Ben M. Chen

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Math 502

Second Midterm Examination, Part I **February 20, 1991**

1. (15 points) Prove that ℓ_1 is separable.

2. (15 points) Let X and Y be normed linear spaces, and let $T : X \to Y$ be a linear operator with $\mathcal{D}(T) = X$. Prove that T is bounded if and only if T is continuous at 0.

3. (20 points) Let X be the space of continuous functions on $[0, 1]$, endowed with the norm $\|x\|_1 = \int_0^1 |x(t)|dt$. Determine whether or not each of the following linear functionals on X is bounded.

   (a) $f(x) = x(.5)$

   (b) $g(x) = \int_0^1 x(t)\sin(\pi t)dt$

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Math 502

Second Midterm Examination, Part II **February 22, 1991**

4. (20 points) Prove that ℓ_3 is complete.

5. (15 points) Let K be a convex set in an inner product space, let $\delta = \inf_{x \in K} \| x \|$, and let (x_n) be a sequence in K such that $\lim_{n \to \infty} \| x_n \| = \delta$. Prove that (x_n) is a Cauchy sequence.

6. (15 points) Let $e_1, e_2, \ldots, e_n$ be orthonormal vectors in a Hilbert space H, and let $Y = \text{span}\{e_1, \ldots, e_n\}$. Given $x \in H$, let

$$y = \sum_{k=1}^{n} \langle x, e_k \rangle e_k.$$

Show that y is the best approximation to x from Y.

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Problem 1 :

Proof: Let $e_1 = (1, 0, 0, 0, \cdots)$

$e_2 = (0, 1, 0, 0, \cdots)$

$e_3 = (0, 0, 1, 0, \cdots)$

$\vdots$

Let S be the set of any finitely many linear combinations e_k with cofficients $\in \mathbb{Q}$. obviously S is countable. Next we will show S is dence in l_1. $\forall\, x = (x_i) \in l_1$. we first show.

$$x = \sum_{i=1}^{\infty} x_i e_i .$$

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Noting that

irrelevant

$$\| x - \sum_{i=1}^{n} x_i e_i \| = \cancel{\| \sum_{i=n+1}^{\infty} x_i e_i \|_1}$$

$$= \sum_{i=n+1}^{\infty} |x_i| \longrightarrow 0$$

since $x \in l_1$. Hence $(\forall\, \varepsilon > 0)\ (\exists\, N \in \mathbb{N})$

$$\| x - \sum_{i=1}^{n} x_i e_i \| < \varepsilon/2 \quad \forall\, n > N$$

Problem 1: Now fix n and for each $i = 1, 2, \cdots, n$.

There exist rational numbers q_i; $i = 1, 2, \cdots, n$

such that

$$|x_i - q_i| < \frac{\varepsilon}{2n}.$$

Now consider

$$\| x - \sum_{i=1}^{n} q_i e_i \| = \| x - \sum_{i=1}^{n} x_i e_i + \sum_{i=1}^{n} x_i e_i - \sum_{i=1}^{n} q_i e_i \|$$

$$\leq \| x - \sum_{i=1}^{n} x_i e_i \| + \| \sum_{i=1}^{n} (x_i - q_i) e_i \|$$

$$< \frac{\varepsilon}{2} + \sum_{i=1}^{n} |x_i - q_i|$$

$$< \frac{\varepsilon}{2} + n \cdot \frac{\varepsilon}{2n} = \varepsilon.$$

This proves S is dense in ℓ_1. Hence, ℓ_1 is separable. #

Problem 2 :

Proof: ($\Rightarrow$) If T is bounded, i.e., $\|T\| \leq M < \infty$,

$$\Rightarrow (\forall \varepsilon > 0) \left(\exists\ \delta = \frac{\varepsilon}{M} > 0\right) \text{ for all } \|x\| < \delta$$

$$\|Tx - T0\| = \|Tx\| \leq M\,\|x\| < M \cdot \delta = M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

$\Rightarrow$ T is continous at 0.

($\Leftarrow$) If T is continous at 0. $\Rightarrow (\forall \varepsilon = 1)\ (\exists \delta > 0)$

such that $\|x\| < \delta$,

$$\|Tx\| = \|Tx - T0\| < \varepsilon = 1.$$

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Now for all $0 \neq z \in X$. Let $\hat{z} = \frac{\delta}{2\|z\|} z$.

$$\Rightarrow \|\hat{z}\| = \frac{\delta}{2} < \delta \Rightarrow \|T\hat{z}\| < 1$$

$$\|Tz\| = \left\|T\left(\frac{2\|z\|}{\delta}\right)\hat{z}\right\| = \frac{2}{\delta}\|z\| \cdot \|T\hat{z}\| < \frac{2}{\delta}\|z\|$$

$$\Rightarrow \|T\| \leq \frac{2}{\delta} \Rightarrow T \text{ is bounded.}$$

\#

Problem 3: (a)

$f(x) = x(0.5)$ is unbound.

Suppose $f(x)$ is bounded, there must exist

a $M > 27$ such that for all $x \in C[0,1]$

$|f(x)| \le M \cdot \|x\|_1$.

Now let us build a continous function

$x_0 \in C[0,1]$ by the following:

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$$x_0(t) = \begin{cases} 0 & 0 \le t \le 0.5 - \frac{1}{M} \\ \sqrt{M}(t - 0.5 + \frac{1}{M}) & 0.5 - \frac{1}{M} < t \le 0.5 \\ \sqrt{M}(0.5 + \frac{1}{M} - t) & 0.5 < t \le 0.5 + \frac{1}{M} \\ 0 & 0.5 + \frac{1}{M} < t \le 1 \end{cases}$$

$$\|x_0(t)\| = \int_0^1 |x_0(t)|\,dt = \cancel{\frac{2}{\sqrt{M}}} \quad \frac{1}{M\sqrt{M}}$$

$$x_0(.5) = \frac{1}{\sqrt{M}}$$

Problem 3: (cont.)

(b) $|g(x)| = \left| \int_0^1 x(t) \sin(\pi t)\, dt \right|$

$$\leq \int_0^1 |x(t) \sin(\pi t)| \cdot dt$$

$$\leq \int_0^1 |x(t)| \cdot |\sin(\pi t)|\, dt$$

$$\leq \int_0^1 |x(t)|\, dt = \|x\|_1 \qquad \text{since} \quad |\sin(\pi t)| \leq 1$$

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$\Rightarrow \|g\| \leq 1 \quad \Rightarrow \quad g$ is bounded.

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Problem 4:

Proof: $x \in \ell_3$

$$\|x\|_3 = \left(\sum_{i=1}^{\infty} |x_i|^3 \right)^{\frac{1}{3}}$$

Let $(x_n)_{n=1}^{\infty} \subset \ell_3$ be a Cauchy sequence in ℓ_3

we will show that (x_n) converges to a point $x \in \ell_3$. $(\forall \varepsilon > 0)$ $(\exists N \in \mathbb{N})$ such that

$$\|x_n - x_m\|_3 < \varepsilon \qquad \forall \quad n, m \geq N.$$

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Now consider a sequence $(x_n^i)_{n=1}^{\infty} \subset \mathbb{F}$ for any fixed $i \in \mathbb{N}$. We have

$$\Longrightarrow \qquad |x_n^i - x_m^i|^3 \leq \|x_n - x_m\|_3 < \varepsilon$$

$\Rightarrow$ (x_n^i) is Cauchy in $\mathbb{F}$. for any fixed $i \in \mathbb{N}$. But $\mathbb{F}$ is complete $\Rightarrow$ $(\exists\ x^i \in \mathbb{F})$

such that $x_n^i \longrightarrow x^i$ as $n \to \infty$.

Now, let $x = (x^1, x^2, \ldots)$

we show that $x \in \ell_3$ and

$$\|x_n - x\|_3 \longrightarrow 0 \quad \text{as} \quad n \to \infty \ \ldots$$

Problem 4: (cont.)

Now for $k \in \mathbb{N}$, we have.

$$\sum_{i=1}^{k} |x^i - x_n^i|^3 \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

No!

This doesn't follow.

$$\Longrightarrow \sum_{i=1}^{\infty} |x^i - x_n^i|^3 = \|x - x_n^i\|_3^3 \to 0$$

as $n \to \infty$

Also fixed n.

$$\|x\|_3 = \|x - x_n + x_n\|_3$$

$$\leq \|x - x_n\|_3 + \|x_n\|_3 < \infty$$

$\Longrightarrow x \in l_3$.

#

Problem 5 :

Proof: $\lim_{n\to\infty} \|x_n\| = \delta \Rightarrow \lim_{n\to\infty} \|x_n\|^2 = \delta^2 \Rightarrow$

$(\forall \varepsilon > 0)\ (\exists N \in \mathbb{N})$ — You cannot subtract a number from a vector.

$$\cancel{\|x_n - \delta\|^2} < \varepsilon/4 \qquad \forall\ n > N$$

$$\Rightarrow \quad \|x_n\|^2 < \delta^2 + \varepsilon/4 \quad \text{o.k.} \quad \forall\ n > N$$

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Using parallelogram law, we have

$$\|x_n - x_m\|^2 = 2(\|x_n\|^2 + \|x_m\|^2) - \|x_n + x_m\|^2$$

$$= 2(\|x_n\|^2 + \|x_m\|^2) - 4\|\tfrac{1}{2}x_n + \tfrac{1}{2}x_m\|^2$$

Since K is convex $\Rightarrow \frac{1}{2}x_n + \frac{1}{2}x_m \in K$ and hence $\|\frac{1}{2}x_m + \frac{1}{2}x_n\| \geq \delta$. Thus

$$\|x_n - x_m\|^2 < 2\left(2\delta^2 + \frac{\varepsilon}{2}\right) - 4\delta^2$$

$$= \varepsilon \qquad \forall\ m, n \geq N$$

Hence (x_n) is Cauchy #

Problem 6:

Proof: H is a Hilbert space and

$Y = \text{span}\{e_1, e_2, \cdots, e_n\}$ is closed.

Thus y is the best approximation to x from Y is equivalent to $x - y \perp Y$. Now consider any $z \in Y \Rightarrow z = \sum_{i=1}^{n} \langle z, e_i \rangle e_i$. and

$$\langle x - y, z \rangle = \left\langle x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k, \sum_{i=1}^{n} \langle z, e_i \rangle e_i \right\rangle$$

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$$= \left\langle x, \sum_{i=1}^{n} \langle z, e_i \rangle e_i \right\rangle - \left\langle \sum_{k=1}^{n} \langle x, e_k \rangle e_k, \sum_{i=1}^{n} \langle z, e_i \rangle e_i \right\rangle$$

$$= \sum_{i=1}^{n} \overline{\langle z, e_i \rangle} \langle x, e_i \rangle - \sum_{k=1}^{n} \langle x, e_k \rangle \left\langle e_k, \sum_{i=1}^{n} \langle z, e_i \rangle e_i \right\rangle$$

$$= \sum_{i=1}^{n} \overline{\langle z, e_i \rangle} \langle x, e_i \rangle - \sum_{k=1}^{n} \langle x, e_k \rangle \cdot \sum_{i=1}^{n} \overline{\langle z, e_i \rangle} \langle e_k, e_i \rangle$$

$$= \sum_{i=1}^{n} \overline{\langle z, e_i \rangle} \langle x, e_i \rangle - \sum_{k=1}^{n} \langle x, e_k \rangle \overline{\langle z, e_k \rangle}$$

$$= 0$$

due to $\langle e_k, e_i \rangle = \delta_{ki}$. Hence $x - y \perp z \Rightarrow x - y \perp Y$

$\Rightarrow y$ is the best approximation to x from Y. #

Math 502

Second Midterm Examination, Part I April 3, 1991

1. (20 points) State and prove the Riesz representation theorem for bounded linear functionals on a Hilbert space.

2. (15 points) Let H be a Hilbert space. Prove that for every $T \in B(H, H)$,

$$\|T^*T\| = \|T\|^2.$$

3. (a) (10 points) Prove that the polynomials are dense in $L_2(-1, 1)$. (Quote the appropriate theorems from Math 501 and use one simple inequality.)

   (b) (5 points) Prove that the polynomials in $L_2(-\infty, \infty)$ are not dense in $L_2(-\infty, \infty)$.

Math 502

Second Midterm Examination, Part II **April 5, 1991**

4. (10 points) Define the canonical map of a normed space into its second dual. Prove that the canonical map is a norm-preserving map. At what point in the proof is the Hahn-Banach theorem used? Define reflexivity.

5. (10 points) Let X be a normed space, and let $T \in B(X, X)$. Prove that $\mathcal{N}(T) = {}^{\perp}\mathcal{R}(T')$. At what point in the proof is the Hahn-Banach Theorem used?

6. (15 points) Let ℓ_∞ denote the vector space of bounded sequences of real numbers. Prove that there exists a linear functional f on ℓ_∞ such that $f(x) = \lim_{n\to\infty} x_n$ for all convergent sequences $x = (x_n)$, and $f(x) \leq \limsup_{n\to\infty} x_n$ for all $x = (x_n) \in \ell_\infty$.

7. (15 points) Let X be a vector space over the complex field, and let X_r denote the same vector space, viewed as a vector space over the real number field. State and prove the relationship between linear functionals on X and linear functionals on X_r.

Problem 1:

Riesz representation Theorem : Given a Hilbert space H. Then $\forall f \in H'$, $(\exists!\ z \in H)$ $f(x) = \langle x, z \rangle$ $\forall x \in H$. Moreover, $\|f\| = \|z\|$.

Proof: If $f = 0$, then pick $z = 0$. It is trivial to see the theorem. Now, Let $f \neq 0$. Since $\eta(f)$ is a closed subspace of H, we have

$$H = \eta(f) \oplus \eta(f)^{\perp}.$$

$f \neq 0$ implies that $\eta(f) \neq H$ and hence $(\exists\ 0 \neq z_0 \in \eta(f)^{\perp})$. Now, consider

$$v = f(x)\, z_0 - f(z_0)\, x$$

Obviously, $f(v) = f(x) f(z_0) - f(z_0) f(x) = 0$ implies $v \in \eta(f)$. Thus

$$0 = \langle v, z_0 \rangle = \langle f(x) z_0 - f(z_0) x , z_0 \rangle$$

$$= f(x) \langle z_0, z_0 \rangle - f(z_0) \langle x, z_0 \rangle.$$

$$\Rightarrow \quad f(x) = \frac{f(z_0)}{\langle z_0, z_0 \rangle} \langle x, z_0 \rangle \qquad \because z_0 \neq 0.$$

Problem 1: (cont.)

Now let $z = \dfrac{\overline{f(z_0)}}{\langle z_0, z_0\rangle}$, we have

$$f(x) = \langle x, z_0 \rangle \quad \forall\, x \in H.$$

② Uniqueness: Suppose $(\exists\, z_1, z_2 \in H)$

$$f(x) = \langle x, z_1\rangle = \langle x, z_2\rangle \quad \forall\, x \in H$$

$$\Rightarrow \quad \langle x, z_1 - z_2\rangle = 0 \quad \forall\, x \in H.$$

Let $x = z_1 - z_2 \Rightarrow \langle z_1 - z_2, z_1 - z_2\rangle = 0$

$$\Rightarrow \quad z_1 = z_2$$

③ $$|f(x)| = |\langle x, z\rangle| \le \|x\| \cdot \|z\| \quad \forall\, x \in H$$

$$\Rightarrow \quad \|f\| \le \|z\|.$$

But $$\frac{|f(z)|}{\|z\|} = \frac{|\langle z, z\rangle|}{\|z\|} = \frac{\|z\|^2}{\|z\|} = \|z\|.$$

$$\Rightarrow \quad \|f\| = \|z\|.$$ Q.E.D.

(Note for the above proof assumes $z \neq 0$. since we assume $f \neq 0$).

Problem 2:

Proof: First we note that $\|T\| = \|T^*\|$ †.

① $\|T^*T\| \leq \|T^*\| \cdot \|T\| = \|T\|^2$.

② Consider

$$
\begin{aligned}
\|Tx\|^2 &= |\langle Tx, Tx\rangle| \\
&= |\langle x, T^*Tx\rangle| \\
&\leq \|x\| \cdot \|T^*Tx\| \\
&\leq \|x\|^2 \cdot \|T^*T\|
\end{aligned}
$$

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$$\Rightarrow \frac{\|Tx\|^2}{\|x\|^2} \leq \|T^*T\| \qquad \forall\, x \neq 0,\ x \in H.$$

$$\Rightarrow \|T\|^2 \leq \|T^*T\|$$

Combine ① and ②, we have $\|T^*T\| = \|T\|^2$.

† This equality is a consequence of Riesz representation theorem for sesquilinear forms. I think we don't have to prove it here.

Problem 3: (a)

Let us denote the polynomials in $L_2(-1,1)$ by $P(-1,1)$. It was proved in Math 501 last semester that

① $P(-1,1)$ is dense in $C(-1,1)$, and in $\|\cdot\|_\infty$

② $C(-1,1)$ is dense in $L_2(-1,1)$. in $\|\cdot\|_2$

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Now, for $x \in L_2(-1,1)$, there exist a sequence in $C(-1,1)$, $\{c_n\}$, which converges to x. More precisely, $(\forall\, \varepsilon>0)$ $(\exists N \in \mathbb{N})$

$$\|x - c_n\|_{L_2} < \varepsilon/2. \quad \forall\, n \geq N$$

Now fix n. $P(-1,1)$ dense in $C(-1,1)$, in $\|\cdot\|_\infty$ implies that $(\exists\, P_k \in P(-1,1)$ that converges to c_n. That is, $(\forall\, \varepsilon>0)$

$(\exists\, K \in \mathbb{N})$ $\|c_n - P_k\|_{L_2} < \frac{\varepsilon}{2}$ $\forall\, k \geq K$.

$$\Rightarrow \|x - P_k\|_{L_2} = \|x - c_n + c_n - P_k\|_{L_2} \leq \|x - c_n\| + \|c_n - P_k\|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, k \geq K.$$

Hence $P(-1,1)$ is dense in $L_2(-1,1)$. Q.E.D

Problem 3: (b)

Polynomials in $L_2(-\infty,\infty)$ are not dense in $L_2(-\infty,\infty)$ since there does not exists a polynomial in $L_2(-\infty,\infty)$ that vanishes nowhere.

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Moreover, the only polynomial in $L_2(-\infty,\infty)$ is $p=0$. which cannot be dense in $L_2(-\infty,\infty)$ obviously.

Problem 4:

Let X = normed space.

The canonical map of X into X'' is defined as follows:

$$C : X \longrightarrow X''$$

$$Cx \longrightarrow g_x \qquad \forall x \in X$$

where $g_x(f) = f(x)$. $\forall f \in X'$.

$$\|g_x\| = \sup_{\substack{f \neq 0 \\ f \in X'}} \frac{\|g_x(f)\|}{\|f\|} = \sup_{\substack{f \neq 0 \\ f \in X'}} \frac{|f(x)|}{\|f\|} = \|x\|$$

$\Rightarrow$ C is norm-preserving map.

$\otimes$ ~~Nowhere~~ in the proof of Hahn-Banach theorem, we use canonical mapping.

$\otimes$ X is said to be reflexivable if $R(C) = X''$.

Problem 5:

Proof:

Recall the definition of $^{\perp}S'$, we can write.

$$
\begin{aligned}
{}^{\perp}R(T') &= \{ x \in X \mid \langle x, f \rangle = 0 \quad \forall f \in R(T') \} \\
&= \{ x \in X \mid \langle x, T'f \rangle = 0 \quad \forall f \in X' \} \\
&= \{ x \in X \mid \langle Tx, f \rangle = 0 \quad \forall f \in X' \} \\
&\quad \Updownarrow \\
&= \{ x \in X \mid Tx = 0 \} \\
&= N(T) \qquad \#
\end{aligned}
$$

Hahn-Banach !

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Again, nowhere in the proof of Hahn-Banach theorem, we use the above result: In fact, Hahn-Banach theorem I, II deal with vector space . .

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Problem 6:

Let $c_0 = \{ x \mid x = (x_n) \text{ is a convergent sequence} \}$

It was proved in a homework problem that c_0 is a subspace of l_∞. Now, define a functional f in c_0 by

$$f(x) = \lim_{n\to\infty} x_n < \infty .$$

obviously $\forall\, \alpha, \beta \in R,\ x, y \in c_0,$

$$f(\alpha x + \beta y) = \lim_{n\to\infty} (\alpha x + \beta y)$$

$$= \alpha \lim_{n\to\infty} x + \beta \lim_{n\to\infty} y$$

$$= \alpha f(x) + \beta f(y)$$

$\Rightarrow f$ is linear.

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Problem 6: (cont.)

Let us defined a sublinear functional p by

$$p(x) = \limsup_{n\to\infty} \cancel{|x_n|} \; x_n$$

Obviously

$$p(x+y) = \limsup_{n\to\infty} |x_n + y_n| \le \limsup_{n\to\infty} |x_n| + \limsup_{n\to\infty} |y_n| = p(x) + p(y)$$

$\forall\, \alpha > 0$

$$p(\alpha x) = \limsup_{n\to\infty} |\alpha x_n| = \alpha \limsup_{n\to\infty} |x_n| = \alpha p(x)$$

$\Rightarrow$ p is a sublinear.

Moreover,

$$f(z) = \lim_{n\to\infty} x_n \le \limsup_{n\to\infty} |x_n| = p(z)$$

$\forall\, z \in c_0$.

Hence, all the hypothesis of Hahn-Banach theorem I are satisfied. $\Rightarrow$

$\exists$ an extension $\hat{f}$ of f such that the properties stated in the problem is true. #

Problem 7:

Let g be a linear functional on X. then g can be expressed as

$$g(x) = g_1(x) - i g_1(ix)$$

where g_1 is a linear functional on X_r.

Conversely if g_1 is a linear functional on X_r. Then

$$g(x) = g_1(x) - i g_1(ix)$$

is a linear functional on X.

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Proof: Let $g(x) = g_1(x) + i g_2(x)$ (*)

where g_1, g_2 are real-valued. $\Rightarrow$

$$i g(x) = g(ix) = g_1(ix) + i g_2(ix)$$

$$\Rightarrow \quad g(x) = g_2(ix) - i g_1(ix) \qquad (**)$$

Problem 7: (cont.)

Compared (*) and (**), we have

$$g_2(x) = -g_1(ix)$$

$$\Rightarrow \quad g(x) = g_1(x) - i g_1(ix).$$

$\forall\ \alpha, \beta \in \mathbb{R},\ x, y \in X_r = X$

$$\begin{aligned} g(\alpha x + \beta y) &= \alpha g(x) + \beta g(y) \\ &= \alpha g_1(x) - \alpha i g_1(ix) \\ &\quad + \beta g_1(y) - \beta i g_1(iy) \\ &= g_1(\alpha x + \beta y) - i g_1(i\alpha x + i\beta y). \end{aligned}$$

$$\Rightarrow \quad g_1(\alpha x + \beta y) = \alpha g_1(x) + \beta g_1(y)$$

$\Rightarrow$ g_1 is a linear functional on X_r.

Problem 7: (cont.)

Suppose that g_1 is a linear functional on X_r. we have. $\forall\, x, y \in X = X_r$

$$g(x+y) = g_1(x+y) - i\, g_1(ix+iy)$$
$$= g_1(x) + g_1(y) - i g_1(ix) - i g_1(iy)$$
$$= (g_1(x) - i g_1(ix)) + (g_1(y) - i g_1(iy))$$
$$= g(x) + g(y)$$

$\forall\, \alpha = \alpha_1 + i\alpha_2 \in \mathbb{C}$, $\forall\, x \in X = X_r$

$$g(\alpha x) = g(\alpha_1 x + i\alpha_2 x)$$
$$= g_1(\alpha_1 x + i\alpha_2 x) - i\, g_1(-\alpha_2 x + i\alpha_1 x)$$
$$= \alpha_1 g_1(x) + \alpha_2 g_1(ix) + \alpha_2 i g_1(x) - \alpha_1 i g_1(ix)$$
$$= (\alpha_1 + i\alpha_2)(g_1(x) - i g_1(ix))$$
$$= \alpha\, g(x)$$

$\Longrightarrow$ g is linear functional on X #