Functional Analysis 502

Homework No: 1

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$\frac{38}{50}$

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Page 71, Problem 8

If in a normed space X, absolute convergence of any series always implies convergence of that series, show that X is complete.

Proof: We show this problem by contradiction. Suppose that absolute convergence of any series alway implies convergence of that series, but X is not complete. Thus, there exists a Cauchy sequence, say (x_n), of X that does not converge in X. First we note that (x_n) is Cauchy implies that $\forall \varepsilon > 0$ $(\exists N \in \mathbb{N})$ such that

$$\|x_m - x_n\| < \varepsilon/2 \qquad \forall\ n, m \geq N.$$

Let $y_1 = x_1$ and $y_n = x_n - x_{n-1}$, $n = 2, 3, \ldots$

We will prove $\sum_{i=1}^{\infty} \|y_i\|$ is convergent by showing that

$$\begin{aligned} \big| \|y_m\| - \|y_n\| \big| &\leq \|y_m\| + \|y_n\| \\ &= \|x_m - x_{m-1}\| + \|x_n - x_{n-1}\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall\ n, m \geq N+1. \end{aligned}$$

This does not imply that $\sum \|y_n\|$ converges

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$\Rightarrow$ $\sum_{i=1}^{\infty} y_i$ is absolutely convergent. However

$$S_n = \sum_{i=1}^{n} y_i = x_n \quad \text{is not convergent,}$$

which is a contradiction. Hence X is complete. ✡

<u>Page 71, Problem 14</u> (Quotient Space)

Let Y be a closed subspace of a normed space $(X, \|\cdot\|)$. Show that a norm $\|\cdot\|_0$ on X/Y is defined by

$$\|\hat{x}\|_0 = \inf_{x\in\hat{x}} \|x\|$$

where $\hat{x} \in X/Y$, that is, $\hat{x}$ is any coset of Y.

These arguements are essentially right but not clear at all.

<u>Proof:</u> (N1) $\|\hat{x}\|_0 \geq 0$ is obvious.

(N2) $\|\hat{x}\|_0 \Leftrightarrow \hat{x} = 0 + Y$.

$\Rightarrow$ If $\hat{x} = 0 + Y$ and since Y is a subspace, hence $x = 0 \in 0 + Y \Rightarrow \|\hat{x}\|_0 = \|0\| = 0$.

On the orther hand, if $\hat{x} = \omega + Y \neq 0 + Y$, then $x = 0 \overset{??}{\notin} \hat{x}$. To see this, we consider the following: suppose $x = 0 \in \hat{x} \Rightarrow -\omega \in Y \Rightarrow \omega \in Y$ $\Rightarrow \hat{x} = \omega + Y = 0 + Y$, which is a contradition.

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details? Since Y is closed and so $\hat{x} = \omega + Y$, $\hat{x}$ cannot have $x = 0$ as its limit point. Hence $\|\hat{x}\|_0 > 0$.

(N3) $\|\alpha \hat{x}\|_0 = |\alpha| \cdot \|\hat{x}\|_0$ ($\alpha = 0$ is trivial. w.l.g. $|\alpha| \neq 0$)

$\hat{x} = x + Y$ is closed $\Rightarrow (\exists\, x_0 \in \hat{x})$

There may not be such an x_0

$(\|\hat{x}\|_0 = \|x_0\| = \inf_{x\in\hat{x}} \|x\|)$. We would to show $\|\alpha x_0\| = \inf_{x\in\alpha\hat{x}} \|x\|$. $(*)$

Suppose $(*)$ is not true. $\Rightarrow (\exists\, \alpha \bar{x}_0 \in \alpha\hat{x})$ $(\|\alpha \bar{x}_0\| < \|\alpha x_0\|)$. why? But

why not $\|\alpha \bar{x}_0\| > \|\alpha x_0\|$?

$$\|\alpha \bar{x}_0\| = |\alpha| \cdot \|\bar{x}_0\| < \|\alpha x_0\| = |\alpha| \cdot \|x_0\| \Rightarrow \|\bar{x}_0\| < \|x_0\|.$$

This is a contradiction. Hence,

$$\|\alpha \hat{x}\|_0 = \|\alpha x_0\| = |\alpha| \cdot \|x_0\| = |\alpha| \cdot \|\hat{x}\|_0.$$

Problem 14 (cont.)

(N4) $\|\hat{x}+\hat{\omega}\|_0 \leq \|\hat{x}\|_0 + \|\hat{\omega}\|_0$

Let $\hat{x} = x + Y$

$\hat{\omega} = \omega + Y$

$\hat{x}+\hat{\omega} = (x+\omega) + Y$

Such vectors may not exist

Let $y_x \in Y$ such that $\|x+y_x\| = \|\hat{x}\|_0$ and

$y_\omega \in Y$ such that $\|\omega + y_\omega\| = \|\hat{\omega}\|_0$ and

$y_0 \in Y$ such that $\|(x+\omega)+y_0\| = \|\hat{x}+\hat{\omega}\|_0$.

Then we have

$$\|\hat{x}+\hat{\omega}\|_0 = \|(x+\omega)+y_0\| = \inf_{y\in Y} \|(x+\omega)+y\|$$

$$\leq \|(x+\omega)+(y_x+y_\omega)\|$$

$$= \|(x+y_x)+(\omega+y_\omega)\|$$

$$\leq \|x+y_x\| + \|\omega+y_\omega\|$$

$$= \|\hat{x}\|_0 + \|\hat{\omega}\|_0$$

$\times$

Page 76, Problem 1

Give examples of subspaces of ℓ^∞ and ℓ^2 which are not closed.

Case 1: ℓ^∞ : Consider a subspace Y of ℓ^∞ in which $\forall y=(\xi_i)\in Y$, there are finitely many nozero ξ_i $(i=1,2,\cdots)$. Next, consider a sequence

$$y_n=(1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots, \tfrac{1}{n}, 0, \cdots), \quad n=1,2,\cdots$$

$\forall \varepsilon>0$ $(\exists \frac{1}{\varepsilon} < N \in \mathbb{N})$ such that $\forall m,n>N$ (w.l.g. $m>n$)

$$\|y_n - y_m\| = \|(0,\cdots,0,\tfrac{1}{n+1},\cdots,\tfrac{1}{m},0,\cdots)\|$$
$$= \frac{1}{n+1} < \varepsilon$$

Hence, (y_n) is Cauchy in ℓ^∞. Obviously, the limit of (y_n) is not in Y since it has infinitely many nozero elements. Thus Y is not complete, by Theorem 2.3-1, Y is not closed.

Case 2: ℓ^2 : Recall that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is covergent. Thus $\forall \varepsilon>0$ $(\exists N\in\mathbb{N})$ such that

$$\sum_{n=m}^{\infty} \frac{1}{n^2} < \varepsilon^2 \quad \forall m \ge N.$$

Now, we consider the same subspace Y and sequence (y_n) as in case 1, but now in ℓ^2. We prove that (y_n) is also Cauchy in ℓ^2 by noting that

$$\|y_n - y_m\| = \sqrt{\sum_{i=n+1}^{m} \frac{1}{i^2}} \le \sqrt{\sum_{i=n+1}^{\infty} \frac{1}{i^2}} < \varepsilon \quad \forall m,n > N$$

Again, Y is not complete in ℓ^2 and hence it is not closed.

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Page 76, Problem 2

What is the largest possible c in (1) if $X = \mathbb{R}^2$ and $x_1 = (1, 0)$ $x_2 = (0, 1)$? If $X = \mathbb{R}^3$ and $x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$?

Recall (1)

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|).$$

Case 1: $X = \mathbb{R}^2$

$$\alpha_1 x_1 + \alpha_2 x_2 = (\alpha_1, \alpha_2)$$

$$\|\alpha_1 x_1 + \alpha_2 x_2\|^2 = |\alpha_1|^2 + |\alpha_2|^2 \geq \frac{1}{2}(|\alpha_1| + |\alpha_2|)^2$$

due to the fact $|\alpha_1| \cdot |\alpha_2| \leq \frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2)$. Hence, $c \geq \frac{\sqrt{2}}{2}$.

But let $\alpha_1 = \alpha_2 = 1$

$$\Rightarrow \quad \|\alpha_1 x_1 + \alpha_2 x_2\| = \sqrt{2} \quad \text{and} \quad (|\alpha_1| + |\alpha_2|) = 2 \quad \Rightarrow \quad c = \frac{\sqrt{2}}{2}.$$

Thus, the largest possible $c = \frac{\sqrt{2}}{2}$.

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Case 2: $X = \mathbb{R}^3$

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3\| = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$$

$$= \frac{1}{3}\left[\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + (\alpha_1^2 + \alpha_2^2) + (\alpha_2^2 + \alpha_3^2) + (\alpha_3^2 + \alpha_1^2)\right]$$

$$\geq \frac{1}{3}\left[\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2|\alpha_1| \cdot |\alpha_2| + 2 \cdot |\alpha_2| \cdot |\alpha_3| + 2|\alpha_3| \cdot |\alpha_1|\right]$$

$$= \frac{1}{3}(|\alpha_1| + |\alpha_2| + |\alpha_3|)^2$$

$$\Rightarrow \quad c \geq \frac{\sqrt{3}}{3}$$

Consider $\alpha_1 = \alpha_2 = \alpha_3 = 1$

$$\Rightarrow \quad \|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3\| = \sqrt{3} \quad \text{and} \quad |\alpha_1| + |\alpha_2| + |\alpha_3| = 3$$

Hence, we have $c = \frac{\sqrt{3}}{3}$.

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Page 76, Problem 8

Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in Prob. 8, Sec. 2.2, satisfy

$$\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1 .$$

Proof: It is simple to see that

$$(|\xi_1|^2+|\xi_2|^2+\cdots+|\xi_n|^2) \leq (|\xi_1|+|\xi_2|+\cdots+|\xi_n|)^2$$

$$\Rightarrow \quad \|x\|_2 \leq \|x\|_1$$

Now, consider

$$|\xi_1|^2+|\xi_2|^2+\cdots+|\xi_n|^2 = \frac{1}{n}(n|\xi_1|^2+n|\xi_2|^2+\cdots+n|\xi_n|^2)$$

$$= \frac{1}{n}\Big[|\xi_1|^2+(|\xi_1|^2+|\xi_2|^2)+(|\xi_1|^2+|\xi_3|^2)+\cdots+(|\xi_1|^2+|\xi_n|^2)$$
$$+|\xi_2|^2+(|\xi_2|^2+|\xi_3|^2)+\cdots+(|\xi_2|^2+|\xi_n|^2)$$
$$\vdots$$
$$+|\xi_n|^2\Big]$$

$$\geq \frac{1}{n}\Big[|\xi_1|^2+2|\xi_1|\cdot|\xi_2|+2|\xi_1|\cdot|\xi_3|+\cdots+2\cdot|\xi_1|\cdot|\xi_n|$$
$$+|\xi_2|^2+2\cdot|\xi_2|\cdot|\xi_3|+\cdots+2\cdot|\xi_1|\cdot|\xi_n|+\cdots+|\xi_n|^2\Big]$$

$$= \frac{1}{n}\Big[|\xi_1|^2+|\xi_1|\cdot|\xi_2|+|\xi_1|\cdot|\xi_3|+\cdots+|\xi_1|\cdot|\xi_n|$$
$$+|\xi_2|\cdot|\xi_1|+|\xi_2|^2+|\xi_2|\cdot|\xi_3|+\cdots+|\xi_2|\cdot|\xi_n|$$
$$\vdots$$
$$+|\xi_n|\cdot|\xi_1|+|\xi_n|\cdot|\xi_2|+\cdots+|\xi_n|^2\Big]$$

$$= \frac{1}{n}(|\xi_1|+|\xi_2|+\cdots+|\xi_n|)^2$$

$$\Rightarrow \quad \|x\|_2 \geq \frac{1}{\sqrt{n}}\|x\|_1 .$$

$$\Longrightarrow \quad \frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1 .$$

Q.E.D.

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Homework No: 2

Functional Analysis 502

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Page 101, Problem 6.

Show that the range $R(T)$ of a bounded linear operator $T: X \to Y$ need not be closed in Y.

Proof: As the hint, let us consider

$$T: \ell^1 \longrightarrow \ell^\infty$$

defined by $y = (\eta_j) = Tx$, $\eta_j = \xi_j / j$, $x = (\xi_j)$.

Let $x, z \in X$ and $x = (\xi_j)$, $z = (\sigma_j)$

$$T(\alpha x + \beta z) = ([\alpha \xi_j + \beta \sigma_j]/j) = \alpha(\xi_j/j) + \beta(\sigma_j/j)$$
$$= \alpha Tx + \beta Tz$$

Hence, T is linear. Now consider

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$$\|Tx\| = \sup_j |\xi_j / j| \leq \sup_j |\xi_j| \leq \sum_{j=1}^{\infty} |\xi_j| = \|x\| \Rightarrow \|T\| \leq 1.$$

$\Rightarrow$ T is bounded. Now, let us define a sequence $(x_n) \in \ell^1$

by $\quad x_n = (\underbrace{1, 1, 1, \cdots, 1}_{n}, 0, \cdots)$, $\quad n = 1, 2, \cdots$

$$\Rightarrow \quad Tx_n = (1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots, \tfrac{1}{n}, 0, \cdots)$$

As $n \to \infty$, $Tx_n \longrightarrow (1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{i}, \cdots) \notin R(T)$ since

$$x = (1, 1, 1, \cdots, 1, \cdots) \notin \ell^1.$$

Hence $R(T)$ is not closed. Q.E.D.

Page 102, Problem 10

On $C[0,1]$ define S and T by

$$y(s) = s\int_0^1 x(t)\,dt, \qquad y(s) = s\,x(s)$$

respectively. Do S and T commute? Find $\|S\|$, $\|T\|$, $\|ST\|$ and $\|TS\|$.

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① S and T do not commute. To show this, consider

$$x(t) = 1 \in C[0,1]$$

$$ST\,x(t) = S[t\,x(t)] = S[t] = t\int_0^1 s\,ds = \tfrac{1}{2}t$$

$$TS\,x(t) = T\left[t\int_0^1 1\,ds\right] = T[t] = t\cdot t = t^2.$$

$\Rightarrow$ $ST\,x \neq TS\,x$ $\Rightarrow$ S and T do not commute.

② $\|Sx\| = \|t\int_0^1 x(s)ds\| = |\int_0^1 x(s)ds|\cdot\|t\| \le \int_0^1 |x(s)|\,ds \le \int_0^1 \|x(s)\|\,ds$

$= \|x\|$ $\qquad \forall x \neq 0.$

$\Rightarrow \|S\| \le 1$. But let $x(t) = 1 \Rightarrow \|Sx\| = \|t\| = \sup_{t\in[0,1]} |t| = 1$

$\Rightarrow \|S\| = 1.$

③ $\|Tx\| = \|t\,x(t)\| = \sup_{t\in[0,1]} |t\,x(t)| \le \sup_{t\in[0,1]} |t|\cdot|x(t)|$ $\qquad \forall x \neq 0$

$\le \sup_{t\in[0,1]} |t|\cdot \sup_{t\in[0,1]} |x(t)| = \|x\|$

$\Rightarrow \|T\| \le 1$. Again, let $x(t) = 1$. $\Rightarrow \|Tx\| = \|t\| = 1.$

$\Rightarrow \|T\| = 1.$

④ Since $Sx \in C[0,1]$ $\forall x \in C[0,1]$ $\Rightarrow \|TS\| \le \|T\| = 1$

Again, consider $x(t) = 1$,

$$\|TS\,x\| = \|T\,t\int_0^1 1\,ds\| = \|Tt\| = \|t^2\| = 1.$$

$\Rightarrow \|TS\| = 1.$

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④ $\|STx\| = \|S\,t\,x(t)\| = \|t\int_0^1 s x(s)ds\|$

$$= |\int_0^1 s\,x(s)ds| \cdot \|t\|$$

$$= |\int_0^1 s\,x(s)ds|$$

$$\leq \int_0^1 |s| \cdot |x(s)|\,ds$$

$$\leq \int_0^1 |s| \cdot \|x\|\,ds$$

$$= \|x\| \cdot \int_0^1 |s|\,ds = \frac{1}{2}\|x\|.$$

$\Rightarrow \|ST\| \leq \frac{1}{2}$.

Again, let $x = 1$,

$$\|STx\| = \|t\int_0^1 s\,x(s)ds\| = \frac{1}{2}\|t\| = \frac{1}{2}$$

$\Rightarrow \|ST\| = \frac{1}{2}$.

Page 110, Problem 11

Show that two linear functionals $f_1 \neq 0$ and $f_2 \neq 0$ which are defined on the same vector space and have the same null space are proportional.

Proof: Let $x_0 \in X - N(f_1) = X - N(f_2)$. Then it is shown in Problem 9 (on page 631) that for any $x \in X$ $(\exists\, y \in N(f_1) = N(f_2))$

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$$x = y + \frac{f_1(x)}{f_1(x_0)} x_0$$

$$\Rightarrow \quad f_2(x) = f_2(y) + \frac{f_2(x_0)}{f_1(x_0)} f_1(x) = \left[\frac{f_2(x_0)}{f_1(x_0)}\right] \cdot f_1(x) \Rightarrow \text{proportional}$$

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{ I guess I don't have to duplicate the proof of Problem 9 on page 631 }.

P111 P14

Show that the norm $\|f\|$ of a bounded linear functional $f \neq 0$ on a normed space X can be interpreted geometrically as the reciprocal of the distance of $\tilde{d} = \inf\{\|x\| \mid f(x) = 1\}$ of the hyperplane $H_1 = \{x \in X \mid f(x) = 1\}$ from the origin.

Proof: If $f = 0$, the result is trivial. Now assume that $f \neq 0$

$$\|f\| = \sup_{f(x)\neq 0} \frac{\|f(x)\|}{\|x\|} = \sup_{f(x)\neq 0} \frac{|f(x)|}{\|x\|}$$

$$= \sup_{f(x)\neq 0} \frac{1}{\left\| \frac{x}{f(x)} \right\|}$$

$$= \frac{1}{\inf_{f(x)\neq 0} \left\| \frac{x}{f(x)} \right\|}.$$

1D
1D

Let $\tilde{x} = \frac{x}{f(x)}$, we have $f(\tilde{x}) = 1$

$$\inf_{f(x)\neq 0} \left\| \frac{x}{f(x)} \right\| = \inf_{f(\tilde{x})=1} \|\tilde{x}\| = \inf\{\|\tilde{x}\| \mid f(\tilde{x}) = 1\} = \tilde{d}$$

Hence,

$$\|f\| = \frac{1}{\tilde{d}}.$$

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P. 126 P. 8

Show that the dual space of the space c_0 is ℓ^1.

Proof: $c_0 = \{ x=(x_i) \mid |x_n| \to 0 \text{ as } n \to \infty \}$. It is shown that c_0 is complete subspace of ℓ^∞. Let us choose a Schauder basis for c_0 as follows:

$$e_1 = (1, 0, 0, 0, \cdots)$$
$$e_2 = (0, 1, 0, 0, \cdots)$$
$$e_3 = (0, 0, 1, 0, \cdots)$$
$$\vdots$$

Then $x \in c_0$, we will show that

$$x = \sum_{i=1}^{\infty} x_i e_i .$$

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$\forall x \in c_0 \quad (\forall \varepsilon > 0) \quad (\exists N \in \mathbb{N})$ such that $|x_i| < \varepsilon \quad \forall i \leq N$.

To show $x = \sum_{i=1}^{\infty} x_i e_i$, let us define

$$S_n = \sum_{i=1}^{n} x_i e_i .$$

w.l.g. let $n \geq m \geq N$, $\|S_n - S_m\| = \| \sum_{i=m+1}^{n} x_i e_i \| = \sup_{m+1 \leq i \leq n} |x_i| < \varepsilon$

$\Rightarrow$ (S_n) is Cauchy and since c_0 is complete. So

$$x = \lim_{n \to \infty} S_n = \sum_{i=1}^{\infty} x_i e_i .$$

Just show that $S_n \to x$.

Let $f \in c_0'$

$$f(x) = f\left(\sum_{i=1}^{\infty} x_i e_i\right) = f\left(\lim_{n\to\infty} \sum_{i=1}^{n} x_i e_i\right) = \lim_{n \to \infty} f\left(\sum_{i=1}^{n} x_i e_i\right)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{\infty} x_i f(e_i).$$

Let $\alpha_i = f(e_i)$ and $\alpha = (\alpha_i)$

$$\Rightarrow \quad f(x) = \sum_{i=1}^{\infty} x_i \alpha_i$$

P. 126 . P.8 (cont.)

$$|f(x)| = \left|\sum_{i=1}^{\infty} x_i \alpha_i\right| \le \|x\|_\infty \cdot \|\alpha\|_1$$

$$\Rightarrow \qquad \|f\| \le \|\alpha\|_1 .$$

Next, we will show $\alpha \in \ell^1$ and $\|f\| = \|\alpha\|_1$.

Define $x^{(n)} = (x_i^{(n)})$, $n = 1, 2, 3, \cdots$ by

$$x_i^{(n)} = \begin{cases} \frac{|\alpha_i|}{\alpha_i} & \text{if } \alpha_i \ne 0 \text{ and } i \le n \\ 0 & \text{if } \alpha_i = 0 \text{ or } i > n \end{cases}$$

$$f(x^{(n)}) = \sum_{i=1}^{n} x_i^{(n)} \alpha_i = \sum_{i=1}^{n} |\alpha_i|$$

$$\|x^{(n)}\|_\infty = \max_{1 \le i \le n} |x_i^{(n)}| = 1$$

$$\|f\| \ge \frac{|f(x^{(n)})|}{\|x^{(n)}\|_\infty} = |f(x^{(n)})| = \sum_{i=1}^{n} |\alpha_i| \qquad \forall\, n$$

$$\therefore \quad \sum_{i=1}^{\infty} |\alpha_i| \le \|f\|$$

$$\Rightarrow \quad \alpha \in \ell^1 \quad \text{and} \quad \|f\| = \|\alpha\|_1 .$$

Define $C : \ell_1' \longrightarrow c_0$ by

$$C(f) = (f(e_1), f(e_2), f(e_3), \cdots)$$

$$\Rightarrow \quad \mathcal{D}(C) = \ell_1' ,$$

$$C(\beta g + \gamma h) = ((\beta g + \gamma h)(e_i)) = (\beta g(e_i) + \gamma h(e_i))$$
$$= \beta (g(e_i)) + \gamma (h(e_i))$$
$$= \beta C(g) + \gamma C(h) \qquad \forall \beta, \gamma \in \mathbb{F},\ g, h \in \ell_1'$$

$\Rightarrow$ C is linear and $\|C(f)\|_1 = \|f\|$ $\forall f \in \ell_1'$.

Now we have to show C is surjective.

P. 126, P.8 (cont.)

Given $\alpha \in l_1$, Define

$$f_\alpha : c_0 \longrightarrow \mathbb{F}$$

by $\quad f_\alpha(x) = \sum_{i=1}^{\infty} x_i \alpha_i \qquad \forall x \in c_0.$

$\Rightarrow \quad \mathcal{D}(f_\alpha) = l_1,$

$$f_\alpha(\beta x + \gamma y) = \sum_{i=1}^{\infty} (\beta x_i + \gamma y_i)\alpha_i = \beta \sum_{i=1}^{\infty} x_i \alpha_i + \gamma \sum_{i=1}^{\infty} y_i \alpha_i$$

$$= \beta f_\alpha(x) + \gamma f_\alpha(y) \qquad \forall \beta, \gamma \in \mathbb{F},\ x, y \in c_0.$$

$\Rightarrow \quad f_\alpha$ is linear.

$$|f_\alpha(x)| = \left| \sum_{i=1}^{\infty} x_i \alpha_i \right| \leq \|x\|_\infty \cdot \|\alpha\|_1$$

$\Rightarrow \quad \|f_\alpha\| \leq \|\alpha\|_1 \quad \Rightarrow \quad f_\alpha$ is bounded.

$\Rightarrow \quad f_\alpha \in l_1'$

and $\quad C(f_\alpha) = \alpha$

$\Rightarrow \quad C$ is surjective.

Hence, $\quad c_0' = l_1.$ $\qquad$ Q.E.D.

Homework No: 3

Functional Analysis 502

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Page 136, Problem 15

If X is a finite dimensional vector space and (e_j) is a basis for X, show that an inner product on X is completely determined by its values $\gamma_{jk} = \langle e_j, e_k \rangle$. Can we choose such scalars γ_{jk} in a completely arbitrary fashion?

Proof: Let $\dim(X) = n$. $\forall x, y \in X$

$$x = \sum_{i=1}^{n} \alpha_i e_i \quad \text{and} \quad y = \sum_{i=1}^{n} \beta_i e_i \qquad \alpha_i, \beta_i \in \mathbb{F}$$

and

$$\langle x, y \rangle = \langle \sum_{j=1}^{n} \alpha_j e_j, \sum_{k=1}^{n} \beta_k e_k \rangle$$

$$= \sum_{j=1}^{n} \alpha_j \langle e_j, \sum_{k=1}^{n} \beta_k e_k \rangle$$

$$= \sum_{j=1}^{n} \alpha_j \sum_{k=1}^{n} \bar{\beta}_k \langle e_j, e_k \rangle$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \bar{\beta}_k \gamma_{jk}$$

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Thus an inner product on X is completely determined by α_j, β_k and γ_{jk}, $j, k = 1, 2, \cdots, n$.

The answer to the second question is no since

$$\gamma_{jk} = \langle e_j, e_k \rangle = \overline{\langle e_k, e_j \rangle} = \bar{\gamma}_{kj}.$$

$$\gamma_{jj} > 0, \quad j = 1, \cdots, n.$$

Page 140, Problem 5

Show that for a sequence (x_n) in an inner product space the conditions $\|x_n\| \longrightarrow \|x\|$ and $\langle x_n, x\rangle \longrightarrow \langle x, x\rangle$ imply convergence $x_n \rightarrow x$.

Proof: $\|x_n\| \longrightarrow \|x\|$ implies that $\langle x_n, x_n\rangle \longrightarrow \langle x, x\rangle$, i.e.

$\forall \varepsilon > 0 \quad (\exists N \in \mathbb{N})$

$$|\langle x, x\rangle - \langle x_n, x_n\rangle| < \frac{\varepsilon}{3}$$

and

$$|\langle x, x\rangle - \langle x_n, x\rangle| < \frac{\varepsilon}{3} \qquad \forall\, n \geq N.$$

Now consider

$$\|x - x_n\|^2 = |\langle x - x_n, x - x_n\rangle|$$

$$= |\langle x, x - x_n\rangle - \langle x_n, x - x_n\rangle|$$

$$= |\langle x, x\rangle - \langle x, x_n\rangle - \langle x_n, x\rangle + \langle x_n, x_n\rangle|$$

$$= |\langle x, x\rangle - \langle x_n, x\rangle + \langle x, x\rangle - \langle x, x_n\rangle + \langle x_n, x_n\rangle - \langle x, x\rangle|$$

$$\leq |\langle x, x\rangle - \langle x_n, x\rangle| + |\langle x, x\rangle - \langle x_n, x_n\rangle| + |\langle x, x\rangle - \langle x, x_n\rangle|$$

$$< \frac{2}{3}\varepsilon + |\langle x, x\rangle - \langle x, x_n\rangle| \qquad \forall\, n \geq N.$$

$$= \frac{2}{3}\varepsilon + \left|\overline{\langle x, x\rangle - \langle x_n, x\rangle}\right|$$

$$= \frac{2}{3}\varepsilon + |\langle x, x\rangle - \langle x_n, x\rangle|$$

$$= \varepsilon \qquad \forall\, n \geq N.$$

Hence, $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

Q.E.D.

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Page 141, Problem 10 (zero operator)

Let $T: X \to X$ be a bounded linear operator on a complex inner product space X. If $\langle Tx, x \rangle = 0$ for all $x \in X$, show that $T = 0$. Show that this does not hold in the case of a real inner product space.

Proof: Complex case

Please give me a hint how to start the proof.

Take $x = u + \alpha v$ for various values of α.

Problem 10 (cont.)

Real case:

Let $X = \mathbb{R}^2$ and T be defined as follows:

$$Tx = T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \qquad \forall\, x \in \mathbb{R}^2.$$

Then $\forall\, x, y \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$,

$$T(\alpha x + \beta y) = T\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix} = \begin{bmatrix} -\alpha x_2 - \beta y_2 \\ \alpha x_1 + \beta y_1 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + \beta \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix} = \alpha Tx + \beta Ty$$

$\Rightarrow$ T is linear. The inner product of $\mathbb{R}^2$ is the usual inner product defined as the following:

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 \Rightarrow \|x\|^2 = x_1^2 + x_2^2$$

$$\Rightarrow \quad \langle Tx, x \rangle = -x_2 x_1 + x_1 x_2 = 0 \qquad \forall\, x \in \mathbb{R}^2$$

$$\|Tx\| = x_1^2 + x_2^2 = \|x\| \qquad \forall\, x \in \mathbb{R}^2 \Rightarrow \|T\| = 1.$$

Hence T is a bounded linear operator on $\mathbb{R}^2$ and satisfies $\langle Tx, x \rangle = 0$ for all $x \in \mathbb{R}^2$. But obviously, $T \neq 0$. ※

Page 150, Problem 7

Let A and $B \supset A$ be nonempty subsets of an inner product space. Show that

(a) $A \subset A^{\perp\perp}$, (b) $B^{\perp} \subset A^{\perp}$, (c) $A^{\perp\perp\perp} = A^{\perp}$

Proof: (a) $x \in A \Rightarrow x \perp A^{\perp} \Rightarrow x \in A^{\perp\perp} \Rightarrow A \subset A^{\perp\perp}$.

(b) $x \in B^{\perp} \Rightarrow x \perp B \supset A \Rightarrow x \perp A \Rightarrow x \in A^{\perp} \Rightarrow B^{\perp} \subset A^{\perp}$.

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(c) $A^{\perp\perp\perp} = (A^{\perp})^{\perp\perp} \supset A^{\perp}$ (see part a).

Since $A \subset A^{\perp\perp}$, then part (b) implies

$(A^{\perp\perp})^{\perp} \subset A^{\perp}$

Hence, $A^{\perp\perp\perp} = A^{\perp}$. Q.E.D.

Page 159, Problem 7

Let (e_k) be any orthonormal sequence in an inner product space X. Show that for any $x, y \in X$,

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle \langle y, e_k\rangle| \le \|x\| \cdot \|y\|$$

Proof: By Cauchy-Schwarz inequality, we have

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle\langle y, e_k\rangle| \le \left(\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \cdot \sum_{k=1}^{\infty} |\langle y, e_k\rangle|^2\right)^{\frac{1}{2}}.$$

By Bessel's inequality, we have

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \le \|x\|^2 \quad \text{and} \quad \sum_{k=1}^{\infty} |\langle y, e_k\rangle|^2 \le \|y\|^2$$

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Hence,

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle\langle y, e_k\rangle| \le \|x\| \cdot \|y\| .$$

Q.E.D.

Functional Analysis 502

Homework No: 4

Ben M. Chen

$\frac{35}{50}$

March 25, 91

P175, Problem 4

Derive from (3) the following formula (which is often called the parseval relation).

$$\langle x, y\rangle = \sum_k \langle x, e_k\rangle \overline{\langle y, e_k\rangle}$$

Proof: Let M be a total orthonormal set in a Hilbert space. Consider any $x, y \in H$ and a subset $\{e_1, e_2, \cdots\} \subseteq M$ such that

$$x = \sum_k \langle x, e_k\rangle e_k ,$$

$$y = \sum_m \langle y, e_m\rangle e_m ,$$

5/10

$$\begin{aligned}
\langle x, y\rangle &= \Big\langle \sum_k \langle x, e_k\rangle e_k , \sum_m \langle y, e_m\rangle e_m \Big\rangle \\
&= \sum_k \langle x, e_k\rangle \Big\langle e_k , \sum_m \langle y, e_m\rangle e_m \Big\rangle \\
&= \sum_k \langle x, e_k\rangle \sum_m \overline{\langle y, e_m\rangle} \langle e_k , e_m\rangle \\
&= \sum_k \langle x, e_k\rangle \overline{\langle y, e_k\rangle} .
\end{aligned}$$

\#

Derive it from Parseval's Thm.

Page 194, Problem 8

Show that any Hilbert space H is isomorphic with its second dual space $H'' = (H')'$.

Proof:

Page 200, Problem 2

Let H be a Hilbert space and $T: H \to H$ a bijective bounded linear operator whose inverse is bounded. Show that $(T^*)^{-1}$ exists and

$$(T^*)^{-1} = (T^{-1})^*.$$

Proof: $\forall x, y \in H$. Consider

$$\langle Ix, y\rangle = \langle x, y\rangle = \langle x, Iy\rangle$$
$$= \langle x, T^{-1}Ty\rangle$$
$$= \langle (T^{-1})^*x, Ty\rangle$$
$$= \langle T^*(T^{-1})^*x, y\rangle$$
$$\Rightarrow \langle [T^*(T^{-1})^* - I]x, y\rangle = 0 \quad \forall, x, y \in H.$$

By 3.9-3 (a). $\Rightarrow T^*(T^{-1})^* - I = 0 \Rightarrow T^*(T^{-1})^* = I$.

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On the other hand, if we consider

$$\langle Ix, y\rangle = \langle x, Iy\rangle$$
$$= \langle x, TT^{-1}y\rangle$$
$$= \langle T^*x, T^{-1}y\rangle$$
$$= \langle (T^{-1})^*T^*x, y\rangle \quad \forall, x, y \in H.$$

$$\Rightarrow (T^{-1})^*T^* = I.$$

Hence, $(T^*)^{-1} = (T^{-1})^*$. Of course, $(T^*)^{-1}$ exists. #

Page 201. Problem 8

Let $S = I + T^*T : H \longrightarrow H$, where T is linear and bounded. Show that $S^{-1} : S(H) \longrightarrow H$ exists.

Proof: By theorem 2.6-10, it is sufficient to show that S is linear and $(Sx = 0 \Rightarrow x = 0)$.

This is clear

(a) $\forall\, x, y \in H,\ \alpha, \beta \in \mathbb{F}$.

$$\begin{aligned} S(\alpha x + \beta y) &= (I + T^*T)(\alpha x + \beta y) \\ &= [\alpha x + T^*T(\alpha x)] + [\beta y + T^*T(\beta y)] \\ &= \alpha[x + T^*Tx] + \beta[y + T^*Ty] \\ &= \alpha Sx + \beta Sy. \end{aligned}$$

Thus, S is indeed linear.

(b) Suppose $Sx = (I + T^*T)x = 0 \Rightarrow x = -(T^*T)x$

Hence, we have

$$\begin{aligned} \langle x, x \rangle &= \langle -(T^*T)x, x \rangle \\ &= -\langle Tx, Tx \rangle \end{aligned}$$

$$\Rightarrow \langle x, x \rangle + \langle Tx, Tx \rangle = 0.$$

Since $\langle x, x \rangle$ and $\langle Tx, Tx \rangle$ are both non-negative.

$$\Rightarrow \langle x, x \rangle = 0$$

By theorem 2.6-10, S^{-1} exists. #

$\frac{10}{10}$

Page 207, Problem 6

If $T: H \longrightarrow H$ is a bounded self-adjoint linear operator and $T \neq 0$, then $T^n \neq 0$. Prove this (a) for $n = 2, 4, 8, 16, \cdots$, (b) for $n \in \mathbb{N}$.

Proof: $T \neq 0 \Rightarrow (\exists x \in H)\ Tx \neq 0$.

(a) $n = 2$.

$\langle T^2 x, x \rangle = \langle Tx, Tx \rangle \neq 0$ since $Tx \neq 0$.

By 3.9-3 (a), $T^2 \neq 0$. Moreover $T^2 x \neq 0$.

$n = 4$.

$\langle T^4 x, x \rangle = \langle T^2 x, T^2 x \rangle \neq 0$ since $T^2 x \neq 0$.

Hence, $T^4 \neq 0$.

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By a trivial induction, we have $T^n \neq 0$, $n = 2, 4, 8, 16, \cdots$

(b) Assume that $T^n = 0$ for $n \in \mathbb{N}$. Let $N = 2^p > n$. Then By (a), $T^N \neq 0$. But $T^N = T^{N-n}(T^n) = 0$ which is a contradiction. Hence $T^n \neq 0$ for all $n \in \mathbb{N}$. ※

Math 502 **Assignment 5**

due Friday, April 12

1. p. 224, # 9 (counts as two problems)
2. p. 224, # 12 ✓
3. p. 246, # 10
4. If $T \in B(X, Y)$, then $T' \in B(Y', X')$ and $T'' \in B(X'', Y'')$. Show that if X and Y are reflexive, then T and T'' are essentially the same operator. (State the result precisely, then prove it.)

Functional Analysis 502

Homework No: 5

Ben M. Chen

40/50

April 12, 1991.

Page 224, Problem 9

Let f be a bouned linear functional on a subspace Z of a separable normed space X. Then there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f on X and $\|\tilde{f}\|_X = \|f\|_Z$. (Prove it without Zorns Lemma).

Proof: X is separable. Thus there exists a countable subset $Y = \{y_n \mid y_n \in X,\ y_n \notin Z,\ n=1,2,3,\cdots\}$ such that $Z \dot{+} \mathrm{span}\{Y\}$ is dense in X.

(1) We first assume that f is real-valued on Z.

Now let $Z_1 = Z \oplus \mathrm{span}\{y_1\} = \{z + \alpha y_1 \mid z \in Z, \alpha \in \mathbb{F}\}$.

Let us extend f to the space Z_1 by setting

$$f_1(z + \alpha y_1) = f(z) + \alpha f_1(y_1)$$

where $f_1(y_1)$ is chosen such that

$$(*) \quad \sup_{\substack{\alpha_2 > 0 \\ z_2 \in Z}} \frac{f(z_2) - \|f\|_Z \cdot \|z_2 - \alpha_2 y_1\|}{\alpha_2} \le f_1(y_1) \le \inf_{\substack{\alpha_1 > 0 \\ z_1 \in Z}} \frac{\|f\|_Z \cdot \|z_1 + \alpha_1 y_1\| - f(z_1)}{\alpha_1}.$$

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First we note that such a $f_1(y_1)$ always exists. To show this, let us consider the following fact:

$$\begin{aligned}
\alpha_1 f(z_2) + \alpha_2 f(z_1) &= f(\alpha_1 z_2 + \alpha_2 z_1) \\
&= f(\alpha z_2 - \alpha_1\alpha_2 y_1 + \alpha_2 z_1 + \alpha_1 \alpha_2 y_1) \\
&\le |f(\alpha z_2 - \alpha_1\alpha_2 y_1 + \alpha_2 z_1 + \alpha_1\alpha_2 y_1)| \\
&\le \|f\|_Z \|\alpha_1(z_2 - \alpha_2 y_1) + \alpha_2(z_1 + \alpha_1 y_1)\| \\
&\le \alpha_1 \|f\|_Z \cdot \|z_2 - \alpha_2 y_1\| + \alpha_2 \|f\|_Z \cdot \|z_1 + \alpha_1 y_1\|
\end{aligned}$$

$$\Rightarrow \quad \frac{f(z_2) - \|f\|_Z \cdot \|z_2 - \alpha_2 y_1\|}{\alpha_2} \le \frac{\|f\|_Z \cdot \|z_1 + \alpha_1 y_1\| - f(z_1)}{\alpha_1}$$

$\forall\ \alpha_1, \alpha_2 > 0,\ z_1, z_2 \in Z.$

Page 224, Problem 9 (cont.)

Hence, we know there always exists such a $f_1(y_1)$ satisfying $(*)$.
It is trivial to see that f_1 is linear, real-valued in Z_1. Moreover from $(*)$ we have

$$f_1(z+\alpha y_1) = f(z)+\alpha f_1(y_1) \le \|f\|_Z \cdot \|z+\alpha y_1\|, \quad \forall z \in Z, \alpha \in \mathbb{F}$$

and

$$-f_1(z+\alpha y_1) = f(-z-\alpha y_1) \le \|f\|_Z \|-z-\alpha y_1\| = \|f\|_Z \cdot \|z+\alpha y_1\|$$

$$\Rightarrow |f_1(z+\alpha y_1)| \le \|f\|_Z \cdot \|z+\alpha y_1\| \quad \forall z\in Z, \alpha\in\mathbb{F}$$

$$\Rightarrow \|f_1\|_{Z_1} \le \|f\|_Z$$

Obviously, $\|f_1\|_{Z_1} \ge \|f\|_Z \Rightarrow \|f_1\|_{Z_1} = \|f\|_Z$.

detail ?

By countably many steps of inductions, we show that there exists an extension $\bar{f}$ of f on $Z\oplus \text{span}\{Y\}$ such that $\|\bar{f}\|_{Z\oplus\text{span}\{Y\}} = \|f\|_Z$. It follows from Theorem 2.7-11 that there exists an extension $\tilde{f}$ of f on $\overline{Z\oplus\text{span}\{Y\}} = X$ such that $\|\tilde{f}\|_X = \|f\|_Z$. Of course, $\tilde{f}$ is bounded and linear.

(2) Now suppose that f is complex-valued. In this case, we can follow the routines given in the class or text. The result follows trivially.

Q.E.D.

$D=\{x_n\}_{n=1}^{\infty}$, Let $Z_0 = Z$. $Z_n = \text{span}\{Z\cup\{x_1,\dots,x_n\}\}$

Page 224, Problem 12

To illustrate Theorem 4.3-3. Let X be the Euclidean plane $\mathbb{R}^2$ and find the functional $\tilde{f}$

Solution: Let $x_0 = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} \in \mathbb{R}^2$, $x_0 \neq 0$.

Let $y_0 \triangleq \begin{pmatrix} x_{02} \\ -x_{01} \end{pmatrix} \in \mathbb{R}^2$. Obviously, x_0, y_0 are linearly independent. $\Rightarrow$ $(\forall x \in \mathbb{R}^2)$ $(\exists \alpha_x, \beta_x \in \mathbb{R})$

$$x = \alpha_x x_0 + \beta_x y_0$$

Now, define a functional $\tilde{f}$ on $\mathbb{R}^2$ by setting

$$\tilde{f}(x) = \alpha_x \|x_0\|.$$

We shall show that $\tilde{f}$ has all the properties we need.

$$\tilde{f}(x+y) = \tilde{f}(\alpha_x x_0 + \beta_x y_0 + \alpha_y x_0 + \beta_y y_0)$$
$$= (\alpha_x + \alpha_y)\|x_0\| = \tilde{f}(x) + \tilde{f}(y)$$

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$$\tilde{f}(\alpha x) = \tilde{f}(\alpha \alpha_x x_0 + \alpha \beta_x y_0) = \alpha \alpha_x \|x_0\| = \alpha \tilde{f}(x)$$

$\forall x, y \in \mathbb{R}^2$, $\forall \alpha \in \mathbb{R}$. $\Rightarrow \tilde{f}$ is linear.

$$|\tilde{f}(x)| = |\tilde{f}(\alpha_x x_0 + \beta_x y_0)| = |\alpha_x| \cdot \|x_0\|$$
$$= \|\alpha_x x_0\| \leq \|\alpha_x x_0\| + \|\beta_x y_0\|$$
$$= \|\alpha_x x_0 + \beta_x y_0\| \quad (\text{due to } x_0 \perp y_0)$$
$$= \|x\|$$

$$\Rightarrow \|\tilde{f}\| \leq 1.$$

Note that $\tilde{f}(x_0) = \|x_0\| \Rightarrow \|\tilde{f}\| = 1.$ Q.E.D.

Page 246, Problem 10

Show that if a normed space X has a linearly independent subset of n elements, so does the dual space X'.

Proof: Let $\{x_1, x_2, \cdots, x_n\}$ be a linearly independent subset of X.

Obviously, $x_i \neq 0$, $i=1,2,\cdots,n$.

Let

$$Y_i = \overline{\text{span}\,\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\}}, \quad i=1,2,\cdots,n.$$

Then by Lemma 4.6-7, there exist a $f_i \in X'$ such that

$$\|f_i\| = 1, \qquad f_i(Y_i) = 0, \qquad f_i(x_i) \neq 0.$$

We claim that such f_i, $i=1,2,\cdots,n$ are linearly independent. Suppose not $\Rightarrow$ ($\exists\, \alpha_i \in \mathbb{F}$, $i=1,2,\cdots,n$) such that at least one of α_i, say α_k, is nonzero. Moreover

$$\sum_{i=1}^{n} \alpha_i f_i = 0.$$

Now consider

$$\left(\sum_{i=1}^{n} \alpha_i f_i\right)(x_k) = \sum_{i=1}^{n} \alpha_i f_i(x_k) = \alpha_k f_k(x_k) \neq 0,$$

which is a contradiction. Hence, we show that X' has a linearly independent subset of n elements, namely, $\{f_1, f_2, \cdots, f_n\}$. Q.E.D.

Problem 4

If $T \in B(X, Y)$, then $T' \in B(Y', X')$ and $T'' \in B(X'', Y'')$.

Show that if X and Y are reflexive, then T and T'' are essentially the same operator, i.e. there exists an isomorphism $Q: B(X, Y) \to B(X'', Y'')$ such that Q is 1-1, $\mathcal{D}(Q) = B(X, Y)$, $R(Q) = B(X'', Y'')$, $Q(T) = T''$, $\|Q(T)\| = \|T''\| = \|T\|$.

Proof: $\forall x \in X$; define $g_x \in X''$ by $g_x(f) = f(x)$ $\forall f \in X'$

$\forall y \in Y$, define $g_y \in Y''$ by $g_y(h) = h(x)$ $\forall h \in Y'$.

Claim: If $X \simeq X''$, $Y \simeq Y''$

Then $Tx = y$ iff $T'' g_x = g_y$.

pf: $\forall f \in Y'$, $\langle f, T'' g_x \rangle = \langle T' f, g_x \rangle$

$= g_x(T'f) = (T'f)(x) = \langle x, T'f \rangle = \langle Tx, f \rangle$

$= \langle y, f \rangle = \langle f, g_y \rangle$ $\therefore$ $T'' g_x = g_y$ //

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Functional Analysis 502

Ben M. Chen.

$\frac{40}{50}$

April 29, 1991

Homework Assignment 6.

Page 255, Problem 10

Let $y = (\eta_j)$, $\eta_j \in \mathbb{C}$, be such that $\sum \xi_j \eta_j$ converges for every $x = (\xi_j) \in c_0$, where $c_0 \subset l^\infty$ is the subspace of all complex sequences converging to zero. Show that $\sum |\eta_j| < \infty$.

Proof: By a previous homework problem, we know that c_0 is a complete subspace of l^∞. Thus c_0 is a Banach space with usual norm in l^∞. Now, let us build a sequence linear operators (also functionals in this case)

$$T_n : c_0 \longrightarrow \mathbb{C} \qquad n = 1, 2, 3, \cdots$$

by

$$T_n x = \sum_{j=1}^{n} \xi_j \eta_j \qquad \text{for every } x = (\xi_j) \in c_0.$$

Obviously, T_n, $n = 1, 2, 3, \cdots$, are linear.

$$|T_n x| = \left| \sum_{j=1}^{n} \xi_j \eta_j \right| \leq \max\{|\xi_1|, |\xi_2|, \cdots, |\xi_n|\} \cdot \sum_{j=1}^{n} |\eta_j| \leq \sum_{j=1}^{n} |\eta_j| \cdot \|x\|$$

It is easy (some simple algebra) to show that $(\exists x \in c_0)$ $\|x\| = 1$ and $|T_n x| = \sum_{j=1}^{n} |\eta_j|$. Thus, $\|T_n\| = \sum_{j=1}^{n} |\eta_j|$ is bounded. Due to $\sum_{j=1}^{\infty} \xi_j \eta_j$ converges, we let $\sum_{j=1}^{\infty} \xi_j \eta_j = g_x$. Then $(\forall x \in c_0)$ $(\exists\, 0 < \varepsilon < \infty)$

$$\left| \sum_{j=1}^{n} \xi_j \eta_j - g_x \right| \leq \varepsilon \qquad \forall\ n = 1, 2, 3, \cdots$$

$$\Rightarrow \quad |T_n x| = \left| \sum_{j=1}^{n} \xi_j \eta_j \right| \leq |g_x| + \varepsilon \triangleq c_x < \infty, \quad n = 1, 2, \cdots$$

By the Uniform Boundedness Theorem, $(\exists\, c)$

$$\|T_n\| = \sum_{j=1}^{n} |\eta_j| < c < \infty, \qquad n = 1, 2, 3, \cdots$$

Hence,

$$\sum_{j=1}^{\infty} |\eta_j| < c < \infty.$$

Q.E.D.

Page 262, Problem 4

Show that $x_n \xrightarrow{w} x_0$ implies $\liminf_{n\to\infty} \|x_n\| \geq \|x_0\|$.

Proof: First we note if $x_0 = 0$, $\liminf_{n\to\infty} \|x_n\| \geq 0 = \|x_0\|$.

Now, suppose $x_0 \neq 0$. By Theorem 4.3-3, there exists a bounded linear functional $\tilde{f}$ on X such that

$$\|\tilde{f}\| = 1 \qquad \text{and} \qquad \tilde{f}(x_0) = \|x_0\|.$$

Thus, $$|\tilde{f}(x_n)| \leq \|\tilde{f}\| \cdot \|x_n\| = \|x_n\|. \quad \cdots (*)$$

$x_n \xrightarrow{w} x_0$ implies that $\tilde{f}(x_n) \longrightarrow \tilde{f}(x_0) = \|x_0\| \Rightarrow$

$$|\tilde{f}(x_n)| \longrightarrow \|x_0\| \quad \text{as } n \to \infty. \quad \cdots (**)$$

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Then it is trivial to see from $(*)$ and $(**)$ that

$$\liminf_{n\to\infty} \|x_n\| \geq \lim_{n\to\infty} |\tilde{f}(x_n)| = \|x_0\|. \qquad \text{Q.E.D.}$$

Page 262, Problem 5

If $x_n \xrightarrow{w} x_0$ in a normed space X, show that $x_0 \in \overline{Y}$, where $Y = \text{span}(x_n)$.

Proof: We prove this problem by contradiction. Suppose that $x_0 \notin \overline{Y}$.

Let $\delta = \inf_{\tilde{y} \in \overline{Y}} \|\tilde{y} - x_0\| > 0$. By theorem 4.6-7, $(\exists \tilde{f} \in X')$

$$\|\tilde{f}\| = 1, \qquad \tilde{f}(\overline{Y}) = 0, \qquad \tilde{f}(x_0) = \delta.$$

Hence, $\tilde{f}(x_n) = 0$, so that $(\tilde{f}(x_n))$ does not converge to $\tilde{f}(x_0) = \delta > 0$. This is a contradiction. Thus, $x_0 \in \overline{Y}$.

Q.E.D.

10/10

Page 262, Problem 6

If (x_n) is a weakly convergent sequence in a normed space X, say, $x_n \xrightarrow{w} x_0$, show that there is a sequence (y_m) of linear comb. of elements of (x_n) which converges strongly to x_0.

Proof: By the previous problem, we know that $x_n \xrightarrow{w} x_0$ implies that

$$x_0 \in \overline{\text{span}(x_n)} .$$

Thus, they exists a sequence $(y_m) \subseteq \text{span}(x_n)$ which converges to x_0, i.e.

$$\| y_m - x_0 \| \longrightarrow 0 \quad \text{as } m \to \infty .$$

$\Rightarrow$ (y_m) converges to x_0 strongly.

Note that $(y_m) \subseteq \text{span}(x_n)$. Hence, each y_m is a linear combination of elements of (x_n). Q.E.D.

Page 262, Problem 7.

Show that any closed subspace Y of a normed space X contains the limits of all weakly convergent sequences of its elements.

Proof: Let $(y_m) \subseteq Y$ be an weakly convergent sequences. Let x_0 be its limit. By Problem 5, $x_0 \in \overline{\text{span}(y_m)} \subseteq Y$. since Y is closed. Q.E.D.

Page 263, Problem 10.

A normed space X is said to be weakly complete if each weak Cauchy sequence in X converges weakly in X. If X is reflexive, show that X is weakly complete.

Proof: First we note that by theorem 4.6-4: X is reflexive implies X is complete.

Now, let (x_n) be a weak Cauchy sequence in X. By

Prove! Problem 8 of this section (x_n) is bounded. Hence, it has a Cauchy (in norm) subsequence (x_{n_k}). No!!!

X is complete $\Rightarrow$ $(\exists\, x_0 \in X)$ such that

$$x_{n_k} \longrightarrow x_0 \qquad \text{as } k \to \infty.$$

$$\Rightarrow \qquad f(x_{n_k}) \to f(x_0) \qquad \forall f \in X' \quad \text{as } k \to \infty$$

Since (x_n) is Cauchy $\Rightarrow$ $\forall f \in X'$, $(f(x_n))$ is Cauchy.

But F is complete $\Rightarrow$ $f(x_n)$ must converge to a element in F which must be $f(x_0)$, i.e.,

$$f(x_n) \longrightarrow f(x_0), \qquad \forall f \in X'.$$

$$\Rightarrow \qquad x_n \xrightarrow{w} x_0 \in X.$$ Hence, X is weakly complete.

Q.E.D.

Make use of X''!!

(e_n) (standard basis of l_2) is weakly Cauchy but has no subsequence that converges in norm.

Page 269, Problem 10.

Let X be a separable Banach space and $M \subset X'$ a bounded set. Show that every sequence of elements of M contains a contains a subsequence which is weak* convergent to an element of X'.

Proof: Since X is separable, there is a countable set $\{x_1, x_2, \cdots, x_n, \cdots\}$, which is dense in X.

Suppose the sequence $(f_n) \subseteq M \subseteq X'$, M is bounded. Then the numerical sequence,

$$f_1(x_1),\ f_2(x_1),\ \cdots,\ f_n(x_1),\ \cdots$$

is bounded. Hence, (f_n) contains a subsequence

$$f_1^{(1)},\ f_2^{(1)},\ \cdots,\ f_n^{(1)},\ \cdots$$

such that the numerical sequence

$$f_1^{(1)}(x_1),\ f_2^{(1)}(x_1),\ \cdots,\ f_n^{(1)}(x_1),\ \cdots$$

converges. By the same token, the subsequence $(f_n^{(1)})$ in turn contains a subsequence $(f_n^{(2)})$ such that the sequence

$$f_1^{(2)}(x_2),\ f_2^{(2)}(x_2),\ \cdots,\ f_n^{(2)}(x_2),\ \cdots$$

converges. Continuning this construction, we get a sequence of sequence $(f_n^{(k)})$, $k=1,2,\cdots$ such that

1) $(f_n^{(k+1)})$ is a subsequence of $(f_n^{(k)})$ for all $k=1,2,\cdots$

2) $(f_n^{(k)})$ converges at all points $x_1, x_2, \cdots, x_k$.

Now, taking the "diagonal sequence"

$$f_1^{(1)},\ f_2^{(2)},\ \cdots,\ f_n^{(n)},\ \cdots,$$

we get a sequence of bounded linear functionals in M such that

Problem 10 (cont.)

$$f_1^{(1)}(x_n),\ f_2^{(2)}(x_n),\ \cdots$$

converges for all n. Hence, by theorem 4.9.-7, i.e.

(1) $(f_n^{(n)}) \subseteq M \Rightarrow$ The sequence $(\|f_n^{(n)}\|)$ is bounded.

(2) $(f_n^{(n)}(x))$ is Cauchy for $\forall x = \{x_1, x_2, \cdots, x_n, \cdots\}$

which is a dense subset of X.

$\Longrightarrow$ $(f_n^{(n)})$ converges weakly* to an $f \in X'$. Q.E.D.