§3.1 Metric Space

- X = nonempty set, metric on X is a function

$$\rho : X \times X \longrightarrow \mathbb{R}$$

such that

$$\begin{cases} \text{(i)} & \rho(x,x) = 0 \quad \forall\ x \in X \\ \text{(ii)} & \rho(x,y) > 0 \quad \text{if } x \neq y \\ \text{(iii)} & \rho(x,y) = \rho(y,x) \quad \forall x, y \in X \\ \text{(iv)} & \rho(x,z) \leq \rho(x,y) + \rho(y,z) \quad \text{(triangular inequality)} \end{cases}$$

- (X, ρ) is a metric space, $a \in X$, $r > 0$, A ball

$$B_r(a) = \{ x \in X \mid \rho(x,a) < r \}$$

- $A \subseteq X$, A is open if $\forall\, a \in A\ (\exists\, r > 0)\ B_r(a) \subseteq A$.

▲ Every Ball is Open

▲ (X, ρ) is metric space. Then

(i) $\emptyset$ and X are open.

(ii) $\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n$ are open $\Rightarrow \bigcap_{i=1}^{n} \mathcal{U}_i$ is open.

(iii) $\mathcal{U}$ is a family of open sets $\Rightarrow \bigcup \mathcal{U}$ is open.

- $A \subseteq X$, diameter $\operatorname{diam}(A) = \sup\{\rho(x,y) \mid x, y \in A\}$

If $\operatorname{diam}(A) < \infty$, A is bounded.

If $\operatorname{diam}(A) = \infty$, A is unbounded.

- $A \subseteq X$, $x \in X$, $\operatorname{dist}(x, A) = \inf\{\rho(x,y) \mid y \in A\}$.

▲ $A \subseteq X$, $x, y \in X \Rightarrow |\text{dist}(x, A) - \text{dist}(y, A)| \leq \rho(x, y)$.

§3.2 Topological Spaces

X = nonempty set

● A topology is a set $\mathcal{T}$ of subsets of X such that

(i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

(ii) $\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n \in \mathcal{T} \Rightarrow \bigcap_{i=1}^{n} \mathcal{U}_i \in \mathcal{T}$.

(iii) $\mathfrak{U} \subseteq \mathcal{T} \Rightarrow \bigcup \mathfrak{U} \in \mathcal{T}$.

[According to the text book, p. 95, members of $\mathcal{T}$ are open]

● $(X, \mathcal{T})$ = topological space, $a \in X$, A neighborhold of a is a open set that contains a.

● $D \subseteq X$, $a \in D$, a is an interior point of D if a has a neighborhold that lies in D. $[(\exists\, \mathcal{U} \in \mathcal{T})\ a \in \mathcal{U} \subseteq D]$.

● Interior of $D = D^\circ = \{a \in D \mid a \text{ is an interior point of } D\}$

▲ D° is an open set

▲ $\mathcal{U}$ is open iff $\forall x \in \mathcal{U}$ ($\exists$ nbhd G of x) $G \subseteq \mathcal{U}$.

▲ $\mathcal{U} \subseteq D$, $\mathcal{U}$ is open $\Rightarrow \mathcal{U} \subseteq D^\circ$.

"D° is the largest open subset of D."

▲ $D^\circ = \bigcup \{\mathcal{U} \mid \mathcal{U} \text{ is open}, \mathcal{U} \subseteq D\}$

▲ D is open iff $D = D^\circ$.

● D is closed if D' is open where $D' = X \setminus D$.

▲ $(X, \mathcal{T})$ = topological space

(i) $\emptyset$ and X are closed.

(ii) $F_1, F_2, \cdots, F_n$ closed $\Rightarrow \bigcup_{i=1}^{n} F_i$ closed.

(iii) $\mathcal{F}$ family of closed sets $\Rightarrow \bigcap \mathcal{F}$ closed.

▲ D° is open

▲ $U \subseteq D$, U open $\Rightarrow U \subseteq D^\circ$

"D° is the largest open set contained in D"

$D^\circ = \bigcup \{ U \mid U \subseteq D, \ U \text{ is open} \}$

▲ D is open iff $D = D^\circ$

● $D \subseteq X$, $a \in X$, a is a limit point of D iff every nbhd of a contains a point of D other than a.

▲ D is closed iff D contains its limit points.

● Closure $\bar{D} = D \cup \{ a \in X \mid a \text{ is a limit point of } D \}$

▲ $\bar{D}$ is closed.

▲ $\bar{D} = \{ a \in X \mid (\forall U \in \mathcal{T})(a \in U \Rightarrow \exists b \in D \cap U) \}$

▲ $D \subseteq F$, F is closed $\Longrightarrow \bar{D} \subseteq F$

$\bar{D}$ is the smallest closed set containing D

$\bar{D} = \bigcap \{ F \mid F \supseteq D, F \text{ is closed} \}$

▲ D is closed iff $D = \bar{D}$.

▲ $(\bar{D})' = (D')^\circ$

● Boundary of $D = \partial D = \bar{D} \cap (\bar{D'})$

▲ ∂D is a closed set and $\partial D = \bar{D} \setminus D^\circ$

● <u>Hausdorff space</u>

$$\begin{cases} \text{A Hausdorff space is topological space such } (\forall a, b \in X) \\ \text{if } a \neq b, \text{ then } \exists \text{ open } \mathcal{U}_a \text{ and } \mathcal{U}_b \text{ such that} \\ a \in \mathcal{U}_a, \quad b \in \mathcal{U}_b \quad \text{and} \quad \mathcal{U}_a \cap \mathcal{U}_b = \phi. \end{cases}$$

▲ Every metric space is a Hausdorff space.

▲ X is a Hausdorff space $\Rightarrow$ finite subsets are closed.

▲ X is a Hausdorff space, $D \subseteq X$, $a \in X$ and a is a limit point of D $\Rightarrow$ $\forall$ open $\mathcal{U}$ containing a, $\mathcal{U} \cap D$ has infinitely many points.

● (X_n) converges to X if every nbhd of X contains a tail of the sequence :

$(\forall \text{ open } \mathcal{U})(x \in \mathcal{U})(\exists N) \Rightarrow (n \geq N \Rightarrow x_n \in \mathcal{U})$

▲ In a Hausdorff space, a sequence can have at most one limit.

▲ (i) In a topoligical space if $(X_n) \subseteq D$ and $X_n \rightarrow X$, $\Rightarrow x \in \bar{D}$.

(ii) In a metric space, if $X \in \bar{D}$, $\Rightarrow (\exists (X_n) \subseteq D)\ X_n \rightarrow X$.

▲ <u>Corollary</u>: In a metric space, $X \in \bar{D}$ iff $(\exists (x_n) \subseteq D)\ x_n \rightarrow x$.

● Subspaces

$(X, \mathcal{T})$ = topological space

$S \subseteq X$, Relative topology on S

$\mathcal{T}_S = \{ G \cap S \mid G \in \mathcal{T} \} \Rightarrow (S, \mathcal{T}_S)$ is a topological space.

(X, ρ) = metric space

$S \subseteq X$, $\rho_S = \rho|_{S \times S}$

(S, ρ_S) is a metric space.

▲ The topology on S reduced by the metric is the same as the relative topology.

● Compactness

$(X, \mathcal{T})$ = topological space, $K \subseteq X$.

An open cover is a family $\mathcal{U} \subseteq \mathcal{T}$ such that $K \subseteq \bigcup \mathcal{U}$.

● K is compact. if every open cover has a finite subcover.

▲ X is compact, $K \subseteq X$, K is closed, Then K is compact.

▲ X is Hausdorff, $K \subseteq X$, K is compact, Then K is closed.

▲ X is metric space, $K \subseteq X$, K is compact, Then K is bounded.

● X is a metric space, $K \subseteq X$, K is said to be totally bounded if $\forall \varepsilon > 0$, K can be covered by a finite number of balls of radii ε.

▲ X = metric space, $K \subseteq X$, K compact, Then K is totally bounded. Totally bounded implies bounded.

▲ $K \subseteq \mathbb{R}^n$, K is bounded implies K is totally bounded.

● <u>Box in $\mathbb{R}^n$</u>

$$\mathbb{B} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \,\middle|\, a_i \le x_i \le b_i,\ i = 1, \cdots, n \right\}$$

▲ Let $\mathbb{B}_1 \supseteq \mathbb{B}_2 \supseteq \mathbb{B}_3 \cdots$ be a sequence of boxes in $\mathbb{R}^n$ such that $\operatorname{diam}(\mathbb{B}_k) \to 0$ as $k \to \infty$. Then

$$\bigcap_{k=1}^{\infty} \mathbb{B}_k = \{x\} \quad (\exists\ x \in \mathbb{R}^n)$$

▲ <u>Heine – Borel Theorem:</u> $F \subseteq \mathbb{R}^n$ is compact iff F is closed & bounded.

▲ X = metric space. Then the following statements are equivalent $(K \subseteq X)$

(i) K is compact.

(ii) Every infinite subset of K has a limit point in K.

(iii) Every sequence in K has a subsequence that converges to a point in K.

● <u>Connectness</u>

$(X, \mathcal{T})$ = topological space

(i) X is disconnected if $(\exists\ \mathcal{U}, V \in \mathcal{T})$ $\mathcal{U} \neq 0$, $V \neq 0$, $\mathcal{U} \cap V = \emptyset$ such that $\mathcal{U} \cup V = X$.

(ii) X is connected if it is not disconnected.

▲ X is connected if its only subsets that are both open and closed are X and $\emptyset$.

● $S \subseteq X$ is connected if $(S, \mathcal{T}_S)$ is a connected space.

▲ $S \subseteq X$ is a disconnected iff $(\exists\ \mathcal{U}, V \in \mathcal{T})$ such that $\mathcal{U} \cap S \neq \emptyset$, $V \cap S \neq \emptyset$, $\mathcal{U} \cap V \cap S = \emptyset$ and $S \subseteq \mathcal{U} \cap V$.

▲ $S \subseteq \mathbb{R}$ is connected iff S is an interval. In particular, $\mathbb{R}$ is connected.

● Completeness: (X, ρ) = metric space

- ● A cauchy sequence in X is a sequence such that $(\forall \varepsilon > 0)(\exists N)(\forall m, n \geq N)\ \rho(x_m, x_n) < \varepsilon$
- ▲ Convergent implies Cauchy but Cauchy does not imply convergent.

(X, ρ) is complete if every Cauchy sequence in X converges to a point in X. ($\mathbb{R}, \mathbb{R}^n, \mathbb{C}, \mathbb{C}^n$ are complete)

▲ X completes, $K \subseteq X$, K is complete iff K is closed.

▲ Every compact metric space is complete.

▲ In metric space, $K \subseteq X$ and K compact $\Leftrightarrow$ K complete and totally bounded.

● S = nonempty set, $B(S)$ = bounded complex-valued functions on S

$f \in B(S)$ iff $\sup\{|f(s)| \mid s \in S\} < \infty$

$\rho(f, g) = \sup\{|f(s) - g(s)| \mid s \in S\}$

▲ $B(S)$ is complete metric space

● A is dense in X if $\overline{A} = X$

A is nowhere dense in X if $(\overline{A})^\circ = \emptyset$

A is of first category in X if A is a countable union of nowhere dense sets.

A is of second category in X if it is not of 1st category.

A set $E \subseteq X$ is called residual in X if E' is of 1st category.

▲ Baire Category Theorem: X is a complete metric space, $A \subseteq X$, A is of 1st category $\Rightarrow$ A' is dense in X.

* Every complete metric space is of 2nd category in itself.

▲ X is complete metric space. Then every countable intersection of open dense sets is dense.

● <u>Limits and Continuity</u>

X and Y are topological spaces, $f: X \to Y$

Let $S \subseteq X$, $a \in X$, a is a limit point of S

$\lim\limits_{\substack{x \to a \\ x \in S}} f(x) = b$, $b \in Y$ means

$\forall$ nbhd V of b ($\exists$ nbhd $\mathcal{U}$ of a) $\Rightarrow \forall x \in \mathcal{U} \cap S \setminus \{a\}$, $f(x) \in V$.

▲ If Y is a Hausdorff space

$$\left.\begin{array}{l} f(x) \longrightarrow b \quad \text{as} \quad x \longrightarrow a \quad (x \in S) \\ f(x) \longrightarrow c \quad \text{as} \quad x \longrightarrow a \quad (x \in S) \end{array}\right\} \Rightarrow b = c.$$

● X and Y are topological spaces, $f: X \to Y$

$p \in X$. Then f is continous at p if

$\forall$ nbhd V of $f(p)$ ($\exists$ nbhd $\mathcal{U}$ of p) $f(\mathcal{U}) \subseteq V$, where

$f(\mathcal{U}) = \{f(x) \mid x \in \mathcal{U}\}$.

* f is continous on X if ($\forall p \in X$) f is continous at p.

▲ f is continous at p iff

(i) p is an isolated point of X

(ii) $\lim\limits_{x \to p} f(x) = f(p)$.

▲ $f: X \to Y$. The following are equivalent

$S \subseteq Y$ $f^{-1}(S) = \{x \in X \mid f(x) \in S\}$

(i) f is continous on X

(ii) V open in $Y \Rightarrow f^{-1}(V)$ open in X.

(iii) B closed in $Y \Rightarrow f^{-1}(B)$ is closed in X.

▲ (X,ρ), (Y,σ) = metric spaces, $f: X \to Y$, the following are equivalent.

(i) f is continous at p.

(ii) $(\forall \varepsilon>0)(\exists \delta>0)(\rho(x,\delta)<\delta \Rightarrow \sigma(f(x), f(p))<\varepsilon)$

(iii) $(\forall (x_n)\subseteq X)\ (x_n \to p \Rightarrow f(x_n)\to f(p))$.

● <u>Uniform Continuity</u>

(X,ρ) and (Y,σ) are metric spaces

$f: X \longrightarrow Y$.

f is uniformly continuous if $(\forall \varepsilon>0)(\exists \delta>0)(\forall x,y\in X)$ $(\rho(x,y)<\delta \Rightarrow \sigma(f(x),f(y))<\varepsilon)$.

▲ X is compact, $f: X \to Y$ is continous $\Rightarrow f$ is uniformly cts.

▲ Compactness, Connectedness and Continuity

X, Y = topological spaces, $f: X \to Y$ is cts on X.

$A \subseteq X$. Then

(i) A is compact $\Rightarrow$ $f(A)$ is compact.

(ii) A is connected $\Rightarrow$ $f(A)$ is connected.

▲ $f: X \to \mathbb{R}$ is cts. and X is compact. $(\exists\ u, v\in X)$ such that $f(u) = \sup\limits_{x\in X} f(x)$ and $f(v) = \inf\limits_{x\in X} f(x)$.

▲ $f: X \to \mathbb{R}$ is cts and X is connected, then $f(x)$ is an interval.

▲ $[a,b]\subseteq \mathbb{R}$, $f:[a,b]\to\mathbb{R}$ and cts, then $(\exists\ c, d\in\mathbb{R})$ such that $f([a,b]) = [c,d]$.

▲ **Intermediate Value Theorem**

$f: [a,b] \to \mathbb{R}$ cts. If y_0 is between $f(a)$ and $f(b)$ then

$(\exists x_0 \in [a,b])\ f(x_0) = y_0$

▲ X = compact, Y = Hausdorff

$f: X \to Y$ is continous and one-to-one, then

$f^{-1}: f(X) \to X$ is continous.

● **Uniform Convergence**

X = nonempty set

$f: X \to \mathbb{C}$

- $f_n \to f$ pointwise if $\forall \varepsilon > 0\ (\forall x \in X)(\exists N)(\forall n \geq N)\ |f_n(x) - f(x)| < \varepsilon$
- $f_n \to f$ uniformly if $\forall \varepsilon > 0\ (\exists N)(\forall x \in X)(\forall n \geq N)\ |f_n(x) - f(x)| < \varepsilon$
- (f_n) is pointwise Cauchy if $(\forall \varepsilon > 0)(\forall x \in X)(\exists N)(\forall m, n \geq N)$ $|f_m(x) - f_n(x)| < \varepsilon$.
- (f_n) is uniformly Cauchy if $(\forall \varepsilon > 0)(\exists N)(\forall x \in X)(\forall m, n \geq N)$ $|f_m(x) - f_n(x)| < \varepsilon$.

▲ In $\mathbb{C}$.

$$\left\{\begin{matrix}\text{pointwise} \\ \text{uniformly}\end{matrix}\right\} \text{convergent} \iff \left\{\begin{matrix}\text{pointwise} \\ \text{uniformly}\end{matrix}\right\} \text{Cauchy}.$$

● **Series of Functions**

$$\sum_{k=1}^{\infty} f_k(x), \quad S_n(x) = \sum_{k=1}^{n} f_k(x)$$

$\sum_{k=1}^{\infty} f_k(x)$ converges $\frac{\text{pointwise}}{\text{uniformly}}$ if $\{S_n(x)\}$ converges $\frac{\text{pointwise}}{\text{uniformly}}$.

▲ <u>Weierstrass M-test</u>

Suppose $(\forall k)(\exists M_k \geq 0)\ |f_k(x)| \leq M_k\ (\forall x \in X)$

and $\sum_{k=1}^{\infty} M_k < \infty$. Then $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly (absolute).

▲ X = topological space, $\{f_n\}$ $f_n : X \to \mathbb{C}$ is cts.

$f_n \longrightarrow f$ uniformly $\Rightarrow$ f is cts.

▲ <u>Dini's Theorem:</u>

X = compact space

Suppose $\{f_n\}$ is a non decreasing sequence of continous functions, i.e.,

$\{f_1(x) \leq f_2(x) \leq \cdots \ \forall x \in X\}$ and $f_n \to f$ pointwise and

f is continous. Then $f_n \to f$ uniformly.

● Spaces $C(X)$ and $C^r(X)$, X = topological space

$B(X)$ = bounded functions $f: X \longrightarrow \mathbb{C}$

$$\rho(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

$f \in B(X)$ iff $\sup\{|f(x)| \mid x \in X\} < \infty$.

$C(X) = \{ f \in B(X) \mid f$ is continuous on $X \}$

= "bounded and continuous functions on X"

$C^r(X) = \{ f \in C(X) \mid f: X \longrightarrow \mathbb{R} \}$.

▲ $C(X)$ and $C^r(X)$ are complete.

● Algebraic Structure:

$f, g \in C^{(r)}(X)$, $(f+g)(x) = f(x) + g(x) \in C^{(r)}(X)$

$(fg)(x) = f(x) g(x) \in C^{(r)}(X)$

$\alpha \in \mathbb{C}$ $(\alpha f)(x) = \alpha f(x) \in C(X)$.

$C(X)$ is an algebra over complex $\mathbb{C}$

$C^r(X)$ is an algebra over $\mathbb{R}$.

● Subalgebra: A subalgebra of $C(X)$ is a subset A that is closed under addition, multiplication and scalar multiplication.

● Special case: X = compact space,

$C(X)$ = all continuous functions on X.

$X = [a, b]$ $C[a, b]$ = continuous functions on $[a, b]$

$P[a, b]$ = polynomials with complex coefficients

$P^r[a, b]$ = polynomials with real coefficients

$P(X)$ is subalgebra of $C(X)$ for any compact X.

▲ <u>Classical Weierstrass Theorem</u>

$P^r[a,b]$ is dense in $C^r[a,b]$.

$P[a,b]$ is dense in $C[a,b]$.

● Let A be a subalgebra of $C(X)$ or $C^r(X)$

A separates points if $(x, y \in X)\ x \neq y$, then $(\exists f \in A)\ f(x) \neq f(y)$.

A vanishes nowhere if $(\forall x \in X)(\exists f \in A)\ f(x) \neq 0$.

▲ <u>Stone–Weierstrass Theorem I</u>

$X =$ compact $\quad A \subseteq C^r(X)$ is a real subalgebra that separates points and vanishes nowhere. Then $\bar{A} = C^r(X)$.

● A is self-adjoint if $f \in A$ implies that $\bar{f} \in A$ (complex conjugate).

▲ <u>Stone–Weierstrass Theorem II</u>

$X =$ compact $\quad A \subseteq C(X)$ is a complex subalgebra that separates points and vanishes nowhere, and is self-adjoint. Then $\bar{A} = C(X)$.

● f and g are real valued functions

upper envelope (join) $= f \vee g(x) = \max\{f(x), g(x)\}$

lower envelope (meet) $= f \wedge g(x) = \min\{f(x), g(x)\}$.

▲ $f \vee g = (f + g + |f - g|)/2 \qquad f \wedge g = (f + g - |f - g|)/2$.

▲ $A \subseteq C^r(X)$ real subalgebra. Then $\bar{A}$ is a subalgebra of $C^r(X)$. Furthermore $f, g \in \bar{A} \Rightarrow |f|, f \vee g, f \wedge g \in \bar{A}$.

▲ $A \subseteq C^r(X)$ is real subalgebra that separates points and vanishes nowhere. Then $(\forall, y, z \in X, y \neq z)(\forall a, b \in \mathbb{R})(\exists f \in A)$ $f(y) = a,\ f(z) = b$.

▲ Stone–Weierstrass (Compact real case)

X is compact, $A \subseteq C^r(X)$ is a real algebra that separates points and vanishes nowhere. Then $\bar{A} = C^r(X)$.

WW 1) If A separates points, then it can vanish at most one point.

2) Let $p \in X$, $C_p = \{f \in C^r(X) \mid f(p) = 0\}$, then C_p is a closed subalgebra of $C^r(X)$. Consequently, if A vanishes at p, then $A \subseteq C_p$, so $\bar{A} \subseteq C_p$.

▲ X = compact, $A \subseteq C^r(X)$ is a real subalgebra that separates points, If A vanishes at p, then $\bar{A} = \{f \in C^r(X) \mid f(p) = 0\}$.

▲ Stone–Weierstrass Theorem (Compact–Complex)

X = compact, $A \subseteq C(X)$ is a complex, self-adjoint subalgebra that separates points: Then

(i) if A vanishes nowhere, $\bar{A} = C(X)$

(ii) if A vanishes at p, $\bar{A} = \{f \in C(X) \mid f(p) = 0\}$.

- **Locally Compact**

  A Hausdorff space X is locally compact if every $x \in X$ has a neighborhood whose closure is compact.

- A locally compact Hausdorff space X, $f \in C(X)$ and f vanishes at infinity if $(\forall \varepsilon > 0)\ \{x \in X \mid |f(x)| \geq \varepsilon\}$ is compact.

  $C_0(X) = \{f \in C(X) \mid f \text{ vanishes at } \infty\}$

  $C_0^r(X) = \{f \in C_0(X) \mid f(X) \subseteq \mathbb{R}\}$

  [$C_0(X)$ and $C_0^r(X)$ are subalgebras of $C(X)$ and $C^r(X)$]

▲ X = locally compact Hausdorff space. Let $A \subseteq C_0(X)$ (or $C_0^r(X)$) be a self-adjoint, subalgebra that saparates point and vanishes nowhere $\Rightarrow \overline{A} = C_0(X)$ (or $\overline{A} = C_0^r(X)$).

- **Equicontinuity**

  Let $\mathcal{F}$ be a set of functions mapping X into Y, $p \in X$

  $\mathcal{F}$ is equicontinuous at p if ($\forall$ nbhd V of $f(p)$) ($\exists$ nbhd $\mathcal{U}$ of p) $f(\mathcal{U}) \subseteq V$ $(\forall f \in \mathcal{F})$.

  $\mathcal{F}$ is equicontinuous on X if $\mathcal{F}$ is equcontinuous at each $p \in X$.

Special cases

$Y = \mathbb{C}$: $\mathcal{F}$ is equicontinuous at p if $(\forall \varepsilon > 0)$ ($\exists$ nbhd $\mathcal{U}$ of p) $\forall y \in \mathcal{U}$ $|f(y) - f(p)| < \varepsilon$ $(\forall f \in \mathcal{F})$

X = metric space: $\mathcal{F}$ is equicts at p if $(\forall \varepsilon > 0)(\exists \delta > 0)$ $\rho(x, p) < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$ $\forall f \in \mathcal{F}$.

● X = metric space

$\mathcal{F}$ is <u>uniformly equicontinuous</u> on X if

$(\forall \varepsilon>0)(\exists \delta>0)(\forall f\in\mathcal{F})(\forall x,y\in X)(\rho(x,y)<\delta \Rightarrow |f(x)-f(y)|<\varepsilon)$.

▲ $K\subseteq C(X)$, K totally bounded $\Rightarrow$ K is equicontinuous on X.

▲ X compact metric space $\Rightarrow$ X separable

▲ X separable, $\{f_n\}\subseteq C(X)$, $\{f_n\}$ bounded and equicontinuous. Then $\{f_n\}$ has a subsequence that converges pointwise on X. The convergence is uniform on compact subsets of X.

▲ <u>Arzelà – Ascoli Theorem</u>

X = compact metric space, $\{f_n\}\subseteq C(X)$, bounded, equicts, $\Rightarrow$ $\{f_n\}$ has a subsequence that converges uniformly to some $f\in C(X)$.

<u>This ends up all material covered by exam no. 1</u>

Lebesgue Integration

§ Lebesgue Outer Measure

- Let I be an interval in $\mathbb{R}^{\#}$ with end points a and b, $a \le b$. The length of I is $|I| = b - a$.

- $E \subseteq \mathbb{R}$, The Lebesgue Outer Measure of E is
$$\lambda(E) = \inf \left\{ \sum_{I \in \ell} |I| : \ell = \text{countable collection of open intervals such that } \cup \ell \supset E \right\}$$

▲ (i) $E \subseteq \mathbb{R} \Rightarrow 0 \le \lambda(E) \le \infty$

(ii) $\lambda(\emptyset) = 0$

(iii) $E \subseteq F \Rightarrow \lambda(E) \le \lambda(F)$

(iv) $\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} \lambda(E_n)$

(v) $C \subseteq \mathbb{R}$, C is countable $\Rightarrow \lambda(C) = 0$

(vi) I = interval $\Rightarrow \lambda(I) = |I|$

(vii) $E + x = \{ y + x \mid x \in E \} \Rightarrow \lambda(E + x) = \lambda(E)$

- If $\lambda(E) = 0$, E is called a <u>λ-null</u> set or a set <u>of measure zero</u>.

▲ (i) F is λ-null, $E \subseteq F \Rightarrow E$ is λ-null.

(ii) C is countable $\Rightarrow C$ is λ-null.

(iii) Countable unions of λ-null sets are λ-null.

(iv) F is λ-null $\Rightarrow \lambda(E \cup F) = \lambda(E)$.

§ Step Function

- X = nonempty set, $E \subseteq X$

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$ characteristic function.

- Step Functions: $\phi(x) = \sum_{j=1}^{n} \alpha_j \chi_{I_j}$, $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$,

  $I_1, I_2, \cdots, I_n$ are disjoint intervals $\subseteq \mathbb{R}$, $\mathbb{R} = \bigcup_{j=1}^{n} I_j$

  and $\alpha_j = 0$ if I_j is unbounded.

- M_0 = set of all step functions.

▲ $\left.\begin{array}{l} \phi, \psi \in M_0 \Rightarrow \phi + \psi \in M_0 \\ \phi \in M_0, \alpha \in \mathbb{R} \Rightarrow \alpha\phi \in M_0 \end{array}\right\} \Rightarrow M_0$ is a real vector subspace.

$\phi \in M_0 \Rightarrow |\phi| \in M_0 \quad \} \Rightarrow M_0$ is a lattice.

- $\phi^+(x) = \begin{cases} \phi(x), & \phi(x) \geq 0 \\ 0 & \phi(x) < 0 \end{cases} \Big\} \quad \phi^+ = \phi \vee 0$

  $\phi^-(x) = \begin{cases} -\phi(x) & \phi(x) \leq 0 \\ 0 & \phi(x) > 0 \end{cases} \Big\} \quad \phi^- = -\phi \vee 0 = -(\phi \wedge 0) \geq 0$

▲ $\phi \in M_0, \Rightarrow \phi^+, \phi^- \in M_0, \quad \phi = \phi^+ - \phi^-$.

- $\phi \in M_0, \quad \int \phi \triangleq \sum_{j=1}^{n} \alpha_j |I_j|$. ($\int$ is well defined).

▲ (a) $\int \phi + \psi = \int \phi + \int \psi$

(b) $\int \alpha \phi = \alpha \int \phi$

(c) $\phi \geq 0 \Rightarrow \int \phi \geq 0$

(d) $\phi \leq \psi \Rightarrow \int \phi \leq \int \psi$.

- Property $S(x)$, $x \in \mathbb{R}$, is true almost everywhere (a.e.) if $\{x \in \mathbb{R} \mid S(x) \text{ is false}\}$ has measure zero.

▲ (ϕ_n) sequence of function in M_0 such that

$$\phi_1 \le \phi_2 \le \phi_3 \le \cdots \qquad \text{(Lemma I)}$$

$$\lim_{n\to\infty} \int \phi_n < \infty \Rightarrow \lim_{n\to\infty} \phi_n(x) < \infty \quad \text{a.e.}$$

▲ (ϕ_n) is sequence in M_0, $\phi_n \ge 0$, $\forall n$

$$\phi_1 \ge \phi_2 \ge \phi_3 \ge \cdots \qquad \text{(Lemma II).}$$

If $\lim_{n\to\infty} \phi_n(x) = 0$ a.e. then $\lim_{n\to\infty} \int \phi_n = 0$.

▲ $(\phi_n), (\psi_m) \subseteq M_0$ nondecreasing sequences.

$\lim_{n\to\infty} \phi_n(x) \le \lim_{m\to\infty} \psi_m(x)$ a.e.

$\Rightarrow \lim_{n\to\infty} \int \phi_n \le \lim_{m\to\infty} \int \psi_m$.

▲ $(\phi_n), (\psi_m) \in M_0$, nondecreasing, $\lim_{n\to\infty} \phi_n(x) = \lim_{m\to\infty} \psi_m(x)$ a.e.

$\Rightarrow \lim_{n\to\infty} \int \phi_n = \lim_{m\to\infty} \int \psi_m$.

- M_1 = the set of functions f such that $(\exists (\phi_n) \subseteq M_0)$ with property
  - (i) $\phi_1 \le \phi_2 \le \phi_3 \le \cdots$
  - (ii) $\lim_{n\to\infty} \phi_n(x) = f(x)$ a.e.
  - (iii) $\lim_{n\to\infty} \int \phi_n < \infty$.

$$\int f = \lim_{n\to\infty} \int \phi_n .$$

▲ $f, g \in M_1$, $\alpha \ge 0$

(i) $f + g \in M_1$ (ii) $\alpha f \in M_1$ $(\alpha \ge 0)$

(iii) $f \vee g \in M_1$ (iv) $f \wedge g \in M_1$. (lattice).

▲ $f: \mathbb{R} \to [0,\infty]$, $C = \{x \in \mathbb{R} : f(x) = \infty$ or f is discts at $x\}$ suppose $\lambda(C) = 0$. Then $\exists (\phi_n) \subseteq M_0$, (ϕ_n) is nondecreasing and $\lim_{n\to\infty} \phi_n(x) = f(x)$ a.e.

<u>The Second Extension</u> — Intergrable Function

● The set of <u>integrable functions</u> $L_1^r = L_1^r(\mathbb{R})$ is the set of all f such that $(\exists g, h \in M_1)$ $f = g - h$ a.e. $\int f = \int g - \int h$.

▲ L_1^r is a real vector lattice and $\int$ is a positive (monotone) linear functional on L_1^r, i.e., $\forall f_1, f_2, f \in L_1^r$, $\alpha \in \mathbb{R}$

$f_1 + f_2 \in L_1^r \qquad \alpha f \in L_1^r$ } vector space.

$f_1 \vee f_2 \in L_1^r \qquad f_1 \wedge f_2 \in L_1^r$ } lattice

$\int f_1 + f_2 = \int f_1 + \int f_2$, $\quad \int \alpha f = \alpha \int f$

$f_1 \le f_2 \Rightarrow \int f_1 \le \int f_2$. $\quad |\int f| \le \int |f|$.

▲ $\forall f \in L_1^r$ $(\exists (\phi_n) \subseteq M_0)$ such that $\lim_{n\to\infty} \phi_n(x) = f(x)$. a.e. and $\lim_{n\to\infty} \int |f - \phi_n| = 0$.

▲ $f \in L_1^r$, $f \ge 0$, a.e. $\varepsilon > 0$. Then $(\exists g, h \in M_1)$ $g \ge 0, h \ge 0$ $f = g - h$ a.e and $\int h < \varepsilon$.

▲ $f \in L_1^r \Rightarrow |f(x)| < \infty$ a.e.

▲ <u>Monotone Converge Theorem for L_1^r</u>.

Let $(f_n) \subseteq L_1^r$ nondecreasing, $\lim_{n\to\infty} \int f_n < \infty$, $f(x) \triangleq \lim_{n\to\infty} f_n(x)$ a.e.

$\Rightarrow f \in L_1^r$ and $\int f = \lim_{n\to\infty} \int f_n$, $\int \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int f_n$.

▲ The MCT holds also for non increasing sequences, provided that $\lim_{n\to\infty} \int f_n > -\infty$.

▲ $f \in L_1^r$. Then $\int |f| = 0$ iff $f = 0$ a.e.

<u>Lebesgue's Dominated Convergence Theorem</u> (LDCT)

▲ $(f_n) \subseteq L_1^r$, $(\exists g \in L_1^r)$, $|f_n| \leq g$ a.e. $\forall n$.

Suppose further $\lim_{n\to\infty} f_n(x) = f(x)$ a.e.

$\Rightarrow f(x) \in L_1^r$ and $\int f = \lim_{n\to\infty} \int f_n$.

<u>Measurable Functions</u>

● f = extended real valued function defined a.e. on $\mathbb{R}$.

f is measurable if $(\exists \phi_n) \subseteq M_0$, $\lim_{n\to\infty} \phi_n(x) = f(x)$ a.e.

M^r = set of measurable functions.

▲ $(f_n) \subseteq M^r$ suppose $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. Then $f \in M^r$.

● $f \in M^r$, If $f \geq 0$ and $f \notin L^r$ define $\int f = \infty$.

Otherwise $f = f^+ - f^-$ $f_+ \geq 0$, $f_- \geq 0$

$\int f = \int f^+ - \int f^-$ provided that $\int f^+ < \infty$ or $\int f^- < \infty$

If $\int f^+ = \infty$ and $\int f^- = \infty$, $\Rightarrow \int f$ is not defined.

▲ $f, g \in M^r$, $f \leq g$ suppose $\int f$ and $\int g$ are defined, $\Rightarrow \int f \leq \int g$.

▲ Monotone Converge Theorem

$(f_n) \subseteq M^r$ nondecreasing, $f_1^- \in L_1^r \Rightarrow \int \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int f_n(x)$.

▲ $f \in M^r$, $\int f$ defined and $\alpha \in \mathbb{R} \Rightarrow \int \alpha f = \alpha \int f$

▲ $(f_n) \subseteq M^r$, $f_n \geq 0 \ \forall n$, $\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$.

▲ Fatou's Lemma: $(f_n) \subseteq M^r$, $f_n \geq 0$, $\forall n$

$$\Rightarrow \int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

● Complex-valued Functions

f = complex valued function defined a.e. on $\mathbb{R}$.

M = the measurable functions $= \{ f = \operatorname{Re} f + i \operatorname{Im} f \mid \operatorname{Re} f \in M^r, \operatorname{Im} f \in M^r \}$

Integral: $L_1 = \{ f \mid \operatorname{Re} f \in L_1^r \text{ and } \operatorname{Im} f \in L_1^r \}$

$$\int f = \int \operatorname{Re} f + i \int \operatorname{Im} f.$$

▲ • If $f \in M^r$ and $(|f(x)| \neq \infty \ \forall x) \Rightarrow f \in M$

• $L_1^r \subseteq L_1$ "almost".

• $f, g \in M$, $\alpha \in \mathbb{C} \Rightarrow f+g \in M$, $\alpha f \in M$, $fg \in M$

$f/g \in M$ provided $\lambda\{x \mid g(x) = 0\} = 0$

$\bar{f} = \operatorname{Re} f - i \operatorname{Im} f \in M$, $|f| \in M$.

• $(f_n) \subseteq M$, $\lim_{n\to\infty} f_n(x) = f(x) \Rightarrow f \in M$.

• $f, g \in L_1$, $\alpha \in \mathbb{C} \Rightarrow f+g \in L_1$, $\alpha f \in L_1$, $\bar{f} \in L_1$.

But $fg \notin L_1$ (may not true in general).

• $f \in L_1 \iff |f| \in L_1$ $(|f| < \infty)$. $\int \bar{f} = \overline{\int f}$.

▲ <u>Lebesgue's Dominated Convergence Theorem</u>

Let $(f_n) \subseteq L_1$. Suppose $f(x) = \lim_{n\to\infty} f_n(x)$, a.e. and $(\exists g \in L_1)$, $|f_n(x)| \le g(x)$ a.e. $\forall n$, Then

$$\int f = \lim_{n\to\infty} \int f_n .$$

▲ $(\mathcal{L}_1, \rho_1)$ is complete. That is if $(f_n) \subseteq L_1$ and (f_n) is Cauchy, then $(\exists f \in L_1)$ such that $\lim_{n\to\infty} \int |f_n - f| = 0$. Moreover, (f_n) has subsequence (f_{n_k}), $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ a.e.

● <u>Measurable Sets</u>

$E \subseteq \mathbb{R}$ is measurable if $\chi_E \in M$.

$\mathcal{M}$ = set of measurable subsets of $\mathbb{R}$.

▲ (i) $\mathbb{R} \in \mathcal{M}$, $\phi \in \mathcal{M}$

(ii) $E \in \mathcal{M} \Rightarrow E' \in \mathcal{M}$

(iii) $(E_n)_{n=1}^{\infty} \subseteq \mathcal{M} \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$ and

(iv) $\bigcap_{n=1}^{\infty} E_n \in \mathcal{M}$

(v) $A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$.

(vi) $\lambda(E) = 0 \Rightarrow E \in \mathcal{M}$

(vii) $\mathcal{U} \subseteq \mathbb{R}$, $\mathcal{U}$ open $\Rightarrow \mathcal{U} \in \mathcal{M}$.

(viii) $F \subseteq \mathbb{R}$, F closed $\Rightarrow F \in \mathcal{M}$.

▲ $E \subset \mathcal{M}$, $\Rightarrow \lambda(E) = \int \chi_E$.

- $\lambda(\phi) = 0$
- $(E_n)_{n=1}^{\infty} \subset \mathcal{M}$, $E_n \cap E_m = \phi$ if $n \ne m$, $\Rightarrow \lambda(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \lambda(E_n)$
- $A, B \in \mathcal{M}$, $A \cap B = \phi \Rightarrow \lambda(A+B) = \lambda(A) + \lambda(B)$.
- $E \subseteq \mathcal{M}$, $x \in \mathbb{R} \Rightarrow E_x = E + x \in \mathcal{M}$.

- $A, B \in \mathcal{M}$, $B \subseteq A$, $\lambda(A \setminus B) = \lambda(A) - \lambda(B)$.
- $(E_n)_{n=1}^{\infty} \subseteq \mathcal{M}$, $E_n \subseteq E_{n+1} \Rightarrow \lambda(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \lambda(E_n)$
- $(E_n)_{n=1}^{\infty} \subseteq \mathcal{M}$, $E_n \supseteq E_{n+1}$ $\forall n$, If $(\exists p\ \lambda(E_p) < \infty)$

  $\lambda(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \lambda(E_n)$.

▲ Outer Regularity

$$E \subseteq \mathbb{R} \Rightarrow \lambda(E) = \inf\{\lambda(\mathcal{U}) \mid E \subseteq \mathcal{U},\ \mathcal{U} \text{ is open}\}$$

Inner Regularity

$$E \subseteq \mathbb{R} \Rightarrow \lambda(E) = \sup\{\lambda(K) \mid K \subseteq E,\ K \text{ is compact}\}.$$

▲ $A \in \mathcal{M}$, $\lambda(A) < \infty$, $E \subseteq A$. $E \subseteq \mathcal{M}$ iff $\lambda(A) = \lambda(E) + \lambda(A \setminus E)$

Remark: $\lambda(A) < \lambda(E) + \lambda(A \setminus E)$.

Structure of Measurable Functions

$f : \mathbb{R} \to \mathbb{R}^{\#}$ a.e. the following are equivalent

(1) f is measurable.

(2) $(\forall \alpha \in \mathbb{R})$ $\{x \in \mathbb{R} \mid f(x) \le \alpha\} \in \mathcal{M}$

(3) $(\forall \alpha \in \mathbb{R})$ $\{x \in \mathbb{R} \mid f(x) > \alpha\} \in \mathcal{M}$

(4) $(\forall \alpha \in \mathbb{R})$ $\{x \in \mathbb{R} \mid f(x) < \alpha\} \in \mathcal{M}$

(5) $(\forall \alpha \in \mathbb{R})$ $\{x \in \mathbb{R} \mid f(x) \ge \alpha\} \in \mathcal{M}$.

Corollaries:
- $f : \mathbb{R} \to \mathbb{R}^c$ a.e. is measurable iff $\forall$ open $\mathcal{U} \subseteq \mathbb{R}^c$ $f^{-1}(\mathcal{U}) \in \mathcal{M}$.
- Every continuous function is Lebesgue measurable.

▲ Egorov's Thm:

$E \in \mathcal{M}$, $\lambda(E) < \infty$, Suppose $(f_n) \subseteq \mathcal{M}$, $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on E

Let $\delta > 0$. Then $(\exists$ compact $K \subseteq E)$ such that $\lambda(E \setminus K) < \delta$ and $f_n \to f$ uniformly on K.

▲ Lusin's Theorem: $f \in M$, $\delta > 0$. Then $\exists$ continuous $g: \mathbb{R} \longrightarrow \mathbb{C}$ such that $\lambda\{x \mid f(x) \neq g(x)\} < \delta$. Moreover, if $|f(x)| \leq \beta$ a.e., then g can be chosen so that $|g(x)| \leq \beta \quad \forall\, x \in \mathbb{R}$.

▲ $f \in M \iff (\exists\ (g_n)_{n=1}^{\infty}$ continuous$)$ such that $\lim_{n\to\infty} g_n(x) = f(x)$ a.e.

● Integration over Measurable sets

Let $E \in \mathfrak{m}$. Let f be defined a.e. on E (and perhaps elsewhere).

$$f\chi_E \triangleq \begin{cases} f(x) & x \in E \\ 0 & x \notin E \end{cases}$$ which is defined a.e. on $\mathbb{R}$.

Definitions: f is measurable on E if $f\chi_E \in M$.

f is integrable over E if $f\chi_E \in L_1$.

$L_1(E)$ = set of functions integrable over E.

$\int_E f = \int f\chi_E$.

▲ Absolute Continuty of Lebesgues Intergral:

Let $f \in L_1$. $(\forall \varepsilon > 0)\ (\exists\, \delta > 0)\ (E \in \mathfrak{m}, \lambda(E) < \delta \Rightarrow \int_E |f| < \varepsilon)$.

● Monotone Functions (chapter 3, P.P. 128 – 129)

- suppose f is defined on $(c, c+\varepsilon)$ $(\varepsilon > 0)$

$$f(c+) = \lim_{x \to c^+} f(x) = \lim_{x \downarrow c} f(x)$$

- If f is defined on $(c-\varepsilon, c)$ $(\varepsilon > 0)$

$$f(c-) = \lim_{x \to c^-} f(x) = \lim_{x \uparrow c} f(x)$$

Prop. $f: (a,b) \longrightarrow \mathbb{C}$, $c \in (a,b)$; f is cts at c iff $f(c-) = f(c) = f(c+)$.

● f has a <u>simple discontinuty</u> at c if both $f(c-)$ and $f(c+)$ exist and are finite.

▲ $f: (a,b) \longrightarrow \mathbb{R}$ nondecreasing, $c \in (a,b)$. Then

(i) $f(c-) = \sup\{ f(x) \mid x < c \} \in \mathbb{R}$.

(ii) $f(c+) = \inf\{ f(x) \mid x > c \} \in \mathbb{R}$.

(iii) $f(c-) \leq f(c) \leq f(c+)$.

▲ $f: (a,b) \longrightarrow \mathbb{R}$ nondecreasing. Then f has at most countably many discontinuities, each of which is simple.

(The same is true if $f = g - h$, where g and h are nondecreasing).

▲ <u>Monotone Functions are Measurable.</u>

● <u>Total Variantion</u> (chapter 3, p. 159)

- A subdivision (partition) of $[a,b]$ is an ordered finite subsets

$$P = \{ a = x_0 < x_1 < x_2 < \cdots < x_n = b \}$$

Let $f: [a,b] \longrightarrow \mathbb{C}$, $V(P,f) \triangleq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$

- A total variation of f on $[a,b]$ is

$$V_a^b f = \sup\{ V(P,f) \mid P \text{ is a partition of } [a,b] \}.$$

- f has finite variation (bounded variation) on $[a,b]$ if $V_a^b f < \infty$

▲ $V_a^b (f+g) \leq V_a^b f + V_a^b g$

$V_a^b f = V_a^c f + V_c^b f \qquad c \in [a,b]$

f unbounded on $[a,b] \Rightarrow V_a^b f = \infty$.

● Lipschitz Continuous $f : [a, b] \to \mathbb{C}$

f satisfies a Lipschitz condition on $[a, b]$ if $(\exists M > 0)$

$|f(x) - f(y)| \leq M \cdot |x - y|$, $\forall x, y \in [a, b]$.

All such functions are uniformly continuous.

▲ Propositions

(i) f is Lipschitz continuous on $[a, b]$, then f has bounded variation on $[a, b]$, i.e., $V_a^b f < \infty$. (checked)

(ii) f is continuously differentiable on $[a, b] \Rightarrow f$ is Lipschitz.

(iii) $f : [a, b] \to \mathbb{R}$, then $f \begin{Bmatrix} \text{nondecreasing} \\ \text{nonincreasing} \end{Bmatrix} \Rightarrow V_a^b f < \infty$.

(iv) Suppose $f = g - h$, where g, h are nondecreasing on $[a, b]$. $V_a^b f < \infty$.

▲ Jordan Decomposition Theorem

$f : [a, b] \to \mathbb{R}$, $V_a^b f < \infty \Rightarrow (\exists$ nondecreasing $g, h)$ $f = g - h$.

corollaries: (i) $f : [a, b] \to \mathbb{C}$, $V_a^b f < \infty$, then $\exists$ nondecreasing $g_1, g_2, h_1, h_2 \ni f = (g_1 - h_1) + i(g_2 - h_2)$

(ii) $V_a^b f < \infty \Rightarrow f$ has at most countably many discontineunity, each of which is simple.

In fact, f is continuous almost everywhere.

f is measurable.

● Differentiability

f is differentiable at x if $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ exists (in $\mathbb{C}$).

▲ If f is differentiable at x, then $f(x)$ is continuous on x.

▲ Lebesgue's Differentiability Theorem

$V_a^b f < \infty \Rightarrow f$ is differentiable a.e. on $[a,b]$.

● Absolute Countinuty (chapter 3, P. 162)

f is absolutely continous on $[a,b]$ if $(\forall \varepsilon > 0)(\exists \delta > 0)$ such that for all finite sets of disjoint open subintervals of $[a,b]$, say, $(a_1,b_1), (a_2,b_2), \ldots, (a_n,b_n)$ if

$\sum_{k=1}^{n}(b_k - a_k) < \delta$, then $\sum_{k=1}^{n}|f(b_k) - f(a_k)| < \varepsilon$

▲ absolutely continuous $\Rightarrow$ uniformly continuous.

▲ f is Lipschitz on $[a,b] \Rightarrow f$ is absolutely continous on $[a,b]$.

▲ $f: [a,b] \to \mathbb{C}$ absolutely continous $\Rightarrow V_a^b f < \infty$.

▲ f is absolutely continuous on $[a,b] \Rightarrow f'$ exists a.e. on $[a,b]$

▲ Let $f \in L_1(a,b)$, Define F by

$$F(x) = \int_a^x f,$$ then F is absolutely continous.

▲ Hierarchy of Continous Functions

class	example
Continously differentiable	e^x, $\sin x$
↓ Lipschitz continous	$f(x) = \lvert x\rvert$ on $[-1,1]$
↓ Absolutely continous	$f(x) = \sqrt{x}$ on $[0,1]$
↙ Bounded Vf ↘ uniformly cts	Lebegues singular function

Step functions

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$$

▲ F is nondecreasing on $[a,b]$. Then $F' \in L_1(a,b)$ and

$$\int_a^b F' \le F(b) - F(a).$$

▲ $V_a^b F < \infty \implies F' \in L_1(a,b)$

▲ $V_a^b F < \infty$, $F(x) - F(a) = \int_a^x F' \quad \forall x \in [a,b]$

$\implies F$ is absolutely continous on $[a,b]$.

▲ Let $f \in L_1(a,b)$, $F(x) = \int_a^x f$. Then $F' = f$ a.e.

▲ Fundamental Theorem of Calculus

Let F be absolutely continous on $[a,b]$. Then $F' \in L_1(a,b)$ and $F(x) - F(a) = \int_a^x F' \quad \forall x \in [a,b]$.

● Integration by parts

▲ If F, G are absolutely continous on $[a,b]$, then FG is absolutely continuous on $[a,b]$ (E.F.M).

▲ F, G is absolutely continous $\Rightarrow (FG)' = F'G + FG'$ a.e.

▲ Integration by Parts Thm: F, G are absolutely continous on $[a,b] \Rightarrow \int_a^b F'G = F(b)G(b) - F(a)G(a) - \int_a^b FG'$.

⌖ F, ϕ are absolutely continous and ϕ is strictly monotone on $[a,b]$, Then $F \circ \phi$ is absolutely continuous on $[a,b]$.

▲ F, ϕ are Lipschitz implies $F \circ \phi$ is Lipschitz.

▲ Chain Rule: F, ϕ are absolutely continous, ϕ is strictly monotone. Let $f = F'$ a.e. Then

$$(F \circ \phi) = (f \circ \phi)\,\phi' \quad \text{a.e.}$$

▲ Substitution Theorem

Let $\phi : [a,b] \to [c,d]$ be absolutely continous and strictly monotone. Let $f \in L_1(c,d)$

$$\int_a^b (f\circ\phi)\,\phi' = \int_{\phi(a)}^{\phi(b)} f.$$

Hölder and Minkowski Inequalities

- $0<p<\infty$, $L_p = \{ f \in M \mid |f|^p \in L_1 \}$
- Conjugate exponents, $p' = \dfrac{p}{p-1} > 1$, $p>1$, $\boxed{\dfrac{1}{p} + \dfrac{1}{p'} = 1.}$

▲ Let $p, p' > 1$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Let $a \geq 0$, $b \geq 0$. Then

$$\boxed{ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}}$$

▲ **Hölder** $\neq$:, $p, p' > 1$. $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. $f, g \in M$

$$\boxed{\int |fg| \leq \sqrt[p]{\int |f|^p} \cdot \sqrt[p']{\int |g|^{p'}}}$$

Remark Hölder is valid for $p=1$ and $p'=\infty$.

▲ Minkowski $\neq$: $1 \leq p < \infty$, $f, g \in M$

$$\boxed{\sqrt[p]{\int |f+g|^p} \leq \sqrt[p]{\int |f|^p} + \sqrt[p]{\int |g|^p}} \qquad \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

▲ $f, g \in L_p$, $\alpha \in \mathbb{C}$.

(i) $\|\alpha f\|_p = |\alpha| \cdot \|f\|_p$

(ii) $\|f\|_p \geq 0$ w/= iff $f = 0$ a.e.

(iii) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. (Minkowski).

● $\rho_p(f,g) = \|f-g\|_p$ is a metric on $\mathcal{L}_p$.

▲ $1 \leq p < \infty$, $(\mathcal{L}_p, \rho_p)$ is complete. That is for every sequence $(f_n) \subseteq L_p$ such that $\lim_{n,m\to\infty} \|f_m - f_n\|_p = 0$, there exists $f \in L_p$ such that $\lim_{n\to\infty} \|f - f_n\|_p = 0$. Furthermore $(\exists\, (f_{n_k}))$ such that

$$\lim_{k\to\infty} f_{n_k}(x) = f(x) \text{ a.e.}$$

● $M_0^c \triangleq$ complex-valued step functions. M_0^c is a subspace of L_p.

<u>Reminder</u> $C_{00}(\mathbb{R})$ = continous function with compact support.

▲ $1 \leq p < \infty$, M_0^c, $C_{00}(\mathbb{R})$ are both dense in L_p.

<u>Integration on $\mathbb{R}^n$</u>

● A **box** (interval) in $\mathbb{R}^n$, $B = I_1 \times I_2 \times \cdots \times I_n$, where I_j, $j = 1, 2, \cdots, n$ are intervals in $\mathbb{R}$.

- The box is open if $I_1, I_2, \cdots, I_n$ are open intervals.
- <u>Volume of B:</u> $|B| = |I_1| \cdot |I_2| \cdots |I_n|$

● <u>n-dimensional Lebesgue Outer measure</u>

$E \subseteq \mathbb{R}^n$, $\lambda(E) = \lambda^n(E) = \inf\left\{ \sum_{B\in\mathcal{B}} |B| \,\middle|\, \mathcal{B} \text{ is countable collection of open boxes} \right\}$

- E is null set if $\lambda(E) = 0$.

▲ i) $0 \le \lambda(E) \le \infty$

ii) $\lambda(\phi) = 0$

iii) $E \subseteq F \Rightarrow \lambda(E) \le \lambda(F)$

iv) $\lambda(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} \lambda(E_n)$

v) E is countable implies $\lambda(E) = 0$

vi) $\lambda(E+a) = \lambda(E)$

vii) $\lambda(N) = 0 \Rightarrow \lambda(E \cup N) = \lambda(E)$.

● Step function

$$\phi = \sum_{k=1}^{n} \alpha_k \chi_{B_k}$$

where $B_1, \cdots, B_n$ are disjoint boxes such that $\bigcup_{k=1}^{n} B_k = \mathbb{R}^n$.

$\alpha_1, \cdots, \alpha_n$ are real and $\alpha_k = 0$ if B_k is unbounded.

- $M_0 = M_0(\mathbb{R}^n)$ = real-valued step function.

▲ M_0 is a real vector, lattice.

● $\int \phi = \sum_{k=1}^{n} \alpha_k |B_k|$ is a monotone, linear functional on M_0.

● $M_1 = M_1(\mathbb{R}^n)$. $f \in M_1$ if $\exists$ nondecreasing sequence $(\phi_n) \subseteq M_0$ such that $\lim_{n\to\infty} \phi_n(x) = f(x)$ a.e. and $\lim_{n\to\infty} \int \phi_n < \infty$.

- $\int f = \lim_{n\to\infty} \int \phi_n$

● $L_1 = L_1(\mathbb{R}^n) = \{ f \mid (\exists g, h \in M_1)\ f = g - h \text{ a.e.} \}$

- $\int f = \int g - \int h$.

▲ L_1 is a real vector lattice.

▲ $\int$ is a monotone linear functional.

● f is <u>measurable</u> if $(\exists(\phi_n)\subseteq M_0)$ such that $\lim_{n\to\infty}\phi_n = f$ a.e. $(f\in m)$

▲ M.C.T., Fatou's Lemma, L.D.C.T.

▲ $E\subseteq m(\mathbb{R}^n) \Rightarrow \lambda^n(E)=\int \chi_E$.

⊕ Egrov, Lusin, Hölder, Minkowski, $L_p(\mathbb{R}^n)$ complete.

step functions are dense in L_p, $C_{00}(\mathbb{R}^n)$ is dense in $L_p(\mathbb{R}^n)$.

* Fundamental Theorem of Calculus does not extend routinely to $\mathbb{R}^n$.

● $x\in(x_1,x_2,\cdots,x_m)\in\mathbb{R}^m$, $y=(y_1,y_2,\cdots,y_n)\in\mathbb{R}^n$

$$z=(x,y)=((x_1,x_2,\cdots,x_m),(y_1,y_2,\cdots,y_n))$$
$$=(x_1,x_2,\cdots,x_m,y_1,y_2,\cdots,y_n)\in\mathbb{R}^{m+n}$$

$f(z)=f(x,y)\in M(\mathbb{R}^{m+n})$

$$\int_{\mathbb{R}^{m+n}} f = \int_{\mathbb{R}^{m+n}} f(z)\,d(z) = \int_{\mathbb{R}^{m+n}} f(x,y)\,d(x,y).$$

▲ <u>Fubini's Theorem</u> $\phi\in M_0(\mathbb{R}^{m+n})$. Then

(i) $\forall x\in\mathbb{R}^m$, $y\mapsto\phi(x,y)$ is in $M_0(\mathbb{R}^n)$.

(ii) $\forall y\in\mathbb{R}^n$, $x\mapsto\phi(x,y)$ is in $M_0(\mathbb{R}^m)$.

(iii) $x\longmapsto\int_{\mathbb{R}^n}\phi(x,y)\,dy$ is in $M_0(\mathbb{R}^m)$

(iv) $y\longmapsto\int_{\mathbb{R}^m}\phi(x,y)\,dx$ is in $M_0(\mathbb{R}^n)$

(v) $\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}\phi(x,y)\,dx\,dy=\int_{\mathbb{R}^{m+n}}\phi(x,y)\,d(x,y)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\phi(x,y)\,dy\,dx$.

▲ $E\subseteq\mathbb{R}^p$. Then $\lambda^p(E)=0$ iff $(\phi_k)\subseteq M_0(\mathbb{R}^n)$, (ϕ_k) is nondecreasing

$$\lim_{k\to\infty}\int_{\mathbb{R}^p}\phi_k(z)\,dz<\infty$$

and $(\forall x\in E)$ $\lim_{k\to\infty}\phi_k(x)=\infty$.

● Notation: $E \subseteq \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$

- Let $x \in \mathbb{R}^m$, $E_x = \{ y \in \mathbb{R}^n \mid (x,y) \in E \}$
- Let $y \in \mathbb{R}^n$, $E^y = \{ x \in \mathbb{R}^m \mid (x,y) \in E \}$

▲ $E \subseteq \mathbb{R}^{m+n}$, $\lambda^{m+n}(E) = 0$. Then

(i) for almost every $x \in \mathbb{R}^m$, $\lambda^n(E_x) = 0$.

(ii) for almost every $y \in \mathbb{R}^n$, $\lambda^m(E^y) = 0$.

▲ Fubini's Theorem for M_1 (for L_1, change M_1 to L_1)

Let $f \in M_1(\mathbb{R}^{m+n})$. Then

(i) for a.e $x \in \mathbb{R}^m$, $y \longmapsto f(x,y)$ is in $M_1(\mathbb{R}^n)$

(ii) for a.e $y \in \mathbb{R}^n$, $x \longmapsto f(x,y)$ is in $M_1(\mathbb{R}^m)$

(iii) $x \longmapsto \int_{\mathbb{R}^n} f(x,y)\,dy$ is in $M_1(\mathbb{R}^m)$

(iv) $y \longmapsto \int_{\mathbb{R}^m} f(x,y)\,dx$ is in $M_1(\mathbb{R}^n)$

(v) $\int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x,y)\,dy\,dx = \int_{\mathbb{R}^{m+n}} f(x,y)\,d(x,y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x,y)\,dx\,dy$

▲ Tonelli's Theorem

Let $f \in M^r(\mathbb{R}^{n+m})$ $\quad 0 \leq f(x,y) \leq \infty \quad \forall (x,y)$

(i) for a.e. $x \in \mathbb{R}^m$ $y \longmapsto f(x,y)$ is in $M^r(\mathbb{R}^n)$

(ii) for a.e $y \in \mathbb{R}^n$ $x \longmapsto f(x,y)$ is in $M^r(\mathbb{R}^m)$

(iii) $x \longmapsto \int_{\mathbb{R}^n} f(x,y)\,dy$ is in $M^r(\mathbb{R}^m)$

(iv) $y \longmapsto \int_{\mathbb{R}^m} f(x,y)\,dx$ is in $M^r(\mathbb{R}^n)$

(v) $\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x,y)\,dx\,dy = \int_{\mathbb{R}^{m+n}} f(x,y)\,d(x,y) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x,y)\,dy\,dx$

▲ $f \in M(\mathbb{R}^{m+n})$

If $\int_{\mathbb{R}^n}\int_{\mathbb{R}^m} |f(x,y)|\,dx\,dy$ or $\int_{\mathbb{R}^m}\int_{\mathbb{R}^n} |f(x,y)|\,dy\,dx < \infty$,

Then $f \in L_1(\mathbb{R}^{m+n})$.

▲ Let $E \in m(\mathbb{R}^{m+n})$. Then

(i) For a.e $x \in \mathbb{R}^m$, $E_x \in m(\mathbb{R}^n)$

(ii) For a.e $y \in \mathbb{R}^n$, $E^y \in m(\mathbb{R}^m)$

Moreover, the following are equivalent

(iii) $\lambda^{m+n}(E) = 0$

(iv) $\forall$ a.e $x \in \mathbb{R}^m$, $\lambda^n(E_x) = 0$.

(v) $\forall$ a.e $y \in \mathbb{R}^n$, $\lambda^m(E^y) = 0$.
