# Math 501

**First Midterm Examination** **October 5, 1990**

1. (20 points) Let $(X, \mathcal{T})$ be a topological space, and let $F \subseteq X$. Prove that if $F$ contains all of its limit points, then $F$ is closed. (Do not invoke theorems; proceed from the definition of *closed*.)

2. Let $(X, \mathcal{T})$ be a locally compact Hausdorff space, and let $p$ be a point not in $X$. Define a topological space $(Y, \mathcal{U})$ by $Y = X \cup \{p\}$ and $\mathcal{U} = \mathcal{T} \cup \{G \cup \{p\} \mid G \in \mathcal{T} \text{ and } X \setminus G \text{ is compact.}\}$. (Reminder: a space is *locally compact* iff every point has a neighborhood whose closure is compact.)

   (a) (20 points) Show that $(Y, \mathcal{U})$ is compact.

   (b) (20 points) Show that $(Y, \mathcal{U})$ is a Hausdorff space.

3. Let $\mathcal{Q}$ be the ~~set~~ following set of even polynomials:

$$\mathcal{Q} = \left\{ \sum_{m=2}^{n} a_m x^{2m} \mid n \geq 2,\ a_2, \ldots, a_n \in \mathbb{R} \right\}$$

   Notice that the sum starts at $m = 2$.

   (a) (10 points) Prove or disprove that $\mathcal{Q}$ is dense in $C^r[0, 1]$.

   (b) (10 points) Prove or disprove that $\mathcal{Q}$ is dense in $C^r[1, 2]$.

4. (20 points) Let $X$ be a topological space, and let $D$ be a dense subset of $X$. Let $\{f_n\}$ be an equicontinuous sequence of functions mapping $X$ into $\mathbb{R}$. Suppose that for each $d \in D$, $\{f_n(d)\}$ is a convergent sequence. Show that for each $x \in X$, $\{f_n(x)\}$ is convergent. (Hint: Show it's Cauchy. Use an $\epsilon/3$ argument.)

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Problem 1:

F is closed if $F'$ is open.

We proof this problem by contradition: Suppose that $F'$ is not open. By definition that there exists $x \in F'$ such that every nbhd $V$ of $x$ is not contained in $F'$ which implies $V$ contains at least one point of $F$. Hence $x$ is a limit point of $F$.

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And this is a contradiction. Hence $F$ is closed.

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Problem 2

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(b) $(X, \tau)$ is a Hausdorff space $\Rightarrow$

$\forall\, x, y \in X \quad x \neq y \quad \exists\, U, V \in \tau.$

such that

$x \in U$ and $y \in V$. but $U \cap V = \phi$.

(i) Now, $\forall\, x, y \in Y$ if $x, y \in X$. we note that

$U, V \in \mathcal{U}$ also. Hence, pick the same

$U, V$ as above will do the job.

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(ii) The second case is that we can only have one of $x$ and $y \notin X$.

W.L.O.G: assume $x \in X$ and $y = p$.

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Problem 3.

$$Q = \left\{ \sum_{m=2}^{n} a_m x^{2m} \;\middle|\; n \geq 2,\ a_2, \dots, a_n \in \mathbb{R} \right\}$$

(a) $Q$ is not dense in $C^r[0,1]$ since $Q$ vanishes at $\{0\} \in [0,1]$.

Does this prove it?

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$$\left. \sum_{m=2}^{n} a_m x^{2m} \right|_{x=0} = 0.$$

(b) $Q$ is dense in $C^r[1,2]$.

① Let $\alpha \in \mathbb{R}$ and $P_n = \sum_{m=2}^{n} a_m x^{2m} \in Q$

$$\Rightarrow \alpha P_n = \sum_{m=2}^{n} (\alpha a_m) x^{2m} \in Q$$

since $\alpha a_m \in \mathbb{R}$, $m = 2, \dots, n$.

② Let $P_n = \sum_{m=2}^{n} a_m^n x^{2m}$ and $P_k = \sum_{m=2}^{k} a_m^k x^{2m}$

With loss of generality, we assume $n \geq k$

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$$P_n + P_k = \sum_{m=2}^{k} (a_m^n + a_m^k) x^{2m} + \sum_{m=k+1}^{n} a_m^n x^{2m} \in Q$$

③ $$P_n \cdot P_k = \sum_{m=2}^{n} a_m^n x^{2m} \cdot \sum_{m=2}^{k} a_m^k x^{2m} \in Q.$$

Moreover. $p = x^4 \in Q$ is separates point and vanishes nowhere. From S-W theorem $Q$ is dense in $C^r[1,2]$.

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Problem 4

① $\{f_n\}$ is an equicontinuous sequence of functions mapping X into $\mathbb{R}$. implies

$(\forall \cancel{\varepsilon/3}\ \varepsilon > 0)(\exists N_1 \in \mathbb{N})(\forall x \in X)(\forall m, n > N_1)$

$|f_n(x) - f_m(x)| < \frac{\varepsilon}{3}$. No!

② $d \in D$ $\{f_n(d)\}$ is a convergent sequence.

Hence

$(\forall \varepsilon/3 > 0)(\exists N_2 \in \mathbb{N})(\forall n > N_2)(d_0 \in \mathbb{R})$

$|f_n(d) - d_0| < \frac{\varepsilon}{3}$

③ Now if $x \in D$, by assumption that $f_n(x)$ is a convergent sequence. If $x \notin D$. since D is dense, x is a limit point of D.

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Hence

$(\forall \cancel{\varepsilon/3}\ \varepsilon > 0)(\exists$ nbhd V of $x)(\exists d \in V)$

$|f_n(x) - f_n(d)| < \frac{\varepsilon}{3}$ $\forall n \in \mathbb{N}$. Yes! Equicontinuity

④ Let $m, n > \max\{N_1, N_2\}$.

$$|f_n(x) - d_0| = |f_n(x) - f_m(x) + f_m(x) - f_m(d) + f_m(d) - d_0|$$ ?

$$\leq |f_n(x) - f_m(x)| + |f_m(x) - f_m(d)| + |f_m(d) - d_0| < \varepsilon.$$

Hence, $\{f_n(x)\}$ is convergent $\forall x \in X$. Q.E.D.

~~79%~~ 82%

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# Math 501

**Second Midterm Examination, Part I** **November 7, 1990**

1. (15 points) Given $E \subseteq \mathbb{R}$ and $a \in \mathbb{R}$, define $E_a \subseteq \mathbb{R}$ by $E_a = E + a = \{x + a \mid x \in E\}$. Show that $E_a \in \mathcal{M}$ if $E \in \mathcal{M}$. (Note: $\mathcal{M}$ denotes the set of Lebesgue measurable sets.)

2. (15 points) Give a sequence of functions $(f_n) \subseteq L_1$ and $f \in L_1$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e., but $\lim_{n\to\infty} \int f_n \neq \int f$.

3. (20 points) State and prove Fatou's lemma.

   (Hint: $\liminf_{n\to\infty} f_n(x) = \lim_{k\to\infty} \inf_{n\geq k} f_n(x)$, so consider the sequence $(g_k)$ defined by $g_k(x) = \inf_{n\geq k} f_n(x)$.)

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Problem 1 :

$E \in \mathfrak{M}$ implies that $\chi_E \in M$. We note that

$$\chi_{E_a}(x) = \chi_E(\overset{x-a}{\cancel{x+a}}) \quad \forall x \in \mathbb{R}.$$

Hence, by structure measurable theorem, we have.

??

$$(\forall \alpha \in \mathbb{R}) \ \{x \in \mathbb{R} : \chi_E(x) < \alpha\}$$

$$= \{x \in \mathbb{R} : \chi_E(\overset{x-a}{\cancel{x+a}}) < \alpha\}$$

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$$= \{x \in \mathbb{R} : \chi_{E_a}(x) < \alpha\} \in \mathfrak{M}$$

Again by structure of measurable sets theorem,

$$\chi_{E_a} \in M$$

Hence, $E_a \in \mathfrak{M}$. Q.E.D

Problem 2:

Let $f_n = n \chi_{[n, n+\frac{1}{n}]}$.

Then $\forall n \in \mathbb{N}$, $f_n \in L_1$ and

$$\int f_n = \int n \chi_{[n, n+\frac{1}{n}]} = \cancel{1}\ .\ n$$

15 ~~12~~ 15

Let $f = 0$.

Then it is obvious that

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \text{a.e.}$$

But

$$\lim_{n \to \infty} \int f_n = \cancel{1}\ \infty \neq \int f = 0.$$

Problem 3:

Fatou's lemma: Given $(f_n)_{n=1}^{\infty} \subseteq \cancel{L_1^r}\ M^r$

$$\int \liminf_{n\to\infty} f_n(x) \leq \liminf_{n\to\infty} \int f_n(x).$$

Proof: Define

$$g_{km}(x) = \inf_{k \leq n \leq k+m} f_n(x).$$

Hence, we have $(g_{km})_{m=1}^{\infty} \subseteq \cancel{L_1^r}$ and

$$g_{k1} \geq g_{k2} \geq \cdots$$

$$\int \lim_{m\to\infty} g_{km} \leq \int g_{k1} < \infty$$

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Hence

$$\lim_{m\to\infty} g_{km} = g_k = \inf_{n \geq k} f_n(x) \in \cancel{L_1^r}$$

Now from above, we have

$$g_1 \leq g_2 \leq g_3 \leq \cdots$$

By $\int \liminf_{n\to\infty} f_n \boxed{= \int \lim_{k\to\infty} g_k \leq \lim_{k\to\infty} \int g_k}$ Why?

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# Math 501

**Second Midterm Examination, Part II** **November 9, 1990**

4. (10 points) Show that if $f \in L_1^r$ and $\alpha \in \mathbb{R}$, then $\alpha f \in L_1^r$. (You may use any relevant facts about $M_1$ without proving them.)

5. (20 points) Show that if $f \in L_1$, then $\lim_{n\to\infty} \int_{-n}^{n^2} f = \int f$.

6. (20 points) Let $(f_n)_{n=1}^{\infty} \subseteq M$ be a sequence that converges uniformly to $f \in M$. Suppose there is a set $E \in \mathcal{M}$ with $\lambda(E) < \infty$, such that for all $n$, $f_n(x) = 0$ if $x \notin E$. Show that $\lim_{n\to\infty} \int |\, f - f_n \,| = 0$.

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Problem 4. $f \in L_1^r$ implies that $(\exists g, h \in M_1)$

$$f = g - h \quad \text{a.e.}$$

Now, if $\alpha \in \mathbb{R}$, $\alpha \geq 0$, then $\alpha g \in M_1$ and $\alpha h \in M_1$. Hence $\alpha f = (\alpha g) - (\alpha h)$ a.e implies that $\alpha f \in L_1^r$.

If $\alpha < 0$,

$$\alpha f = \alpha g - \alpha h \quad \text{a.e}$$

$$= (-\alpha) h - (-\alpha) g \quad \text{a.e}$$

Since $-\alpha h \in M_1$ and $-\alpha g \in M_1$ due to $-\alpha > 0$, hence $\alpha f \in L_1^r$. Q.E.D.

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Problem 5. $f \in L_1 \Rightarrow |f| \in L_1$

Now let us define a sequence

$$f_n = f \cdot \chi_{[-n, n^2]} \quad , n = 1, 2, 3, \cdots$$

Hence,

$$|f_n| = |f \cdot \chi_{[-n, n^2]}| \leq \overset{\in L_1}{|f|} \quad \forall n \in \mathbb{N}.$$

Moreover

$$\lim_{n \to \infty} f_n = f \quad \text{a.e.}$$

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Thus apply LDCT, we have

$$\int f = \lim_{n\to\infty} \int f_n$$

$$= \lim_{n\to\infty} \int f \cdot \chi_{[-n, n^2]}$$

$$= \lim_{n\to\infty} \int_{-n}^{n^2} f \ldots \qquad \text{Q.E.D.}$$

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$\lambda(E)=0?$ (3)

Problem 6 $(f_n)_{n=1}^{\infty} \subseteq M$, $f_n(x)=0 \quad \forall\, x\in \mathbb{R}\setminus E \quad \forall n$

and $f_n(x) \to f(x)$ uniformly implies that

$$f(x) \equiv 0 \quad \forall\, x \in \mathbb{R}\setminus E.$$

Now we focus our attention on the set $E$. Let

$\lambda(E) = a < \infty$. We note that

$$f_n(x) \longrightarrow f(x) \text{ uniformly on } E.$$

Hence, $(\forall \varepsilon>0)\ (\exists N\in\mathbb{N})\ \forall\, n\geq N$

20/20

$$|f_n(x) - f(x)| < \frac{\varepsilon}{a} \quad \forall\, x\in E.$$ (what if a=0?)

Let $g(x) \triangleq \begin{cases} \frac{\varepsilon}{a} & \forall\, x\in E \\ 0 & \forall\, x\notin E \end{cases}$ $g\in M$.

Hence $|f_n(x) - f(x)| \leq g(x)$

$$\Rightarrow \int |f_n - f| \leq \int g = \int_E \frac{\varepsilon}{a}\chi_E = \varepsilon.$$

Hence $\lim_{n\to\infty} \int |f_n - f| = 0$. Q.E.D.

Grade in Math 501: A

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I hope to see you in 502!

171/175 ← Outstanding!

# Math 501

**Final Examination** **December 18, 1990**

**There are 175 points possible.**

1. (10 points) Prove that every countable subset of $\mathbb{R}$ has Lebesgue outer measure zero.

2. (a) (10 points) Give an example of a sequence $(f_n) \subseteq L_1$ such that $\lim_{n\to\infty} f_n(x) \to 0$ a.e., but $\lim_{n\to\infty} \int |f_n| \neq 0$.

   (b) (10 points) Give an example of a sequence $(f_n) \subseteq L_1$ such that $\lim_{n\to\infty} \int |f_n| = 0$, but $(f_n)$ does not converge to 0 uniformly.

3. (a) (10 points) What does Lusin's Theorem say?

   (b) (10 points) Give an example that shows that Lusin's Theorem does **not** imply that for every $\epsilon > 0$ there is a set $S_\epsilon \in \mathcal{M}$ such that $\lambda(S_\epsilon) < \epsilon$ and $f$ is continuous at each point of $\mathbb{R} \setminus S_\epsilon$.

4. (a) (10 points) Show that $L_2(0,1) \subseteq L_1(0,1)$.

   (b) (10 points) Show that $L_2(1,\infty) \not\subseteq L_1(1,\infty)$.

5. (20 points) Let $(f_n)$ be a sequence in $C[0,1]$, and suppose $f_n \to f$ uniformly on $[0,1]$. Prove that $f$ is uniformly continuous on $[0,1]$.

6. Let $\mathcal{Q}$ be the set of all even polynomials:

$$\mathcal{Q} = \left\{ \sum_{m=0}^{n} a_m x^{2m} \mid n \geq 0,\ a_0, \ldots, a_n \in \mathbb{C} \right\}$$

   (a) (10 points) Prove or disprove that $\mathcal{Q}$ is dense in $C[0,1]$.

   (b) (10 points) Prove or disprove that $\mathcal{Q}$ is dense in $C[-1,1]$.

7. (15 points) Use the appropriate convergence theorem to prove the following result directly: If $(E_n)_{n=1}^{\infty} \subseteq \mathcal{M}$, $E_n \supseteq E_{n+1}$ for all $n$, $\lambda(E_1) < \infty$, and $E = \bigcap_{n=1}^{\infty} E_n$, then $\lambda(E) = \lim_{n\to\infty} \lambda(E_n)$.

8. Let $f,\ g \in L_1(\mathbb{R})$.

   (a) (10 points) Show that the function $h(x,y) = f(x-y)g(y)$ is in $L_1(\mathbb{R}^2)$.

   (b) (15 points) Define the convolution $f * g$ by

   $$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

   Show that $f * g \in L_1(\mathbb{R})$ and $\| f * g \|_1 \leq \| f \|_1 \| g \|_1$.

   In each part state clearly which theorem(s) you are using.

9. Let $f$ and $g$ be absolutely continuous, real-valued functions defined on $[0,1]$.

   (a) (5 points) Then, of course, $f$ and $g$ are bounded. Explain why this is so.

   (b) (10 points) Let $M$ be an upper bound for both $|f(x)|$ and $|g(x)|$ on $[0,1]$. Derive an upper bound for the expression

   $$|f(x)g(x) - f(y)g(y)| \qquad (x,\ y \in [0,1])$$

   that will be useful to you in part (c).

   (c) (10 points) Show that the product $fg$ is absolutely continuous.

**Final Exam** **Real Analysis 501** **Ben M. Chen**

Problem 1

We first show that $\{a\}$ has a Lebesgue outer measure zero. By definition

$$\lambda\{a\} = \inf \Big\{ \sum_{I_i \in \mathcal{I}} |I_i| \;\Big|\; \{a\} \subseteq \bigcup I_i \text{ and } \mathcal{I} \text{ is a family of countably many open sets} \Big\}$$

Now, let us construct a sequence of open intervals

$$I_n = \left(a - \tfrac{1}{n}, a + \tfrac{1}{n}\right), \quad n = 1, 2, \ldots$$

we note $\{a\} \subseteq \left(a - \frac{1}{n}, a + \frac{1}{n}\right) \quad \forall n \in \mathbb{N}$.

Then $\forall \varepsilon > 0 \quad \exists N = \lceil \frac{1}{\varepsilon} \rceil \cdot 2 \in \mathbb{N}$ such that

$$\lambda\{a\} < \lambda\left(a - \tfrac{1}{n}, a + \tfrac{1}{n}\right) = \tfrac{1}{2n} \leq \varepsilon \quad \forall n \geq N.$$

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Thus $\lambda\{a\} = 0$.

Let $A = \{a_1, a_2, \cdots\}$, we have

$$\lambda(A) \leq \sum_{i=1}^{\infty} \lambda\{a_i\} = 0.$$

Q.E.D.

Problem 2.

(a)
$$f_n = \chi_{[n,\, n+1]}, \quad n = 1, 2, \cdots$$

Then $\lim\limits_{n\to\infty} f_n(x) \to 0$ a.e. and

$$\int f_n = \int \chi_{[n,\, n+1]} = 1 \Rightarrow f_n \in L_1$$

$\frac{10}{10}$

But $\lim\limits_{n\to\infty} \int |f_n| = 1 \neq 0$.

(b)
$$f_n = \chi_{[n,\, n+\frac{1}{n}]}, \quad n = 1, 2, \cdots$$

$$\int |f_n| = \int \chi_{[n,\, n+\frac{1}{n}]} = \frac{1}{n} \Rightarrow f_n \in L_1$$

$\frac{10}{10}$

and

$$\lim_{n\to\infty} \int |f_n| = \lim_{n\to\infty} \frac{1}{n} = 0.$$

But $(f_n)$ does not converge to 0 uniformly.

Problem 3.

(a) Lusin's Theorem: For any $f \in M$ and $\delta > 0$, There exists a continuous function $g: \mathbb{R} \to \mathbb{C}$ such that $\lambda\{x \in \mathbb{R} : f(x) \neq g(x)\} < \delta$. Moreover, if $|f(x)| \le \beta$ a.e., then $g$ can be choosen such that $|g(x)| \le \beta$.

$\frac{10}{10}$

(b) Let $f = \chi_{\mathbb{Q}}$, where $\mathbb{Q}$ is the set of rational number, we have $\lambda(\mathbb{Q}) = 0 < \varepsilon$.

$\frac{10}{10}$

But $f$ is not continous anywhere ~~in $\mathbb{R}\setminus\mathbb{Q}$~~. In fact, $f|_{\mathbb{R}\setminus\mathbb{Q}}$ is continous.

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Problem 4

(a) Let $f \in L_2(0,1)$ which implies that

$$\int |f|^2 \chi_{(0,1)} < \infty \implies \left(\int |f|^2 \cdot \chi_{(0,1)}\right)^{\frac{1}{2}} < \infty$$

Now, consider

$$\int |f| \chi_{(0,1)} = \int |f \chi_{(0,1)}| = \int |f \chi_{(0,1)} \cdot \chi_{(0,1)}|$$

Let $p=2$, $p'=2$ which satisfy $\frac{1}{p} + \frac{1}{p'} = 1$.

Thus by Hölder inequality, we have

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$$\int |f \chi_{(0,1)} \cdot \chi_{(0,1)}| \leq \left(\int |f|^2 \chi_{(0,1)}\right)^{\frac{1}{2}} \underbrace{\left(\int |\chi_{(0,1)}|^2\right)^{\frac{1}{2}}}_{=1}$$

$$= \left(\int |f|^2 \chi_{(0,1)}\right)^{\frac{1}{2}} < \infty$$

Hence, $f \in L_1(0,1)$ and $L_2(0,1) \subseteq L_1(0,1)$ #

Problem 4

(b) Let us consider

$$f(x) = \frac{1}{x} \qquad x \in (1, \infty).$$

Then

$$\int_1^\infty |f(x)|^2 dx = \int_1^\infty \frac{1}{x^2} dx = -\frac{1}{x}\Big|_1^\infty = 1$$

Thus $f \in L_2(1, \infty)$.

But

$$\int_1^\infty |f(x)| dx = \int_1^\infty \frac{1}{x} dx = \ln x\Big|_1^\infty = \infty$$

$\Rightarrow$ $f \notin L_1(1, \infty)$

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Hence $L_2(1, \infty) \not\subseteq L_1(1, \infty)$. #

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Problem 5

Since $f_n \to f$ uniformly on a compact set $[0,1]$

Hence $f$ is continous on $[0,1]$. Next we will

I'm asking you to prove this.

show that $f$ is uniformly continous on $[0,1]$.

And you do!

$f_n \to f$ uniformly implies that

$$(\forall \varepsilon>0)\ (\exists N\in\mathbb{N})\ (\forall x\in[0,1])\ (\forall n\geq N)$$

$$|f_n(x)-f(x)|<\frac{\varepsilon}{3}$$

$f_n \in C[0,1]$, Hence $f_n$ is uniformly cts on $[0,1]$

since $[0,1]$ is compact. $\Rightarrow$ Fixed $n\geq N$

$$(\forall \varepsilon>0)\ (\exists \delta>0)\ (\forall x,y\in[0,1])\ |x-y|<\delta.$$

$$|f_n(x)-f_n(y)|<\frac{\varepsilon}{3}.$$

Now consider,

$$|f(x)-f(y)|=|f(x)-f_n(x)+f_n(x)-f_n(y)+f_n(y)-f(y)|$$

$$\leq|f(x)-f_n(x)|+|f_n(x)-f_n(y)|+|f_n(y)-f(y)|$$

$$<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

Hence $f$ is uniformly continous on $[0,1]$. ※

**Final Exam** **Real Analysis 501** **Ben M. Chen**

Problem 6

(a) $Q$ is dense in $C[0,1]$. Because

(i) $f(x)=1=\sum_{m=0}^{0}(1)\cdot x^{2m} \in Q$ vanishes nowhere.

(ii) $f(x)=x^2=\sum_{m=0}^{1} a_i x^{2m} \in Q$, where $a_0=0, a_1=1$

separates points in $[0,1]$.

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(iii) $f=\sum_{m=0}^{n} a_m x^{2m} \in Q \Rightarrow \bar{f}=\sum_{m=0}^{n} \bar{a}_m x^{2m} \in Q$

(iv) $Q$ is an algebra.

Hence by S-W theorem (complex), $Q$ is dense in $C[0,1]$. #

(b) $Q$ is not dense in $C[-1,1]$.

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First we not any function $f \in Q$ is a even function, i.e., $f(x)=f(-x)$ $\forall$ $x \in [0,1]$. Thus $Q$ does not ~~contain any function that~~ seperates point in $[-1,1]$.

For instant, $g(x)=x \in C[-1,1]$ cannot be approached by any sequence in $Q$. #

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Problem 7

Let $f_n = -\chi_{E_n}$, $f_n \in M^r$ since $E_n \in \mathfrak{m}$.

- $f_1 \in L_1^r$ since $\lambda(E_1) < \infty$. Hence $f_1^- = -f_1 \in L_1^r$.

- $f_n$ is nondecreasing, i.e

$$f_n \leq f_{n+1}, \quad n = 1, 2, \ldots, \quad \text{since } E_n \supseteq E_{n+1}.$$

- Let $f = -\chi_E$. we have

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$$\lim_{n\to\infty} f_n = f$$

Apply MCT theorem (for measurable functions),

$$-\lambda(E) = \int -\chi_E = \int f = \lim_{n\to\infty} \int f_n$$
$$= \lim_{n\to\infty} \int -\chi_{E_n}$$
$$= -\lim_{n\to\infty} \lambda(E_n).$$

Hence

$$\lambda(E) = \lim_{n\to\infty} \lambda(E_n)$$

Q.E.D.

Problem 8

(a) $$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)|\cdot|g(y)|\,dx\,dy$$

$$= \int_{\mathbb{R}} |g(y)| \int_{\mathbb{R}} |f(x-y)|\,dx\,dy$$

Since $f \in L_1(\mathbb{R}) \Rightarrow \int_{\mathbb{R}} |f(x-y)|\,dx = M < \infty$

Hence

$$\int_{\mathbb{R}} |g(y)| \int_{\mathbb{R}} |f(x-y)|\,dx\,dy$$

$$\le M \int_{\mathbb{R}} |g(y)|\,dy < \infty$$

due to $g \in L_1(\mathbb{R})$. Thus,

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$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)|\cdot|g(y)|\,dx\,dy < \infty.$$

By Tonelli ~~Fubini's~~ theorem (corollary)

$$h(x,y) = f(x-y)g(y) \in L_1(\mathbb{R}^2).$$

#

Problem 8

(b) $\|f*g\|_1 = \int_{\mathbb{R}} |f*g|\,dx$

$$= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y)g(y)\,dy \right| dx$$

$$\le \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y|\cdot|g(y)|\,dy\,dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)|\cdot|g(y)|\,dx\,dy \quad \text{(Fobini's)}$$

$\frac{10}{10}$

$$= \int_{\mathbb{R}} |g(y)| \int_{\mathbb{R}} |f(x-y)|\,dx\,dy$$

$f \in L_1 \Rightarrow \int_{\mathbb{R}} |f(x-y)|\,dx = \int |f(x)|\,dx = \|f\|_1$

Hence

$$\|f*g\|_1 = \int_{\mathbb{R}} |f*g|\,dx$$

$$\le \|f\|_1 \int_{\mathbb{R}} |g(y)|\,dy = \|f\|_1 \cdot \|g\|_1 < \infty$$

$\Rightarrow f*g \in L_1(\mathbb{R})$ and $\|f*g\|_1 \le \|f\|_1 \cdot \|g\|_1$

\#

**Final Exam** **Real Analysis 501** **Ben M. Chen**

Problem 9

(a) Absolutely cts implies uniformly cts and any uniformly cts function on a compact set is bounded. Hence $f, g$ are bounded.

$\frac{5}{5}$

(b) $|f(x)g(x) - f(y)g(y)|$

$$= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|$$

$$\leq |f(x)| \cdot |g(x) - g(y)| + |f(x) - f(y)|\, g(y)$$

$\frac{10}{10}$

$$\leq M \cdot \left( |f(x) - f(y)| + |g(x) - g(y)| \right)$$

\#

(c) $\forall\, \varepsilon > 0$ and for any finitely many open intervals

$(a_1, b_1)$, $(a_2, b_2)$, $\cdots$ $(a_n, b_n)$

such that $\sum_{k=1}^{n} (b_k - a_k) < \delta$ and

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \frac{\varepsilon}{2M} \quad \text{and} \quad \sum_{k=1}^{n} |g(b_k) - g(a_k)| < \frac{\varepsilon}{2M}$$

Now consider

$\frac{10}{10}$

$$\sum_{k=1}^{n} |f(b_k)g(b_k) - f(a_k)g(a_k)|$$

$$\leq \sum_{k=1}^{n} M \left( |f(b_k) - f(a_k)| + |g(b_k) - g(a_k)| \right) \quad \text{(part b)}$$

$$< M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.$$

Hence $fg$ is absolutely continuous on $[0, 1]$. \#