Page 111, Problem 4

25/50

Let X be a metric space and let A and B be nonvoid subsets of X. Define

$$\operatorname{dist}(A,B) = \inf\{\rho(x,y) : x \in A,\ y \in B\}.$$

Then

(a) If A and B are compact, then there exist $a \in A$ and $b \in B$ such that $\operatorname{dist}(A,B) = \rho(a,b)$.

(b) There exist disjoint nonvoid closed subsets A and B of $\mathbb{R}$ for which $\operatorname{dist}(A,B) = 0$.

Proof: (a) By the definition of dist(A,B), it is obvious to show that we can construct the following sequences:

$$\forall \text{ given } \gamma_n = \operatorname{dist}(A,B) + \frac{1}{n}, \quad n = 1, 2, \cdots$$

$$\exists\ a_n \in A \text{ and } b_n \in B, \quad n = 1, 2, \cdots$$

$$\ni\ \rho(a_n, b_n) \le \gamma_n, \quad n = 1, 2, \cdots$$

5/10

Now, since A and B are compact, [using theorem (3.43) that every sequence in a compact set (metric space) has a subsequence that coverges to a point of this compact set] we know there exist subsequences of $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, say $(a_{n_k})_{k=1}^{\infty}$ and $(b_{n_k})_{k=1}^{\infty}$, which converge to a point of A (say a) and a point of B (say b), respectively. Then it is triavial to see that

Careful! Take subsequences one at a time.

proof?

$$\operatorname{dist}(A,B) \le \rho(a,b) \le \lim_{n_k\to\infty} \left[\rho(A,B) + \frac{1}{n_k}\right] = \operatorname{dist}(A,B)$$

Hence, $\rho(a,b) = \operatorname{dist}(A,B)$.

Problem 4 (cont.)

(b)

$$A = \{2, 3, 4, 5, \ldots\}$$

$$B = \{2+\tfrac{1}{2}, 3+\tfrac{1}{3}, 4+\tfrac{1}{4}, \ldots\}$$

$\frac{0}{10}$

Page 112 Problem 8

If X is a metric space, then X is second coutable if and only if X has a countable dense subset.

Proof: ($\Leftarrow$) If X has a countable dense subset

$$S = \{ x_1, x_2, \cdots, x_n, \cdots \}$$

Then we construct a countable f^a^mily $\mathcal{B}$ of open sets as follows:

$$\mathcal{B} = \left\{ \begin{array}{l} B_{\frac{1}{m}}(x_1), \ m=1,2,\cdots, \\ B_{\frac{1}{m}}(x_2), \ m=1,2,\cdots, \\ \quad \vdots \\ B_{\frac{1}{m}}(x_n), \ m=1,2,\cdots, \\ \quad \vdots \end{array} \right\}$$

3/10

Then given any open set $V \subset X$ and $x \in V$, there is an open ball $B_{\frac{1}{m}}(x_n)$ such that $x \in B_{\frac{1}{m}}(x_n) \subset V$ for suitable m and n. Since if this argument is false, then we have either 1) $x \notin B_{\frac{1}{m}}(x_n)$ for any m and n which implies S is not dense and is a contradiction or 2) $\forall$ m and n, $\ni x \in B_{\frac{1}{m}}(x_n)$ but $B_{\frac{1}{m}}(x_n) \not\subset V$. This simply implies that V is not open which is also a contradiction.

No!

No!

Problem 8 (cont.)

10/10

($\Longrightarrow$) If X is second countable, then by definition there exists some countable family $\mathcal{B}$ of open sets such that each open set $V \subset X$ can be expressed as the union of some subfamily of $\mathcal{B}$.

Let $\mathcal{B} = \{B_1, B_2, \cdots, B_n, \cdots\}$ and choose a point x_n in each B_n. Then the set

$$S = \{x_1, x_2, \cdots, x_n, \cdots\}$$

is countable. Moreover, S is dense in X, since otherwise the nonempty open set $X \setminus \bar{S}$ would contain ~~no~~ points of S. But $X \setminus \bar{S}$ is open, $X \setminus \bar{S}$ can be expressed as the union of some subfamily of $\mathcal{B}$ (or some sets of B_n) and B_n contains the point $x_n \in S$.

Hence, S is countable dense subset of X.

Page 112, Problem 14.

Let A be any given bounded subset of $\mathbb{R}$ that is not closed. Construct explicitly an open cover of A that has no finite subcover.

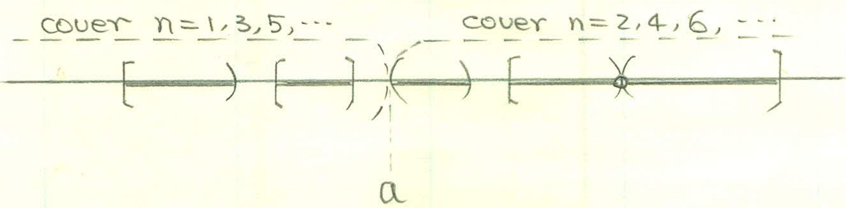
Proof: Since A is bounded subset of $\mathbb{R}$ and since A is not closed, there must exist a limit point of A, say a and $-\infty < a < +\infty$, such that $a \notin A$. Then construct the following open cover

$$\mathcal{U} = \left\{ \begin{array}{ll} (-\infty, a-\frac{1}{n}) : & n = 1, 3, 5, \cdots \\ (a+\frac{1}{n}, \infty) : & n = 2, 4, 6, \cdots \end{array} \right\}$$

which will cover A. However, it is obvious that any finite sub~~cover~~ family of $\mathcal{U}$ cannot cover A. ※

Prove it anyway.

7/10



This point cannot be covered by finite subcover of $\mathcal{U}$.

This picture is not a proof.

Real Analysis 501

Homework No. 2

Ben M. Chen

September 21, 1990

$\frac{50}{50}$

Page 120, Problem 3. (a) (b) (c)

Let X be a topological space, let Y be a metric space with metric ρ, and let f be a function from X into Y. Define the oscillation function ω of f on X by

$$\omega(x) = \inf\{\operatorname{diam} f(\mathcal{U}) : \mathcal{U} \text{ is a neighborhood of } x\}.$$

Then (a) f is continuous at $p \in X$, if and only if $\omega(p) = 0$.

Proof: ($\Rightarrow$) Since Y is a metric space, $\forall \varepsilon > 0$, we can define a neighborhood V_ε of $f(p)$ as follows:

$$V_\varepsilon = B_{\frac{\varepsilon}{2}}(f(p)) = \{y \in Y \mid \rho(y, f(p)) < \tfrac{\varepsilon}{2}\}.$$

Now since f is continuous at p, then by definition, there exists a neighborhood $\mathcal{U}_\varepsilon$ of p such that

$$f(\mathcal{U}_\varepsilon) \subseteq V_\varepsilon.$$

10/10

By definition of diameter, we have

$$\operatorname{diam} f(\mathcal{U}_\varepsilon) \leq \operatorname{diam} V_\varepsilon < \varepsilon.$$

By definition of $\omega(x)$, we have $\omega(x) < \varepsilon$, $\forall \varepsilon > 0$. This implies $\omega(x) = 0$.

[Suppose $\omega(x) = a > 0$, then let $\varepsilon = a$, we have $\omega(x) < \varepsilon = a$, which is a contradiction.]

Problem 3 (a) (cont.)

($\Leftarrow$) Since Y is a metric space, then for any neighborhood V of $f(p)$, there exists $\gamma > 0$ such that

$$B_\gamma(f(p)) = \{ y \in Y \mid \rho(y, f(p)) < \gamma \} \subseteq V.$$

[a neighborhood V of $f(p)$ is an open set containing $f(p)$. And in a metric space, V is open $\Rightarrow$ $(\exists \gamma > 0)$ $B_\gamma(f(p)) \subseteq V$.]

Now, $\omega(p) = \inf \{ \operatorname{diam} f(\mathcal{U}) : \mathcal{U}$ is a neighborhood of $p \} = 0$ implies that there exists a neighborhood $\mathcal{U}_\gamma$ of p such that $\operatorname{diam} f(\mathcal{U}_\gamma) < \gamma$.

By the definition of diameter, we have $\forall\, y \in f(\mathcal{U}_\gamma)$ $\rho(y, f(p)) \le \operatorname{diam} f(\mathcal{U}_\gamma) < \gamma$ which implies $y \in B_\gamma(f(p)) \subseteq V$.

Hence, $f(\mathcal{U}_\gamma) \subseteq B_\gamma(f(p)) \subseteq V$: That is

$\forall$ nbhd V of $f(p)$ ($\exists$ nbhd $\mathcal{U}_\gamma$ of p) such that $f(\mathcal{U}_\gamma) \subseteq B_\gamma(f(p)) \subseteq V$.

By the definition, we conclude that f is continuous at p.

Q.E.D.

Page 120, Problem 3 (b)

(b) for each $\beta \leq \infty$, the set $\{x \in X : \omega(x) < \beta\}$ is open in X.

Proof: Let us define

$$S_\beta = \{x \in X : \omega(x) < \beta\}.$$

If $\beta \leq 0$, then $S_\beta = \emptyset$ and $\emptyset$ is open. In case of $\beta > 0$, $\forall\, s \in S_\beta$, we have

$$\omega(s) = \beta_s < \beta$$

which implies that ($\exists$ a nbhd G of s) such that

$$\operatorname{diam} f(G) = \beta_G \quad \text{and} \quad \beta_s \leq \beta_G < \beta.$$

Now, $\forall\, g \in G$, we note that G is also a nbhd of g and moreover

$$\omega(g) \leq \operatorname{diam} f(G) = \beta_G < \beta$$

which implies $g \in S_\beta$ and hence $G \subseteq S_\beta$. Thus S_β is open for any $\beta \leq \infty$.

Q.E.D.

10/10

Page 120, Problem 3(c)

(c) the set $\{p \in X : f \text{ is continuous at } p\}$ is a G_δ set in X.

Proof: It is shown in Problem 3(b) on page 03 that

$$S_\beta = \{x \in X : \omega(x) < \beta\}$$

is an open set for each $\beta \leq \infty$. Let us now construct a sequence of open sets by letting

$$\beta = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots.$$

In this way we obtain

$$S_1 \supseteq S_{\frac{1}{2}} \supseteq S_{\frac{1}{3}} \supseteq \cdots \supseteq S_{\frac{1}{n}} \supseteq \cdots.$$

Now, let us define

$$S_0 = \{p \in X : f \text{ is continuous at } p\}.$$

From Problem 3(a) on page 01, we have

$$S_0 = \{p \in X : \omega(p) = 0\}.$$

Now, we need to show $S_0 = \bigcap_{n=1}^{\infty} S_{\frac{1}{n}}$. It is simple to see that $S_0 \subseteq \bigcap_{n=1}^{\infty} S_{\frac{1}{n}}$ since $0 < \frac{1}{n}$, $n = 1, 2, 3, \ldots$.

Next, we have to show $\bigcap_{n=1}^{\infty} S_{\frac{1}{n}} \subseteq S_0$. We show this argument by contradiction. Suppose $(\exists s \in \bigcap_{n=1}^{\infty} S_{\frac{1}{n}})\ s \notin S_0$ which implies $\omega(s) = \beta_s > 0$. But

$$s \notin S_{\frac{1}{n}} \quad \text{for } n > \frac{1}{\beta_s}.$$

Thus $s \notin \bigcap_{n=1}^{\infty} S_{\frac{1}{n}}$ which is a contradiction. Hence,

$S_0 = \bigcap_{n=1}^{\infty} S_{\frac{1}{n}}$ and by definition S_0 is a G_δ in X.

Q.E.D.

Page 121 Problem 8

Suppose that f and g are continuous from a topological space X into a Hausdorff space Y and that $f(d)=g(d)$ for all $d \in D$, where D is a dense subset of X. Then $f(x)=g(x)$ for all $x \in X$.

Proof: Since D is dense in X, $\bar{D}=X$. And since we know $f(d)=g(d)$ $\forall\, d \in D$, hence we can prove that $f(x)=g(x)$ $\forall\, x \in X$ by showing

$$f(s)=g(s) \quad \forall\, s \in X \setminus D = \bar{D} \setminus D.$$

By definition, we know that $\forall s \in \bar{D} \setminus D$, s is a limit point of D. From the definition of limit, we have

$$f(x) \longrightarrow b_f \in Y \quad \text{as} \quad x \to s \quad (x \in D)$$

$$g(x) \longrightarrow b_g \in Y \quad \text{as} \quad x \to s \quad (x \in D).$$

10/10

Since f and g are continuous functions on X, we have

$$f(x) \to f(s) \quad \text{as} \quad x \to s \quad (x \in D)$$

$$g(x) \to g(s) \quad \text{as} \quad x \to s \quad (x \in D).$$

Moreover, since $f(x)=g(x)$ $\forall\, x \in D$, we have

$$f(x) = g(x) \longrightarrow g(s) \quad \text{as} \quad x \to s \quad (x \in D).$$

and since Y is Hausdorff, Theorem 3.64 says

$$f(s)=g(s) \quad \forall\, s \in X \setminus D.$$

Thus, we have

$$f(x)=g(x), \quad \forall\, x \in D \cup (X \setminus D) = X. \qquad \text{Q.E.D.}$$

Page 124, Problem 1

If X is a compact metric space and $A \subseteq X$ is nonvoid, then there is some $u \in X$ such that $\mathrm{dist}(u, A) \geq \mathrm{dist}(x, A)$ for all $x \in X$.

Proof: Let us first define a function $f: X \longrightarrow \mathbb{R}$ as follows

$$f(x) = \mathrm{dist}(x, A), \quad \forall x \in X \quad (A \subseteq X \text{ is nonvoid}).$$

We note that $f(X) = S$ with normal metric of $\mathbb{R}$ reduced on S is a metric space. Then for any $p \in X$ and $\forall$ nbhd V of $f(p)$, $(\exists\, B_r(f(p)))$ for some $r > 0$ such that

$$B_r(f(p)) = \{ s \in S \mid |s - f(p)| < r \} \subseteq V.$$

Now, assume that the metric space X with a metric ρ and consider a nbhd of p to be a ball,

$$B_r(p) = \{ x \in X \mid \rho(x, p) < r \}.$$

Then, $\forall\, g \in B_r(p)$, it follows from theorem 3.8 in text that

$$|f(g) - f(p)| = |\mathrm{dist}(g, A) - \mathrm{dist}(p, A)| \leq \rho(g, p) < r.$$

Hence $f(g) \in B_r(f(p)) \subseteq V$. This shows that f is continous on X. Since X is compact, it follows from theorem 3.78 that $f(X) = S \subseteq \mathbb{R}$ is compact.. ~~Moreover, it's closed and bounded~~.

Thus $(\exists\, S_0 \in S)$ such that $s \leq S_0$ $\forall s \in S$, which implies that there exists a $u = f^{-1}(S_0) \in X$ such that

$$f(u) = \mathrm{dist}(u, A) = S_0 \geq f(x) = \mathrm{dist}(x, A), \quad \forall x \in X.$$

Q.E.D.

10/10

Page 124, Problem 3

Let f and g be continuous real-valued function on $[a,b] \subset \mathbb{R}$ such that $f(a) < g(a)$ and $f(b) > g(b)$. Then $f(x_0) = g(x_0)$ for some $x_0 \in [a,b]$.

Proof: Let us define a new function

$$h = f - g.$$

Hence, $h(a) = f(a) - g(a) < 0$ and $h(b) = f(b) - g(b) > 0$.

Next, we need to show that h is continuous on $[a,b]$.

Since f and g are continuous on $[a,b]$, $\forall p \in [a,b]$ and $\forall \varepsilon > 0$, ($\exists\, \delta_f$ and δ_g) such that

$$\forall x \in [a,b] \text{ and } |x-p| < \delta_f, \quad |f(x) - f(p)| < \frac{\varepsilon}{2}$$

and

$$\forall x \in [a,b] \text{ and } |x-p| < \delta_g, \quad |g(x) - g(p)| < \frac{\varepsilon}{2}.$$

Hence, we have $\forall x \in [a,b]$ and $|x-p| < \min(\delta_f, \delta_g)$

$$|f(x) - f(p)| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - g(p)| < \frac{\varepsilon}{2}.$$

Now we consider

$$\begin{aligned} |h(x) - h(p)| &= |f(x) - g(x) - f(p) + g(p)| \quad \forall\, |x-p| < \min(\delta_f, \delta_g) \\ &\le |f(x) - f(p)| + |g(x) - g(p)| \\ &< \varepsilon \end{aligned}$$

10/10

Hence, h is continuous on $[a,b]$ and $h([a,b])$. Since $h(a) < 0$ and $h(b) > 0$, it follows from theorem 3.82 (Intermediate Value Theorem) that $0 = h(x_0)$ for some $x_0 \in [a,b]$. That $f(x_0) = g(x_0)$.

Q.E.D.

Real Analysis 501

Homework No. 3

Ben M. Chen

October 8, 1990

47/50

Page 125 13 (a) (b)

(a) A topological space X is connected if and only if every continuous function from X into the discrete two element space $\{0, 1\}$ is a constant.

PROOF: ($\Rightarrow$) (Proof by contradiction). Suppose that X is connected and f is a continuous function from X into the discrete two element space $\{0, 1\}$ is not a constant. That is $f(X) = \{0, 1\}$. Now, it follows from theorem (3.78) that X is connected implies $f(X) = \{0, 1\}$ is connected. But consider subsets $\{0\}$ and $\{1\}$ of $\{0, 1\}$, $\{0\} \cap \{1\} = \emptyset$ and $\{0\} \cup \{1\} = \{0, 1\}$, which implies $\{0, 1\}$ is disconnected and is a contradiction. Hence $f(X)$ is a constant.

10/10

($\Leftarrow$) (Proof By Contradiction) Suppose every continuous function from X into $\{0, 1\}$ is constant, but X is disconnected. That is there exist open sets $\mathcal{U}$ and V of X, such that $\mathcal{U} \neq \emptyset \neq V$, $\mathcal{U} \cap V = \emptyset$, $X = \mathcal{U} \cup V$. Now let us define a function from X in $\{0, 1\}$ as follows:

$$f(x) = \begin{cases} 0 & \forall\, x \in \mathcal{U} \\ 1 & \forall\, x \in V. \end{cases}$$

f is a continuous function but $f(X)$ is not a constant. This is a contradiction. Hence, X is connected.

Q.E.D.

Page 125 Problem 13 (b)

Let X be a topological space, let $A \subseteq X$ be connected and let $A \subseteq B \subseteq \bar{A}$. Then B is connected.

Proof: (Proof By Contradiction)

Suppose that B is disconnected. That is there exist open sets $\mathcal{U}$ and V such that $B \cap \mathcal{U} \neq \phi \neq B \cap V$, $B \cap \mathcal{U} \cap V = \phi$ and $B \subseteq \mathcal{U} \cup V$. Now observe that

$$A \cap \mathcal{U} \cap V \subseteq B \cap \mathcal{U} \cap V = \phi \Rightarrow A \cap \mathcal{U} \cap V = \phi \quad \cdots (*)$$

and

$$A \subseteq B \subseteq \mathcal{U} \cup V \Rightarrow A \subseteq \mathcal{U} \cup V. \quad \cdots (**)$$

The remaining task is to show $A \cap \mathcal{U} \neq \phi \neq A \cap V$.

Let $x \in B \cap \mathcal{U}$ and since $B \subseteq \bar{A}$. Hence, either $x \in A$ or x is a limit point of A. (1) If $x \in A \Rightarrow$ $\underline{A \cap \mathcal{U} \neq \phi}$.

(2) If x is a limit point of A and since $\mathcal{U}$ is open containing x, then by definition, we know $\mathcal{U}$ contains at least one point y of A. Hence, $\underline{A \cap \mathcal{U} \neq \phi}$. In both cases, we have $A \cap \mathcal{U} \neq \phi$. By similar argument, we will also obtain $A \cap V \neq \phi$. Together with the facts of $(*)$ and $(**)$, we show A is disconnected and which is a contradiction. Hence, B is connected. ($\bar{A}$ is connected either).

Q.E.D.

10/10

Page 125 Problem 17 (a)

Let X be a metric space and let f and g be complex-valued functions on X that are uniformly continuous on X. Then

$f + g$ is uniformly continuous on X.

Proof: f is uniformly continuous on X. $\Rightarrow$

Let $\varepsilon > 0$ $(\forall \cancel{\tfrac{\varepsilon}{2}} > 0)(\exists \delta_f)(\forall x, y \in X)$ such that

$$\rho(x,y) < \delta_f \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Similarly, g is uniformly continuous on X $\Rightarrow$

$(\forall \cancel{\tfrac{\varepsilon}{2}} > 0)(\exists \delta_g)(\forall x, y \in X)$ such that

$$\rho(x,y) < \delta_g \Rightarrow |g(x) - g(y)| < \frac{\varepsilon}{2}$$

$\frac{10}{10}$

Now $(\forall \varepsilon > 0)(\exists \delta = \min(\delta_f, \delta_g))(\forall x, y \in X)$

$\rho(x,y) < \delta \Rightarrow$

$$\begin{aligned}
|(f+g)(x) - (f+g)(y)| &= |f(x) + g(x) - f(y) - g(y)| \\
&\leq |f(x) - f(y)| + |g(x) - g(y)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&< \varepsilon .
\end{aligned}$$

Hence, by definition, $f+g$ is uniformly continuous on X.

Q.E.D.

Problem 3 on Page 156

If X is a topological space and $f: X \to \mathbb{C}$ is a function, we define the support of f to be the set $\overline{\{x \in X : f(x) \neq 0\}}$. Write

$$C_{00}(X) = \{ f \in C(X) : f \text{ has compact support} \}.$$

If X is a locally compact metric space, then $C_{00}(X)$ is dense in $C_0(X)$.

PROOF: We will show $C_{00}(X)$ is dense in $C_0(X)$ by showing that $C_{00}(X) \subseteq C_0(X)$ is a self-adjoint, sub algebra that separates point and vanishes nowhere.

(i) $f \in C_{00}(X) \Rightarrow f \in C(X) \Rightarrow (\forall \varepsilon > 0)\ \{x \in X : |f(x)| \geq \varepsilon\}$ is closed

more over $\{x \in X : |f(x)| \geq \varepsilon\} \subseteq \overline{\{x \in X : f(x) \neq 0\}}$ which is compact by definition $\Rightarrow (\forall \varepsilon > 0)\ \{x \in X : |f(x)| \geq \varepsilon\}$ is compact.

Hence $f \in C_0(X)$ and $C_{00}(X) \subseteq C_0(X)$.

(ii) If $f \in C_{00}(X) \Rightarrow \overline{\{x \in X : f(x) \neq 0\}}$ is compact.

we note that $\bar{f} \in C(X)$ and $\bar{f}(x) = 0$ iff $f(x) = 0\ \forall x$. Hence

$$\overline{\{x \in X : \bar{f}(x) \neq 0\}} = \overline{\{x \in X : f(x) \neq 0\}} \text{ is compact.}$$

By definition, $\bar{f} \in C_{00}(X)$. (self-adjoint)

6/10

(iii) $\forall f \in C_{00}(X)$ and $\alpha \in \mathbb{C}$

$$\overline{\{x \in X : \alpha f(x) \neq 0\}} = \phi \text{ is compact if } \alpha = 0$$

$$\overline{\{x \in X : \alpha f(x) \neq 0\}} = \overline{\{x \in X : f(x) \neq 0\}} \text{ is compact.}$$

Hence, $\alpha f \in C_{00}(X)$. $\forall \alpha \in \mathbb{C}$ and $\forall f \in C_{00}(X)$.

$\forall f, g \in C_{oo}(X)$

$$\{x\in X : f(x)g(x)\neq 0\} = \{x\in X: f(x)\neq 0\} \cap \{x\in X : g(x)\neq 0\}$$
$$\subseteq \overline{\{x\in X : f(x)\neq 0\}} \cap \overline{\{x\in X : g(x)\neq 0\}}$$

which is compact and closed. Hence

$$\overline{\{x\in X: f(x)g(x)\neq 0\}} \subseteq \overline{\{x\in X: f(x)\neq 0\}} \cap \overline{\{x\in X: g(x)\neq 0\}}$$

is compact. Thus, $(f\cdot g) \in C_{oo}(X)$.

$$\{x\in X : f(x)+g(x)\neq 0\} \subseteq \{x\in X: f(x)\neq 0\} \cup \{x\in X: g(x)\neq 0\}$$

Hence

$$\overline{\{x\in X : f(x)+g(x)\neq 0\}} \subseteq \overline{\{x\in X : f(x)\neq 0\} \cup \{x\in X : g(x)\neq 0\}}$$
$$= \overline{\{x\in X : f(x)\neq 0\}} \cup \overline{\{x\in X : g(x)\neq 0\}}$$

which is again compact. Hence $\overline{\{x\in X: f(x)+g(x)\neq 0\}}$ is compact and $(f+g)\in C_{oo}(X)$.

Thus, <u>$C_{oo}(X)$ is a subalgebra</u>.

(iv) Let ρ be the metric of metric space X. Let $x_0 \in X$ and

$$f(x) \triangleq \rho(x_0, x) \quad \forall\, x\in X. \qquad f \notin C_{oo}(X)$$

f separates points and

$$\overline{\{x\in X : f(x)\neq 0\}}$$

Page 157, Problem 7 (a)

Let X be a compact space and let $L \subseteq C^r(X)$ be a lattice [$f, g \in L$ implies $f \wedge g$, $f \vee g \in L$].

(a) If $f \in C^r(X)$ and if for each $x, y \in X$ and $\varepsilon > 0$ there is some $h \in L$ such that $|f(z) - h(z)| < \varepsilon$ for $z = x$ and $z = y$, then there is an $h \in L$ such that $|f(z) - h(z)| < \varepsilon \ \forall z \in X$.

PROOF: $\forall x, y \in X$, $(\forall \varepsilon > 0)(\exists h_{x,y} \in L)(|f(z) - h_{x,y}(z)| < \varepsilon,\ \forall z = x, y)$

Let $V_{x,y} := \{ z \in X \mid h_{x,y}(z) < f(z) + \varepsilon \}$ $\quad x, y \in V_{xy}$

As shown in the class, $V_{x,y}$ is open. Now fix x, then $\{ V_{x,y} \mid y \in X \}$ is open cover of X and since X is compact. There exist a finite subcover

Why?

$V_{x,y_1}, V_{x,y_2}, \cdots, V_{x,y_n}$. Define

$$g_x \triangleq h_{xy_1} \wedge h_{xy_2} \wedge \cdots \wedge h_{xy_n} \in L$$

$$\Rightarrow \quad g_x(z) < f(z) + \varepsilon \qquad \forall z \in X.$$

8/10

Let $W_x = \{ z \in X \mid g_x(z) > f(z) - \varepsilon \}$. Again W_x is open and $x \in W_x$. Moreover $\{ W_x \mid x \in X \}$ is another open cover of X. Hence there exists a finite subcover of X

Why?

$W_{x_1}, W_{x_2}, \cdots, W_{x_m}$. Now define

$$g \triangleq g_{x_1} \vee g_{x_2} \vee \cdots \vee g_{x_n} \in L$$

Then

$$f(z) - \varepsilon < g(z) < f(z) + \varepsilon \qquad \forall z \in X.$$

Q.E.D.

Why?

Real Analysis 501

Homework No. 4

October 29, 1990

Ben M. Chen

$\frac{40}{50}$

Page 277. Problem 13 (a)

If $f \in L_1^r$ and $\phi \in M_0$, then $f\phi \in L_1^r$.

Proof: Let $\phi = \sum_{j=1}^{n} \alpha_j \xi_{I_j} \in M_0$, where $\alpha_j \in \mathbb{R}$, $j = 1, 2, \cdots, n$.

Then $f\phi = \sum_{j=1}^{n} \alpha_j f \xi_{I_j}$.

Now we need to prove that $f\xi_{I_j} \in L_1^r$ for $j = 1, 2, \cdots, n$.

$f \in L_1^r$ implies that $(\exists g, h \in M_1 \in L_1^r)$ such that $f = g - h$ a.e.

and $f\xi_{I_j} = g\xi_{I_j} - h\xi_{I_j}$, a.e.. By definition of M_1, there exist nondecreasing sequences (ϕ_n) and $(\psi_n) \subseteq M_0$ such that

$$\lim_{n\to\infty} \phi_n(x) = g(x) \text{ a.e and } \lim_{n\to\infty} \psi_n(x) = h(x) \text{ a.e.}$$

10/10

Then it is simple to see that $(\phi_n \xi_{I_j})$ and $(\psi_n \xi_{I_j}) \subseteq M_0 \subseteq L_1^r$

Moreover, they are nondecreasing with $|\phi_n \xi_{I_j}| \le g$, $|\psi_n \xi_{I_j}| \le h$ and

$$\lim_{n\to\infty} \phi_n \xi_{I_j} = g\xi_{I_j} \text{ a.e and } \lim_{n\to\infty} \psi_n \xi_{I_j} = h\xi_{I_j} \text{ a.e.}$$

Apply LDCT, we have $g\xi_{I_j} \in L_1^r$ and $h\xi_{I_j} \in L_1^r$ and

Hence $f\xi_{I_j} = g\xi_{I_j} - h\xi_{I_j} \in L_1^r$, $j = 1, 2, \cdots, n$, which implies

$$f\phi = \sum_{j=1}^{n} \alpha_j f \xi_{I_j} \in L_1^r.$$

Q.E.D.

Page 277, Problem 15 (a)

Suppose that f is a real-valued function defined on the plane $\mathbb{R}^2$ such that, for each fixed x, the function $y \to f(x,y)$ is in L_1^r and each fixed y, the function $x \to f(x,y)$ is continuous on $\mathbb{R}$. Define ϕ on $\mathbb{R}$ by

$$\phi(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$

(a) If there is some function $g \in L_1^r$ such that $|f(x,y)| \le |g(y)|$ for all $y \in \mathbb{R}$ and all x in some open interval $I \subseteq \mathbb{R}$, then ϕ is continuous on I

Proof: $\forall x \in I$, we will show ϕ is continuou at x by showing that $(\forall (x_n) \subseteq I)(x_n \to x$ implies $\phi(x_n) \to \phi(x))$.

Since for each fixed y, the function $x \to f(x,y)$ is continuous.

Hence, $(\forall (x_n) \subseteq I)(x_n \to x$ implies $f(x_n, y) \to f(x,y))$.

10/10

For each fixed x_n, $f(x_n, y) \in L_1^r$. Moreover,

$$|f(x_n, y)| \le |g(y)| \in L_1^r \quad (g \in L_1^r \Rightarrow |g| < L_1^r)$$

Apply LDCT, we have $f(x,y) \in L_1^r$. Furthermore,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x_n, y)\,dy = \lim_{n\to\infty} \phi(x_n) = \int_{-\infty}^{\infty} f(x,y)\,dy = \phi(x).$$

Hence, ϕ is continuous on I.

Q.E.D.

P277 16.

The sequence of functions f_n defined by $f_n(x) = \sin(nx)$ has no subsequence which converges a.e. on $\mathbb{R}$.

Proof: We prove this problem by contradiction. Suppose f_n has a subsequence $(f_{n_k})_{k=1}^{\infty}$ which converges some function f a.e. on $\mathbb{R}$.

Let

$$g_k = (f_{n_k} - f_{n_{k+1}})^2.$$

Then

$$\lim_{k\to\infty} g_k = \lim_{k\to\infty} (f_{n_k} - f_{n_{k+1}})^2$$
$$= (f-f)^2 = 0 \qquad \text{a.e.} \quad \text{on } [0, 2\pi]$$

Let

$$h(x) = 4\cdot\chi_{[0,2\pi]}$$

Then it is simple to see that $h \in L_1^r$ and

$$|g_k \chi_{[0,2\pi]}| < h \in L_1^r$$

Apply LDCT, we have

10/10

$$\lim_{k\to\infty} \int g_k \chi_{[0,2\pi]} = \int_0^{2\pi} 0 = 0.$$

But

$$\int g_k \chi_{[0,2\pi]} = \int_0^{2\pi} [\sin(n_k x) - \sin(n_{k+1} x)]^2\, dx.$$
$$= \cancel{1}\ 2\pi$$

Hence $\lim_{k\to\infty} \int g_k \chi_{[0,2\pi]} = 1$ and this is a contradiction.

Thus, the sequence of $f_n(x) = \sin(nx)$ has no subsequence which converges a.e. on $\mathbb{R}$.

Q.E.D.

P278, Problem 19 (a)

If $a<b$ in $\mathbb{R}^{\#}$ and if $(u_n)_{n=1}^{\infty}$ is a sequence of functions in $L^r_i(a,b)$ satisfying

$$\sum_{n=1}^{\infty} \int_a^b |u_n(x)|\,dx < \infty,$$

then the series

$$f(x) = \sum_{n=1}^{\infty} u_n(x)$$

converges absolutely for almost every $x \in (a,b)$, and the function f thereby defined is in $L^r_i(a,b)$ and satisfies

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx.$$

PROOF: Let $g_k(x) = \sum_{n=1}^{k} |u_n(x)|$. We note that g_k is non-decreasing

$$\int_a^b g_k = \int_a^b \sum_{n=1}^{k} |u_n(x)|\,dx = \sum_{n=1}^{k} \int_a^b |u_n(x)|\,dx \le \sum_{n=1}^{\infty} \int_a^b |u_n(x)|\,dx < \infty$$

Hence, $g_k \in L^r_i(a,b)$. Apply MCT, we have

$$g = \lim_{k\to\infty} g_k = \sum_{n=1}^{\infty} |u_n| \in L^r_i(a,b), \quad \text{a.e.}$$

Hence, $\sum_{n=1}^{\infty} u_n(x)$ converges absolutely a.e. on (a,b) to $g(x)$.

Now, let $f_k(x) = \sum_{n=1}^{k} u_n(x)$. Note that

$$|f_k| = \left|\sum_{n=1}^{k} u_n\right| \le \sum_{n=1}^{k} |u_n| = g_k < g \quad \text{a.e,} \quad g \in L^r_i(a,b)$$

10/10

Apply LDCT, we have

$$\lim_{k\to\infty} f_k(x) = \sum_{n=1}^{\infty} u_n(x) = f(x) \quad \text{a.e.} \quad f \in L^r_i(a,b)$$

Moreover,

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} \int_a^b f_k(x)\,dx$$

$$= \lim_{k\to\infty} \int_a^b \sum_{n=1}^{k} u_n(x)\,dx$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} \int_a^b u_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx.$$ Q.E.D

P279, 22 (b)

If $f \in L_1^r$ is defined everywhere on $\mathbb{R}$ and is uniformly continuous on $\mathbb{R}$, then $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$, and so f is bounded.

Proof: $f \in L_1^r \Rightarrow |f| \in L_1^r$. o.k. Moreover $|f|$ is defined everywhere on $\mathbb{R}$ and is uniformly continuous on $\mathbb{R}$. Let

$$g_n(x) = |f(x)| \chi_{[-n,n]}.$$

Then we have $g_n \in L_1^r$, $\forall n \in \mathbb{N}$, and $\lim_{n\to\infty} g_n(x) = |f(x)|$.

Moreover, $|g_n| \le |f|$. Apply LDCT, we obtain

$$\int |f| = \lim_{n\to\infty} \int g_n$$

or

$$\lim_{n\to\infty} \int |f| - g_n = 0.$$

$\Rightarrow (\forall \varepsilon > 0)(\exists N \in \mathbb{N})$ such that for all $n \ge N$,

$$\int |f| - g_n < \varepsilon \quad \text{------} \quad (*)$$

Since $|f|$ is uniformly continuous on $\mathbb{R} \Rightarrow (\forall \varepsilon > 0)(\exists \delta > k\varepsilon > 0)$ $(|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$.

No!

Now Suppose that there exist a $x_1 \in \mathbb{R}$ $(|x_1| \ge N)$ and

$$|f(x_1)| = a > \left(\frac{\delta + 1}{\delta}\right) \cdot \varepsilon \quad \cdots$$

0/10

Dr. Watkins: I can't finish this problem. Please give me a hint.

Work with $|f|$. Suppose $\lim_{x\to\infty} |f(x)| \ne 0$.

Then $(\exists \varepsilon > 0)(\forall n \in \mathbb{N})(\exists x \ge n)$ $|f(x)| \ge \varepsilon$.

$(\exists \delta > 0)$ $(|x-y| < \delta \Rightarrow ||f(x)| - |f(y)|| < \varepsilon/2)$

Find $x_1 < x_2 < x_3 < x_4 < \ldots$ such that $x_{i+1} > x_i + \delta$ $\forall i$ and $|f(x_i)| \ge \varepsilon$.
Show that $\int |f| = \infty$.

Real Analysis 501

Homework No. 5

November 16, 1990

Ben M. Chen

44

50

Page 310. Problem 14

If $f \in L_1$, and $\varepsilon > 0$, then there exists a continuous function $g: \mathbb{R} \to \mathbb{C}$ which vanishes identically outside some bounded interval and satisfies $\int |f-g| < \varepsilon$.

10/10

Proof: First we consider the special case where $f \in L_1^r$. Then it follows from Theorem (6.18) that there exists a sequence $(\phi_n)_{n=1}^{\infty} \subseteq M_0$ such that

$$\lim_{n\to\infty} \int |f - \phi_n| = 0 \quad \text{and} \quad \lim_{n\to\infty} \phi_n(x) = f(x) \text{ a.e.}$$

Hence $(\forall \varepsilon > 0)$ $(\exists N \in \mathbb{N}$ and $\phi_N \in M_0)$ such that

$$\int |f - \phi_N| < \frac{\varepsilon}{2}.$$

From the definition of step function, we can rewrite

$$\phi_N(x) = \sum_{m=1}^{M} \alpha_m \chi_{I_m} \quad \text{for some } M \in \mathbb{N} \text{ and disjoint intervals}$$

I_m, $m=1,\cdots,M$. Also, we know ϕ_N vanishes identically outside some bounded interval, i.e., $\alpha_m = 0$ if I_m is unbounded. (note that $\bigcup_{m=1}^{M} I_m = \mathbb{R}$). Now let $\alpha = \max\{|\alpha_1|, \cdots, |\alpha_M|\}$ and

$$D = \{ x \in \mathbb{R} \mid \phi_N(x) \text{ is discontinuous on } x \}.$$

We note that D contains at most M points. Let us define

$$D_e = \bigcup_{m=1}^{M} \left(x_m - \frac{\varepsilon}{4M\alpha}, x_m + \frac{\varepsilon}{4M\alpha} \right), \quad x_m \in D$$

and

$$g(x) = \begin{cases} \phi_N(x) & \text{if } x \notin D_e \\ \frac{2M\alpha}{\varepsilon}\left[\phi_N\left(x_m + \frac{\varepsilon}{4\alpha M}\right) - \phi_N\left(x_m - \frac{\varepsilon}{4\alpha M}\right)\right]\cdot\left(x - x_m + \frac{\varepsilon}{4\alpha M}\right) & \\ \quad + \phi_N\left(x_m - \frac{\varepsilon}{4\alpha M}\right) & \text{if } x \in \left(x_m - \frac{\varepsilon}{4M\alpha}, x_m + \frac{\varepsilon}{4\alpha M}\right) \\ & \text{and } x_m \in D. \end{cases}$$

Problem 14 (cont.)

Hence, $g(x)$ is continuous on $\mathbb{R}$ and vanishes identically outside some bounded interval and satisfies

$$\int |\phi_N - g| \leq \alpha \cdot M \cdot \left(\frac{\varepsilon}{4\alpha M} + \frac{\varepsilon}{4\alpha M}\right) = \frac{\varepsilon}{2}.$$

Thus,

$$\int |f - g| = \int |f - \phi_N + \phi_N - g| \leq \int (|f - \phi_N|) + (|\phi_N - g|)$$

$$= \int |f - \phi_N| + \int |\phi_N - g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

Now, let us consider the general case, i.e, $f \in L_1$. Hence, we have $f = \mathrm{Re}\, f + i \cdot \mathrm{Im}\, f$; where $\mathrm{Re}\, f \in L_1^r$ and $\mathrm{Im}\, f \in L_1^r$. Then it follows from the special case we proved above that there exist continuous real valued functions $\mathrm{Re}\, g$ and $\mathrm{Im}\, g$ which vanish identically outside some bounded interval and satisfy

$$\int |\mathrm{Re} f - \mathrm{Re}\, g| < \frac{\varepsilon}{2} \quad \text{and} \quad \int |\mathrm{Im} f - \mathrm{Im}\, g| < \frac{\varepsilon}{2}.$$

Let $\quad g = \mathrm{Re}\, g + i\, \mathrm{Im}\, g$, then

$$\int |f - g| = \int |\mathrm{Re} f - \mathrm{Re}\, g + i(\mathrm{Im} f - \mathrm{Im}\, g)|$$

$$\leq \int |\mathrm{Re} f - \mathrm{Re}\, g| + |i| \cdot \int |\mathrm{Im} f - \mathrm{Im}\, g|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Q.E.D.

Page 314, Problem 28 (d)

Let $(E_n)_{n=1}^{\infty}$ be any sequence of measurable sets. Define

$\underline{\lim}_{n\to\infty} E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. Then $\lambda(\underline{\lim}_{n\to\infty} E_n) \leq \underline{\lim}_{n\to\infty} \lambda(E_n)$.

Proof: $\forall k \in \mathbb{N}$, since $\bigcap_{n=k}^{\infty} E_n \subseteq E_n$, $n = k, k+1, \dots, \dots$

hence $\lambda(\bigcap_{n=k}^{\infty} E_n) \leq \lambda(E_n)$, $n = k, k+1, \dots$.

Thus we have

$$\lambda(\bigcap_{n=k}^{\infty} E_n) \leq \inf_{n\geq k} \lambda(E_n).^{\dagger} \quad \dots\dots \quad \cancel{(*)}$$ obvious!!

Now, let us consider

$$\lambda(\underline{\lim}_{n\to\infty} E_n) = \lambda(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n) = \lambda(\bigcup_{k=1}^{\infty} D_k)$$

10/10

where

$$D_k = \bigcap_{n=k}^{\infty} E_n.$$

We note that $D_k \subseteq D_{k+1}$ $\forall k \in \mathbb{N}$. Hence

$$\lambda(\underline{\lim}_{n\to\infty} E_n) = \lim_{k\to\infty} \lambda(D_k) = \lim_{k\to\infty} \lambda(\bigcap_{n=k}^{\infty} E_n).$$

Using $(*)$, we have

$$\lambda(\underline{\lim}_{n\to\infty} E_n) \leq \lim_{k\to\infty} \inf_{n\geq k} \lambda(E_n) = \underline{\lim}_{n\to\infty} \lambda(E_n). \qquad \text{Q.E.D.}$$

- -

† Subproof of equation $(*)$: By contradiction. Suppose $\lambda(\bigcap_{n=k}^{\infty} E_n) > \inf_{n\geq k} \lambda(E_n)$ which implies that (By definition of infimum) $(\exists E_N, N \geq k)$ such that

$$\lambda(E_N) < \inf_{n\geq k} \lambda(E_N) + \frac{1}{2}\left[\lambda(\bigcap_{n=k}^{\infty} E_n) - \inf_{n\geq k} \lambda(E_n)\right] < \lambda(\bigcap_{n=k}^{\infty} E_n).$$

This is a contradiction since $\lambda(E_N) \geq \lambda(\bigcap_{n=k}^{\infty} E_n)$. q.e.d.

Page 315, Problem 29(a)

Let $(f_n)_{n=1}^{\infty} \subseteq M$ be given. Suppose that

$$\lim_{n,m\to\infty} \lambda(\{x\in\mathbb{R} : |f_m(x)-f_n(x)| \geq \delta\}) = 0$$

for every $\delta>0$. In this case we say that $(f_n)_{n=1}^{\infty}$ is Cauchy in measure. Then there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ and an $f\in M$ such that $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ a.e. Moreover, the original sequence converges in measure to f which means that

$$\lim_{n\to\infty} \lambda(\{x\in\mathbb{R} : |f(x)-f_n(x)| \geq \delta\}) = 0$$

for every $\delta>0$.

Proof: Based on the fact that $\forall\ \delta>0$,

$$\lim_{n,m\to\infty} \lambda(\{x\in\mathbb{R} : |f_m(x)-f_n(x)| \geq \delta\}) = 0,$$

let us construct a sequence as follows:

(i) Let $\delta = \varepsilon = 2^{-1}$, $(\exists\, n_1\in\mathbb{N})$ such that $\forall\, m>n_1$

$$\lambda(\{x\in\mathbb{R} : |f_m(x)-f_{n_1}(x)| \geq \delta = 2^{-1}\}) < \varepsilon = 2^{-1}.$$

(ii) Let $\delta = \varepsilon = 2^{-2}$, $(\exists\, n_2\in\mathbb{N},\ n_2>n_1)$ such that $\forall\, m>n_2$

$$\lambda(\{x\in\mathbb{R} : |f_m(x)-f_{n_2}(x)| \geq \delta = 2^{-2}\}) < \varepsilon = 2^{-2}.$$

(iii) . . . . . .

By induction, we obtain a sequence f_{n_k}, $k=1,2,\ldots$, which satisfies

$$\lambda(E_k) < 2^{-k}$$

where

$$E_k = \{x\in\mathbb{R} : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 2^{-k}\}.$$

We now prove that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ a.e. for some $f \in M$.

To do this suppose that

$$P_j = \bigcup_{k=j}^{\infty} E_k \quad , \quad E = \bigcap_{j=1}^{\infty} P_j = \overline{\lim_{k\to\infty}}\, E_k .$$

Then since $\sum_{k=1}^{\infty} \lambda(E_k) = \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty$, by 28 (g), $\lambda(E) = 0$.

We will show that for any $x_0 \in \mathbb{R}\setminus E$, $f_{n_k}(x_0)$ converges.

$\forall\, x_0 \in \mathbb{R}\setminus E$, $(\exists\, j_0 \in \mathbb{N})$ such that $x_0 \notin P_{j_0}$, so that from the definitions of P_j and E_k

$$x_0 \notin E_k = \{x \in \mathbb{R} : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 2^{-k}\}$$

for all $k \geq j_0$. It follows that

$$|f_{n_{k+1}}(x_0) - f_{n_k}(x_0)| < 2^{-k}$$

Since $\sum_{k=j_0}^{\infty} |f_{n_{k+1}}(x_0) - f_{n_k}(x_0)| \leq \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty$, hence

$$f_{n_k}(x_0) \quad , \quad k = 1, 2, \cdots$$

15/20

converges, i.e,

Why?

$$\lim_{k\to\infty} f_{n_k}(x_0) = f(x_0) . \quad \forall\, x_0 \in \mathbb{R}\setminus E$$

which implies (due to $\lambda(E) = 0$)

$$\lim_{k\to\infty} f_{n_k}(x) = f(x) \quad \text{a.e.}$$

Since $f_{n_k} \in M$, $f \in M$. Notice that

Verify this! $\{x \in \mathbb{R} : |f(x) - f_n(x)| \geq \delta\} \subseteq \underline{\lim_{k\to\infty}} \{x \in \mathbb{R} : |f_{n_k}(x) - f_n(x)| \geq \frac{\delta}{2}\}$

Using (28) (d), we have

$$\lambda\{x \in \mathbb{R} : |f(x) - f_n(x)| \geq \delta\} \leq \underline{\lim_{k\to\infty}}\, \lambda\{x \in \mathbb{R} : |f_{n_k}(x) - f_n(x)| \geq \tfrac{\delta}{2}\} < \varepsilon$$

for all suffiently large n. Hence f_n converges in measure to f. Q.E.D.

Why?

Page 315, Problem 29 (e)

If $\lim_{n,m\to\infty} \int |f_m - f_n| = 0$, then (f_n) is Cauchy in measure.

Proof:

9/12

We will prove this problem by contradiction. Suppose (f_n) is not Cauchy in measure, i.e. for some $\delta > 0$

$$\lim_{n,m\to\infty} \lambda(\{x\in\mathbb{R} : |f_m(x) - f_n(x)| \geq \delta\}) \neq 0.$$

This implies that ($\exists \alpha > 0$) there exist some $m, n \geq N$ for any $N \in \mathbb{N}$ such that

$$\lambda(\{x\in\mathbb{R} : |f_m(x) - f_n(x)| \geq \delta\}) \geq \alpha > 0$$

Let us define

$$D_{mn} = \{x\in\mathbb{R} : |f_m(x) - f_n(x)| \geq \delta\}$$

and a new function

$$g_{mn}(x) = \begin{cases} \delta & \text{if } x \in D \\ 0 & \text{if } x \notin D. \end{cases}$$

Hence

$$|g(x)| \leq |f_m(x) - f_n(x)| \quad \forall x \in \mathbb{R}$$

Moreover

$$\int |f_m - f_n| \geq \int g = \int \delta \chi_{D_{mn}} = \delta\lambda(D_{mn}) \geq \alpha\delta > 0$$

Independent of m and n

which implies that $\lim_{n,m\to\infty} \int |f_m - f_n| \neq 0$. This is a contradiction. Hence the result.

Q.E.D.

Real Analysis 501

38/50

Homework No. 6

December 10, 1990

Ben M. Chen

Page 163, Problem 1

Let $[a,b] \subset \mathbb{R}$ and let $BV([a,b])$ be the set of all complex-valued functions f on $[a,b]$ such that $V_a^b f < \infty$. For $f \in BV([a,b])$ define the variation norm of f to be the number

$$\|f\|_v = |f(a)| + V_a^b f$$

and let

$$\|f\|_u = \sup\{|f(x)| : x \in [a,b]\}$$

denote, as usual, the uniform norm of f. If $f, g \in BV([a,b])$ and $\alpha \in \mathbb{C}$, then

(e) $fg \in BV([a,b])$ and $\|fg\|_v \le \|f\|_u \|g\|_v + \|f\|_v \|g\|_u$.

Proof: Consider a subdivision of $[a,b]$

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}.$$

We have

$$\begin{aligned}
&|f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})| \\
&= |f(x_k)g(x_k) - f(x_{k-1})g(x_k) + f(x_{k-1})g(x_k) - f(x_{k-1})g(x_{k-1})| \\
&\le |g(x_k)| \cdot |f(x_k) - f(x_{k-1})| + |f(x_{k-1})| \cdot |g(x_k) - g(x_{k-1})| \\
&\le \|f\|_u \cdot |g(x_k) - g(x_{k-1})| + \|g\|_u \cdot |f(x_k) - f(x_{k-1})|.
\end{aligned}$$

10/10

This implies $V_a^b(fg) \le \|f\|_u \cdot V_a^b g + \|g\|_u \cdot V_a^b f < \infty$, since $f, g \in BV([a,b])$ implies that $\|f\|_u < \infty$, $V_a^b g < \infty$, $\|g\|_u < \infty$ and $V_a^b f < \infty$. Hence, $fg \in BV([a,b])$. Then we have

$$\begin{aligned}
\|fg\|_v &= |f(a)| \cdot |g(a)| + V_a^b(fg) \le \|f\|_u \cdot |g(a)| + |f(a)| \cdot \|g\|_u + V_a^b(fg) \\
&\le \|f\|_u \cdot |g(a)| + |f(a)| \cdot \|g\|_u + \|f\|_u \cdot V_a^b g + \|g\|_u \cdot V_a^b f \\
&= \|f\|_u \cdot \|g\|_v + \|f\|_v \cdot \|g\|_u.
\end{aligned}$$

Q.E.D.

P163, Problem 1 (g)

(g) Defining $\rho(f,g) = \|f-g\|_v$ makes $BV([a,b])$ a complete metric space.

Proof: First we show that ρ is a metric. It is simple to verify that

(i) $\rho(f,f) = \|0\|_v = 0 \quad \forall f \in BV([a,b])$.

(ii) $\rho(f,g) = \|f-g\|_v$

$= |f(a)-g(a)| + V_a^b(f-g) > 0$ if $f \neq g$.

(iii) $\rho(f,g) = \|f-g\|_v = |f(a)-g(a)| + V_a^b(f-g)$

$= |g(a)-f(a)| + V_a^b(g-f)$

$= \rho(g,f) \quad \forall f,g \in BV([a,b])$.

(iv) $\rho(f,g) = |f(a)-g(a)| + V_a^b(f-g)$

$= |f(a)-h(a)+h(a)-g(a)| + V_a^b(f-h+h-g)$

$\leq |f(a)-h(a)| + V_a^b(f-h) + |h(a)-g(a)| + V_a^b(h-g)$

$= \rho(f,h) + \rho(h,g)$.

Thus ρ is a metric in $BV([a,b])$.

$\frac{0}{10}$

Page 164, Problem 4.

Let $f:[a,b] \to \mathbb{R}$ be of finite variation on $[a,b]$. Define g on $[a,b]$ by $g(x) = V_a^x f$. Then f is continous at $x \in [a,b]$ if and only if g is continous at x.

Proof: ($\Leftarrow$) Assume that g is continous at x. Then $\forall \varepsilon > 0$ $(\exists \delta > 0)$ such that for all $y \in [a,b]$, $|y-x| < \delta$ $|g(y) - g(x)| < \varepsilon$.

Case 1: $y \geq x$, $|g(y) - g(x)| = |V_a^y f - V_a^x f|$

$= V_x^y f < \varepsilon$ implies that

$|f(y) - f(x)| \leq V_x^y f < \varepsilon$

Case 2: $y < x$, $|g(y) - g(x)| = |V_a^x f - V_a^y f|$

$= V_y^x f < \varepsilon$ which implies that

$|f(y) - f(x)| \leq V_y^x f < \varepsilon$.

5/10

Hence f is continuous at x.

($\Rightarrow$) Since g is nondecreasing with $g(b) = V_a^b f < \infty$, g has atmost countably many simple discontinuities. Now suppose that x is one of such discontinous point on $[a,b]$ and also without loss of generality we assume $g(x+) \neq g(x)$. [note that in case of $g(x-) \neq g(x)$, same argument applies].

We note that since $f(x+)$ exists due to $V_a^b f < \infty$,

$$0 < g(x+) - g(x) = \lim_{\delta \to 0+} V_x^{x+\delta} f = |f(x+) - f(x)|.$$

Thus f is also discontinous at x. Q.E.D.

The whole problem is to prove this.

Page 343, Problem 1.

Let $0 < r < s < \infty$ be given.

(a) If $f \in M$ and $E \in \mathfrak{m}$, then

$$\left(\int_E |f|^r\right)^{\frac{1}{r}} \leq \left(\int_E |f|^s\right)^{\frac{1}{s}} (\lambda(E))^{\frac{1}{r}-\frac{1}{s}}.$$

Proof: Let $p = \frac{s}{r} > 1$. Then

$$p' = \frac{p}{p-1} = \frac{\frac{s}{r}}{\frac{s}{r}-1} = \frac{s}{s-r} = \frac{1}{1-\frac{r}{s}} > 1.$$

Using Hölder inequality, we have

$$\int_E |f|^r = \int \left| |f|^r \cdot \chi_E \right| = \int \left| |f \cdot \chi_E|^r \cdot \chi_E \right|$$

$$\text{(Hölder)} \leq \left(\int |f \cdot \chi_E|^{r \cdot p}\right)^{\frac{1}{p}} \cdot \left(\int |\chi_E|^{p'}\right)^{\frac{1}{p'}}$$

$$= \left(\int |f \cdot \chi_E|^{r \cdot \frac{s}{r}}\right)^{\frac{r}{s}} \cdot \left(\int \chi_E\right)^{1-\frac{r}{s}}$$

5/5

$$= \left(\int_E |f|^s\right)^{\frac{r}{s}} \cdot (\lambda(E))^{r\left(\frac{1}{r}-\frac{1}{s}\right)} \quad \cdots \quad (*)$$

This implies that

$$\left(\int_E |f|^r\right)^{\frac{1}{r}} \leq \left(\int_E |f|^s\right)^{\frac{1}{s}} \cdot (\lambda(E))^{\frac{1}{r}-\frac{1}{s}}.$$

Q.E.D.

P. 343, Problem 1 (cont.)

(b) If $E \in \mathfrak{m}$ and $\lambda(E) < \infty$, then $L_s(E) \subset L_r(E)$.

Proof: $\forall f \in L_s(E)$. By definition, we have

$$\int_E |f|^s < \infty$$

which implies that

$$\left(\int_E |f|^s\right)^{\frac{r}{s}} < \infty .$$

Recall (*) in the proof of part (a),

$$\int_E |f|^r \le \left(\int_E |f|^s\right)^{\frac{r}{s}} \cdot (\lambda(E))^{1-\frac{r}{s}}$$

$$< \infty ,$$

5/5

which implies that $f \in L_r(E)$ and hence

$L_s(E) \subseteq L_r(E)$. Q.E.D.

P. 343 Problem 1 (cont.)

(c) $L_r([0,1]) \setminus L_s([0,1]) \neq \emptyset$.

Proof: Let

$$f(x) = x^{-\frac{1}{s}} \quad \forall\, x \in [0,1].$$

Then

$$\int_0^1 |f|^s = \int_0^1 \frac{1}{x} = \infty \Rightarrow f \notin L_s([0,1])$$

But since $r < s$,

$$\int_0^1 |f|^r = \int_0^1 x^{-\frac{r}{s}} = \frac{1}{1-\frac{r}{s}} \cdot x^{(1-\frac{r}{s})}\Big|_0^1$$

$$= \frac{s}{s-r} < \infty$$

5/5

Hence, $f \in L_r([0,1])$. Moreover,

$$f \in L_r([0,1]) \setminus L_s([0,1]).$$

Thus

$$L_r([0,1]) \setminus L_s([0,1]) \neq \emptyset.$$

Q.E.D.

Page 343, Problem 3(a)

If $0<r<s<t<\infty$ and $f\in L_r\cap L_t$, then $f\in L_s$ and

$\|f\|_s^s \le \|f\|_r^{r\alpha}\cdot\|f\|_t^{t(1-\alpha)}$, where $s=\alpha r+(1-\alpha)t$

Proof: $s=\alpha r+(1-\alpha)t$ implies that

$$0<\alpha=\frac{t-s}{t-r}<1.$$

Let $p=\frac{1}{\alpha}>1$ and $p'=\frac{p}{p-1}=\frac{1}{1-\alpha}$.

Now consider

$$\|f\|_s^s=\int|f|^s=\int\left||f|^{\alpha r}\cdot|f|^{(1-\alpha)t}\right|$$

$$\text{(Hölder)}\quad \le\left(\int|f|^{\alpha r p}\right)^{\frac{1}{p}}\cdot\left(\int|f|^{(1-\alpha)tp'}\right)^{\frac{1}{p'}}$$

$$=\left(\int|f|^r\right)^{\alpha}\cdot\left(\int|f|^t\right)^{(1-\alpha)}\quad\cdots\quad(*)$$

5/5

Since $f\in L_r\cap L_t\Rightarrow\int|f|^r<\infty,\ \int|f|^t<\infty$.

Hence $\int|f|^s\le\left(\int|f|^r\right)^{\alpha}\cdot\left(\int|f|^t\right)^{(1-\alpha)}<\infty$.

$\Rightarrow f\in L_s$. Moreove, it follows from $(*)$ that

$$\|f\|_s^s\le\|f\|_r^{r\alpha}\cdot\|f\|_t^{t(1-\alpha)}.\qquad \text{Q.E.D.}$$