E-E 509 TEST ONE

Adaptive Control Systems

BEN M CHEN

PROBLEM STATEMENT (No. 4 On Page 137)

When an identification or an adaptive control problem yields an error differential equation

$$\dot{e} = -e + \phi\, x_p,$$

the adaptive law $\dot{\phi}=-ex_p$ was chosen in Chapter 3. This makes $V(e,\phi)=(e^2+\phi^2)/2$ a Lyapunov function since $V>0$ and $\dot{V}=-e^2\leq 0$. If the adaptive law is

$$\dot{\phi} = -\phi - e\, x_p,$$

$V(e,\phi)$ would still be a Lyapunov function with $\dot{V}=-\phi^2-e^2<0$ assuring the uniform asymptotic stability in the large of the origin in the (e,ϕ) space. Why then is such an adaptive law not used?

Solution : Since $\phi(t)=\hat{a}_p(t)-a_p$, the new adaptive law can then be rewritten as

$$\dot{\hat{a}}_p(t) = -\hat{a}_p(t) - e(t)\, x_p(t) + a_p.$$

where a_p is an <u>unknown</u> constant. Hence, the above adaptive law is physically unrealizable and it is useless. **Q.E.D.**

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Ben M. Chen

EE509

MID TERM EXAM

Problem #2

Are the following transfer functions PR? Are they SPR?

(a) $\frac{1}{S}$

(b) $\frac{-1}{S}$

(c) $\frac{S^3+3S^2+5S+1}{S^3+2S^2+S+2}$ $-1.3855 \pm j1.5639$, -0.2291
-2, $\pm j$

(d) $\frac{S+3}{(S+1)\ (S+2)}$

(e) $\frac{S+4}{(S+3)\ (S+5)}$

(f) $\frac{S^2+S-2}{S^3+4S^2+3S+2}$ -2, 1
-3.2695 ; $-0.3652 \pm j0.6916$

(g) $\frac{S^3+4S+2}{S^5+5S^4+10S^3+8S^2+S+6}$ $0.2367 \pm j2.0416$, -0.4735
-2.2952, $-1.6024 \pm j1.5128$, $0.25 \pm j0.6898$

Excellent

25/25

Problem # 3

Assume the transfer functions $G_1(S)$ and $G_2(S)$ are PR. Check whether the following transfer functions are PR in general.

1. $G_a(S) = G_1(S) + G_2(S)$

2. $G_m(S) = G_1(S)\ G_2(S)$

3. $G_f(S) = G_1(S)\ \left(1+G_2(S)\right)^{-1}$

Repeat 1, 2, 3 assuming $G_1(S)$ and $G_2(S)$ are SPR.

Problem 2 Are the following functions PR? Are they SPR?

(a) $1/s$, Answer: PR, but not SPR.

Reasons for PR:

① $1/s$ is real for real s.

② $1/s$ is analytic in $Re(s)>0$ and the residue associated to the only $j\omega$-axis pole $=1>0$.

③ $Re[1/j\omega]=0$ for $\omega\neq 0$.

Reason for not SPR:

③ $n^*=1$, but $\lim_{|\omega|\to\infty}\omega^2 Re[1/j\omega]=0 \not> 0$.

(b) $-1/s$, Answer: not PR, not SPR.

Reason for not PR:

② the residue associated to the $j\omega$-axis pole $=-1<0$.

Reason for not SPR:

Because it is not PR.

(c) $g(s)=\dfrac{s^3+3s^2+5s+1}{s^3+2s^2+s+2}=\dfrac{s^3+3s^2+5s+1}{(s+2)(s+j)(s-j)}$ Answer: P.R., not S.P.R.

Reasons for P.R. (it is obvious to see $g(s)$ is not S.P.R.)

① $g(s)$ is real whenever s is real.

② $g(s)$ is analytic in $Re(s)>0$ and has two simple $j\omega$-axis poles at $s=\pm j$, the associated residues are.

$$R_{+j}=\frac{-j-3+5j+1}{2j(2+j)}=1, \quad R_{-j}=1$$

③ $Re[g(j\omega)]=\alpha(\omega)(\omega^6-3\omega^2+2)$ with $\alpha(\omega)>0$.

$$\frac{d}{d\omega}[\omega^6-3\omega^2+2]=6\omega[\omega^4-1]=0 \Rightarrow \omega_1=0,\ \omega_2=1$$

$$Re[g(j0)]=2, \quad Re[g(j1)]=0$$

Hence, $Re[g(j\omega)]\geq 0$, $g(s)$ is P.R.

(d) $g(s) = \dfrac{(s+3)}{(s+1)(s+2)}$ Answer: PR, but not SPR.

$$g(j\omega) = \frac{j\omega + 3}{(j\omega+1)(j\omega+2)}$$

$$= \frac{j\omega + 3}{(2-\omega^2) + j3\omega} = \frac{(6-3\omega^2+3\omega^2) + \text{Imag}}{(\omega^4 + 5\omega^2 + 4)}$$

$$\therefore \ \mathrm{Re}[g(j\omega)] = 6/(\omega^4+5\omega^2+4) > 0 \Rightarrow \text{P.R.}$$

$$\text{But} \ \lim_{|\omega|\to\infty} \omega^2 \mathrm{Re}[g(j\omega)] = 0 \Rightarrow \text{not. S.P.R.}$$

(e) $g(s) = \dfrac{s+4}{(s+3)(s+5)}$ Answer: P.R. and S.P.R.

$$g(j\omega) = \frac{j\omega + 4}{(j\omega+3)(j\omega+5)} = \frac{j\omega+4}{(15-\omega^2)+j8\omega}$$

$$= \frac{4\omega^2 + 60}{\omega^4 + 34\omega^2 + 225} + \text{Imag.}$$

$$\mathrm{Re}[g(j\omega)] > 0 \Rightarrow \text{P.R.}$$

$$\lim_{|\omega|\to\infty} \omega^2 \mathrm{Re}[g(j\omega)] = 4 > 0 \Rightarrow \text{S.P.R.}$$

(f) $g(s) = \dfrac{s^2 + s - 2}{s^3 + 4s^2 + 3s + 2}$,   Answer: not P.R, not S.P.R.

$$\mathrm{Re}[g(0.5)] = -0.27027\cdots < 0 \Rightarrow \text{not P.R.} \Rightarrow \text{not S.P.R.}$$

(g) $g(s) = \dfrac{s^3 + 4s + 2}{s^5 + 5s^4 + 10s^3 + 8s^2 + s + 6}$ Answer: not P.R., not S.P.R.

Check: $g(j1) = \dfrac{2 + 3j}{j + 5 - 10j - 8 + j + 6}$

$$= \frac{2 + j3}{3 - j8}$$

$$= \frac{-18}{73} + \text{Imag.}$$

Good

$$\mathrm{Re}[g(j1)] < 0 \Rightarrow \text{not P.R.}$$

$$\Rightarrow \text{not S.P.R.}$$

Problem 3.

Assume the transfer function $G_1(s)$ and $G_2(s)$ are P.R. Check whether the following transfer functions are P.R. in general.

1. $G_a(s) = G_1(s) + G_2(s)$ Answer: YES

① $G_a(s)$ is real for real s because $G_1(s)$ and $G_2(s)$ are real.

② $Re[G_a(s)] = Re[G_1(s)] + Re[G_2(s)] \geq 0$ for all $Re(s) > 0$

because $Re[G_1(s)] \geq 0$, $Re[G_2(s)] \geq 0$.

2. $G_m(s) = G_1(s)G_2(s)$ Answer: No

Counter Example:

$G_1(s) = G_2(s) = \frac{1}{s}$ which is shown in 2(a) are P.R.

$G_m(s) = \frac{1}{s^2}$ has multiple poles at $s=0$.

3. $G_f(s) = G_1(s)\cdot[1+G_2(s)]^{-1}$ Answer: No

Let $G_2(s) = \frac{1}{s}$ which is shown to be P.R.

$G_1(s) = s$ is also P.R. by Corollary 1.3.1.

$$G_f(s) = s\cdot[1+\tfrac{1}{s}]^{-1} = \frac{s^2}{s+1}$$

$$G_f(j\omega) = \frac{-\omega^2}{1+j\omega} = \frac{-\omega^2}{\omega^2+1} + \text{Imag.}$$

$Re[G_f(j\omega)] < 0$ for all ω $\Rightarrow$ not P.R.

Repeat 1, 2, 3 assuming $G_1(s)$ and $G_2(s)$ are SPR.

1. $G_a(s) = G_1(s) + G_2(s)$ Answer: P.R. (YES) and S.P.R.

Proof: According to (ii) in text on page 65.

2. $G_m(s) = G_1(s) \cdot G_2(s)$ Answer: Not P.R, not SPR in general

Counter Example

$$G_1(s) = \frac{1}{s+1}, \quad G_2(s) = \frac{1}{s+1}$$

$$\text{Re}[G_1(j\omega)] = \frac{1}{\omega^2+1} > 0, \quad \lim_{|\omega|\to\infty} \omega^2/(\omega^2+1) = 1 > 0.$$

Thus, $G_1(s)$ and $G_2(s)$ are SPR.

But $G_m(s) = \dfrac{1}{(s+1)^2}$

$$G_m(j\omega) = \frac{1}{(1+j\omega)^2} = \frac{1}{(1-\omega^2)+j2\omega}$$

$$= \frac{(1-\omega^2)-j2\omega}{\omega^4+2\omega^2+1}$$

$\text{Re}[G_m(j\omega)] < 0$ for $\omega > 1$.

3. $G_f(s) = G_1(s) \cdot [1+G_2(s)]^{-1}$. Answer: Not P.R, not SPR

Counter Example.

$$G_1(s) = \frac{1}{s+2}, \quad G_2(s) = s+1$$

They are SPR as shown in 2. and Corollary 1.3.1.

$$G_f(s) = \frac{1}{s+2} \cdot [1+s+1]^{-1}$$

$$= \frac{1}{(s+2)^2}$$

$$G_f(j\omega) = \frac{1}{(4-\omega^2)+j4\omega}$$

$$\text{Re}[G_f(j\omega)] = \frac{4-\omega^2}{\omega^4+8\omega^2+16} < 0 \text{ for } \omega > 2.$$

Adaptive Control

MID-TERM EXAMINATION PART 3

Ben M Chen

Show that there is some thing wrong in the selections of $\lambda(s)$, $C^*(s)$ and $D^*(s)$ in Equation 5.17 on page 194 in text.

Answer: In case of n being the upper bound of the order of plant (not the order of the plant), the results in book are incorrect.

Recall the difinitions of

$$\lambda(s) = \text{the charactistic function of } (sI-\Lambda)^{-1}, \text{ where } \Lambda \in R^{(n-1)x(n-1)}.$$

$$\frac{C(s)}{\lambda(s)} = \theta_1^T*(sI-\Lambda)^{-1}l \quad \text{and} \quad \frac{D(s)}{\lambda(s)} = \theta_o+\theta_2^T*(sI-A)^{-1}l.$$

which imply that $\lambda(s)$, $C(s)$ and $D(s)$ are at most the orders of (n-1), (n-2) and (n-1), respectively. From the selections in text, we know

$\lambda(s) = Z_m(s)$ implies that $Z_m(s)$ is a monic polynormial of order (n-1).

$Z_p(s) = \lambda(s)-C^*(s)$ implies that $Z_p(s)$ must be a polynomial of order (n-1).

$R_p(s) = R_m(s)+k_pD^*(s)$ implies that $R_p(s)$ must be a polynomial of order n.

For the order of the plant is less than n, the results are wrong. ✓ **Q.E.D.**

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Adaptive Control

MID-TERM EXAMINATION PART 4

Ben M Chen

Under what condition, the adaptive parameters converge to true values for the adaptive control problem with $n^*=1$.

ANSWER: The result is given in Section 6.5.2. on pages 261-262 in text. Following I will do some analysis, because this is a 10-point exam problem.

From the class notes as well as the text, we can easily obtain following equations:

$$\dot{e}(t) = A\ e(t) + b\ \omega^T(t)\ \phi(t)$$

$$\dot{\phi}(t) = -\ \mathrm{sgn}(k_p)\ \omega(t)\ h^T\ e(t)$$

Then according to Theorem 2.17 in text, we know that $[e^T\ \phi^T]=0$ is uniformly asymptotically stable if, and only if, ω satisfies condition (2.33), which is denoted by $\Omega_{(2n,T)}$ in Chapter 6. As it done in the text, we can rewrite ω as $\omega = \omega^* + C\ e$, where

$$\omega^* = [\ r,\ \omega_1^{*T},\ y_m,\ \omega_2^{*T}\ ]^T$$

Then by choosing the Lyapunov function as $V=[e^TPe+\phi^T\phi]/2$, we can make $e \longrightarrow 0$ as $t \longrightarrow \infty$. Hence, this implies that $\omega \in \Omega_{(2n,t1,T)}$ if, and only if, $\omega^* \in \Omega_{(2n,to,T)}$ where $t_1 \geq t_o$. Since ω^* corresponds to the controllable part of the state x_{mn}, parameter convergence follows, if, and only if, r has at least 2n spectral lines, e.g., r must have at 2n different frequency components.

0/10

Adaptive Control

final Exam.

Ben M. Chen

12-21-89

Excellent

#1	17/20
#2	20/20
#3	20/20
#4	20/20
#5	20/20
#6	20/20
#7	15/20

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A

Problem 1: If $W(s)$ is a strictly proper transfer function that is asymptotically stable with input x and output y, show that

(a) $y(t)$ is uniformly bounded if $x(t)$ is uniformly bounded.

(b) $y(t) \to 0$ as $t \to \infty$ if $x(t) \to 0$ as $t \to \infty$.

(c) $x \in L_1$ or $\in L_2 \Rightarrow y(t) \to 0$ as $t \to \infty$.

Answer: (a) Assuming (C, A, B) is minimal realization of $W(s)$. Then follows the proved Lemma 2.6, we have

$$|y(t)| = O[x_s(t)]$$

Thus, if $x(t)$ is bounded, $y(t)$ is bounded.

(b) Refer to the proof in: Desor and Wu, Stability of Linear Time-Invariant Systems, IEEE Tran. on AC, No: 3 Sep. 1968, PP 245 − 250.

(1) Let $h(t) = \mathcal{L}^{-1}[W(s)] = Ce^{At}B$ and A stable

$$\Rightarrow \int_0^\infty |h(t)| < \infty. \quad \Rightarrow h \in L_1 \cap L_2 \cap L_\infty$$

Now $x(t) \to 0$ as $t \to \infty$ $\boxed{\text{(assume } x \in L_\infty)}$

$\forall \varepsilon > 0$, $\exists$ $T_x(\varepsilon) > 0$ such that for $t > T_x(\varepsilon)$, $|x(t)| < \varepsilon$.

Since $h \in L_1 \cap L_\infty$, for any $\varepsilon > 0$, there exists $T_h(\varepsilon)$ such that for $t > T_h(\varepsilon)$

$$\int_t^\infty |h(t)|\, dt < \varepsilon$$

Let $t > T_h(\varepsilon) + T_x(\varepsilon)$, then

$$|y(t)| = \left| \int_0^t h(t-\tau)x(\tau)d\tau \right|$$
$$\leq \int_0^t |h(t-\tau)| \cdot |x(\tau)|\, d\tau$$

(cont.)

1 (b) (cont.)

$$|y(t)| \leq \int_0^{t-T_h} |h(t-\tau)|\,|x(\tau)|\,d\tau + \int_{t-T_h}^{t} |h(t-\tau)|\cdot|x(\tau)|\,d\tau$$

In the first integral the argument of h varies from T_h to t.

Therefore, since $u \in L_\infty$, the first integral satisfies

$$\int_0^{t-T_h} |h(t-\tau)|\cdot|x(\tau)|\,d\tau \leq \varepsilon\, x_M$$

In the second integral the argument of x varies from $t - T_h > T_x$ to t, hence, over this inteval $|x(t)| < \varepsilon$ and the second integral satisfies

$$\int_{t-T_h}^{t} |h(t-\tau)|\cdot|x(\tau)|\,d\tau \leq \varepsilon \cdot h_M \quad (h \in L_\infty)$$

Thus, we have shown that $t > T_h(\varepsilon) + T_x$ implies

$$|y(t)| \leq \varepsilon\,(x_M + h_M).$$

good

Therefore, we have proved $y(t) \to 0$ as $t \to \infty$. Q.E.D.

(c) : $y = h \circledast x$, $h \in L_1 \cap L_2 \cap L_\infty$

$$= \int_0^t h(t-\tau)\,x(\tau)\,d\tau$$

① if $x \in L_1$,

$$|y| \leq \int_0^t |h(t-\tau)|\cdot|x(\tau)|\,d\tau$$

$$\leq h_M \cdot \int_0^t |x(\tau)|\,d\tau < \infty \Rightarrow y \in L_\infty.$$

② if $x \in L_2$, follows Hölder's inequality

$$|y| \leq \int_0^t |h(t-\tau)|\cdot|x(\tau)|\,d\tau$$

$$\leq \left(\int_0^t |h(t-\tau)|^2 d\tau\right)^{1/2} \cdot \left(\int_0^t |x(\tau)|^2 d\tau\right)^{1/2} < \infty \Rightarrow y \in L_\infty.$$

(To be continued.)

1. (c) (cont.)

Lemma 1: If $x \in L_1$, then $y \in L_1$

Proof: $\int_0^\infty |y(t)|\,dt \le \int_0^\infty \left[\int_0^\infty |h(t-\tau)|\cdot|u(\tau)|\,d\tau\right]dt$

$$\le \int_0^\infty |u(\tau)|\,d\tau \left[\int_0^\infty |h(t-\tau)|\,dt\right]$$

$$= H_1 \int_0^\infty |u(\tau)|\,d\tau = M < \infty \Rightarrow y \in L_1.$$

Lemma 2: $x \in L_p$, then $y \in L_p$

Proof: Consider $1 < p < \infty$; we have

$$|y(t)| \le \int \underbrace{|h(t-\tau)|^{1/p} |x(\tau)|}_{1} \cdot |h(t-\tau)|^{1/q}\,d\tau$$

$x \in L_p$ implies that the first factor is L_p [by Lemma 1].
and the second is in L_q; Hence, by Hölder inequality

$$|y(t)| \le \left[\int |h(t-\tau)|\,|x(\tau)|^p d\tau\right]^{1/p} \cdot \left[\int |h(t-\tau)|\,d\tau\right]^{1/q}$$

Note that the last factor is bounded by $h_M^{1/q}$. Taking L_p norms of both sides,

$$\|y\|_p \le h_M^{1/q} \left\{\int \left[\int |h(t-\tau)|\cdot|x(\tau)|^p d\tau\right] dt\right\}^{1/p}$$

$$\le h_M^{1/q} \cdot h_M^{1/p} \cdot \|u\|_p = h_M \cdot \|u\|_p < \infty \Rightarrow y \in L_p$$

where we used again Lemma 1. Good

Now, y is the output of LTI system. $\dot{y}$ exists and y is continuous. Hence, together with the results we obtained above, we show

$y(t) \to 0$ as $t \to \infty$. ?

Q.E.D

17/20

2.

11. **(a)** In Lemma 2.8, if A is an asymptotically stable matrix, show that $u_s \sim \|x\|_s$.
(b) If $W(s)$ is an asymptotically stable transfer function with all its zeros in $\mathbb{C}^-$, show that the input and the output grow at the same rate.
(c) Give a simple example to show that (b) is violated if at least one zero of $W(s)$ lies in the open right half plane.

Answer: (a) First, we note that both Lemma 2.6 and Lemma 2.8 are proven in the text. Thus, we can use these results without any problem.

① A is stable, from the proof of Lemma 2.6, we have

$u \in \mathcal{E}$

$$\|x(t)\| \le C_1 \sup_{\tau \le t} |u(\tau)| + C_2 \Rightarrow \|x(t)\| = O[u_s(t)]$$

$$\Rightarrow \|x(t)\|_s = O[u_s(t)]$$

② from Lemma 2.8, we have

$$u_s(t) = O[\|x(t)\|_s]$$

good

Hence, $u_s \sim \|x\|_s$ by definition. Q.E.D.

(b) We assume $W(s)$ is strictly proper. (If it is not proper, one simple counter example such as $W(s) = s+1$, $u = e^{t^2} \Rightarrow y = (2t+1)e^{t^2}$, not the same rate. If it is non-strictly proper, result follows proved corollary 2.5).

Let $W(s) = \dfrac{p(s)}{z(s)} = \dfrac{p(s)}{z_1(s)} \cdot \dfrac{1}{z_2(s)}$ and $u \in \mathcal{E}$

where $z_1(s) \cdot z_2(s) = z(s)$ and p and z_1 have the same degree

$$y = W(s)u = \frac{p(s)}{z_1(s)} \cdot \frac{1}{z_2(s)} u = \frac{p(s)}{z_1(s)} \cdot u_1$$

$$u_1 = \frac{1}{z_2(s)} u$$

Follows the proof of (ii) on page 81, we have

$$u_s = O[u_{1s}(t)]$$

(to be cont.)

2. (b) (cont.)

Since $p(s)/q_1(s)$ is of relative degree zero and $p(s)$ and $q_1(s)$ are Hurwitz, from Corollary 2.5, it follows that

$$u_{1s} = O[y_s(t)].$$

Hence, $u_s = O[y_s(t)]$. On the other hand, $W(s)$ is stable, from lemma 2.6, we have $|y(t)| = O[u_s(t)]$

$\Rightarrow$ $y_s = O[u_s(t)]$, Hence, $u_s \sim y_s$ Q.E.D.

(c) Counter Example:

$u \in \mathcal{E}$ ← important

$$W(s) = \frac{s-1}{(s+1)(s+2)}, \quad u = e^t \quad u(s) = \frac{1}{s-1}$$

$$y(s) = W(s)u(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\Rightarrow y(t) = e^{-t} - e^{-2t} \longrightarrow 0 \text{ as } t \to \infty.$$

(We assume that all the initial conditions are identically zero.)

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[illegible]

3.

9. In the system shown in Fig. 3.12, determine rules for adjusting the parameters k_1 and k_2 so that the output error e_1 tends to zero.

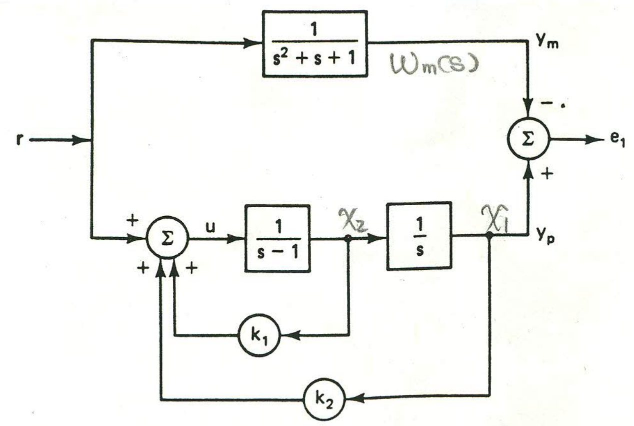

Figure 3.12 Adaptive control using state feedback.

Answer: First, we realize $W_m(s)$ as follow:

$$\dot{x}_m = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}}_{A_m} x_m + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_m} r, \quad y_m = \underbrace{[1 \quad 0]}_{h^T} x_m$$

And then realize the plant as:

$$\dot{x}_p \triangleq \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{A_p} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B_p} u, \quad y_p = \underbrace{[1 \quad 0]}_{h^T} x_p$$

$$u = (k_2 \quad k_1)\, x_p + r$$

Of course, it is easy to verify that there exists

$$K^* = (k_2^* \quad k_1^*) \quad \text{such that} \quad A_m = A_p + B_p K^*$$

* For this system, we can assume the structure of the plant is known, e.g. $B_p = B_m$, and only A_p is unknown.

Define error:

$$e_1 \triangleq y_m - y_p = h^T (x_m - x_p)$$

$$e \triangleq x_m - x_p, \quad e_1 = h^T e$$

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* In this system with this structure, we are not able to do anything if we do not have such assumption.

3 (cont.)

$$\dot{e} = \dot{x}_m - \dot{x}_p \qquad (x_m \text{ is known})$$

$$= A_m x_m + B_m r - (A_p + B_m K(t))x_p - B_m r$$

$$= A_m e + (A_m - A_p)x_p - B_m K(t)\, x_p$$

$$= A_m e + [B_m K^* - B_m K(t)]\, x_p$$

$$= A_m e + B_m [K^* - K(t)]\, x_p$$

$$= A_m e - B_m \phi\, x_p \quad , \quad \text{where } \phi \triangleq K - K^*$$

Since, we use state feedback, we must know x_p, so is e.

Choose Lyaponov function as

$$V = \frac{1}{2}[e^T P e + tr\{\phi^T \phi\}] > 0, \quad P = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.0 \end{bmatrix} > 0$$

$$\dot{V} = \frac{1}{2}[\dot{e} P e + e^T P \dot{e}] + tr\{\phi^T \dot{\phi}\}$$

$$= \frac{1}{2}[e^T A_m^T P e - x_p^T \phi^T B_m^T P e + e^T P A_m e - e^T P B_m \phi x_p] + tr\{\phi^T \dot{\phi}\}$$

$A_m^T P + P A_m = -I \Rightarrow$ $$= -\frac{1}{2} e^T e + tr\{\phi^T \dot{\phi}\} - tr\{x_p^T \phi^T B_m^T P e\}$$

$$= -\frac{1}{2} e^T e + tr\{\phi^T \dot{\phi}\} - tr\{\phi^T B_m^T P e x_p^T\}$$

$$= -\frac{1}{2} e^T e + tr\{\phi^T [\dot{\phi} - B_m^T P e x_p^T]\}$$

if we choose the adaptive law

$$\dot{\phi} = \dot{K} = B_m^T P e\, x_p^T$$

$$(\dot{k}_1 \ \ \dot{k}_2) = (0 \ \ 1)\begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}(x_1 \ \ x_2)$$

$$\underline{(\dot{k}_1 \ \ \dot{k}_2) = [0.5 e_1 + e_2]\cdot(x_1 \ \ x_2)}$$

20/20

$$\Rightarrow \dot{V} = -\frac{1}{2} e^T e \le 0 \Rightarrow e, \phi \in L_\infty$$

$$\int_0^\infty \dot{V}\, dt = V(\infty) - V(0) < \infty \Rightarrow e \in L_2$$

$$x_p = \underset{\in L_\infty}{x_m} - \underset{\in L_\infty}{e} \in L_\infty \Rightarrow \dot{e} \in L_\infty \Rightarrow e \to 0 \text{ as } t \to \infty$$

Hence, $e_1 = h^T e \longrightarrow 0$ as $t \to \infty$. #

4.

1. **(a)** If in error model 1, with the error equation $\phi^T u = e_1$, the adaptive law is modified as

$$\dot{\phi} = -\Gamma e_1 u \qquad \Gamma = \Gamma^T > 0$$

show that (i) if $u \in \mathcal{L}^\infty$, then $\phi \in \mathcal{L}^\infty$, and (ii) if, in addition, $\dot{u} \in \mathcal{L}^\infty$, then $\lim_{t\to\infty} e_1(t) = 0$.

(b) If in (a), $\dot{u}$ is not bounded, give an example to show that $e_1(t)$ does not tend to zero as $t \to \infty$.

Answer: (a) Since $\Gamma = \Gamma^T > 0$ implies $\forall \quad x \neq 0$

$$x^T\Gamma^{-1}x = (x^T\Gamma^{-T})\Gamma(\Gamma^{-1}x) > 0.$$

This implies that $\Gamma^{-1} > 0$. ($\Gamma^{-1} = \Gamma^{-T}$ also).

Choose Lyapunov function

$$V = \phi^T\Gamma^{-1}\phi > 0 \quad \text{and ridially unbounded.}$$

$$\dot{V} = \dot{\phi}^T\Gamma^{-1}\phi + \phi^T\Gamma^{-1}\dot{\phi}$$

$$= -u^T e_1 \Gamma^T\Gamma\phi - \phi^T\Gamma^{-1}\Gamma e_1 u$$

$$= -2e_1^2 \leq 0$$

Hence, we have $\underline{\phi \in L_\infty}$. and if $u \in L_\infty$

$\Rightarrow$ $e_1 \in L_\infty$. and $\dot{\phi} = -\Gamma e_1 u \in L_\infty$

$$\int_0^\infty \dot{V}\,dt = -2\int_0^\infty e_1^2(t)\,dt = V(\infty) - V(0) < \infty$$

$\Rightarrow$ $e_1 \in L_2$

if $\dot{u} \in L_\infty$, $\dot{e}_1 = \dot{\phi}^T u + \phi^T \dot{u} \in L_\infty$

$e_1 \in L_2 \cap L_\infty$, $\dot{e}_1 \in L_\infty$ implies $e_1 \to 0$ as $t \to \infty$

(According to the well-known Barbalat's Lemma).

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4. (b) A Counter Example

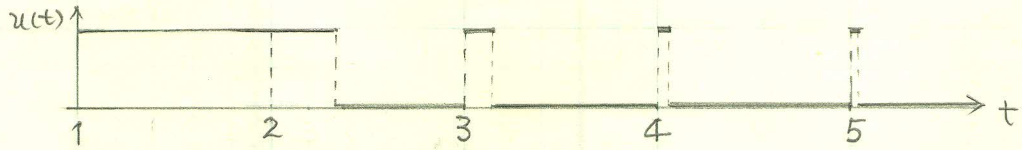
Let, u and ϕ be scalars and let $\Gamma = 1$, we have

$$\dot{\phi} = -u^2 \phi$$

Choose $u(t)$ as follow:

$$u(t) = \begin{cases} 1 & n \le t < n + \frac{1}{n^2} \\ 0 & n + \frac{1}{n^2} \le t < n+1 \end{cases}$$

for all positive integers n. Then $u \in L_\infty$, but $\dot{u} \notin L_\infty$



Assume the initial condition $\phi(2) = 1$, $t_0 = 2$, Thus

① for $2 \le t < 2.25$, we have

$$\dot{\phi} = -\phi$$

$$\phi(t) = e^{-(t-2)} \cdot 1 = e^{2} \cdot e^{-t}, \Rightarrow e_1(t) = e^{2} \cdot e^{-t}$$

② for $2.25 \le t < 3$, we have $\phi(2.25) = e^{-0.25}$

$$\dot{\phi} = 0 \Rightarrow \phi(t) = e^{-0.25} \Rightarrow e_1(t) = 0$$

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③ for $3 \le t < 3 + \frac{1}{3^2}$, we have $\phi(3) = e^{-\frac{1}{2^2}}$

$$\phi(t) = e^{-(t-3)} \cdot e^{-\frac{1}{2^2}} \Rightarrow e_1(t) = e^{-\frac{1}{2^2}} \cdot e^{-(t-3)}$$

$$\Rightarrow \phi(3 + \tfrac{1}{3^2}) = e_1(t + \tfrac{1}{3^2}) = e^{-(\frac{1}{2^2} + \frac{1}{3^2})}$$

very good

Continuing on the same arguments, we have

$$e_1(n + \tfrac{1}{n^2}) = e \cdot e^{-\sum_{i=1}^{n} \frac{1}{n^2}}$$

And we know that $1.5 < \sum_{i=1}^{\infty} \frac{1}{n^2} \triangleq m < 2$. Hence, we have

$$\lim_{n \to \infty} e_1(n + \tfrac{1}{n^2}) = e^{1-m} > 0 \Rightarrow \underline{e_1(t) \not\to 0 \text{ as } t \to \infty}. \quad \#$$

5.

2. In error model 3, the errors e_1 and ϕ are related by the differential equation $\dot{e}_1 = -\alpha e_1 + \phi^T u$ where $\alpha > 0$. If the adaptive law is modified as

$$\dot{\phi} = -\Gamma e_1 u \qquad \Gamma = \Gamma^T > 0$$

show that (i) if $u \in \mathcal{L}^\infty$, then $\phi \in \mathcal{L}^\infty$, and $\lim_{t\to\infty} e_1(t) = 0$.

<u>Answer:</u> As it is shown in Problem 4. $\Gamma^{-1} = \Gamma^{-T} > 0$

Choose $V = e_1^2 + \phi^T \Gamma^{-1} \phi > 0$, radially unbounded.

$$\dot{V} = 2e_1\dot{e}_1 + 2\dot{\phi}^T\Gamma^{-1}\phi$$

$$= 2e_1[-\alpha e_1 + \phi^T u] - 2u^T e_1 \phi$$

$$= -2\alpha e_1^2 \leq 0$$

mistake

Hence, $[e_1 \ \ \phi]^T = 0$ is a.s. in the large $\Rightarrow \phi \in L_\infty$

stable not a.s. !!

and $e_1 \in L_\infty$. Consider that

$$\int_0^\infty \dot{V}\,dt = -2\alpha\int_0^\infty e_1^2\,dt = V(\infty) - V(0) < \infty$$

$$\Rightarrow \quad e_1 \in L_2$$

If $u \in L_\infty$, we have.

$$\dot{e}_1 = -\alpha e_1 + \phi^T u \in L_\infty.$$

$$e_1 \in L_2 \cap L_\infty, \ \dot{e}_1 \in L_\infty \Rightarrow e_1 \to 0 \text{ as } t \to \infty.$$

(Once again, we use Barbalat's Lemma).

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6.

5. In Problem 1, if u is the output of an LTI system, which is unstable, how would you modify the adaptive law? Determine the nature of e_1 in this case.

Answer: I would like to choose $\Gamma = I$ and modify the adaptive law as:

$$\dot{\phi} = -\frac{e_1 u}{1+u^T u}$$

which is working for any u. Since, we don't trust any result in the text and book. Following we will re-do the analysis.

The output of an unstable LTI system could be bounded, e.g. the input to the system and initial values are zero. But, this is not the case we are interested. Without loss of generality, we assume that u is unbounded. And it is well known that

$$u(t) = Ce^{A(t-t_0)}x_0 + C\int_{t_0}^{t} e^{A(t-\tau)} B\, r(\tau)\, d\tau$$

where, (C, A, B) are the unstable LTI system. Hence,

$u(t) \in P.C.[0,\infty]$

Choose $\quad V = \phi^T\phi > 0$

$$\dot{V} = 2\phi^T\dot{\phi} = -2\cdot\frac{e_1\phi^T u}{1+u^T u} = \frac{-2e_1^2}{1+u^T u} \leq 0$$

Hence, $\quad \phi \in L_\infty$

$$\int_0^\infty \dot{V}\, dt = -2\int_0^\infty \left(\frac{e_1}{\sqrt{1+u^T u}}\right)^2 dt = V(\infty) - V(0) < \infty$$

$$\Rightarrow \quad \beta(t) \triangleq \frac{e_1}{\sqrt{1+u^T u}} \in L_2 \quad \text{or} \quad e_1 = \underbrace{\beta(t)}_{\in L_2}\sqrt{1+u^T u}$$

6 (cont.) And then

$$\dot{\phi} = -\underbrace{\frac{e_1}{\sqrt{1+u^Tu}}}_{\in L_2} \cdot \underbrace{\frac{u}{\sqrt{1+u^Tu}}}_{\in L_\infty} \Rightarrow \dot{\phi} \in L_2$$

$$\dot{\phi} = -\underbrace{\frac{u}{\sqrt{1+u^Tu}}}_{\in L_\infty} \cdot \underbrace{\frac{u^T}{\sqrt{1+u^Tu}}}_{\in L_\infty} \cdot \underbrace{\phi}_{\in L_\infty} \Rightarrow \dot{\phi} \in L_\infty$$

$\Rightarrow \dot{\phi} \to 0$

$$\frac{e_1}{\sqrt{1+u^Tu}} = \underbrace{\frac{u^T}{\sqrt{1+u^Tu}}}_{\in L_\infty} \cdot \underbrace{\phi}_{\in L_\infty} \Rightarrow \beta(t) \in L_\infty$$

Now, if $\|\dot{u}\| = O[\|u\|]$, we have

$$\frac{d}{dt}\beta(t) = \frac{d}{dt}\left[\frac{e_1}{\sqrt{1+u^Tu}}\right]$$

$$= \frac{\dot{e}_1\sqrt{1+u^Tu} - e_1 \times 0.5\,(1+u^Tu)^{-0.5} \cdot 2u^T\dot{u}}{(1+u^Tu)}$$

$$= \frac{(1+u^Tu)(\phi^T\dot{u} + \dot{\phi}^Tu) - eu^T\dot{u}}{(1+u^Tu)^{3/2}}$$

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$$= \underbrace{\phi^T}_{\in L_\infty} \cdot \underbrace{\frac{\dot{u}}{\sqrt{1+u^Tu}}}_{\in L_\infty} + \underbrace{\dot{\phi}^T}_{\in L_\infty}\underbrace{\frac{u}{\sqrt{1+u^Tu}}}_{\in L_\infty} - \underbrace{\frac{e}{\sqrt{1+u^Tu}}}_{\in L_\infty} \cdot \underbrace{\frac{u^T}{\sqrt{1+u^Tu}}}_{\in L_\infty} \cdot \underbrace{\frac{\dot{u}}{\sqrt{1+u^Tu}}}_{\in L_\infty}$$

$\Rightarrow \dot{\beta}(t) \in L_\infty$ and $\beta(t) \in L_2 \cap L_\infty \Rightarrow \beta(t) \to 0$ as $t \to \infty$

i.e. $\dfrac{e_1}{\sqrt{1+u^Tu}} \to 0$

Recall

$$e_1 = \beta(t) \cdot \sqrt{1+u^Tu}$$

By definition, we have

$$e_1(t) = o[u(t)]$$

Q.E.D.

Problem 7: Let $g(t) = \begin{bmatrix} f_1(t) \\ f_1(t) + f_2(t) \end{bmatrix}$, where $f_1(t)$ is P.E. and $f_2(t) \in L_2$

Determine whether $g(t)$ is P.E. or not. (Prove it).

Dis-proof: Consider the example given on page 248 – 249.

Let $f_1(t) = \sin t$, $R_{f_1}(0) = \frac{1}{2}\cos(0) = 0.5 > 0$

$\Rightarrow f_1(t)$ is P.E.

Let $f_2(t) = e^{-t}$, $f_2 \in L_2$ (Actually $f_2 \in L_p$, $p \in [1, \infty]$

Then

$$g(t) = \begin{bmatrix} \sin t \\ \sin t + e^{-t} \end{bmatrix}$$

Consider the projection of $u(t)$ along the direction $W = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^T$. Then, we have

$$g^T(t) \cdot W = \frac{\sqrt{2}}{2} e^{-t}$$

Hence

$$\frac{1}{T_0} \int_t^{t+T_0} \left|\frac{\sqrt{2}}{2} e^{-\tau}\right| d\tau$$

$$= \frac{\sqrt{2}}{2T_0} \left(-e^{-\tau}\right)\Big|_t^{t+T_0} = \frac{\sqrt{2}}{2T_0}[1 - e^{-T_0}] \cdot e^{-t} \longrightarrow 0 \quad \text{as } t \to \infty$$

for $T_0 > 0$.

Hence, $g(t)$ can not satisfy condition (6.9).

$g(t)$ is not P.E. Q.E.D.

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You showed that there are f_1 & f_2 such that g is not PE. However you are suppose to show that g is not PE for any f_1 & f_2.