\ y \in Y)\ \|x - y\| = \inf_{\hat{y} \in Y} \|x - \hat{y}\|$.

y is characterized by $x - y \perp Y$

Thm: X = Hilbert space Y = closed subspace.

$\Rightarrow X = Y \oplus Y^{\perp}$

Cor: If Y is closed subspace of X, then, $Y = Y^{\perp\perp}$

P.f: E.F.Y.

Thm: V = subspace of X, which is Hilbert space

$\Rightarrow$ ($\overline{V}$ = subspace)

(i) $V^{\perp} = \overline{V}^{\perp}$

(ii) V is dense in X iff $V^{\perp} = \{0\}$

E.F.Y.

Projectors: X = vector space over $\mathbb{F}$

S_1, S_2 complementary subspace.

$$X = S_1 \oplus S_2$$

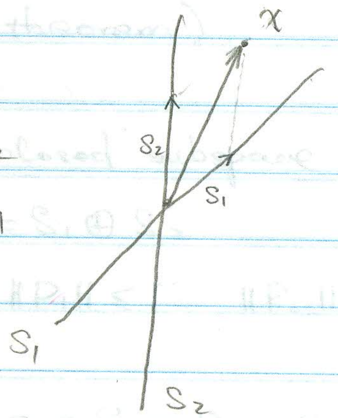
$x \in X$, $x = s_1 + s_2$

Define: $P_1 : X \to X$, $P_2 : X \to X$

$P_1 x = s_1$ $\qquad$ $P_2 x = s_2$

P_1 and P_2 are linear.

$\overset{P_2}{P_1}$ is called the projection of X onto $\overset{S_2}{S_1}$ in the direction of $\overset{S_1}{S_2}$



$\mathcal{R}(P_1) = S_1$, $\mathcal{N}(P_1) = S_2$

$\mathcal{R}(P_2) = S_2$, $\mathcal{N}(P_2) = S_1$

$P_1 + P_2 = I$

Def: $A : X \to X$

$A^2 : X \to X \Rightarrow A^2 x = AAx$

def: $A : X \to X$ is called ~~an~~ an idempotent if $A^2 = A$.

E.F.Y.

Projectors are idempodente.

Thm: Every idempotent is a projector.

p.f: Let P be an idempodent $P^2 = P$

Let $S_1 = R(P)$, $S_2 = \eta(P)$

E.F.Y. show that $R(P) = \{x \in X \mid Px = x\}$

Show that $X = S_1 \oplus S_2$ and P is the projector onto S_1 along S_2. ※

Thm: $X =$ Banach space, S_1 & S_2 are closed subspaces

$X = S_1 \oplus S_2 \Rightarrow P_1, P_2$ are bounded.

p.f: ch 4 (closed graph theorem).

Thm: $X =$ Hilbert space, $S =$ closed subspace,

$S_1 = S$, $S_2 = S^\perp$, $X = S_1 \oplus S_2$

$\Rightarrow P_1, P_2$ are bounded, $\|P_1\| \le 1$, $\|P_2\| \le 1$.

P.f: $x \in X$, $x = s_1 + s_2$, $s_1 \in S_1 = S$, $s_2 \in S_2 = S^\perp$

$\|x\|^2 = \|s_1\|^2 + \|s_2\|^2$ (Pythagoream Thm).

$\|s_1\| \le \|x\|$, $\|s_2\| \le \|x\|$

$\|P_1 x\| \le \|x\|$, $\|P_2 x\| \le \|x\|$: orthoprojectors.

prop. P orthoprojector, $P \ne 0 \Rightarrow \|P\| = 1$.

§3.4 Orthonormal Families $X =$ inner product space

Def: orthonormal family

$(x_\alpha)_{\alpha \in I}$, $x_\alpha \in X$

$\uparrow$ index set.

$$\langle x_\alpha, x_\beta \rangle = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta . \end{cases}$$

def: orthonormal sequence

$(x_k)_{k \in \mathbb{N}}$

Thm $(x_\alpha)_{\alpha \in I}$ orthonormal $\Rightarrow$ $(x_\alpha)_{\alpha \in I}$ is linearly independent.

p.f. E.F.Y.

Ex: $\mathbb{F}^n \Rightarrow (1, 0, \cdots 0), (0, 1, 0, \cdots 0), \cdots (0, 0, \cdots 1)$

Ex: ℓ^2 , $e_1 = (1, 0, 0, \cdots), e_2 = (0, 1, 0, \cdots) \cdots$

Ex: $L_2(0, 2\pi)$ $u_0(t) = \frac{1}{\sqrt{2\pi}}$

$u_n(t) = \frac{1}{\sqrt{\pi}} \cos nt$ $n \in \mathbb{N}$

$v_n(t) = \frac{1}{\sqrt{\pi}} \sin nt$ $n \in \mathbb{N}$,

$(u_0, v_1, u_1, v_2, u_2, \cdots)$ is o.n. Fourier series

Ex: $L_2(0, 2\pi)$

$$e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int} \quad n \in \mathbb{Z}. \quad \text{is o.n.}$$

Thm: $e_1, \cdots, e_n$ orthonormal family.

$$Y = \text{span}\{e_1, e_2, \cdots, e_n\}, \quad y \in Y$$

$$y = \sum_{k=1}^{n} \alpha_k e_k$$

Then $\alpha_i = \langle y, e_i \rangle \quad i = 1, \cdots, n.$

Thus $y = \sum_{k=1}^{n} \langle y, e_k \rangle e_k$

P.f: $\langle y, e_i \rangle = \langle \sum_{i=1}^{n} \alpha_k e_k, e_i \rangle$

$$= \sum_{i=1}^{n} \alpha k \underbrace{\langle e_k, e_i \rangle}_{\delta_{ik}} = \alpha_i$$

Thm: $e_1, e_2, \cdots, e_n$ orthonormal. $Y = \text{span}\{e_1, \cdots, e_n\}$
$x \in X$. Let $y = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$

Then y is the best approximation to X from Y.

P.f: S.T.P $x - y \perp Y$. s.t.p. $\langle x-y, e_i \rangle = 0$, $i = 1, \cdots, n$.

$$\langle x-y, e_i \rangle = \langle x, e_i \rangle - \langle \sum_{i=1}^{n} \langle x, e_i \rangle e_k, e_i \rangle$$

$$= \langle x, e_i \rangle - \sum_{i=1}^{n} \langle x, e_i \rangle \langle e_k, e_i \rangle$$

$$= 0$$

#

Wed.
13-02-91

Remark X = Banach space
S_1 = closed subspace
$(\exists\, S_2)$ $X = S_1 \oplus S_2$? (No!)

True in Hilbert space by taking $S_2 = S_1^\perp$
and finite dimensional space.

thm. $x \in Y \Rightarrow \sum_{i=1}^{n} \langle x, e_k \rangle e_k$
$x \in X \Rightarrow y = \sum_{i=1}^{n} \langle x, e_k \rangle e_k$
y is the best approximation to x from Y.

Let (e_k) be o.n. sequence, $x \in X$

$$Y_n = \operatorname{span}\{e_1, e_2, \cdots, e_n\}, \quad n = 1, 2, \cdots$$

Let $y_n = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$ (best approx to x from Y_n)

$$y_{n+1} = \sum_{k=1}^{n+1} \langle x, e_k \rangle e_k = y_n + \langle x, e_{n+1} \rangle e_{n+1}$$

$x = y_n + z_n$, where $y_n \perp z_n$ $(x - y_n \perp Y_n)$

$$\|x\|^2 = \|y_n\|^2 + \|z_n\|^2 \geq \|y_n\|^2$$

$$\Rightarrow \|y_n\|^2 \leq \|x\|^2 \quad \forall n.$$

$$\|y_n\|^2 = \left\|\sum_{k=1}^{n} \langle x, e_k \rangle e_k\right\|^2 = \left\langle \sum_{k=1}^{n} \langle x, e_k \rangle e_k, \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\rangle$$

$$= \cdots = \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$$

$$* \quad \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \le \|x\|^2 \qquad \forall\, n$$

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \le \|x\|^2 \qquad \approxeq \text{ Bessel's inequality.}$$

<u>Gram-Schmidt Process:</u>

Let (x_k) be a linearly independent sequence.

Let $Y_n = \text{span}\{x_1, \ldots, x_n\}$, $n = 1, 2, 3, \ldots$

Gram-Schmidt process produces o.n. sequences (e_k) such that

$$\text{spam}\{e_1, e_2, \ldots, e_n\} = Y_n \quad \forall\, n = 1, 2, \ldots$$

$x_1 \ne 0$, Let $e_1 = \dfrac{x_1}{\|x_1\|}$ $\quad \text{span}\{e_1\} = \text{span}\{x_1\}$

Suppose we have constructed $e_1, \ldots, e_n$ o.n. such that $\text{span}\{e_1, \ldots, e_k\} = Y_k$, $k = 1, 2, \ldots, n$

$x_{n+1} \notin Y_n$. Let u_{n+1} be the best approx. to x_{n+1} from Y_n $\Rightarrow u_{n+1} = \sum_{k=1}^{n} \langle x_{n+1}, e_k \rangle e_k$

Let $v_{n+1} = x_{n+1} - u_{n+1} \ne 0$. $\Rightarrow v_{n+1} \perp Y_n$

Let $e_{n+1} = \dfrac{v_{n+1}}{\|v_{n+1}\|}$

$$\text{span}\{e_1, \ldots, e_{n+1}\} = \text{span}\{x_1, \ldots, x_{n+1}\}$$

$$\left\{\begin{array}{l}\text{for } n=0,1,2,\ldots \\ \quad V_{n+1} = x_{n+1} - \sum_{k=1}^{n} \langle x_{n+1}, e_k \rangle e_k \\ \quad \text{if } (V_{n+1}=0) \text{ stop. } (x_1, \ldots, x_n \text{ linearly dependent}) \\ e_{n+1} = V_{n+1}/\|V_{n+1}\|. \end{array}\right.$$

§3.5 Series related to orthonormal sequences.

Thm: H = Hilbert space, (e_k) o.n. sequence in H

$\sum_{k=1}^{\infty} \alpha_k e_k$ converges iff $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges.

Let

$$S_n = \sum_{k=1}^{n} \alpha_k e_k \qquad \beta_n = \sum_{k=1}^{n} |\alpha_k|^2$$

For $n \geq m$ (w.l.g.)

$$S_n - S_m = \sum_{k=m+1}^{n} \alpha_k e_k$$

$$\|S_n - S_m\|^2 = \|\sum_{k=n+1}^{n} \alpha_k e_k\|^2 = \sum_{k=m+1}^{n} |\alpha_k|^2 = \beta_n - \beta_m$$

$$\|S_n - S_m\|^2 = |\beta_n - \beta_m|^2, \quad \forall, n, m.$$

$\therefore$ (S_n) is Cauchy Convergence iff (β_n) is Cauchy convergent.

Thm: If $\sum_{k=1}^{\infty} \alpha_k e_k$ converges and $x = \sum_{k=1}^{\infty} \alpha_k e_k$

then $\alpha_k = \langle x, e_k \rangle$.

continuity of $\langle , \rangle$

P.f: $\langle x, e_j \rangle = \langle \sum_{k=1}^{\infty} \alpha_k e_k ; e_j \rangle = \sum_{k=1}^{\infty} \langle \alpha_k e_k, e_j \rangle$

$= \sum_{k=1}^{\infty} \alpha_k, \delta_{kj} = \alpha_j$.

$\langle \sum_{k=1}^{\infty} \alpha_k e_k, e_j \rangle = \langle \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k e_k, e_j \rangle$

$\Rightarrow \boxed{f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)}$

$= \lim_{n\to\infty} \langle \sum_{k=1}^{n} \alpha_k e_k, e_j \rangle$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \langle \alpha_k e_k, e_j \rangle$

$= \sum_{k=1}^{\infty} \alpha_k \langle e_k, e_j \rangle$

$x = \sum_{k=1}^{\infty} \underbrace{\langle x, e_k \rangle}_{} e_k$ Generalized Fourier series.

Generalized Fourier cofficients

~~Thm~~ Let $Y = \overline{\text{span}\{e_1, e_2, e_3, \dots\}}$

Thm Let $x \in X$, let $y = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$

Then y is the best approximation to x from Y.

P.f. E.F.Y. $x - y \perp Y$

Thm: $x \in X \Rightarrow \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ converges.

P.f: It converges iff $\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$ converges

But Bessel's inequality shows that

$$\sum |\langle x, e_k \rangle|^2 \leq \|x\|^2 < \infty.$$

Thm. The value of $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ is independent of the orders of (e_k).

P.f: $Y = \text{span}\{e_1, e_2, \cdots, \}$ is indp. of order

$y = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ is the best approx. to x from Y. This is unique. $\#$

Friday Feb. 15

Uncountable Orthonormal Families

X = inner product space, $(e_\beta)_{\beta \in I}$ o.n. family in X. Let $x \in X$, Generalized Fourier coffs $\langle x, e_\beta \rangle$ $\beta \in I$

Thm: At most countably many $\langle x, e_\beta \rangle$ are non-zero.

P.f: Let $S = \{\beta \in I \mid \langle x, e_\beta \rangle \neq 0\}$

Let $S_n = \{\beta \in I \mid |\langle x, e_\beta \rangle| \geq \frac{1}{n}\}$

$\Rightarrow S = \bigcup_{n=1}^{\infty} S_n$

Claim: each S_n is finite. Suppose not, say $S_{\hat{n}}$ is infinite. Then

$S_{\hat{n}}$ contains a countable subset

$\beta_1, \beta_2, \beta_3, \ldots$

$|\langle x, e_{\beta_i}\rangle| \geq \frac{1}{\hat{n}}$

$$\sum_{i=1}^{m} |\langle x, e_{\beta_i}\rangle|^2 \geq \frac{m}{\hat{n}^2} \quad \forall m$$

$$\sum_{i=1}^{\infty} |\langle x, e_{\beta_i}\rangle|^2 = \infty.$$

This violates Bessel's inequality $\divideontimes$

Consequently

* $\sum_{\beta \in I} |\langle x, e_\beta\rangle|^2$ is meaningful (a series of nonnegative terms, so the order of summation doesn't matter).

$$\sum_{\beta \in I} |\langle x, e_\beta\rangle|^2 \leq \|x\|^2 \text{ (Bessel)}$$

* $\sum_{\beta \in I} \langle x, e_\beta\rangle e_\beta$ is also meaningful

It converges (if X is complete)

It is the best approximation to x from

$$\overline{Y = \text{span}\{e_\beta \mid \beta \in I\}} \quad \text{(E.F.Y.)}$$

the order of summation is immeturral.

§ 3.6 Total Orthonormal Families

H = Hilbert space, $(e_\beta)_{\beta \in I}$ = orthonormal family int

Def: $(e_\beta)_{\beta \in I}$ is a total orthonormal family if $\text{span}\{e_\beta \mid \beta \in I\}$ is dense in H, i.e.,

$$\overline{\text{span}\{e_\beta \mid \beta \in I\}} = H$$

Thm: $(e_\beta)_{\beta \in I}$ is total o.n. family iff $(\forall x \in H)$

$$x = \sum_{\beta \in I} \langle x, e_\beta \rangle e_\beta.$$

Def: Orthonormal Basis = total orthonormal family (Hilbert Basis)

Remark: This is a Schauder basis, not a Hamel basis

Thm: $(e_\beta)_{\beta \in I}$ is am o.n. basis iff

$$\{e_\beta \mid \beta \in I\}^\perp = \{0\}$$

Pf: E.F.Y.

Thm: Parseval Theorem:

$(e_\beta)_{\beta\in I}$ is an o.n. basis iff $(\forall x \in H)$

$$\|x\|^2 = \sum_{\beta\in I} |\langle x, e_i\rangle|^2$$

Pf E.F.Y.

Thm: Every Hilbert space has an o.n. basis.

Pf: Zorn's lemma

Pf: (separable case) H separable $\Rightarrow \exists$ coutable dense subset $(x_i)_{i=1}^{\infty}$. Apply Gram-Schmidt to this sequence, with one modification. If $x_i \in \operatorname{span}\{x_1, \dots, x_{i-1}\}$, throw it out

This gives a finite or countable o.n. family.

$\Rightarrow$ (E.F.Y.) (e_i) is an o.n. Basis of H.

Thm: Any two o.n. bases of H have the same cardinality.

Pf: Transfinite arithmetic.

Def: the dimension of H is the # of element in a basis.

Def: H_1, H_2 = Hilbert spaces. An isomorphism of H_1 and H_2 is a linear map $T: H_1 \to H_2$ $\mathcal{D}(T) = H_1$, $R(T) = H_2$

$$\langle Tx, Ty \rangle = \langle x, y \rangle \quad \forall\, x, y \in H_1$$

consequences: $\|Tx\| = \|x\| \quad \forall\, x \in H_1$

T is 1-1.

Thm: H_1, H_2 = Hilbert space over $\mathbb{F}$. Then H_1 and H_2 are isomorphic iff the have the same dimension.

Pf: $(\Rightarrow)$ H_1, H_2 are isomorphic. $\exists$ isomorphism $T: H_1 \to H_2$ choose basis in $H_1 \to H_2$.

$(\Leftarrow)$ $\dim(H_1) = \dim(H_2)$ Let $(e_\beta)_{\beta \in I}$ be an o.n. basis of H_1. Let $(f_\beta)_{\beta \in I}$ be an o.n. basis of H_2. define

$$T e_\beta = f_\beta \qquad \forall\, \beta \in I.$$

Extend to $G = \operatorname{span}\{e_\beta \mid \beta \in I\}$ by linearity

$$T\left(\sum_{i=1}^{n} \alpha_i e_{\beta_i}\right) = \sum_{i=1}^{n} \alpha_i T e_{\beta_i}$$

$T: G \to H_2$ is linear and preserves the linear product $\langle Tg_1, Tg_2 \rangle = \langle g_1, g_2 \rangle \ \forall\, g_1, g_2 \in G$

$\therefore \Rightarrow \|Tg\| = \|g\| \quad \forall g \in G.$

$\therefore$ T is Lipschitz continuous.

G is dense in H_1,

Extend T to H_1 by continuity

$T: H_1 \to H_2$

T is an isomorphism. #

Monday
Feb. 18

Midterm $38+50 = 88/2 = 44$.

Thm: H = Hilbert space. H is separable iff its dimension is at most countably infinite.

Pf: ($\Leftarrow$) Suppose dim (H) is countable,

$\exists$ o.n. basis $e_1, e_2, e_3, e_4, \ldots$

Let S = set of finite linear comb. of $e_1, e_2, \ldots$ with rational coefficients.

S is countable.

Claim S is dense in it. Let $x \in H$, $\varepsilon > 0$.

$$x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i = \sum_{i=1}^{\infty} \alpha_i e_i$$

$$(\exists N) \quad \left\| x - \sum_{i=1}^{N} \alpha_i e_i \right\| < \frac{\varepsilon}{2}$$

For $i = 1, \ldots, N$, $(\exists\, q_i \in \mathbb{Q})$ $|\alpha_i - q_i| < \frac{\varepsilon}{\sqrt{4N}}$

Then $\| \sum_{i=1}^{N} \alpha_i e_i - \sum_{i=1}^{N} q_i e_i \|^2$

$$= \sum_{i=1}^{N} |\alpha_i - q_i|^2 \leq \sum_{i=1}^{N} \frac{\varepsilon^2}{4N} = \frac{\varepsilon^2}{4}$$

($\Rightarrow$) ① orthonormalized countable dense set by Gram-schmidt.

② Suppose $\dim(H)$ is uncountable $\Rightarrow$ ($\exists$ o.n. basis) $(e_\alpha)_{\alpha \in I}$, where I is uncountable.

if $\alpha \neq \beta$ $\quad \|e_\alpha - e_\beta\|^2 = \|e_\alpha\|^2 + \|e_\beta\|^2 = 2$

$\Rightarrow \|e_\alpha - e_\beta\| = \sqrt{2}$

Let S be any dense subset of H $\quad (\forall \alpha \in I)$

$(\exists S_\alpha \in S)$ $\quad \|S_\alpha - e_\alpha\| < \frac{\sqrt{2}}{2}$

If $\alpha \neq \beta$, $\quad S_\alpha \neq S_\beta$

If $S_\alpha = S_\beta$, then $\|e_\alpha - e_\beta\| \leq \|e_\alpha - S_\alpha\| + \|S_\beta - e_\beta\|$

$$< \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \quad \circledast$$

$\alpha \mapsto S_\alpha$ is a 1-1 map of I onto S

I is uncountble $\Rightarrow$ S is uncountable

#

101, 6

$T: (x_1, x_2, x_3, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$

Let $y = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \cdots) \in l_\infty$

$y \notin R(T)$. $\Rightarrow y \in \overline{R(T)}$

$B(X, Y)$ is complete if Y is complete
$C[a, b]$ is complete
l^∞, l^1, l^2 complete

locally compact

Hamel $\Rightarrow$ vector space $x = \sum_{i=1}^{n} \alpha_i e_{\beta_i}$

$\{e_\beta \mid \beta \in I\}$

Schauder $\Rightarrow$ normed space $\{e_\beta \mid \beta = I\}$

$x = \sum_{i=1}^{\infty} \alpha_i e_i$

Bounded linear operator $\Leftrightarrow$ cts linear operator

Linear operator in finite dimension space.

$N(T)$ is closed if $\|T\| < \infty$.

dual of c_0 . . . .

$\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$.

$\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$

l_p is not inner product space using parallel law.

Schwartz inequality. triangle.

inner product is a cts function.

Hilbert o.n. basis. complementary

<u>closed convex set</u>

unique ~~b~~best approximation

~~pa~~ parallel gram.

$(x-y) \perp Y$

$\Rightarrow$. $\quad P^2 = P$.

o.n is linearly independent.

Bessell inequality . . . .