Ben M. Chen

MATH 415 CLASS EXAMINATION.

19 FEBRUARY 1990.

ANSWER ANY ONE OF THE FOLLOWING QUESTIONS.

(1) (i) For a given nonlinear system $\ddot{x} + h(x,\dot{x}) + g(x) = 0$, one can associate a kinetic energy $\frac{1}{2}\dot{x}^2$ and a potential energy $\int g(x)\,dx$. Obtain the expression for the rate of change of total energy along a phase path. Hence, show that the change in total energy as time moves from t_0 to t_1 is given by $-\int_{t_0}^{t_1} \dot{x}\,h(x,\dot{x})\,dt$.

(ii) A simple harmonic oscillator is described by the equation $\ddot{x} + x = 0$. Find the equilibrium point and analyse its stability. Draw the corresponding phase diagram.

(iii) Using (i) and (ii) above, or otherwise, obtain the approximate amplitude of the limit cycle of the equation

$$\ddot{x} + \varepsilon(x-3)(x+1)\dot{x} + x = 0 \quad , \quad \text{where } 0 < \varepsilon < 0.001.$$

Determine the stability of this limit cycle and comment on the stability of the equilibrium point of this system. Sketch the phase diagram.

(2) (i) A linear system is given by $\dot{\underline{x}} = A\underline{x}$ where $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. ($a, b, c$ + d are constants.)

Obtain the condition for A to have eigenvalues which are complex.

If the corresponding complex eigenvalues are $u \pm iv$, using the transformations $\xi = cx + (u-a)y$ and $\eta = vy$ analyse the behaviour of the phase paths near the equilibrium point $(0,0)$. Draw the corresponding phase diagram.

(ii) Find the equilibrium points of the nonlinear system

$$\dot{x} = y$$
$$\dot{y} = 2x - x^2.$$

Determine the stability of these equilibrium points and sketch the complete phase diagram.

80/100

(1) (i) By the definition given in the statement,

the total energy is,

$$E(t) = \frac{1}{2}\dot{x}^2 + \int g(x)\,dx$$

$$\Rightarrow \frac{dE(t)}{dt} = \dot{x}\ddot{x} + g(x)\cdot\dot{x} = \dot{x}(\ddot{x} + g(x))$$

From statement, we have

$$\ddot{x} + g(x) = -h(x, \dot{x})$$

10 Hence, the rate of change of total energy along a phase path is

$$\frac{dE(t)}{dt} = -\dot{x}\,h(x, \dot{x})$$

The change in total energy as time moves from t_0 to t_1 is then given as

$$E_{change} = E(t_1) - E(t_0) = \int_{t_0}^{t_1} dE(t)$$

$$= \int_{t_0}^{t_1} [-\dot{x}\,h(x, \dot{x})]\,dt$$

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$$= -\int_{t_0}^{t_1} \dot{x}\,h(x, \dot{x})\,dt.$$

Q.E.D.

(1) (ii) $\ddot{x} + x = 0$

Let $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$

Then it is simple to see that the equilibrium

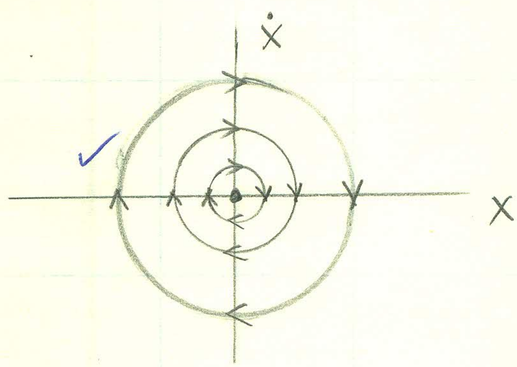
10 point is $(x, y) = (0, 0)$.

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y\,dy = -x\,dx$$

$$\Rightarrow \int y\,dy = -\int x\,dx + \tfrac{1}{2}C \quad (C \text{ is a constant})$$

$$\Rightarrow \tfrac{1}{2}y^2 = -\tfrac{1}{2}x^2 + \tfrac{1}{2}C \Rightarrow x^2 + y^2 = C$$

The phase diagram:



10 The direction is determined as follow:

Let $y = 0$, $x > 0$

$\Rightarrow \dot{y} = -x < 0$

and $y = 0$, $x < 0 \Rightarrow \dot{y} > 0$. and so on.

The equilibrium point for this system is a center and hence it is stable.

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(iii)

$$\ddot{x} + \varepsilon(x-3)(x+1)\dot{x} + x = 0$$

$$\varepsilon h(x,\dot{x}) = \varepsilon(x-3)(x+1)\dot{x} = \varepsilon(x^2-2x-3)y$$

Let $\varepsilon = 0$, we have

$$\ddot{x} + x = 0, \Rightarrow \begin{cases} x = a\cos t \\ y = -a\sin t \end{cases}$$

If there is a limit cycle, then

$$0 = g(a) = -\int_0^{2\pi} \varepsilon y h(x,y)\,dt$$

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$$= \int_0^{2\pi} \varepsilon a h(a\cos t, -a\sin t)\sin t\,dt$$

sign?

$$= -\varepsilon a\int_0^{2\pi} (a^2\cos^2 t - 2a\cos t - 3)a\sin^2 t\,dt$$

$$= -\varepsilon a\Big[a^3\int_0^{2\pi} \frac{1}{4}(1+\cos 2t)(1-\cos 2t)\,dt$$

$$- 2a^2\int_0^{2\pi} \cancel{\sin^2 t\,d(\sin t)}^{0}$$

$$- 3a\int_0^{2\pi} \frac{1}{2}(1-\cancel{\cos 2t}^{0})\,dt\Big]$$

$$= -\varepsilon a\Big[a^3\cdot\frac{2\pi}{4} - \frac{a^3}{4}\int_0^{2\pi}\cos^2 2t\,dt$$

$$- \frac{3a}{2}\cdot 2\pi\Big]$$

$$= -\varepsilon a\Big[\frac{a^3\pi}{2} - 3a\pi - \frac{a^3}{4}\int_0^{2\pi}\frac{1}{2}[1+\cancel{\cos 4t}^{0}]\,dt\Big]$$

$$= -\varepsilon a\Big[\frac{a^3\pi}{2} - 3a\pi - \frac{a^3}{4}\cdot\frac{1}{\cancel{2}}\cdot\cancel{2}\pi\Big]$$

$$= -\varepsilon a\pi\Big[\frac{a^3}{4} - 3a\Big] = \varepsilon a^2\pi\Big[\frac{a^2}{4} - 3\Big]$$

(iii) (cont.)

$$g(a) = -\varepsilon\pi a^2\left[\frac{a^2}{4} - 3\right] = 0$$

$$\Rightarrow a_o^2 = 12 \Rightarrow a_o = \sqrt{12}.$$

10

Sign?

$$g'(a) = -\varepsilon\pi\left[a^3 - 6a\right]$$

$$g'(a_o) = \varepsilon\pi\left[41.5692 - 6\cdot\sqrt{12}\right] \cong 20.7846\,\varepsilon\pi > 0 \quad (\varepsilon > 0)$$

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$\Rightarrow$ limit cycle is unstable x

and Hence the equilibrium point (0, 0) is stable (asymptotically).

$\dot{x} = y$

Sign?

$\dot{y} = -x - \varepsilon(x^2 - 2x + 3)y$

$\dot{y} = -x - 3\varepsilon y$

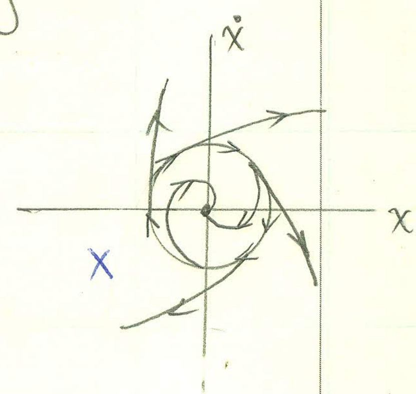
$\begin{bmatrix} 0 & 1 \\ -1 & -3\varepsilon \end{bmatrix}$

$\lambda(\lambda + 3\varepsilon) + 1 = 0$

$\lambda^2 + 3\varepsilon\lambda + 1 = 0$

$-3\varepsilon \pm \sqrt{9\varepsilon^2 - 4}$

The phase diagram:



X

Linerized system at (0, 0)

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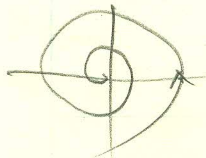
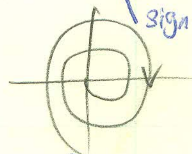
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & +3\varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

sign





the direction can be determined as before: Let $y = 0$, $x > 0 \Rightarrow \dot{y} < 0$.

You can see that (0,0) is an unstable spiral.

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MATH 415 CLASS EXAMINATION.

14 MARCH 1990.

Answer any one of the following questions.

(1) (i) The equation $\ddot{x} - \varepsilon x\dot{x} + x = 0$ models unforced vibrations of a system. When ε is small, it can be assumed that the frequency of the vibration, ω is given by,

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots\cdots$$

How do you justify this assumption?

(ii) One could reduce the given system to

$$\omega^2 x'' + x - \varepsilon\omega x x' = 0 \, , \text{ by a suitable}$$

transformation. What is this transformation? What is the advantage of solving the new system compared to the old system, if we are looking for periodic solutions?

(iii) Using the conditions $x(0) = 1$, $\dot{x}(0) = 0$ and neglecting terms of order ε^2 and higher, obtain the approximate periodic solutions of the original system. What are the corresponding frequencies?

(2) (i) A non-linear system is described by the equation

$$\ddot{x} + 0.001\,\dot{x}^3 + x = 0.$$

What is the approximate amplitude of the limit cycle of this system?

(ii) Assuming slowly varying amplitude obtain approximate solutions to the equation.

(iii) Using (ii) or otherwise, determine the type of equilibrium point the system has. Comment on the stability of this equilibrium point.

(Hint: Use $da/dt = -\varepsilon p_0(a)$ and $d\theta/dt = -1 - \frac{\varepsilon}{a} r_0(a)$,

where, $$p_0(a) = \frac{1}{2\pi}\int_0^{2\pi} h(a\cos u, a\sin u)\sin u \, du$$

and $$r_0(a) = \frac{1}{2\pi}\int_0^{2\pi} h(a\cos u, a\sin u)\cos u \, du.)$$

(3) (i) A pendulum equation with a forcing term is given by,

$$\ddot{x} + (1.009)x - 0.004x^3 = 0.006 \cos t.$$

What is special about this equation?

(ii) Using the perturbation method, show that there is a periodic solution of period 2π, if the amplitude of the zero-order solution is unity.

(iii) What are the other possible solutions, if one neglect terms of order ε^2 and higher?

Problem 1:

100/100

(1) By letting $\varepsilon = 0$, we have

$$\ddot{x} + x = 0$$

and

$$x(t) = a_0 \cos t + b_0 \sin t$$

which have a period $T_0 = 2\pi \Rightarrow \omega_0 = 1$.

Hence, when ε is small in the original system, we can expect that the actual period of solution for original system is in the neighbourhood of ω_0. This is

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$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots$$

With $\omega_0 = 1$, we have the assumption in part (a).

(2) $\boxed{\tau = \omega t} \Rightarrow d\tau = \omega\, dt \Rightarrow d\tau^2 = \omega^2 dt^2$

$$\Rightarrow \frac{d^2x}{dt^2} - \varepsilon x \cdot \frac{dx}{dt} + x = 0$$

$$\Rightarrow \frac{\omega^2 d^2x}{\omega^2 \cdot dt^2} - \varepsilon x \cdot \frac{\omega\, dx}{\omega \cdot dt} + x = 0 \Rightarrow \frac{\omega^2 d^2x}{d\tau^2} + x - \varepsilon\omega x \frac{dx}{d\tau} = 0$$

$$\Rightarrow \omega^2 x'' + x - \varepsilon\omega x x' = 0.$$

If we are looking for the periodic solutions, the advantage of solving the new system is that ω shows up explicitly in the equation. Hence, we are able to find the approximate ω.

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(3) let $\omega = 1 + \varepsilon\omega_1 + O(\varepsilon^2)$

$$x(\tau) = x_0 + \varepsilon x_1 + O(\varepsilon^2)$$

$[1+2\varepsilon\omega_1]$

$$\Rightarrow \left[x_0'' + \varepsilon x_1'' + O(\varepsilon^2)\right]\cdot\left[1+\varepsilon\omega_1 + O(\varepsilon^2)\right]^2$$

$$+\left\{x_0 + \varepsilon x_1 + O(\varepsilon^2)\right\} - \left[1+\varepsilon\omega_1 + \cdots\right]\cdot\left[x_0 + \varepsilon x_1 + O(\varepsilon^2)\right]\cdot\varepsilon\cdot$$

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$$\cdot\left[x_0' + \varepsilon x_1'' + O(\varepsilon^2)\right] = 0$$

$$\Rightarrow \varepsilon^0 : \quad x_0'' + x_0 = 0$$

$$\varepsilon^1 : \quad x_1'' + x_1 = -2\omega_1 x_0'' + x_0 x_0'$$

$\vdots$

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$\because \quad x(0) = 1, \quad \dot{x}(0) = 0$

$$\Rightarrow x_0(0) = 1, \quad \dot{x}_0(0) = 0 \ ; \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0$$

and $\quad x_0'' + x_0 = 0$

$$\Rightarrow \quad x_0(\tau) = a_0 \cos\tau + b_0 \sin\tau$$

$$x_0(0) = a_0 = 1$$

$$\dot{x}_0 = -a_0 \sin\tau + b_0 \cos\tau$$

$$\dot{x}_0(0) = b_0 = 0 \,..$$

$$\Rightarrow \boxed{x_0(\tau) = \cos\tau}$$

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$$\Rightarrow x_0' = -\sin\tau \quad \Rightarrow x_0'' = -\cos\tau.$$

$$\Rightarrow x_1'' + x_1 = -2\omega_1(-\cos\tau) + \cos\tau(-\sin\tau)$$

$$= 2\omega_1\cos\tau - \sin\tau\cos\tau$$

$$= 2\omega_1\cos\tau - \tfrac{1}{2}\sin 2\tau$$

In order to have a periodic solution

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for x_1, $2\omega_1 \equiv 0 \Rightarrow \boxed{\omega_1 = 0.}$

and $\quad x_1'' + x_1 = -\frac{1}{2}\sin 2\tau$

$$\Rightarrow x_1(\tau) = a_1\cos\tau + b_1\sin\tau - \frac{1}{2}\cdot\frac{\sin 2\tau}{-4+1}$$

$$= a_1\cos\tau + b_1\sin\tau + \frac{1}{6}\sin 2\tau$$

$$x_1(0) = 0 \Rightarrow a_1 = 0$$

$$\dot{x}_1(\tau) = -a_1\sin\tau + b_1\cos\tau + \frac{1}{3}\cos 2\tau$$

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$$\dot{x}_1(0) = b_1 + \frac{1}{3} = 0 \Rightarrow b_1 = -\frac{1}{3}$$

$$\Rightarrow \boxed{x_1(\tau) = -\frac{1}{3}\sin\tau + \frac{1}{6}\sin 2\tau}$$

$$\Rightarrow \boxed{\begin{aligned} x(\tau) &= \cos\tau + \varepsilon\left(\tfrac{1}{6}\sin 2\tau - \tfrac{1}{3}\sin\tau\right) + O(\varepsilon^2) \\ \omega &= 1 + \varepsilon\omega_1 + O(\varepsilon^2) = 1 + O(\varepsilon^2) \end{aligned}}$$

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100/100

May 11, 8:00–10:00 a.m. Final

Ben Chen

mean = 79

MATH 415 CLASS EXAMINATION

20 APRIL 1990.

ANSWER ANY ONE OF THE FOLLOWING QUESTIONS.

(1) a) Construct the fundamental matrix for the system

$$\dot{x}_1 = x_2 - x_3 , \quad \dot{x}_2 = x_3 \quad \text{and} \quad \dot{x}_3 = x_2 .$$

b) Hence, deduce the solution of $\dot{x}_1 = x_2 - x_3$, $\dot{x}_2 = x_3 + e^{-t}$, $\dot{x}_3 = x_2$ with $\underline{x}(0) = \begin{bmatrix} 0 \\ 3/2 \\ 0 \end{bmatrix}$, where $\underline{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$.

(2) a) Find the phase paths for $\dot{x} = x$, $\dot{y} = y \ln y$ in the half plane $y > 0$. Which paths are Poincaré stable?

b) Obtain a suitable Liapunov function to study the stability of the zero solution of the system $\dot{x} = -y - 2x^3$, $\dot{y} = 2x - x^2 y$. What type of stability will you get? Comment on the stability of the solutions of the system $\dot{x} = -y - (2\alpha x^3 + \beta)$, $\dot{y} = 2x$, where α and β are constants.

$\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

$x \quad 1$

$-2 \quad \lambda \qquad \lambda^2 + 2 =$

Problem 1:

100/100

(a) $\dot{x}_1 = x_2 - x_3$

$\dot{x}_2 = x_3$

$\dot{x}_3 = x_2$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\dot{\underline{x}} = A \underline{x}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & -1 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix}$$

$$= -\lambda^3 + \lambda = -\lambda(\lambda^2 - 1) = 0$$

$$\Rightarrow \boxed{\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = -1}$$

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$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \lambda \cdot \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} \gamma_2 - \gamma_3 &= \lambda \gamma_1 \\ \gamma_3 &= \lambda \gamma_2 \\ \gamma_2 &= \lambda \gamma_3 \end{aligned}}$$

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If $\lambda = \lambda_1 = 0 \Rightarrow$

$$\begin{aligned} \gamma_2 - \gamma_3 &= 0 \\ \gamma_3 &= 0 \\ \gamma_2 &= 0 \end{aligned} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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If $\lambda = \lambda_2 = 1 \Rightarrow$

$$\begin{aligned} \gamma_2 - \gamma_3 &= \gamma_1 \\ \gamma_3 &= \gamma_2 \\ \gamma_2 &= \gamma_3 \end{aligned} \Rightarrow \gamma_1 = 0 \Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

If $\lambda = \lambda_3 = -1 \Rightarrow$

$$\begin{aligned} \gamma_2 - \gamma_3 &= -\gamma_1 \\ \gamma_3 &= -\gamma_2 \\ \gamma_2 &= -\gamma_3 \end{aligned} \Rightarrow 2\gamma_2 = -\gamma_1 \Rightarrow v_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence, the fundamental matrix of the system is

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$$\Phi(t) = \begin{bmatrix} 1 & 0 & 2e^{-t} \\ 0 & e^{t} & -e^{-t} \\ 0 & e^{t} & e^{-t} \end{bmatrix}.$$

(b) $\dot{x}_1 = x_2 - x_3$

$\dot{x}_2 = x_3 + e^{-t}$

$\dot{x}_3 = x_2$

$\Rightarrow \ddot{x}_3 = \dot{x}_2 = x_3 + e^{-t}$

$\Rightarrow \ddot{x}_3 - x_3 = e^{-t}$

$(D^2 - 1)\, x_3 = e^{-t}$

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$$x_3 = \frac{1}{D^2 - 1}\{e^{-t}\}$$

$$= \frac{t e^{-t}}{-2} = \underline{-0.5 t e^{-t}}$$

$x_2 = \dot{x}_3 = \underline{-0.5\,(e^{-t} - t e^{-t})}$

$\dot{x}_1 = x_2 - x_3 = -0.5(e^{-t} - t e^{-t}) + 0.5 t e^{-t}$

$= -0.5 e^{-t} + t e^{-t}$

$x_1 = 0.5 e^{-t} - t e^{-t} - e^{-t}$

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$= \underline{-0.5 e^{-t} - t e^{-t}}$

$\Rightarrow$ Particalar integral

$$= \begin{bmatrix} -0.5e^{-t}(1+2t) \\ -0.5e^{-t}(1-t) \\ -0.5te^{-t} \end{bmatrix}$$

$\Rightarrow$ Solution of the system is

$$X(t) = \begin{bmatrix} 1 & 0 & 2e^{-t} \\ 0 & e^{t} & -e^{-t} \\ 0 & e^{t} & e^{-t} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} - 0.5e^{-t} \begin{bmatrix} 1+2t \\ 1-t \\ t \end{bmatrix}$$

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$$X(0) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} - 0.5 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/2 \\ 0 \end{bmatrix}$$

We have $\alpha_1 = 2.5$, $\alpha_2 = 1$, $\alpha_3 = -1$

Hence, the solution is

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$$\boxed{\begin{aligned} x_1(t) &= 2.5 - 2e^{-t} - 0.5e^{-t}(1+2t) \\ x_2(t) &= e^{t} + e^{-t} - 0.5e^{-t}(1-t) \\ x_3(t) &= e^{t} - e^{-t} - 0.5te^{-t} \end{aligned}}$$

100/100

60/60

Ben M. Chen
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MATH 415 FINAL EXAMINATION

11 MAY 1990 8.00 – 10.00 am

Answer any three of the following questions.

(1) (i) A linear system is given by $\dot{\underline{x}} = A\underline{x}$, where $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, ($a, b, c$ and d are constants). Obtain the condition for A to have eigenvalues which are complex.

(ii) Find the equilibrium points of the nonlinear system
$\dot{x} = y$
$\dot{y} = 2x - x^2$. Determine the stability of these equilibrium points and sketch the complete phase diagram.

(2) (i) A non-linear system is described by the equation
$\ddot{x} + 0.001\dot{x}^3 + x = 0$. What is the approximate amplitude of the limit cycle of this system?

(ii) Assuming slowly varying amplitude, obtain approximate solutions to the equation.

(iii) Using (ii) or otherwise, determine the type of equilibrium point the system has. Comment on the stability of this equilibrium point.

(3) (i) Construct the fundamental matrix for the system
$$\dot{x}_1 = x_2 - x_3 ,\quad \dot{x}_2 = x_3 \quad \text{and} \quad \dot{x}_3 = x_2 .$$

(ii) Hence, deduce the solution of $\dot{x}_1 = x_2 - x_3$
$\dot{x}_2 = x_3 + e^{-t}$ and $\dot{x}_3 = x_2$ with $\underline{x}(0) = \begin{bmatrix} 0 \\ 3/2 \\ 0 \end{bmatrix}$, where
$\underline{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$.

(4) (i) Find the phase paths for $\dot{x} = x$, $\dot{y} = y \ln y$ in the half plane $y > 0$. Which paths are Poincaré stable?

(ii) Obtain a suitable Liapunov function to study the stability of the zero solution of the system $\dot{x} = -y - 2x^3$, $\dot{y} = 2x - x^2 y$. What type of stability will you get? Comment on the stability of the solutions of the system $\dot{x} = -y$, $\dot{y} = 2x$.

(5) (i) Find the bifurcation points of the system
$$\dot{x} = -\lambda x + y$$
$$\dot{y} = -\lambda x - 3y$$
, where λ is a parameter.

(ii) Determine whether these bifurcation points are Hopf bifurcation points.

Problem 1.

(a) $\det(A-\lambda I) = \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= ad - (a+d)\lambda + \lambda^2 - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad-bc)$$

✓

$$\Delta = (a+d)^2 - 4(ad-bc)$$

$$= a^2 + d^2 + 2ad - 4ad + 4bc$$

$$= a^2 + d^2 - 2ad + 4bc$$

2 $$= (a-d)^2 + 4bc < 0$$

The condition for A to have eigenvalues which are complex is ✓ $(a-d)^2 + 4bc < 0$.

(b)

$$\dot{x} = y \qquad\qquad y = 0$$
$$\Rightarrow$$
$$\dot{y} = 2x - x^2 \qquad\qquad 2x - x^2 = x(2-x) = 0$$

✓

Thus, the equilibrium points are:

3 $\{0, 0\}$ and $\{2, 0\}$

✓

Let us first linearize the system at (0,0),

$$\dot{x} = y$$

$$\dot{y} = (2-2x)\big|_{(0,0)} x + 0 = 2x$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \Rightarrow \begin{vmatrix}-\lambda & 1\\ 2 & -\lambda\end{vmatrix} = \lambda^2 - 2 = 0$$

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$\Rightarrow \lambda_{1,2} = \pm\sqrt{2} \Rightarrow$ saddle and this equilibrium point is unstable.

Next, linearize it at (2, 0)

$$\dot{x} = y$$

$$\dot{y} = (2-2x)\big|_{(2,0)} x = -2x$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \Rightarrow \begin{vmatrix}-\lambda & 1\\ -2 & -\lambda\end{vmatrix} = \lambda^2 + 2 = 0$$

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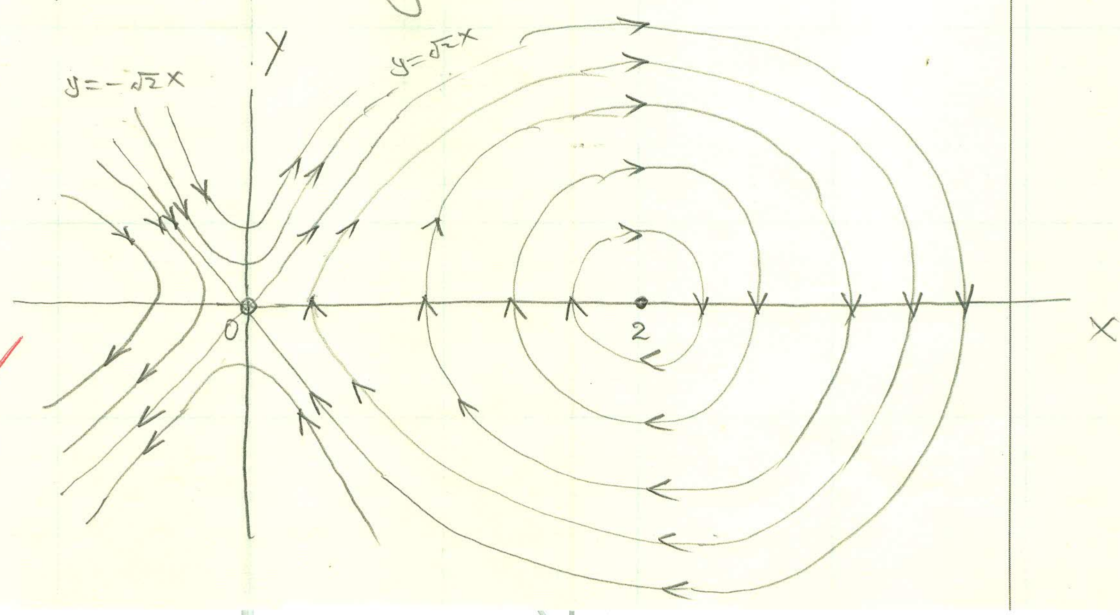
$\Rightarrow \lambda_{1,2} = \pm\sqrt{2}\, i \Rightarrow$ center and this equilibrium point is stable (normally one e.p is unstable the other is stable)

The complete phase diagram is:

$y > 0 \quad x \nearrow$

$y < 0 \quad x \searrow$



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Problem 3.

(a) $\dot{x}_1 = x_2 - x_3$

$\dot{x}_2 = x_3$

$\dot{x}_3 = x_2$

$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & -1 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix}$$

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$$= -\lambda \cdot (\lambda^2 - 1) = 0$$

$$\Rightarrow \lambda_1 = -1, \quad \lambda_2 = 0, \quad \lambda_3 = 1$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} \cdot \lambda$$

$$\Rightarrow \quad \gamma_2 - \gamma_3 = \lambda \gamma_1$$

$$\gamma_3 = \lambda \gamma_2$$

$$\gamma_2 = \lambda \gamma_3$$

For $\lambda_1 = -1$, $\Rightarrow$

$$\gamma_2 - \gamma_3 = -\gamma_1$$

$$\gamma_3 = -\gamma_2 \quad \Rightarrow \quad \underline{R}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\gamma_2 = -\gamma_3$$

For $\lambda_2 = 0$ $\Rightarrow$

$$\gamma_2 - \gamma_3 = 0$$

$$\gamma_3 = 0 \quad \Rightarrow \quad \underline{R}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\gamma_2 = 0$$

For $\lambda_3 = 1 \Rightarrow$

$$\gamma_2 - \gamma_3 = \gamma_1$$

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$$\gamma_3 = \gamma_2 \quad \Rightarrow \quad \underline{R_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\gamma_2 = \gamma_3$$

The fundamental matrix is

2

$$\underline{\Phi} = \begin{bmatrix} 2e^{-t} & 1 & 0 \\ -e^{-t} & 0 & e^{t} \\ e^{-t} & 0 & e^{t} \end{bmatrix}.$$

(b) $\dot{x}_1 = x_2 - x_3$

$\dot{x}_2 = x_3 + e^{-t}$

$\dot{x}_3 = x_2$

$$\Rightarrow \quad \ddot{x}_3 = \dot{x}_2 = x_3 + e^{-t}$$

2

$$\Rightarrow \quad \ddot{x}_3 - x_3 = e^{-t}$$

$$\Rightarrow \quad x_3 = \frac{1}{D^2 - 1}\{e^{-t}\}$$

$$= \frac{t\,e^{-t}}{-2} = \underline{-0.5\,t\,e^{-t}}$$

$$x_2 = \dot{x}_3 = -0.5\,e^{-t} + 0.5\,t\,e^{-t}$$

$$= \underline{0.5\,(t-1)\,e^{-t}}$$

$$\Rightarrow \quad \dot{x}_1 = x_2 - x_3 = 0.5\,(t-1)\,e^{-t} + 0.5\,t\,e^{-t}$$

$$= t\,e^{-t} - 0.5\,e^{-t}$$

$$x_1 = \int te^{-t}\,dt - 0.5\int e^{-t}\,dt$$

$$= -\int t\,de^{-t} + 0.5e^{-t}$$

$$= -te^{-t} + \int e^{-t}\,dt + 0.5e^{-t}$$

$$= -te^{-t} - e^{-t} + 0.5e^{-t}$$

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$$= -te^{-t} - 0.5e^{-t}$$

Thus the particular integral is

$$\begin{cases} x_1 = -0.5e^{-t} - te^{-t} \\ x_2 = -0.5e^{-t} + 0.5te^{-t} \\ x_3 = \qquad\qquad -0.5te^{-t} \end{cases}$$

Hence, the closed solution for whole system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 2e^{-t} & 1 & 0 \\ -e^{-t} & 0 & e^{t} \\ e^{-t} & 0 & e^{t} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} -0.5 - t \\ -0.5 + 0.5t \\ -0.5t \end{bmatrix} e^{-t}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} -0.5 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.5 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \alpha_1 = -1, \quad \alpha_2 = 2.5 \text{ and } \alpha_3 = 1$$

Problem 3 | Ben M. Chen | 3.4

4.1

The answer is:

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2e^{-t} & 1 & 0 \\ -e^{-t} & 0 & e^{t} \\ e^{-t} & 0 & e^{t} \end{pmatrix} \begin{pmatrix} -1 \\ 2.5 \\ 1 \end{pmatrix} + \begin{pmatrix} -0.5 - t \\ -0.5 + 0.5t \\ -0.5t \end{pmatrix} e^{-t}.$$

Problem 4.

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= y \ln y \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y \ln y}{x}$$

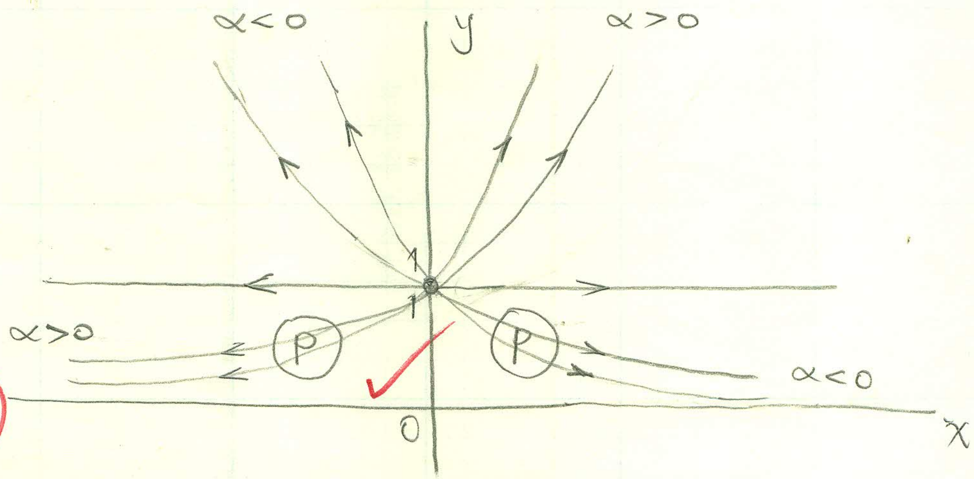
$$\Rightarrow \quad \frac{dy}{y \ln y} = \frac{dx}{x} \quad \Rightarrow \quad \frac{d \ln y}{\ln y} = \frac{dx}{x}$$

$$\Rightarrow \quad \ln(\ln y) = \ln x + \ln \alpha = \ln(\alpha x)$$

6

$$\Rightarrow \quad \ln y = \alpha x \quad \Rightarrow \quad y = e^{\alpha x}$$

The phase diagram: equilibrium point $(0, 1)$



4

The above phase paths marked by Ⓟ are Poincaré stable.

(b) $$\dot{x} = -y - 2x^3 \quad \Rightarrow$$
$$\dot{y} = 2x - x^2 y$$

Consider a Lyapunov function:

$$V(x,y) = x^2 + \tfrac{1}{2}y^2$$

We note that

$V(x,y) > 0$ if $(x,y) \neq (0,0)$
$V(x,y) = 0$ if $(x,y) = (0,0)$
$\Big\} \Rightarrow V(x,y)$ is valid.

$$\dot{V}(x,y) = 2x\dot{x} + y\dot{y}$$
$$= 2x(-y - 2x^3) + y(2x - x^2 y)$$
$$= -2xy - 4x^4 + 2xy - x^2y^2$$
$$= -4x^4 - x^2y^2 \le 0.$$

6

$\Rightarrow$ the zero solution of the system is asymptotically stable.

for $\dot{x} = -y$
$\dot{y} = 2x$

which is simply the linearized system of the original one. Hence, it is stable. But it is simple to check it is only a center, not asymptotically stable.

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