Very Good

MIDTERM EXAM # 1

EE507 - Spring 1989

Time allowed : 1 Hr.

Open Book and Open Notes

91

$\frac{89+2}{100}$

#1 19/20

#2 15+2/20

#3 20/20

#4 15/20

#5 20/20

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Problem 1 : (*20 pts.*) The joint *pdf* of random variables X and Y is

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}, & \text{if } \substack{0 \le x \le y \\ 0 \le y \le 2}; \\ 0, & \text{otherwise.} \end{cases}$$

a. Find the marginal *pdf*'s, $f_X(x)$ and $f_Y(y)$.

b. Find the conditional *pdf*'s $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

c. Find $E\{X|\,Y = 1\}$ and $E\{X|\,Y = 0.5\}$.

d. Are X and Y statistically independent?

Problem 2: (*20 pts.*) The RV's X and Y are independent with exponential densities

$$f_X(x) = \alpha \exp(-\alpha x)U(x),$$

$$f_{\cancel{X}}(\cancel{x}) = \beta \exp(-\beta y)U(y).$$

Y y

Find the *pdf* of $Z = \min(X,Y)$.

Problem 3: (*20 pts.*) X_1 and X_2 are two independent RV's each with the following density function

$$f_{X_i}(x_i) = \begin{cases} \exp(-x_i), & x_i > 0\,; \\ 0, & x_i \le 0\,. \end{cases}, \qquad i = 1, 2.$$

Let

$$Y_1 = X_1 + X_2$$

$$Y_2 = \frac{X_1}{X_1 + X_2}.$$

a. Find $f_{Y_1Y_2}(y_1,y_2)$.

b. Find $f_{Y_1}(y_1)$, $f_{Y_2}(y_2)$ and show that Y_1 and Y_2 are independent.

Problem 4: (*20 pts.*) The time-to failure in months, X, of light bulbs produced at two manufacturing plants A and B obey, respectively , the following PDF's

$$F_{X_A}(x) = (1 - e^{-x/5})u(x) \text{ for plant } A$$

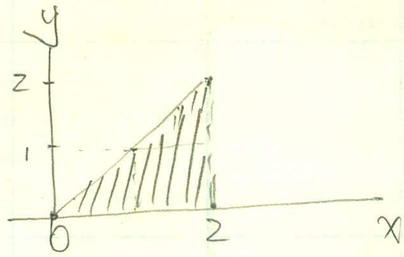
$$F_{X_B}(x) = (1 - e^{-x/2})u(x) \text{ for plant } B.$$

Plant B produces three times as many bulbs as plant A.The bulbs, indistinguishable to the eye, are intermingled and sold. What is the probability that a bulb purchased at random will burn at least (a) two months; (b) five months ?

Problem 5: (*20 pts.*) X is a *Gaussian* random variable with mean zero and variance σ_x^2. Find the *pdf* of Y if

$$Y = \frac{1}{2}[X + |X|].$$

1.



a. $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_x^2 \frac{1}{2}\,dy = \frac{1}{2}(2-x)$ for $0 \le x \le 2$

$$\Rightarrow \quad f_X(x) = \begin{cases} \frac{1}{2}(2-x) & 0 \le x \le 2 \\ 0 & \text{elesewhere} \end{cases}$$

5/5

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^y \frac{1}{2}\,dx$$

$$= \frac{1}{2}y \qquad \text{for} \qquad 0 \le y \le 2$$

$$\Rightarrow \quad \boxed{f_Y(y) = \begin{cases} \frac{1}{2}(2-y) & 0 \le y \le 2 \\ 0 & \text{elesewhere} \end{cases}}$$

?..

b. $$f_{Y|X}(x|y) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$= \begin{cases} \frac{1}{2-x} & x \le y \le 2 \quad 0 \le x < 2 \\ 0 & \text{eleswhere} \end{cases}$$

5/5

$$f_{X|Y}(y|x) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \frac{1}{y} & 0 \le x \le y,\ 0 \le y < 2 \\ 0 & \text{eleswhere} \end{cases}$$

1 (c)

$$E[X \mid Y=1] = \int_0^1 x \cdot \frac{1}{1} \cdot dx$$

$$= \frac{1}{2}x^2 \Big|_0^1 = 0.5$$

$$E[X \mid Y=0.5] = \int_0^{0.5} x \cdot \frac{1}{0.5}\, dx$$

3/5

$$= \frac{1}{2} \cdot \frac{1}{2} x^2 \cancel{\Big|_0^{0.5} = \frac{1}{16}}$$

1 (d) X and Y are not independent.

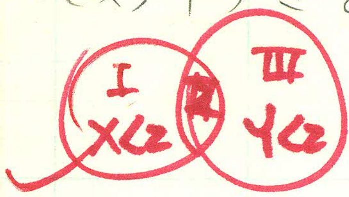
because $f_X(x) \cdot f_Y(y) \neq f_{XY}(x,y)$

5/5

19/20

2. $P[Z \le z] = P[\min(X, Y) \le z]$

$\overset{\mathrm{II}}{=} P[X \le z, Y \le z]$



$+ \underset{\mathrm{I}}{P[X \le z, Y > z]} + \underset{\mathrm{III}}{P[X > z, Y \le z]}$

$z \ge 0$ $\quad = \int_0^z \int_0^z \alpha\beta e^{-(\alpha x + \beta y)}\, dx\, dy$

$+ \int_0^z \int_z^\infty \alpha\beta e^{-(\alpha x + \beta y)}\, dx\, dy$

$+ \int_z^\infty \int_0^z \alpha\beta e^{-(\alpha x + \beta y)}\, dx\, dy$

error

$= \int_0^z \alpha e^{-\alpha x}\, dx + \int_0^z \alpha e^{-\alpha x} \left[\int_z^\infty \beta e^{-\beta y}\, dy\right] dx$

$= -e^{-\alpha x}\Big|_0^z + \int_0^z \alpha e^{-\alpha x} \cdot \left(-e^{-\beta y}\Big|_z^\infty\right) dx$

$= 1 - e^{-\alpha z} + \int_0^z \alpha e^{-\alpha x} e^{-\beta z}\, dx$

$= 1 - e^{-\alpha z} - e^{-\beta z}(1 - e^{-\alpha z})$

$= (1 - e^{-\alpha z})(1 + e^{-\beta z})$

2.2 (CONT.)

$$\Rightarrow \quad F_Z(z) = (1 - e^{-\alpha z})(1 - e^{-\beta z})$$

$$\Rightarrow \quad \cancel{f_Z}(z) = \alpha e^{-\alpha z}(1 - e^{-\beta z}) + \beta e^{-\beta z}(1 - e^{-\alpha z})$$

$$\Rightarrow \quad f_Z(z) = \begin{cases} \alpha e^{\alpha z}(1 - e^{-\beta z}) + \beta e^{\beta z}(1 - e^{-\alpha z}) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

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3.

$$Y_1 = X_1 + X_2 \qquad \Rightarrow \begin{cases} Y_1 = X_1 + X_2 \\ Y_1 Y_2 = X_1 \end{cases}$$
$$Y_2 = \frac{X_1}{X_1 + X_2}$$

$$\Rightarrow \begin{cases} X_1 = Y_1 Y_2 \\ X_2 = Y_1 - Y_1 Y_2 \end{cases}$$

$$\tilde{J} = \begin{bmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{bmatrix} \qquad |\tilde{J}| = |-y_2 y_1 \cancel{+} - y_1 + y_1 y_2| = y_1$$

a) $f_{Y_1Y_2}(y_1, y_2) = f_{X_1X_2}(y_1 y_2 \; , \; y_1 - y_1 y_2) \cdot y_1$

$$= e^{-y_1 y_2} \cdot e^{-y_1 + y_1 y_2} \cdot y_1$$

$$= e^{-y_1} \cdot y_1 = y_1 e^{-y_1} \qquad \text{for } \begin{matrix} 0 < y_1 < \infty \\ 0 < y_2 < \infty \end{matrix}$$

0 eleswhere

b) $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1Y_2}(y_1, y_2)\, dy_2 = y_1 e^{-y_1} \qquad 0 < y_1 < \infty$

$$f_{Y_2}(y_2) = \int_{0^+}^{\infty} y_1 e^{-y_1}\, dy_1 = \int_0^{\infty} -y_1\, d e^{-y_1}$$

$$= -y_1 e^{-y_1} \Big|_{0^+}^{\infty} + \int_0^{\infty} e^{-y_1}\, dy_1$$

20/20

$$= 0 + 1 = 1 \qquad \text{for } 0 < y_2 < \infty$$

Thus

$$f_{Y_1Y_2}(y_1, y_2) = \begin{cases} y_1 e^{-y_1} = f_{Y_1}(y_1) f_{Y_2}(y_2) & \begin{matrix} 0 < y_1 < \infty \\ 0 < y_2 < \infty \end{matrix} \\ 0 = f_{Y_1}(y_1) f_{Y_2}(y_2) & \text{elsewhere} \end{cases}$$

$\Rightarrow$ Y_1 and Y_2 are independent.

4-

a) the Probability that a bulb from plant A will burn at least two months is

$$P_{Aa} = F_X(2) = 1 - e^{-2/5}$$

the probability that a bulb from plant B will burn at least two months is

$$P_{Ba} = F_X(2) = 1 - e^{-1}$$

$\Rightarrow$ A bulb will burn at least two month is

$$P_a = \frac{3}{4} P_{Aa} + \frac{1}{4} P_{Ba}$$

$$= \frac{3}{4} \times (1 - e^{-\frac{2}{5}}) + \frac{1}{4}(1 - e^{-1}) = 0.879$$

$$=$$

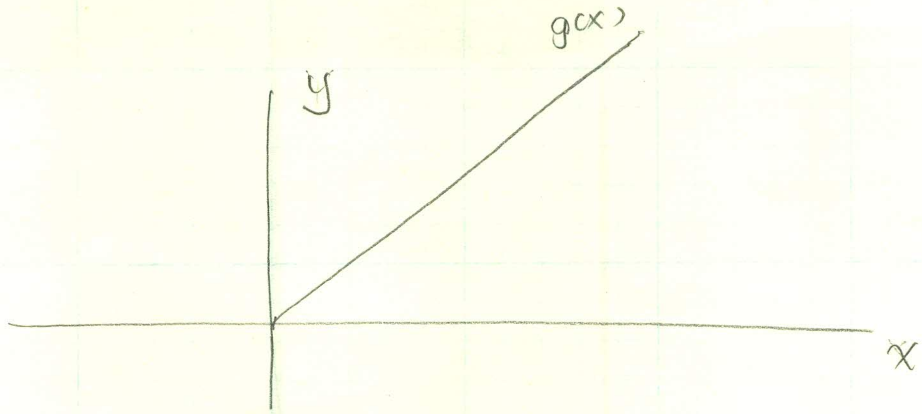
b) By the same method, we have a bulb will burn at least five month is

$$P_b = \frac{3}{4}(1 - e^{-1}) + \frac{1}{4}(1 - e^{\frac{5}{2}})$$

$$= 0.703$$

15/20

5.



$$Y = \frac{1}{2}[X + |X|] = \begin{cases} X & X \geq 0 \\ 0 & X < 0 \end{cases}$$

$$\Rightarrow \quad Y = X\, u(X)$$

$$\Rightarrow \quad F_Y(y) = 0 \quad \text{if} \quad y < 0$$

$$F_Y(y) = F_X(y) \quad \text{if} \quad y \geq 0$$

$$\Rightarrow \quad F_Y(y) = F_X(y)\, u(y)$$

$$\Rightarrow \quad f_Y(y) = f_X(y)\, u(y) + F_X(y)\, \delta(y)$$

$$= f_X(y)\, u(y) + F_X(0)\, \delta(y)$$

$$= \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{y^2}{2\sigma_X^2}} u(y) + 0.5\, \delta(y)$$

20/20

MIDTERM EXAM # 2

EE507 - Spring 1989

Time allowed : 1 Hr.

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#1 20/20

#2 15/25

#3 15/15

#4 25/25

#5 14/15

NAME : Ben M. Chen

89/100

April 6, 1989

Problem 1 : (*20 pts.*)

Let $\{X_n\}$ be a sequence of RV's defined by

$$X_n = \begin{cases} e^n & \text{with probability } 1/n^2; \\ 0 & \text{with probability } 1 - (1/n^2). \end{cases}$$

Let $X \triangleq 0$.

a. Does $X_n \to X$ in probability?

b. Does $X_n \to X$ in mean-square sense?

Problem 2: (*25 pts.*)

$X(t)$ is a complex-valued random process defined as

$$X(t) = A\exp(2\pi jYt + j\Theta)$$

where A, Y, and Θ are independent random variables. Moreover Θ is uniformly distributed on $(-\pi, \pi)$. Find μ_x ,and $R_{xx}(t, t+\tau)$. Is $X(t)$ WSS random process?

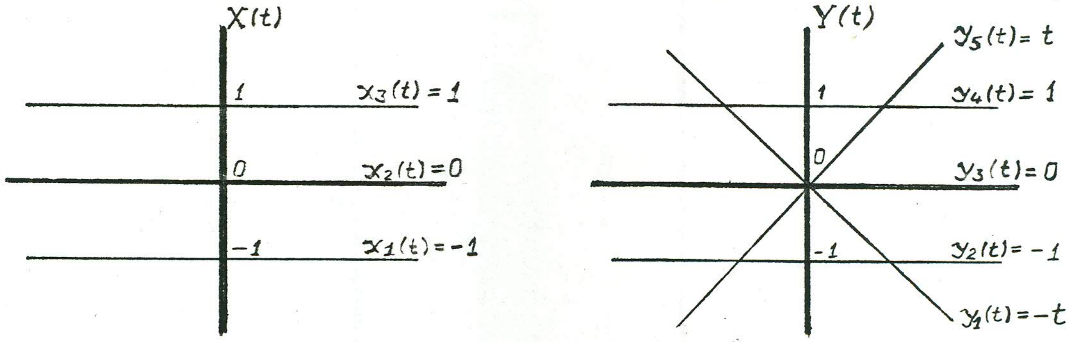
Problem 3: (*15 pts.*)

Using the Markov property, show that if $X(n)$, $n = 1, 2, 3, \ldots$, is Markov, then

$$E\{X(n+1)|X(1), X(2), \ldots, X(n)\} = E\{X(n+1)|X(n)\}.$$

Problem 4: *(25 pts.)*

The member functions of two random process $X(t)$ and $Y(t)$ are shown in below figure.



Assume that the member functions have equal probabilities of occurence.

a. Find μ_x, and $R_{xx}(t, t+\tau)$. Is $X(t)$ *WSS*?

b. Find μ_y, and $R_{yy}(t, t+\tau)$. Is $Y(t)$ *WSS*?

c. Find $R_{xy}(0, 1)$ assuming that the underlying random experiments are independent.

Problem 5: *(15 pts.)*

a) *(10pts.)* $X(t)$ is a *WSS* process and let $Y(t) = X(t+a) - X(t-a)$. Show that

$$R_{yy}(\tau) = 2R_{xx}(\tau) - R_{xx}(\tau + 2a) - R_{xx}(\tau - 2a).$$

b) *(5pts.)* Determine whether the function

$$2 \sin 2\pi(1000)\tau$$

can be the autocorrelation function of real valued *WSS* random process.

Problem 1 :

$\forall\ \varepsilon > 0$

(a) $P[\,|X_n - 0| > \varepsilon\,] = P[\,|X_n| > \varepsilon\,]$

$\leq \frac{1}{n^2} \to 0$. as $n \to \infty$.

$\therefore$ $X_n \to X$ in probability .

(b) $E[\,|X_n - X|^2\,] = E[\,X_n^2\,]$

$= \frac{1}{n^2} e^{2n} \to \infty$ as $n \to \infty$.

$\therefore$ X_n does not converge to X in m.s.

20/20

Problem 2:

$$X(t) = A\exp(2\pi j Y t + j\theta) = A\, e^{j2\pi Yt} \cdot e^{j\theta}$$

$$\mu_X = E[X(t)] = \mu_A \cdot \int_{-\infty}^{\infty} e^{j2\pi Yt} f_Y(y)\,dy \int_{-\pi}^{\pi} \frac{1}{2\pi} e^{j\theta}\,d\theta$$

$$= \mu_A \cdot \frac{1}{j2\pi} \cdot e^{j\theta}\Big|_{-\pi}^{\pi} \cdot \int_{-\infty}^{\infty} e^{j2\pi Yt} f_Y(y)\,dy.$$

$$= \mu_A \cdot \frac{1}{j2\pi}\left(e^{j\pi} - e^{-j\pi}\right) \int_{-\infty}^{\infty} e^{j2\pi Yt} f_Y(y)\,dy$$

$$= \frac{1}{\pi}\mu_A \sin\pi \cdot \int_{-\infty}^{\infty} e^{j2\pi Yt} f_Y(y)\,dy = 0.$$ 10

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= E[A^2 e^{j2\pi Yt} \cdot e^{j2\pi Y(t+\tau)} e^{j2\theta}]$$

$$= E[A^2] \cdot E[e^{j2\pi Y(2t+\tau)}] \cdot E[\cancel{e^{j2\theta}}^{\,0}]$$

$$= 0.$$

$\therefore$ X(t) is W.S.S.

15/25

Problem 3: $X(n)$ is Markov then

$$f_X(x_{n+1} | x_n, \cdots x_1)$$

$$= f_X(x_{n+1} | x_n)$$

$$E[X(n+1) | X(1), X(2), \cdots, X(n)]$$

$$= \int_{-\infty}^{\infty} x_{n+1} f_X(x_{n+1} | x_n, \cdots, x_1)\, dx_{n+1}$$

$$= \int_{-\infty}^{\infty} x_{n+1} f_X(x_{n+1} | x_n)\, dx_{n+1}$$

$$= E[X(n+1) | X(n)]$$ Q.E.D

15/15

Problem 4:

a) $\mu_X = E[X(t)] = 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} - 1 \cdot \frac{1}{3} = 0$.

$R_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)]$

$= 1^2 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + (-1)^2 \cdot \frac{1}{3} = \frac{2}{3}$

$X(t)$ is W.S.S.

b) $\mu_Y = E[Y(t)] = -t \cdot \frac{1}{5} - 1 \cdot \frac{1}{5} + 0 \cdot \frac{1}{5} + 1 \cdot \frac{1}{5} + t \cdot \frac{1}{5} = 0$.

$R_{YY}(t, t+\tau) = E[Y(t)\, Y(t+\tau)]$

$= \frac{1}{5}(-t)(-t-\tau) + \frac{1}{5}(-1)^2 + \frac{1}{5}0^2 + \frac{1}{5} \cdot 1^2$

$+ \frac{1}{5}(t)(t+\tau)$

25/25

$= \frac{2}{5} t(t+\tau) + \frac{2}{5}$

$Y(t)$ is not W.S.S.

c) $R_{XY}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$

$= E[X(t_1)]\, E[Y(t_2)] = 0$.

$R_{XY}(0, 1) = 0$

Problem 5:

a) $Y(t) = X(t+a) - X(t-a)$

$$R_{YY}(t, t+\tau) = E[Y(t)\,Y(t+\tau)]$$

$$= E\big[[X(t+a) - X(t-a)][X(t+\tau+a) - X(t+\tau-a)]\big]$$

$$= E[X(t+a)X(t+\tau+a)] - E[X(t-a)X(t+\tau+a)]$$

$$- E[X(t+a)X(t+\tau-a)]$$

$$+ E[X(t-a)X(t+\tau-a)]$$

$$= R_{xx}(\tau) - R_{xx}(\tau+2a) - R_{xx}(\tau-2a) + R_{xx}(\tau)$$

$$= 2R_{xx}(\tau) - R_{xx}(\tau+2a) - R_{xx}(\tau-2a)$$ Q.E.D

b) $2\sin 2\pi(1000)\tau$ can not be autocorrelation function of real valued WSS. R.P.

~~Assuming~~ $R(\tau) = 2\sin 2\pi(1000)\tau$.

$$R(0) = 0.$$

condition $|R(\tau)| \le R(0)$ is destroied.

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