B+

Problem 1. $\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + x = 0$

Using polar coordinate, find a particular solution to this equation. Hence, sketch the associated phase diagram.

Let

$$y = \dot{x} \Rightarrow \dot{y} = -x - (x^2 + y^2 - 1)y$$

$$\Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -x - (x^2 + y^2 - 1)y \end{cases} \Rightarrow \text{equilibrium point} = \begin{Bmatrix} x = 0 \\ y = 0 \end{Bmatrix}$$

Let

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad x^2 + y^2 = r^2$$

$$\Rightarrow \quad \dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{xy - xy - (x^2 + y^2 - 1)y^2}{r}$$

$$= -\frac{(r^2 - 1)r^2\sin^2\theta}{r} = r\sin^2\theta(1 - r^2).$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{-x^2 - (x^2 + y^2 - 1)xy - y^2}{r^2}$$

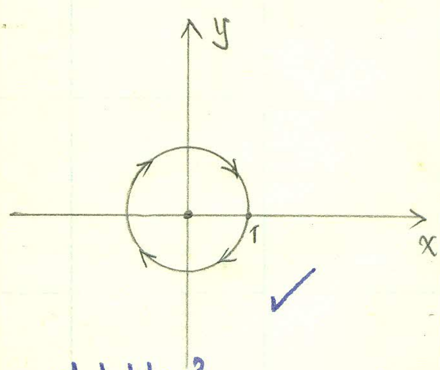
$$= \frac{-r^2 - (r^2 - 1)r^2\sin\theta\cos\theta}{r^2} = (1 - r^2)\sin\theta\cos\theta - 1$$

$$\Rightarrow \begin{cases} \dot{r} = r(1 - r^2)\sin^2\theta \\ \dot{\theta} = (1 - r^2)\sin\theta\cos\theta - 1 \end{cases}$$

If we choose $r = 1$, we will obtain a particular solution

$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = -1 \end{cases}$$

Hence, the phase diagram $\Rightarrow$



stability ?

Problem 2

$$\begin{cases} \dot{x} = -\beta x y + \mu \\ \dot{y} = \beta x y - \gamma y \end{cases} \quad \text{where } \beta, \gamma, \mu > 0.$$

Obtain the equilibrium point of the system. Draw the phase paths near this equilibrium point and determine the stability, if $\beta\mu < 4\gamma^2$.

$$\begin{cases} f(x,y) = -\beta x y + \mu = 0 \\ g(x,y) = \beta x y - \gamma y = 0 \end{cases} \Rightarrow \beta x y = \mu = \gamma y \Rightarrow y = \frac{\mu}{\gamma}$$

✓

$$\Rightarrow \quad x = \frac{\mu}{\beta y} = \frac{\gamma}{\beta}$$

Hence, the equilibrium point is $\left\{ x = \frac{\gamma}{\beta},\ y = \frac{\mu}{\gamma} \right\}$. ✓

Linearize this system at the equilibrium point. ✓

$$\frac{\partial f}{\partial x} = -\beta y \Big|_{x=\frac{\gamma}{\beta},\, y=\frac{\mu}{\gamma}} = -\frac{\beta\mu}{\gamma}$$

$$\frac{\partial f}{\partial y} = -\beta x \Big|_{x=\frac{\gamma}{\beta},\, y=\frac{\mu}{\gamma}} = -\gamma$$

$$\frac{\partial g}{\partial x} = \beta y = \frac{\beta\mu}{\gamma}, \qquad \frac{\partial g}{\partial y} = \beta x - \gamma = 0$$

$\Rightarrow$ Linearized system is :

$$\begin{cases} \dot{x} = -\frac{\beta\mu}{\gamma} x - \gamma y \\ \dot{y} = \frac{\beta\mu}{\gamma} x \end{cases}$$

✓

or:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\frac{\beta\mu}{\gamma} & -\gamma \\ \frac{\beta\mu}{\gamma} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \qquad A = \begin{pmatrix} \frac{-\beta\mu}{\gamma} & -\gamma \\ \frac{\beta\mu}{\gamma} & 0 \end{pmatrix}$$

$$\det\{A-\lambda I\} = \left(-\frac{\beta\mu}{\gamma}-\lambda\right)(0-\lambda)+\beta\mu = 0$$

$$\Rightarrow \quad \lambda^2 + \left(\frac{\beta\mu}{\gamma}\right)\lambda + \beta\mu = 0$$

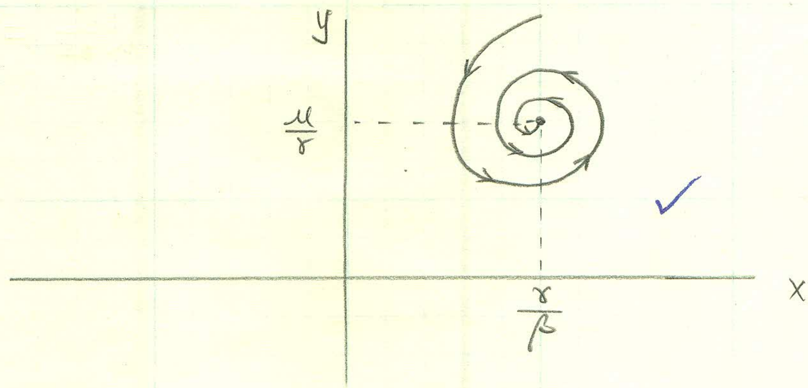
$$\lambda_1, \lambda_2 = -\left(\frac{\beta\mu}{2\gamma}\right) \pm \frac{1}{2}\sqrt{\left(\frac{\beta\mu}{\gamma^2}\right)(\beta\mu - 4\gamma^2)}$$

Under condition $\beta\mu < 4\gamma^2$, λ_1 and λ_2 are complex conjugated.

Then from the class note, this is Case 2 of Linear system.

You need to show this!

we have following phase diagram:



The equilibrium point is asymptotically stable.

Problem 1 $\ddot{\theta} + 2k\dot{\theta} + g\sin\theta = 0$, $k>0$, $g>0$ $(g>k^2)$

Determine the equilibrium point, stability and phase diagram.

Let

$$x = \theta, \quad y = \dot{\theta} \quad \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -2ky - g\sin x \end{cases}$$

$\Rightarrow$ equilibrium points are: $(k\pi, 0)$, k are any integer.

$f(x, y) = y$.

$g(x, y) = -2ky - g\sin x$

- following devide the equilibrium points into two groups:

group 1: $(2n\pi, 0)$

$n = 0, \pm 1, \pm 2, \cdots$

group 2: $(2n\pi + \pi, 0)$

Case I: Now let us linearize the non-linear system at neighborhood of the equilibrium points in group 1. We have

$$\dot{x} = y$$

$$\dot{y} = (-g\cos x)\Big|_{(2n\pi, 0)} x + (-2k)\cdot y$$

$\Rightarrow$ Linearized system for group 1:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -gx - 2ky \end{cases} \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g & -2k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 + 2k\lambda + g$$

$$\Rightarrow \lambda_1, \lambda_2 = -k \pm i\sqrt{g - k^2}$$

which will give a stable spiral in the equilibrium points in group 1.

Case II: We linerize the non-linear system at the equilibrium points in group 2. we have

$$\dot{x} = y$$

$$\dot{y} = (-g\cos x)\Big|_{(2n\pi+\pi,0)} x - (2k)\, y$$

$\Rightarrow$ Linearized system for group 2:

$$\begin{cases} \dot{x} = y \\ \dot{y} = gx - 2ky \end{cases} \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ g & -2k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 + 2k\lambda - g$$

$$\Rightarrow \lambda_1, \lambda_2 = -k \pm \sqrt{k^2 + g}$$

which will give an unstable saddle.

Now, we construct the phase diagram:

① In order to determine the type of spiral in case, we let $y=0$, $x>0$ we find $\dot{y} = -gx < 0$, indicating that the rotation is clockwise.

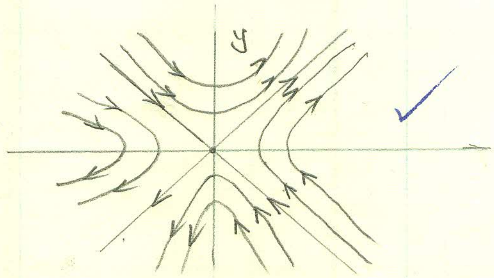
② for case 2, we use

$$\frac{dy}{dx} = \frac{gx - 2ky}{y}, \quad \text{choose } y = px$$

$$\Rightarrow p = \frac{gx - 2kpx}{px} = \frac{g - 2kp}{p} \Rightarrow p^2 + 2kp - g = 0$$

$$\Rightarrow y = (-k - \sqrt{k^2+g})\, x \text{ and } y = (-k + \sqrt{k^2+g})\, x$$

Phase diagram at the equilibrium point of group 2



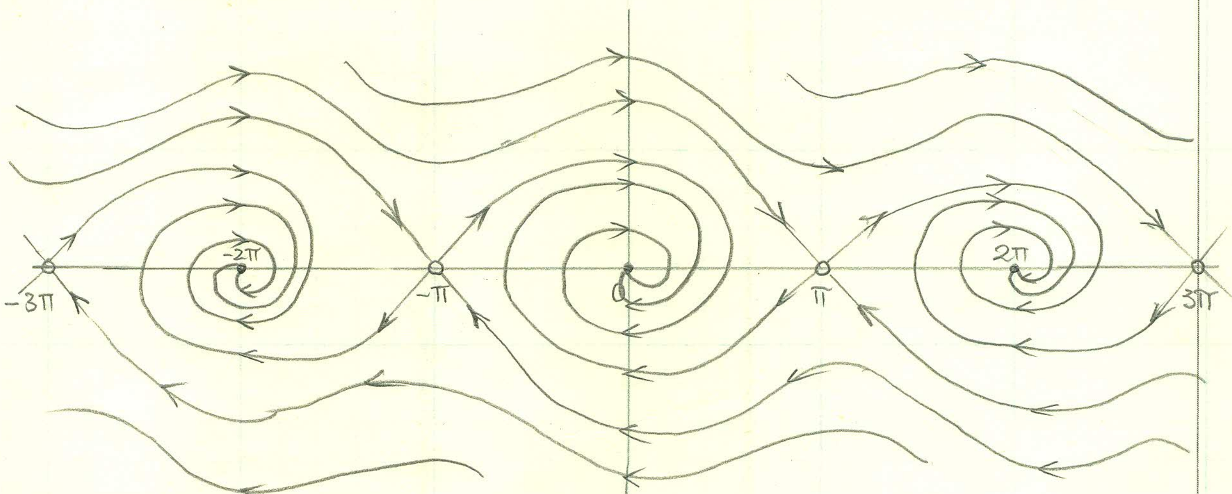
{ Test : $y = -(k+\sqrt{k^2+q})\,x$ and $x>0$

$$\dot{x} = -(k+\sqrt{k^2+q})\,x < 0$$

$$\dot{y} = qx + 2k(k+\sqrt{k^2+q})\,x > 0$$

Hence, we the directions of the saddle in previous page. }

The complete phase diagram:



Stability : the equibrium points in group 1 are asymptotically stable. ✓

And those in group 2 are unstable.

Problem 2. $\frac{d^2u}{d\theta^2} + u = \frac{\gamma}{h^2} u^{\alpha-2}$ $(h, \gamma, \alpha > 0)$

Classify the equilibrium points and determine the stability.

Let $\begin{array}{l} x = u \\ y = \dot{u} \end{array} \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -x + \frac{\gamma}{h^2} x^{\alpha-2} \end{cases}$

$\Rightarrow \dot{x} = 0 \Rightarrow y = 0.$

$\dot{y} = 0 \Rightarrow \frac{\gamma}{h^2} x^{\alpha-2} - x = 0 \Rightarrow \gamma x^{\alpha-2} - h^2 x = 0$

$\Rightarrow x(\gamma x^{\alpha-3} - h^2) = 0$

Hence, we have following equilibrium points: (we assume $\alpha \neq 3$)

① $\{x=0, y=0\}$, ② $\{x = (\frac{h^2}{\gamma})^{\frac{1}{\alpha-3}}, y=0\}$

{According to problem 18 on page 57, $u = r^{-1}$, $r > 0 \Rightarrow x > 0$.
If $\alpha = 3$, there is only one equilibrium point at $(0,0)$}

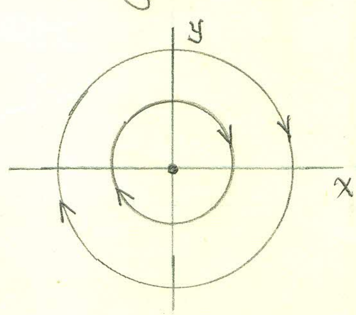
Case I: Linearization at first equilibrium point $(0, 0)$

$$\dot{x} = y$$
$$\dot{y} = \left[-1 + \frac{\gamma}{h^2}(\alpha-2)x^{\alpha-3}\right]\Big|_{(0,0)} x = -x$$

$$\Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda_1, \lambda_2 = \pm i$$

This gives a centre.



The direction is determined as follows:

Let $y = 0$ and $x > 0$

$\dot{y} = -x < 0 \Rightarrow$ clockwise.

Case II: Linearization for the second equilibrium point $\left(\left[\frac{\hbar^2}{\gamma}\right]^{\frac{1}{\alpha-3}}, 0\right)$.

$$\dot{x} = y$$

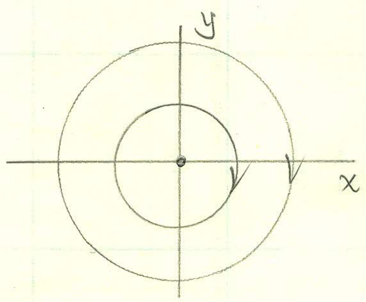
$$\dot{y} = \left[-1 + \frac{\gamma}{\hbar^2}(\alpha-2)x^{\alpha-3}\right]\Big|_{\left(\left[\frac{\hbar^2}{\gamma}\right]^{\frac{1}{\alpha-3}},0\right)} x = (\alpha-3)x$$

$$\Rightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 1\\ \alpha-3 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$ ✓

$$\Rightarrow \det(A-\lambda I) = \lambda^2 - (\alpha-3) = 0$$

Sub-case (i) $\alpha < 3$, $\Rightarrow \lambda_1, \lambda_2 = \pm i \cdot \sqrt{3-\alpha}$

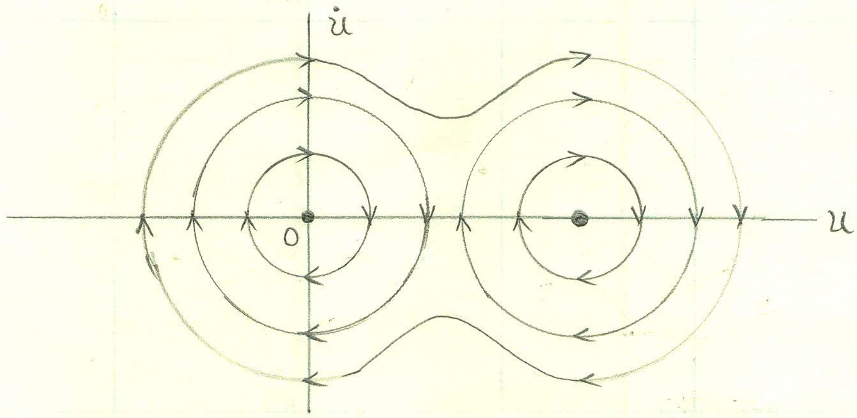
which will give a centre as following ✓



direction: Let $y=0$, $x>0$

$\dot{y} = (\alpha-3)x < 0$

The complete phase diagram for $\alpha < 3$



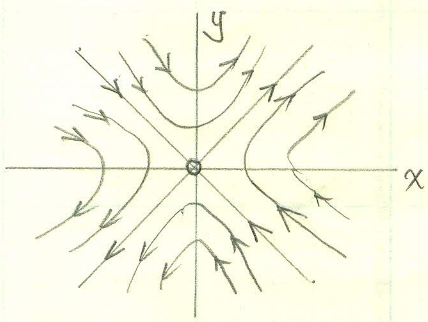
in this case, both equilibrium points are centers. Hence, they are stable.

sub-case ii: $\alpha > 3 \Rightarrow \lambda_1, \lambda_2 = \pm\sqrt{\alpha - 3}$

which is a saddle (hence, unstable)

$$\frac{dy}{dx} = \frac{(\alpha-3)x}{y} \qquad \text{choose } y = kx$$

$$\Rightarrow \quad k = \frac{(\alpha-3)x}{kx} \Rightarrow k^2 = (\alpha-3) \Rightarrow k_1 = \sqrt{\alpha-3}\,,\ k_2 = -\sqrt{\alpha-3}$$

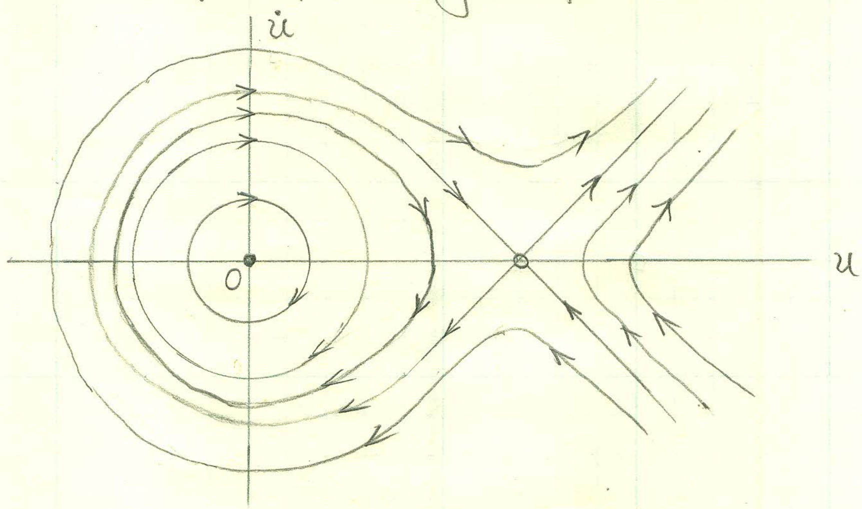


choose: $y = \sqrt{\alpha-3}\, x$,

$\dot{x} = \sqrt{\alpha-3}\; x > 0$

$\dot{y} = (\alpha-3)\, x > 0$

The complete phase diagram for $\alpha > 3$



Hence, the equilibrium point $(0,0)$ is stable. and

$\left(\left[\frac{h^2}{\gamma}\right]^{\frac{1}{\alpha-3}}, 0\right)$ is unstable.

Problem 3. $\ddot{x} + \varepsilon (x^4 - 1)\dot{x} + x = 0$

Find limit cycle and its amplitude, stability.

Let $y = \dot{x} \Rightarrow \dot{y} = -\varepsilon(x^4-1)y - x$

$\Rightarrow h(x,y) = (x^4-1)y$, $h(0,0) = 0 \Rightarrow (0,0)$ is an equilibrium point

Consider

$$0 = \int_0^{2\pi} h(a\cos t, -a\sin t)\sin t\,dt$$

$$= \int_0^{2\pi} (a^4\cos^4 t - 1)(-a\sin^2 t)\,dt$$

$$= \int_0^{2\pi} a\sin^2 t\,dt - a^5\int_0^{2\pi} \cos^4 t\sin^2 t\,dt$$

$$= \int_0^{2\pi} \frac{a}{2}(1-\cos 2t)\,dt - a^5\int_0^{2\pi} \left[\tfrac{1}{2}(1+\cos 2t)\right]^2\left[\tfrac{1}{2}(1-\cos 2t)\right]dt$$

$$= a\cdot\pi - \frac{a^5}{8}\int_0^{2\pi}\left[1+\cos 2t - \cos^2 2t - \cos^3 2t\right]dt$$

$$= a\pi - \frac{a^5\pi}{4} + \frac{a^5}{8}\int_0^{2\pi}(\cos^2 2t + \cos^3 2t)\,dt$$

$$= a\pi - \frac{a^5\pi}{4} + \frac{a^5}{8}\int_0^{2\pi}\left[\tfrac{1}{2}(1+\cos 4t)\right]dt + \frac{a^5}{16}\int_0^{2\pi} \overset{0}{\cancel{\cos^2 2t}}\, d\sin 2t$$

$$= a\pi - \frac{a^5\pi}{4} + \frac{a^5\pi}{8} = 0 \Rightarrow a^4 = 8 \Rightarrow \boxed{a_0 = \sqrt[4]{8} = 1.682}$$

$$\Rightarrow g(a) = \varepsilon a^2\pi - \frac{\varepsilon a^6\pi}{8}$$

$$g'(a_0) = 2\varepsilon\pi a_0 - \frac{5}{8}\varepsilon\pi a_0^5 = \varepsilon\pi\left(2a_0 - \frac{5}{8}a_0^5\right) = -5.045\,\varepsilon\pi < 0$$

which implies that the limit cycle is stable. if $\varepsilon > 0$

To find a complete phase diagram for this problem.

Linearize the non-linear system at (0, 0), we have

$$\dot{x} = y$$

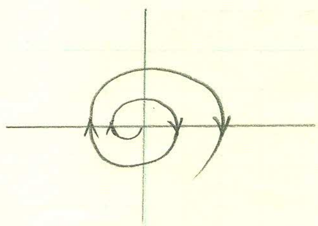
$$\dot{y} = [-1 - 4\varepsilon x^3 y]\Big|_{(0,0)} \cdot x + [-\varepsilon(x^4 - 1)]\Big|_{(0,0)} \cdot y$$

$$= -x + \varepsilon y$$

$$\Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \det(\lambda I - A) = \lambda^2 - \varepsilon\lambda + 1 = 0$$

$$\Rightarrow \lambda_1, \lambda_2 = \frac{1}{2}(\varepsilon \pm i\sqrt{4 - \varepsilon^2})$$

✓

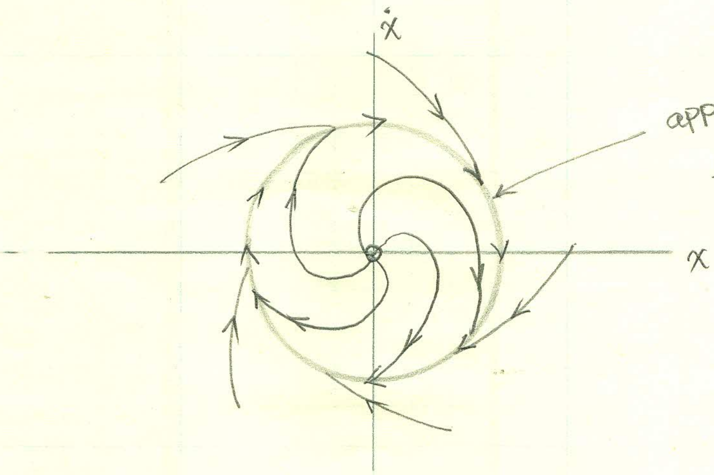


Let $y = 0$, $x > 0$

$$\Rightarrow \dot{y} = -x < 0$$

This analysis is redundant!

Hence, we have following phase diagram:



Thus, we can say, the limit cycle is stable. ✓

The equilibrium point is unstable.

(1) $\ddot{x} + \varepsilon(x^2 + \dot{x}^2 - 4)\dot{x} + x = 0$

A

$x = 2\cos t$ is a limit cycle.

Stability ; path close to limit cycle; period with an error $O(\varepsilon^2)$.

Solution:

$$h(x, \dot{x}) = (x^2 + \dot{x}^2 - 4)\dot{x}$$

$$h(a\cos\theta, a\sin\theta) = (a^2\cos^2\theta + a^2\sin^2\theta - 4)a\sin\theta$$

$$= (a^2 - 4)a\sin\theta$$

$$\Rightarrow P_0(a) = \frac{1}{2\pi}\int_0^{2\pi} h(a\cos\theta, a\sin\theta)\sin\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (a^2 - 4)a\sin^2\theta\, d\theta$$

$$= \frac{a(a^2-4)}{2\pi}\int_0^{2\pi} \frac{1}{2}(1 - \cos 2\theta)\, d\theta = \frac{1}{2}a(a^2 - 4).$$

$$\Rightarrow \gamma_0(a) = \frac{1}{2\pi}\int_0^{2\pi} h(a\cos\theta, a\sin\theta)\cos\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} a(a^2-4)\sin\theta\cos\theta\, d\theta = 0$$

$$\Rightarrow \quad da = -\varepsilon P_0(a)\, dt$$

$$\frac{da}{a(a^2-4)} = -\frac{\varepsilon}{2}\, dt \Rightarrow \int \frac{da}{a(a^2-4)} = -\frac{\varepsilon}{2}\int dt + C$$

$$\Rightarrow \frac{1}{a(a^2-4)} = \frac{A}{a} + \frac{B}{a+2} + \frac{C}{a-2} = -\frac{1}{4a} + \frac{1}{8}\frac{1}{a+2} + \frac{1}{8}\frac{1}{a-2}$$

$$\int \frac{da}{a(a^2-4)} = -\frac{1}{4}\ln a + \frac{1}{8}\ln(a+2) + \frac{1}{8}\ln(a-2) = -\frac{\varepsilon}{2}t + C$$

Let $a(0) = a_0 \Rightarrow$

$$C = -\frac{1}{4}\ln a_0 + \frac{1}{8}\ln(a_0+2) + \frac{1}{8}\ln(a_0-2)$$

$$\Rightarrow -2\ln a + \ln(a^2-4) = -4\varepsilon t - 2\ln a_0 + \ln(a_0^2-4)$$

$$\ln\left\{\frac{a^2-4}{a^2}\right\} = -4\varepsilon t + \ln\left\{\frac{a_0^2-4}{a_0^2}\right\}$$

$$\Rightarrow \left(\frac{a^2-4}{a^2}\right)\left(\frac{a_0^2}{a_0^2-4}\right) = e^{-4\varepsilon t} \Rightarrow a^2\cdot a_0^2 - 4a_0^2 = (a_0^2-4)e^{-4\varepsilon t}a^2$$

$$\Rightarrow \boxed{a^2 = \frac{4a_0^2}{a_0^2-(a_0^2-4)e^{-4\varepsilon t}}}$$

If $\varepsilon>0$, $\lim_{t\to\infty} a^2(t) = 4 \Rightarrow a(\infty)=2$ which is the radii of the limit cycle.

$$\frac{d\theta}{dt} = -1 - \frac{\varepsilon}{a}\gamma_0(a) = -1 \Rightarrow \boxed{\theta = -t + C_0}$$

Hence the solution of path close to limit cycle is:

$$\boxed{x(t) = a(t)\cos\theta(t) = \sqrt{\frac{4a_0^2}{a_0^2-(a_0^2-4)e^{-4\varepsilon t}}}\cos(-t+C_0)}$$

And since $a(\infty)=2 \Rightarrow$ limit cycle is stable if $\varepsilon>0$.

the equilibrium point at $(0,0)$ is unstable, if $\varepsilon>0$

$$\omega \cong 1 + \frac{\varepsilon}{2\pi a_0}\int_0^{2\pi} h(a\cos\theta, a\sin\theta)\cos\theta\, d\theta$$

$$= 1 + \text{zero} + O(\varepsilon^2)$$

$$\Rightarrow T = \frac{2\pi}{\omega} \cong 2\pi + O(\varepsilon^2).$$

If $\varepsilon<0$, the limit cycle is unstable.

(2) $\ddot{x} - 0.1\, x^3 + 0.6\, x = \cos t$

Find first two harmonics of the solution of period 2π

Assume the solution we are looking for is

$$x(\varepsilon, t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$$

Then we have. $\ddot{x} + 0.6x + \varepsilon x^3 = \cos t$ ✓

$$\ddot{x}_0 + \varepsilon \ddot{x}_1 + O(\varepsilon^2) + \varepsilon\left(x_0^3 + 3\varepsilon x_0^2 x_1 + O(\varepsilon^2)\right)$$
$$+ 0.6\left(x_0 + \varepsilon x_1\right) = \cos t$$

$$\Rightarrow \quad \ddot{x}_0 + 0.6\, x_0 = \cos t$$
$$\ddot{x}_1 + 0.6\, x_1 = -\, x_0^3$$
$$\vdots$$

① $\ddot{x}_0 + 0.6\, x_0 = 0 \Rightarrow x_0(t) = A\cos 0.7746t + B\sin 0.7746t$

which can not have a period of $2\pi \Rightarrow A = B = 0.$

② $x_0 = \dfrac{1}{D^2+0.6}[\cos t] = \dfrac{1}{-1^2+0.6}\cos t = -2.5\cos t$ ✓

$$\Rightarrow \ddot{x}_1 + 0.6\, x_1 = 15.625\cos^3 t$$

① $\ddot{x}_1 + 0.6\, x_1 = 0 \Rightarrow$ no solution of period 2π. ✓

② $x_1 = \dfrac{1}{D^2+0.6}\left[15.625\cos^3 t\right]$

$$= \frac{15.625}{D^2+0.6}\left[\cos t\,(1+\cos 2t)/2\right]$$

$$= \frac{15.625}{D^2+0.6}\left[\tfrac{1}{2}\cos t + \tfrac{1}{4}(\cos t + \cos 3t)\right]$$ ✓

$$= \frac{11.71875}{D^2+0.6}[\cos t] + \frac{3.90625}{D^2+0.6}[\cos 3t]$$

$$= -29.296875\cos t - 0.46503\cos 3t$$

Thus, we have

$$x(\varepsilon,t) = -2.5\cos t - \varepsilon\left(29.296875\cos t + 0.46503\cos 3t\right) + O(\varepsilon^2)$$

And

$$\boxed{x(-0.1,t) \cong 0.4296875\cos t + 0.046503\cos 3t}$$

(3) The Duffing equation with weak excitation

$$\ddot{x} + \Omega^2 x = \varepsilon(\gamma\cos t + x^3)$$

Ω is near resonance at 3. Show that there are solutions of period 2π if the amplitude of the zero-order solution is 0 or ? Find the solution to order ε in the latter case.

Let $\Omega^2 = 9 + \varepsilon\beta$, then we have

$$\ddot{x} + 9x + \varepsilon\beta x = \varepsilon(\gamma\cos t + x^3)$$

$$\Rightarrow \ddot{x} + 9x + \varepsilon(\beta x - x^3 - \gamma\cos t) = 0$$

We are looking for a solution of form

$$x(\varepsilon,t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2) \quad \text{with } T = 2\pi$$

$$\Rightarrow \left(\ddot{x}_0 + \varepsilon\ddot{x}_1 + O(\varepsilon^2)\right) + 9\left(x_0 + \varepsilon x_1 + O(\varepsilon^2)\right)$$
$$+ \varepsilon\left[\left(\beta x_0 + \varepsilon\beta x_1 - x_0^3 - 3\varepsilon x_0^2 x_1 - \gamma\cos t + O(\varepsilon^2)\right]\right. = 0$$

ε^0: $\ddot{x}_0 + 9x_0 = 0$

ε^1: $\ddot{x}_1 + 9x_1 = \gamma\cos t - \beta x_0 + x_0^3$

$\vdots$

from $\ddot{x}_0 + 9x_0 = 0 \Rightarrow x_0 = A_0\cos 3t + B_0\sin 3t$

$$\Rightarrow \quad \ddot{x}_1 + 9x_1 = \gamma\cos t - \beta\,(A_0\cos 3t + B_0\sin 3t) + (A_0\cos 3t + B_0\sin 3t)^3$$

$$= \gamma\cos t - A_0\beta\cos 3t - B_0\beta\sin 3t + B_0^3\sin^3 3t$$

$$+ A_0^3\cos^3 3t + 3A_0^2B_0\cos^2 3t\sin 3t + 3A_0B_0^2\cos 3t\sin^2 3t$$

$$= \gamma\cos t - A_0\beta\cos 3t - B_0\beta\sin 3t$$

$$+ B_0^3\left[\sin 3t\,(1-\cos 6t)/2\right] + A_0^3\left[\cos 3t\,(1+\cos 6t)/2\right]$$

$$+ 3A_0^2B_0\left[\tfrac{1}{2}(1+\cos 6t)\sin 3t\right]$$

$$\checkmark \quad + 3A_0B_0^2\left[\tfrac{1}{2}(1-\cos 6t)\cos 3t\right]$$

$$= \gamma\cos t - A_0\beta\cos 3t - B_0\beta\sin 3t$$

$$+ \frac{1}{4}B_0^3\left[3\sin 3t - \sin 9t\right] + \frac{1}{4}A_0^3\left[3\cos 3t + \cos 9t\right]$$

$$+ \frac{3}{4}A_0^2B_0\left[\sin 3t + \sin 9t\right] + \frac{3}{4}A_0B_0^2\left[\cos 3t - \cos 9t\right]$$

$$= \gamma\cos t + \left[\frac{3}{4}A_0^3 + \frac{3}{4}A_0B_0^2 - A_0\beta\right]\cos 3t$$

$$+ \left[\frac{3}{4}B_0^3 + \frac{3}{4}A_0^2B_0 - B_0\beta\right]\sin 3t$$

$$+ \left[-\frac{1}{4}B_0^3 + \frac{3}{4}A_0^2B_0\right]\sin 9t$$

$$+ \left\{\frac{1}{4}A_0^3 - \frac{3}{4}A_0B_0^2\right\}\cos 9t$$

$$x_1(t) = \frac{\gamma}{D^2+9}\{\cos t\} + \left[\frac{3}{4}A_0^3 + \frac{3}{4}A_0B_0^2 - A_0\beta\right]\frac{1}{D^2+9}\{\cos 3t\}$$

$$+ \left[\frac{3}{4}B_0^3 + \frac{3}{4}A_0^2B_0 - B_0\beta\right]\frac{1}{D^2+9}\{\sin 3t\}$$

$$+ \left[\frac{3}{4}A_0^2B_0 - \frac{1}{4}B_0^3\right]\frac{1}{D^2+9}\{\sin 9t\} + \left[\frac{1}{4}A_0^3 - \frac{3}{4}A_0B_0^2\right]\frac{1}{D^2+9}\{\cos 9t\}$$

$$= -\frac{\gamma}{8}\cos t + \left[\frac{3}{4}A_0^3 + \frac{3}{4}A_0B_0^2 - A_0\beta\right]\cdot\frac{3t}{18}\sin 3t$$

$$\checkmark \quad - \left[\frac{3}{4}B_0^3 + \frac{3}{4}A_0^2B_0 - B_0\beta\right]\cdot\frac{3t}{18}\cos 3t$$

$$-\frac{1}{72}\left[\frac{3}{4}A_0^2B_0-\frac{1}{4}B_0^3\right]\cdot\sin 9t-\frac{1}{72}\left[\frac{1}{4}A_0^3-\frac{3}{4}A_0B_0^2\right]\cos 9t$$

In order to have a period 2π solution, we must have

$$\frac{3}{4}A_0^3+\frac{3}{4}A_0B_0^2-A_0\beta=0$$

$$\Rightarrow A_0=0 \quad \text{and} \quad 3A_0^2+3B_0^2=4\beta$$

And

$$\frac{3}{4}B_0^3+\frac{3}{4}A_0^2B_0-B_0\beta=0$$

$$\Rightarrow B_0=0 \quad \text{and} \quad 3A_0^2+3B_0^2=4\beta$$

Thus if $A_0=B_0=0$, we have a period 2π solution.

And if $A_0^2+B_0^2=\frac{4}{3}\beta$, we also have a 2π period solution.

and in this case we have $A_0^2=\frac{4}{3}\beta-B_0^2 \Rightarrow A_0=\pm\sqrt{\frac{4}{3}\beta-B_0^2}$

$$x_0= \quad A_0 \quad \cos 3t+B_0\sin 3t$$

$$x_1=-\frac{\gamma}{8}\cos t+\frac{1}{72}\left[B_0^3-\beta B_0\right]\sin 9t+\frac{1}{72}A_0\left[B_0^2-\frac{1}{3}\beta\right]\cos 9t$$

Solution 1: $\quad |B_0|\leqslant\sqrt{\frac{4}{3}\beta}$

$$x(\varepsilon,t)=\sqrt{\frac{4}{3}\beta-B_0^2}\cos 3t+B_0\sin 3t-\frac{\gamma}{8}\cos t$$
$$+\frac{1}{72}\left[B_0^3-\beta B_0\right]\sin 9t+\frac{1}{72}\sqrt{\frac{3}{4}\beta-B_0^2}\left[B_0^2-\frac{1}{3}\beta\right]\cos 9t$$

Solution 2: $\quad |B_0|\leqslant\sqrt{\frac{4}{3}\beta}$

$$x(\varepsilon,t)=-\sqrt{\frac{4}{3}\beta-B_0^2}\cos 3t+B_0\sin 3t-\frac{\gamma}{8}\cos t+\frac{1}{72}(B_0^3-\beta B_0)\sin 9t$$
$$-\frac{1}{72}\sqrt{\frac{4}{3}\beta-B_0^2}\left(B_0^2-\frac{1}{3}\beta\right)\cos 9t.$$

If $\beta<0 \Rightarrow$ no solution.

A

1. Given

$$\dot{x} = x$$
$$\dot{y} = y \log y \quad , \quad y > 0$$

Find the phase path and determine if Poincare's stable ?

① $\dot{x} = x \Rightarrow \frac{dx}{x} = dt \Rightarrow \int \frac{dx}{x} = \int dt$

$\Rightarrow \ln x = t + C \Rightarrow x = e^{t+c} \triangleq Ae^t \quad (A = e^c)$

$$\boxed{x = Ae^t}$$ ✓

② $\dot{y} = y \log y \; , \; y > 0 \Rightarrow \int \frac{dy}{y \ln y} = \int dt$

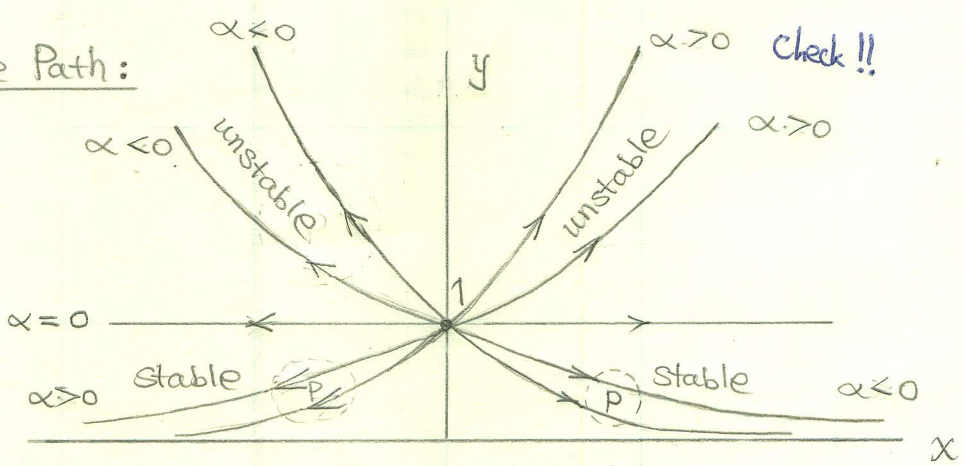
$\Rightarrow \int \frac{d \ln y}{\ln y} = \int dt \Rightarrow \ln(\ln y) = t + \text{Const.}$

$\Rightarrow \ln y = Be^t \Rightarrow \boxed{y = e^{Be^t}}$ ✓

Hence, we obtain

$$\boxed{y = e^{BX/A} \triangleq e^{\alpha x}}$$ ✓

Phase Path:



The phase pathes in region marked by (P) are Poincare's stable.

And unstable elsewhere. (Change the signs of α.)

2. Determine the stability of the solutions

(a) $\dot{x}_1 = x_2 \sin t$, $\dot{x}_2 = 0$

$\dot{x}_2 = 0 \Rightarrow x_2 = \alpha$ (a constant).

$\Rightarrow \dot{x}_1 = \alpha \sin t \Rightarrow x_1 = -\alpha \cos t + \beta$

$$\begin{cases} x_1 = -\alpha \cos t + \beta \\ x_2 = \alpha \end{cases}$$

$$\Phi = \begin{bmatrix} 1 & -\cos t \\ 0 & 1 \end{bmatrix}$$

All columns of Φ are bounded, the solution is stable.

(b) $\dot{x}_1 = 0$, $\dot{x}_2 = x_1 + x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\det(\lambda I - A) = \lambda(\lambda - 1) = 0 \Rightarrow \lambda_1 = 0$, $\lambda_2 = 1$

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \Phi = \begin{bmatrix} 1 & 0 \\ -1 & e^t \end{bmatrix} \Rightarrow \text{the solution is unstable.}$$

We must say some the solutions of the system is unstable. Or, say not all the solutions of the system are stable.

2. (c)

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}1\\ -2\end{pmatrix}e^t$$

According to Theorem 8.1 of the text, all solution of the regular system $\dot{x}=A(t)x+f(t)$ have the same stability property as that of the zero solution of $\dot{\xi}=A(t)\xi$. Thus, let us consider

$$\checkmark \begin{pmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{pmatrix}=\begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

$\det(\lambda I-A)=(\lambda+2)^2-1=(\lambda+3)(\lambda+1)=0 \Rightarrow \lambda_1=-1$ and $\lambda_2=-3$

✓

Without finding the solution, we know the solutions to this system is stable.

(d) $\ddot{x}+e^{-t}\dot{x}+x=e^t$, Define $x_1=x$, $x_2=\dot{x}_1$

$\Rightarrow$ $\dot{x}_1=x_2$ and $\dot{x}_2=\ddot{x}=-x_1-e^{-t}x_2+e^t$

or

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & -e^{-t}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}e^t$$

Follow (c), we only need to determine the stability of

$$\checkmark \begin{pmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & -e^{-t}\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \quad \cdots \quad (*)$$

Let us choose a Liaponuv function

$$V=(\xi_1\ \ \xi_2)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\xi_1^2+\xi_2^2>0 \quad (=0 \text{ if } \xi_1=\xi_2=0)$$

Then $\frac{dV}{dt}$ calculated along the trajectory (*) satisfies

$$\dot{V}=(\dot{\xi}_1\ \ \dot{\xi}_2)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}+(\xi_1\ \ \xi_2)\begin{pmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{pmatrix}=(\xi_1\ \ \xi_2)\begin{pmatrix}0 & 0\\ 0 & -2e^{-t}\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

$$\checkmark \quad =-2e^{-t}\xi_2^2\le 0$$

which implies that the solution of the system is stable (Liaponuv)

3. Find the fundamental matrix of the system characterized by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -2x_1 + x_2 + 2x_3 \end{cases}$$

And then find the solutions for the system below:

$$\begin{cases} \dot{x}_1 = x_2 + e^t \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -2x_1 + x_2 + 2x_3 \end{cases} \qquad \underline{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2 & -1 & \lambda-2 \end{pmatrix}$$

$$= \lambda^2(\lambda-2) + 2 - \lambda = (\lambda-2)(\lambda^2-1)$$

$$= (\lambda-2)(\lambda-1)(\lambda+1) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = -1$$

✓

And from

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \lambda$$

$$\Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Thus

$$\Phi = \begin{bmatrix} e^t & e^{2t} & e^{-t} \\ e^t & 2e^{2t} & -e^{-t} \\ e^t & 4e^{2t} & e^{-t} \end{bmatrix}.$$

✓

3. (second part). It is simple to verify that the particular solution of the regular linear system is

$$x_1 = te^t$$
$$x_2 = te^t$$
$$x_3 = te^t + e^t$$

Then we have the solution of the second system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} e^t & e^{2t} & e^{-t} \\ e^t & 2e^{2t} & -e^{-t} \\ e^t & 4e^{2t} & e^{-t} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \begin{pmatrix} te^t \\ te^t \\ te^t + e^t \end{pmatrix}$$

Since,

$$\underline{x_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \underline{x_0} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

we have $\alpha_1 = \frac{3}{2}$, $\alpha_2 = -\frac{2}{3}$, $\alpha_3 = \frac{1}{6}$

Hence,

$$\boxed{\begin{aligned} x_1(t) &= \tfrac{3}{2}e^t - \tfrac{2}{3}e^{2t} + \tfrac{1}{6}e^{-t} + te^t \\ x_2(t) &= \tfrac{3}{2}e^t - \tfrac{4}{3}e^{2t} - \tfrac{1}{6}e^{-t} + te^t \\ x_3(t) &= \tfrac{5}{2}e^t - \tfrac{8}{3}e^{2t} + \tfrac{1}{6}e^{-t} + te^t \end{aligned}}$$