① I. Find the Laplace Transforms of the following functions:

(A) very Good!

a) $g(t) = t \cdot e^{-2t}$

$$G(s) = \int_0^\infty g(t) e^{-st} dt$$

$$= \int_0^\infty t \cdot e^{-2t} \cdot e^{-st} dt$$

$$= \int_0^\infty t \cdot e^{-(s+2)t} \cdot dt = -\frac{1}{s+2} \int_0^\infty t \cdot d e^{-(s+2)t}$$

$$= \underbrace{-\frac{1}{s+2} t \cdot e^{-(s+2)t} \Big|_0^\infty}_{=0} + \left[\int_0^\infty e^{-(s+2)t} dt\right] \cdot \frac{1}{s+2}$$

$$= -\frac{1}{(s+2)^2} e^{-(s+2)t} \Big|_0^\infty = \boxed{\frac{1}{(s+2)^2}} \Leftarrow \text{answer}$$

b) $g(t) = t \cos 5t$

$$\cos 5t = \frac{1}{2}(e^{j5t} + e^{-j5t})$$

$$G(s) = \int_0^\infty g(t) e^{-st} dt = \frac{1}{2} \int_0^\infty [t e^{-(s-j5)t} + t \cdot e^{-(s+j5)t}] dt$$

from a), we know

$$\int_0^\infty t e^{-(s-j5)t} dt = \frac{1}{(s-j5)^2}$$

$$\int_0^\infty t e^{-(s+j5)t} dt = \frac{1}{(s+j5)^2}$$

Thus

$$G(s) = \frac{1}{2}\left[\frac{1}{(s-j5)^2} + \frac{1}{(s+j5)^2}\right]$$

$$= \boxed{\frac{s^2 - 25}{(s^2+25)^2}} \Leftarrow \text{Answer}$$

c) $g(t) = e^{-t}\sin\omega t$

$$\sin\omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

Thus

$$g(t) = e^{-t}\cdot\sin\omega t = \frac{1}{2j}[e^{(j\omega-1)t} - e^{-(j\omega+1)t}]$$

$$G(s) = \int_0^\infty g(t)e^{-st}dt$$

$$= \frac{1}{2j}[\int_0^\infty e^{(j\omega-1-s)t}dt - \int_0^\infty e^{-(j\omega+1+s)t}dt]$$

$$= \frac{1}{2j}[\frac{1}{j\omega-1-s}e^{(j\omega-1-s)t}\Big|_0^\infty + \frac{e^{-(j\omega+1+s)t}}{j\omega+1+s}\Big|_0^\infty]$$

$$= \frac{1}{2j}\left[\frac{1}{s+1-j\omega} - \frac{1}{s+1+j\omega}\right]$$

$$= \boxed{\frac{\omega}{(s+1)^2+\omega^2}} \Leftarrow \text{Answer}$$

---

II. Find the inverse Laplace Transforms of following functions.

a) $G(s) = \frac{1}{(s+2)(s+3)}$

$$= \frac{1}{s+2} - \frac{1}{s+3}$$

$$g(t) = \mathcal{L}^{-1}[G(s)] = \mathcal{L}^{-1}[\frac{1}{s+2}] - \mathcal{L}^{-1}[\frac{1}{s+3}] = e^{-2t} - e^{-3t}, \quad t \geq 0$$

b) $G(s) = \frac{1}{(s+1)^2\cdot(s+4)}$

$$= \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$\Rightarrow A = +\frac{1}{3}, \quad C = \frac{1}{9}, \quad B = -\frac{1}{9}$$

$$\therefore G(s) = \frac{1}{9}\left[\frac{3}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+4}\right]$$

$$g(t) = \frac{1}{9}[3te^{-t} - e^{-t} + e^{-4t}]u(t)$$

c) $G(s) = \dfrac{10}{s(s^2+4)(s+1)}$

$$= \frac{A}{s} + \frac{Bs+C}{s^2+4} + \frac{D}{s+1}$$

$$\Rightarrow A = \frac{5}{2}, \quad D = -2$$

$$\therefore G(s) = \frac{10}{s(s^2+4)(s+1)} = \frac{2.5}{s} - \frac{2}{s+1} + \frac{B\cdot s + C}{s^2+4}$$

$$= \frac{0.5s + 2.5}{s^2+s} + \frac{B\cdot s + C}{s^2+4}$$

$$= \frac{(0.5+B)\cdot s^3 + (B+C+2.5)\cdot s^2 + (2+C)\cdot s + 10}{s(s^2+4)(s+1)}$$

$$\Rightarrow B = -0.5, \quad C = -2$$

$$\therefore G(s) = \frac{2.5}{s} - \frac{0.5s+2}{s^2+4} - \frac{2}{s+1}$$

$$g(t) = 2.5\,u(t) - 2e^{-t} - 0.5\cos 2t - \sin 2t, \quad t \geq 0$$

d) $G(s) = \dfrac{2(s+1)}{s(s^2+s+2)}$

$$= \frac{A}{s} + \frac{B\cdot s + C}{s^2+s+2} \triangleq \text{RHS (Right hand side)}$$

$$\Rightarrow A = 1$$

$$\text{RHS} = \frac{s^2+s+2+B\cdot s^2 + Cs}{s(s^2+s+2)} = \frac{2s+2}{s(s^2+s+2)}$$

$$\Rightarrow B = -1, \quad C = 1$$

$$\therefore G(s) = \frac{1}{s} - \frac{s-1}{s^2+s+2} = \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2 + (1.32288)^2} - \frac{1.5}{(s+0.5)^2+(1.32288)^2}$$

$$g(t) = u(t) - e^{-0.5t}\cos(1.32288t) - 1.13389\,e^{-0.5t}\sin(1.32288t)$$

$$, \quad t \geq 0$$

III. a) Find the poles and zeros of the functions given in II, including the ones at infinity.

II.a) poles: $s_1 = -2$ , $s_2 = -3$

zeros: $s_{1,2} = \infty$

b) poles: $s_{1,2} = -1$ , $s_3 = -4$

zeros: $s_{1,2,3} = \infty$

c) poles: $s_1 = 0$, $s_{2,3} = \pm j2$ , $s_4 = -1$

zeros: $s_{1,2,3,4} = \infty$

d) poles: $s_1 = 0$ , $s_{2,3} = -0.5 \pm j\,1.32288$

zeros: $s_1 = -1$, $s_{2,3} = \infty$

✓

b) Find the initial and final values of these function (given II), using appropriate Laplace-Transform Theorems.

Initial Value : $f(0) = \lim_{s\to\infty} sF(s)$

Final Value : $f(\infty) = \lim_{s\to 0} sF(s)$ ; $\mathrm{Re}[\text{pole of } sF(s)] < 0$

II. a) $sG(s) = \dfrac{s}{(s+2)(s+3)}$

$g(0) = 0$ and $g(\infty) = 0$

b) $sG(s) = \dfrac{s}{(s+1)^2(s+4)}$

$g(0) = 0$ and $g(\infty) = 0$

c) $sG(s) = \dfrac{10}{(s^2+4)(s+1)}$

$g(0) = 0$ and no final value because $sG(s)$ has poles on $j\omega$.

d) $sG(s) = \dfrac{2(s+1)}{s^2+s+2}$

$g(0) = 0$ and $g(\infty) = 1$

✓

IV. Solve the differential equation by means of Laplace Transform.

Assume Initial Conditions to be zero.

$$\frac{d^2 f(t)}{dt^2} + 5 \cdot \frac{df(t)}{dt} + 4 f(t) = u_s(t)$$

Solution: Taking Laplace transforms to both sides, we have

$$S^2 F(S) + 5F(S) + 4F(S) = U(S) = \frac{1}{S}$$

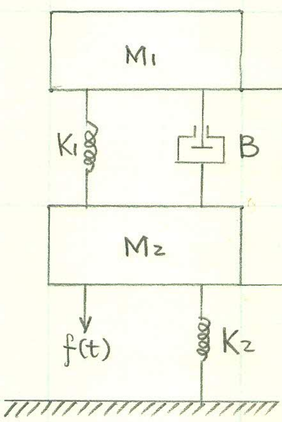
$$(S^2 + 5 + 4) F(S) = \frac{1}{S}$$

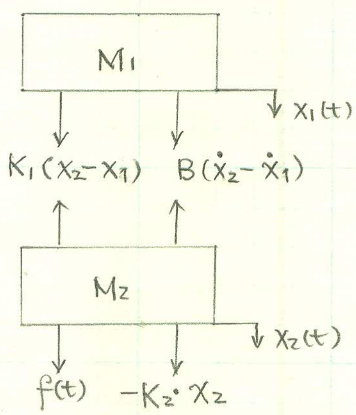
$$F(S) = \frac{1}{S(S+1)(S+4)}$$

$$= \frac{0.25}{S} - \frac{1/3}{S+1} + \frac{1/12}{S+4}$$

$$\therefore \quad f(t) = [\tfrac{1}{4} - \tfrac{1}{3} e^{-t} + \tfrac{1}{12} e^{-4t}] u(t)$$ Answer.

② For a mechanical system given below find $X_1(S)/F(S)$ and $X_2(S)/F(S)$




Equations:

$$\begin{cases} M_1 \ddot{x}_1 + K_1(x_2 - x_1) + B(\dot{x}_2 - \dot{x}_1) = 0 \\ M_2 \ddot{x}_2 + f(t) - K_2 x_2 = -K_1(x_2 - x_1) - B(\dot{x}_2 - \dot{x}_1) \end{cases}$$

$$\Rightarrow \begin{cases} M_1 \cdot S^2 X_1(S) + K_1 X_2(S) - K_1 X_1(S) + B \cdot S X_2(S) - B \cdot S X_1(S) = 0 \\ M_2 \cdot S^2 X_2(S) + F(S) - K_2 X_2(S) = -K_1 X_2(S) + K_1 X_1(S) - B \cdot S X_2(S) + B \cdot S X_1(S) \end{cases}$$

$$\Rightarrow \begin{cases} (M_1 \cdot S^2 - B \cdot S - K_1) X_1(S) = -(B \cdot S + K_1) X_2(S) \\ (M_2 \cdot S^2 + B \cdot S + K_1 - K_2) X_2(S) = (B \cdot S + K_1) X_1(S) - F(S) \end{cases}$$

Define :
$$A(s) = M_1 \cdot s^2 - B \cdot s - K_1$$
$$B(s) = M_2 \cdot s^2 + B \cdot s + K_1 - K_2$$
$$C(s) = +(B \cdot s + K_1)$$

Then, we have

$$\begin{cases} A(s)X_1(s) = -C(s)X_2(s) \\ B(s) \cdot X_2(s) = C(s)X_1(s) - F(s) \end{cases}$$

$$\Rightarrow \quad X_2(s) = -\frac{A(s)}{C(s)} X_1(s)$$

$$\therefore \quad -B(s) \cdot \frac{A(s)}{C(s)} \cdot X_1(s) = C(s) \cdot X_1(s) - F(s)$$

$$\Rightarrow \quad F(s) = \frac{C^2(s) + A(s)B(s)}{C(s)} \cdot X_1(s)$$

$$\therefore \quad X_1(s)/F(s) = \frac{C(s)}{A(s)B(s) + C^2(s)}$$

$$= \frac{B \cdot s + K_1}{(M_1 \cdot s^2 - B \cdot s - K_1)(M_2 \cdot s^2 + B \cdot s + K_1 - K_2) + (B \cdot s + K_1)^2}$$

$$\frac{X_2(s)}{F(s)} = \frac{-A(s)}{C(s)} \cdot \frac{X_1(s)}{F(s)} = -\frac{A(s)}{A(s)B(s) + C^2(s)}$$

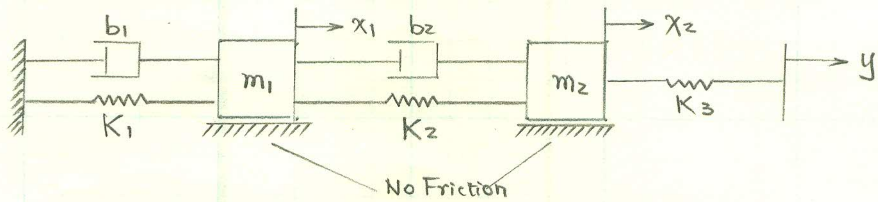
$$= -\frac{M_1 \cdot s^2 - B \cdot s - K_1}{(M_1 \cdot s^2 - B \cdot s - K_1)(M_2 \cdot s^2 + B \cdot s + K_1 - K_2) + (B \cdot s + K_1)^2}$$

✓

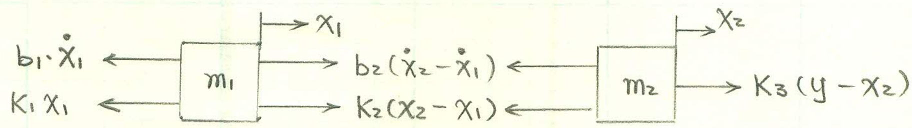
③ From Text: "Feedback Control of Dynamic Systems."

PROBLEM # 2.1  a) Write the differential equations for the mechanical system shown in Fig. 2.49.

b) Obtain the transfer functions from $y$ to $x_1$ and $x_2$.




Solution:




(Remark: $K_i$'s like resistors in electrical circuits, $b_1, b_2$ like inductors $x_1, x_2, y$ are like currents, And then treat forces as voltages)

a)
$$\begin{cases} m_1 \cdot \ddot{x}_1 = -b_1 \cdot \dot{x}_1 - b_2(\dot{x}_2 - \dot{x}_1) + K_1 x_1 - K_2(x_2 - x_1) \\ m_2 \cdot \ddot{x}_2 = -b_2(\dot{x}_2 - \dot{x}_1) \mp K_2(x_2 - x_1) - K_3(y - x_2) \end{cases}$$

b)
$$\begin{cases} m_1 \cdot s^2 X_1(s) = (b_1 - b_2)\cdot s \cdot X_1(s) - b_2 \cdot s X_2(s) + K_1 X_1(s) - K_2 X_2(s) + K_2 X_1(s) \\ m_2 \cdot s^2 \cdot X_2(s) = b_2 \cdot s \cdot X_2(s) - b_1 s X_1(s) + K_2 X_2(s) - K_2 X_1(s) - K_3 Y(s) + K_3 X_2(s) \end{cases}$$

$$\Rightarrow \begin{cases} [m_1 \cdot s^2 + (b_2 - b_1)\cdot s - (K_1 + K_2)] X_1(s) = -(b_2 \cdot s + K_2) X_2(s) \\ [m_2 \cdot s^2 - b_2 \cdot s - (K_2 + K_3)] X_2(s) - (b_1 \cdot s + K_2) X_1(s) = -K_3 Y(s) \end{cases}$$

Let $A = m_1 \cdot s^2 + (b_2 - b_1)\cdot s - (K_1 + K_2)$, $B = m_2 \cdot s^2 - b_2 \cdot s - (K_2 + K_3)$

$C = b_1 \cdot s + K_2$, $D = -(b_2 \cdot s + K_2)$

$$\Rightarrow \begin{cases} A X_1(s) = D X_2(s) \\ B X_2(s) - C X_1(s) = -K_3 Y(s) \end{cases}$$

$$\Rightarrow \quad X_2(S) = \frac{A}{D} X_1(S)$$

$$\therefore \quad B \cdot \frac{A}{D} X_1(S) - C\,X_1(S) = -K_3\,Y(S)$$

$$\frac{X_1(S)}{Y(S)} = \frac{-K_3}{BA/D - C} = \frac{K_3 D}{CD - AB}$$

$$= \frac{K_3(b_2 \cdot S + K_2)}{(b_1 \cdot S + K_2)(b_2 \cdot S + K_2) + (m_1 \cdot S^2 + (b_2 - b_1)\cdot S - (K_1 + K_2)) \cdot [m_2 \cdot S^2 - b_2 S - (K_2 + K_3)]}$$

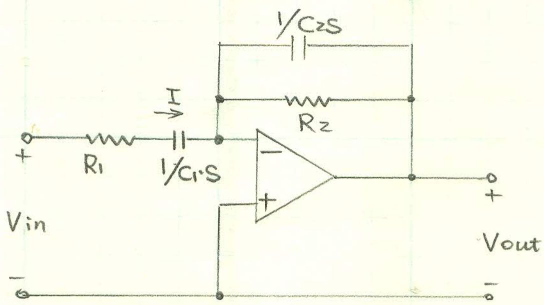
$$\frac{X_2(S)}{Y(S)} = \frac{K_3 \cdot A}{CD - AB}$$

$$= \frac{-K_3 \cdot [m_1 \cdot S^2 + (b_2 - b_1)\cdot S - (K_1 + K_2)]}{(b_1 \cdot S + K_2)(b_2 \cdot S + K_2) + [m_1 \cdot S^2 + (b_2 - b_1)\cdot S - (K_1 + K_2)] \cdot [m_2 \cdot S^2 - b_2 \cdot S - (K_2 + K_3)]}$$

---

2.3 Use node analysis to compute the transfer functions of the circuits indicated. Assume ideal operational amplifiers in every case. For (b) and (c) determine the location of the poles.

a) The lead network in Fig. 2.51

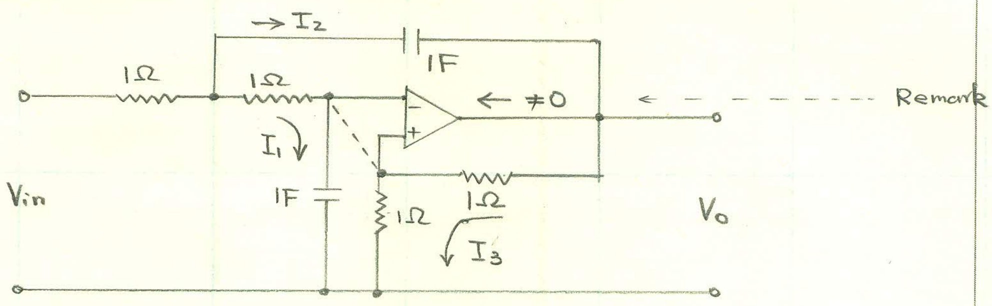



$$V_{in} = I\left(R_1 + \frac{1}{C_1 \cdot S}\right) = I \cdot \frac{R_1 C_1 \cdot S + 1}{C_1 \cdot S}$$

$$V_{out} = -I\left(R_2 \parallel \frac{1}{C_2 S}\right) = -I \cdot \frac{R_2}{R_2 C_2 \cdot S + 1}$$

$$\frac{V_{out}(S)}{V_{in}(S)} = -\frac{R_2 \cdot C_1 \cdot S}{(R_2 C_2 \cdot S + 1)(R_1 C_1 \cdot S + 1)}$$

b) The Sallen-Key circuit in Fig. 2.52, choose R = 1




$$V_{in} = (I_1 + I_2) + I_1 + I_1 \cdot \frac{1}{s} = (2 + \tfrac{1}{s}) I_1 + I_2$$

$$V_o = 2 I_3$$

Since for an ideal OP-Ampilifier, the two input terminals are virtual short-circuit (as indicated in the circuit above), Thus

$$I_1 \cdot \frac{1}{s} = I_3 \quad \Rightarrow I_1 = I_3 \cdot s$$

$$I_3 = -I_2/s + I_1 \quad \Rightarrow \quad I_3 \cdot s = -I_2 + I_1 \cdot s$$

$$\Rightarrow I_2 = I_1 (s-1) = I_3 \cdot s(s-1)$$

$$\therefore V_{in} = (2 + \tfrac{1}{s}) \cdot I_3 \cdot s + I_3 \cdot s(s-1)$$

$$= [(2s+1) + s(s-1)] \cdot I_3$$

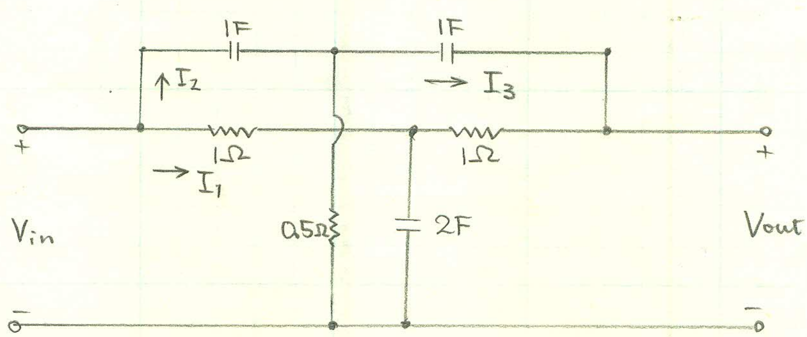
$$= (s^2 + s + 1) I_3$$

$$\frac{V_o}{V_{in}} = \frac{2}{s^2 + s + 1}, \quad \text{poles: } s_{1,2} = -0.5 \pm j0.866$$

✓

---

c) The twin tee in Fig. 2.53



To simplify the problem, we assume $V_{in}=1$. Thus we have

$$\begin{cases} I_1+(I_1+I_3)\cdot\frac{1}{2S} = V_{in}=1 \\ I_2\cdot\frac{1}{S}+(I_2-I_3)\times 0.5 = V_{in}=1 \\ I_2\cdot\frac{1}{S}+I_3\cdot\frac{1}{S}+I_3+(I_1+I_3)/2S = V_{in}=1 \end{cases}$$

$$\Rightarrow \begin{cases} I_1(2S+1)+I_3=2S \\ I_2(S+2)-SI_3=2S \\ I_1+2I_2+(2S+3)I_3=2S \end{cases}$$

Solve for $I_1, I_3$, $(I_2)$ we have

$$I_1=\frac{S^2+3S}{S^2+4S+1}$$

$$I_3=\frac{S^2-S}{S^2+4S+1}$$

$$V_{out}=I_3+(I_1+I_3)\cdot/2S$$

$$=\frac{S^2-S}{S^2+4S+1}+\frac{2S^2+2S}{(S^2+4S+1)\cdot 2S}$$

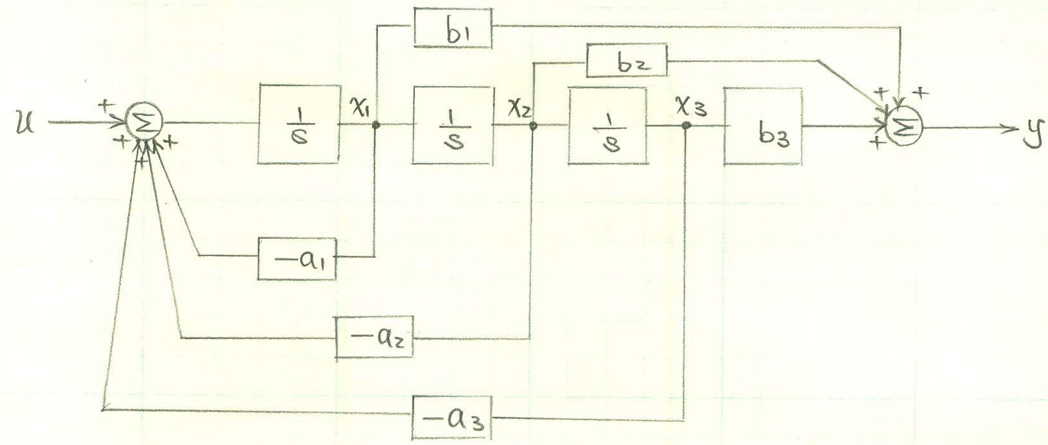
$$=\frac{S^2+1}{S^2+4S+1}$$ ✓

Poles: $S_{1,2}=-2\pm\sqrt{3}$

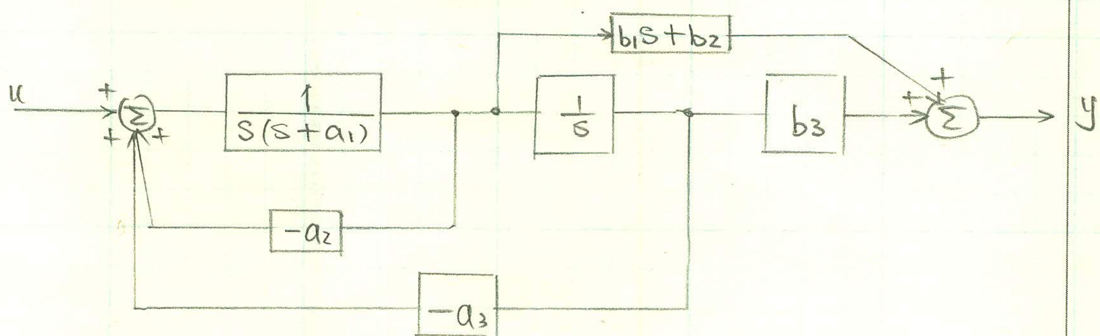
2.11 a) Compute the transfer function of the block diagram shown in Fig. 2.60 by successive application of the results of Fig. 2.30 and Mason's rule.

b) Write the third-order differential equation relating $y$ and $u$.

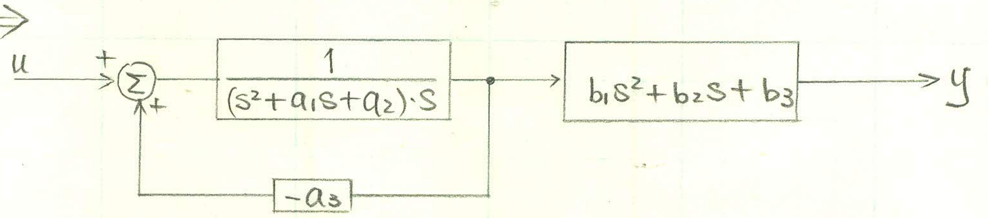
c) Write three simultaneous first-order (state-variable) differential equations using variables $x_1, x_2$, and $x_3$ as defined in Fig. 2.60.

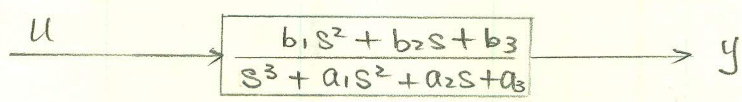



a) ⇒



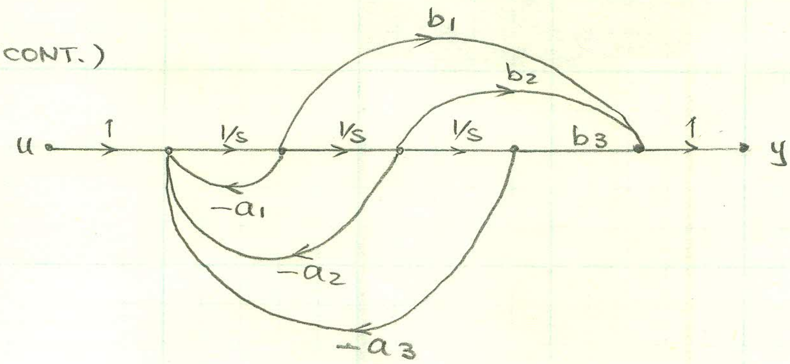

⇒




T.F. $G(s) = \frac{b_1 \cdot s^2 + b_2 \cdot s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$ ✓

a) (CONT.)




Mason's rule: $G(s) = \frac{1}{\Delta} \sum_i G_i \Delta_i$

$$G_1 = \frac{b_3}{s^3}, \quad G_2 = \frac{b_2}{s^2}, \quad G_3 = \frac{b_1}{s}$$

$$\Delta = 1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3}\right) = \frac{1}{s^3}\left(s^3 + a_1 \cdot s^2 + a_2 \cdot s + a_3\right)$$

$$\Delta_1 = \Delta_2 = \Delta_3 = 1 - (0)$$

$$G(s) = \frac{s^3}{s^3 + a_1 \cdot s^2 + a_2 s + a_3} \cdot \left(\frac{b_3}{s^3} + \frac{b_2}{s^2} + \frac{b_1}{s}\right)$$

$$= \frac{b_1 \cdot s^2 + b_2 s + b_3}{s^3 + a_1 \cdot s^2 + a_2 s + a_3} = \frac{Y(s)}{U(s)}$$

Q.E.D.

b) $$s^3 \cdot Y(s) + a_1 \cdot s^2 Y(s) + a_2 \cdot s Y(s) + a_3 Y(s)$$

$$= b_1 \cdot s^2 \cdot U(s) + b_2 \cdot s \cdot U(s) + b_3 U(s)$$

We have

$$\frac{d^3 y(t)}{dt^3} + a_1 \cdot \frac{d^2 y(t)}{dt^2} + a_2 \cdot \frac{dy(t)}{dt} + a_3 y(t)$$

$$= b_1 \cdot \frac{d^2 u(t)}{dt} + b_2 \frac{du(t)}{dt} + b_3 u(t)$$

✓

c) $$\dot{x}_1 = -a_1 \cdot x_1 - a_2 x_2 - a_3 x_3 + u$$

$$\dot{x}_2 = x_1, \quad \dot{x}_3 = x_2, \quad y = b_1 x_1 + b_2 x_2 + b_3 x_3$$

State Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot u$$

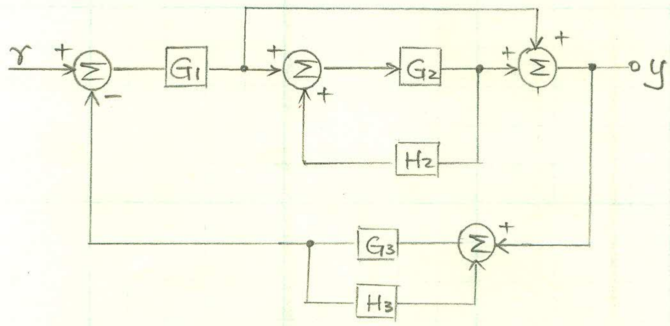
$$y = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

✓

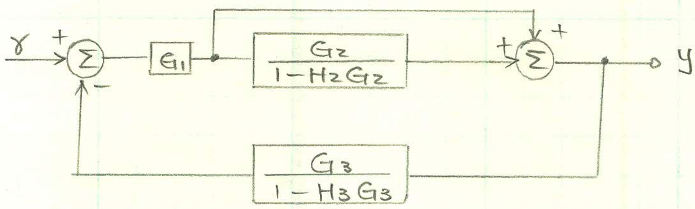
2.12 Find the transfer functions of the block diagrams in Fig. 2.61(a), (b), (c), and (d) by the rules of Fig. 2.30 and by Mason's rule.

---

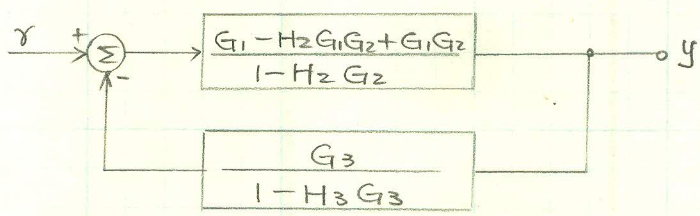
(a)




⇒



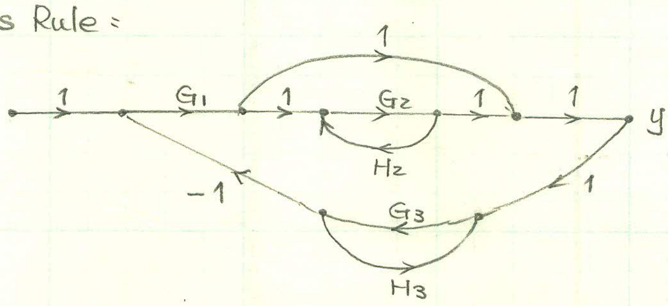

⇒




$$r \longrightarrow \boxed{\frac{G_1-H_2G_1G_2+G_1G_2-H_3G_1G_3+H_2H_3G_1G_2G_3-H_3G_1G_2G_3}{1-H_2G_2-H_3G_3+H_2H_3G_2G_3+G_1G_3+G_1G_2G_3-H_2G_1G_2G_3}} \longrightarrow y$$

Transfer function

Mason's Rule:



$$g_1 = G_1G_2 \quad , \quad g_2 = G_1$$

$$\Delta = 1 - (H_2G_2 + H_3G_3 - G_1G_2G_3 - G_1G_3) + (H_3G_3H_2G_2 - H_2G_2G_1G_3)$$

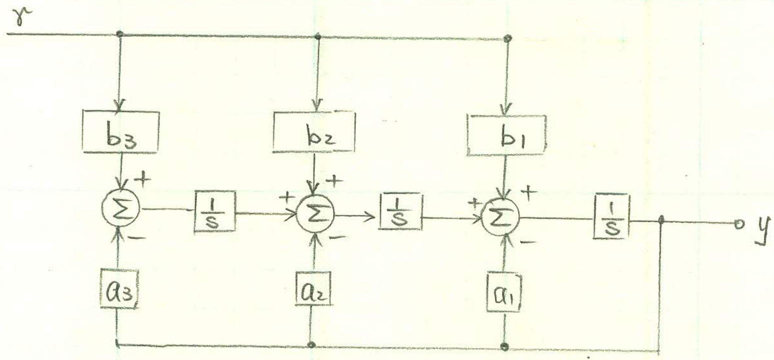
$$\Delta_1 = 1 - H_3G_3$$

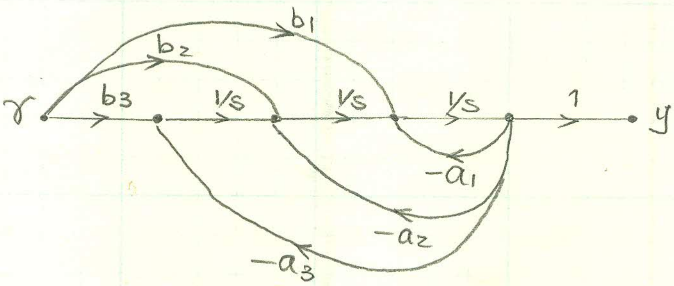
$$\Delta_2 = 1 - H_2G_2 - H_3G_3 + H_2H_3G_2G_3$$

$$G(s) = \frac{G_1G_2 - H_3G_1G_2G_3 + G_1 - H_2G_1G_2 - H_3G_1G_3 + H_2H_3G_1G_2G_3}{1 - H_2G_2 - H_3G_3 + G_1G_2G_3 + G_1G_3 + H_2H_3G_2G_3 - H_2G_1G_2G_3}$$

Q.E.D.

(b)




$$g_1 = \frac{b_3}{s^3} \quad , \quad g_2 = \frac{b_2}{s^2} \quad , \quad g_3 = \frac{b_1}{s}$$

$$\Delta = 1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3}\right) = \frac{1}{s^3}\left(s^3 + a_1s^2 + a_2s + a_3\right)$$

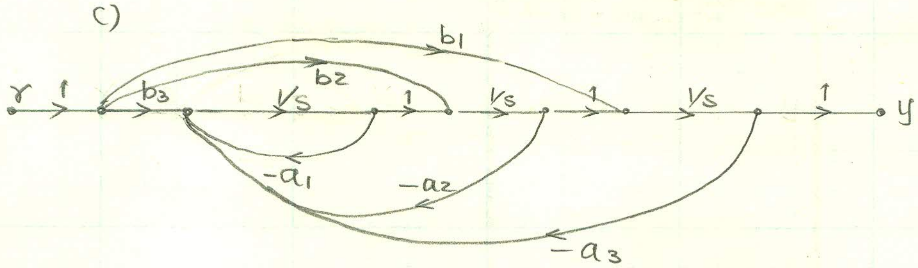
$$\Delta_1 = 1, \quad \Delta_2 = 1, \quad \Delta_3 = 1$$

$$G(s) = \frac{1}{\Delta}\left(g_1\Delta_1 + g_2\Delta_2 + g_3\Delta_3\right)$$

$$= \frac{b_1 \cdot s^2 + b_2 \cdot s + b_3}{s^3 + a_1 \cdot s^2 + a_2 \cdot s + a_3}$$

✓

c)




$$G_1 = \frac{b_3}{s^3}, \quad G_2 = \frac{b_2}{s^2}, \quad G_3 = \frac{b_3}{s}$$

$$\Delta = 1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3}\right) = \frac{1}{s^3}\left(s^3 + a_1 s^2 + a_2 \cdot s + a_3\right)$$

$$\Delta_1 = 1, \quad \Delta_2 = 1 + \frac{a_1}{s}$$

$$\Delta_3 = 1 + \frac{a_1}{s} + \frac{a_2}{s^2}$$

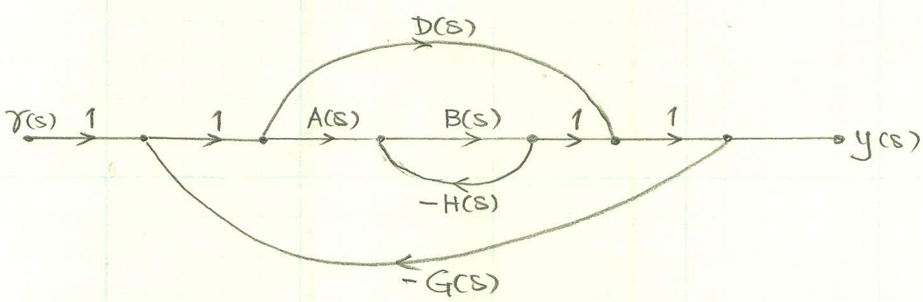
$$G_1 \cdot \Delta_1 + G_2 \Delta_2 + G_3 \Delta_3 = \frac{b_3}{s^3} + \frac{b_2}{s^2}\left(1 + \frac{a_1}{s}\right) + \frac{b_3}{s}\left(1 + \frac{a_1}{s} + \frac{a_2}{s^2}\right)$$

$$= \frac{1}{s^3} \cdot \left[ b_3(s^2 + a_1 s + a_2) + b_2(s + a_1) + b_3 \right]$$

$$G(s) = \frac{b_3(s^2 + a_1 s + a_2) + b_2(s + a_1) + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Q.E.D.

d)




$$G_1 = A(s)B(s), \quad G_2 = D(s)$$

$$\Delta = 1 + H(s)B(s) + G(s)A(s)B(s) + G(s)D(s) + G(s)D(s)H(s)B(s)$$

$$\Delta_1 = 1; \quad \Delta_2 = 1 + H(s)B(s)$$

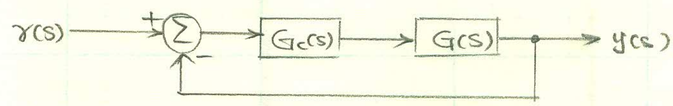
Thus, the transfer function

$$T(s) = \frac{A(s)B(s) + H(s)B(s)D(s) + D(s)}{1 + H(s)B(s) + G(s)A(s)B(s) + G(s)D(s) + G(s)D(s)H(s)B(s)}$$

2.13 Consider the system shown in Fig. 2.62: Let

$$G(s) = \frac{1}{(s+3)\cdot s} \qquad G_c(s) = \frac{K(s+z)}{(s+p)}$$

Find K, z, and p such that the closed loop system has a 5% overshoot to a step input and a settling time of 4/3 (2%)




$$y(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)} \cdot r(s)$$

$$= \frac{\frac{K(s+z)}{(s+p)} \cdot \frac{1}{(s+3)\cdot s}}{1 + \frac{K(s+z)}{s+p} \cdot \frac{1}{(s+3)\cdot s}} \cdot r(s)$$

$$= \frac{K(s+z)}{s^3 + (3+p)\cdot s^2 + (3p+K)s + K\cdot z} \cdot r(s)$$

Let $z=0$ (theoretically, it is O.K. But not pratical)

$$y(s) = \frac{K}{s^2 + (3+p)s + (3p+K)} \cdot r(s)$$

5% overshoot $\Rightarrow$ $\xi = 0.7$

$$e^{-\xi\omega_n \cdot t_s} = e^{-0.7\cdot\omega_n\cdot 4/3} = 0.02 \Rightarrow \omega_n = 4.191453$$

Designed C.E. $= s^2 + 5.868\,s + 17.568$

Let $\begin{cases} 3+p = 5.868 \\ 3p+K = 17.568 \end{cases} \Rightarrow \begin{cases} p = 2.868 \\ K = 8.964 \end{cases}$ & $z = 0$

---

Note: This is only a theoretical design. In practice, we can never have a perfect cancellation of pole $s=0$ of $G(s)$ with zero $s_1 = -z = 0$ of $G_c(s)$. But by doing so, life will be much easier.

2.14 Given the G(S) below, sketh the step response.

$$G(s) = \frac{s/2 + 1}{(s/40+1)[(s/4)^2 + s/4 + 1]}$$

$$= \frac{320(s+2)}{(s+40)(s^2+4s+16)}$$

Step response, $u(s) = \frac{1}{s}$

$$y(s) = \frac{320(s+2)}{s(s+40)(s^2+4s+16)}$$

$$= \left[\frac{A}{s} + \frac{B}{s+40} + \frac{C\cdot s + D}{s^2+4s+16}\right]$$

$\Rightarrow$ $A = 1$, $B = \frac{7}{37} = 0.207$

Let $s = 1$, we have

$C + D = 2.317732$ ---- (1)

Let $s = -1$, we have

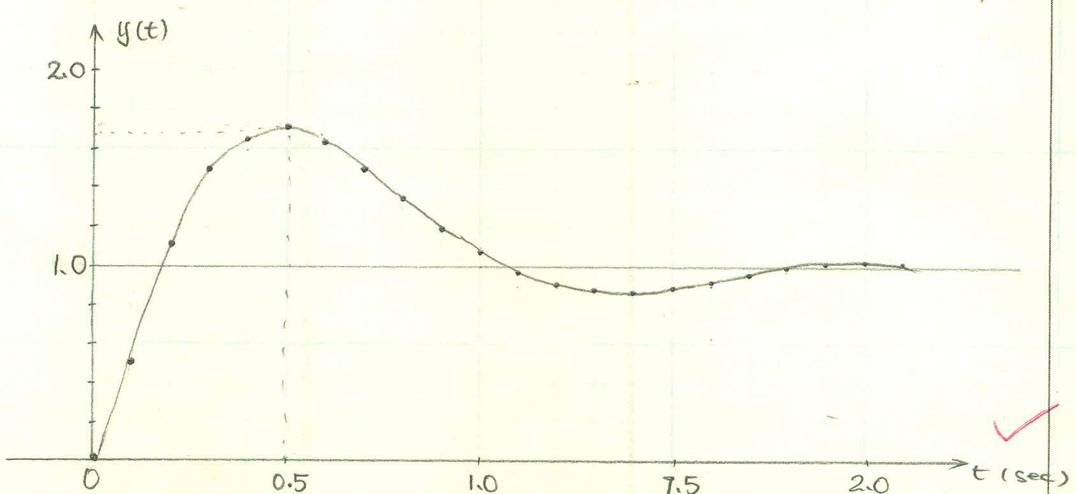
$D - C = 4.7318089$ ---- (2)

$\Rightarrow$ $C = -1.20704$ ; $D = 3.52477$

$$y(s) = \frac{1}{s} + \frac{0.189189}{s+40} - \frac{1.20704(s+2)}{(s+2)^2 + (3.4641)^2} + \frac{5.93885}{(s+2)^2 + (3.4641)^2}$$

$$y(t) = 1 + 0.207e^{-40t} - 1.20704e^{-2t}\cos(3.4641t) + 1.7144e^{-2t}\sin(3.4641\cdot t)$$

$t \geq 0$



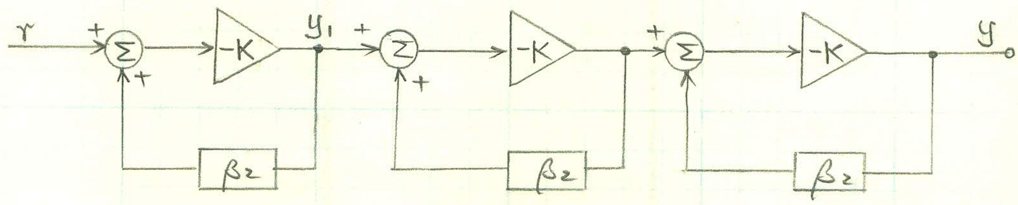
3.1 Bode defined the sensitivity function of a transfer function G to one of its parameters $k$ as the ratio of percent change in $k$ to percent in G. We define the reciprocal of Bode's function as

$$S_k^G = \frac{dG/G}{dk/k} = \frac{k}{G}\cdot\frac{dG}{dk}$$

(c) Compute the sensitivities of Fig. 3.21 (b) and (c) to $\beta_2$ and $\beta_3$. Comment on the relative demands for precision in sensors and actuators from these cases.

---

Fig. 3.21 (b)




$$\frac{y_1}{r} = \frac{-K}{1-\beta_2(-K)} = \frac{-K}{1+\beta_2 K}$$

Thus

$$\frac{y}{r} = G = \frac{-K^3}{(1+\beta_2 K)^3}$$

$$S_{\beta_2}^G = \frac{\beta_2}{G}\cdot\frac{3K^3}{(1+\beta_2 K)^4}\cdot K$$

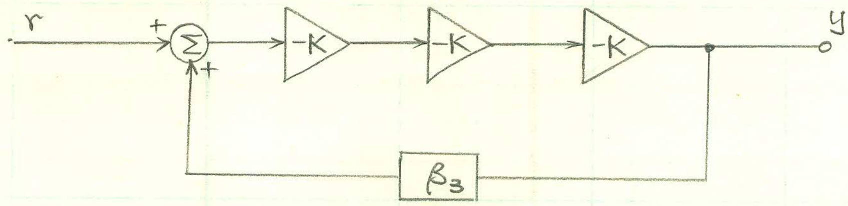
$$= \frac{\beta_2 (1+\beta_2 K)^3}{-K^3}\cdot\frac{3K^4}{(1+\beta_2 K)^4} = -\frac{3\beta_2}{1+\beta_2 K}.$$

$$\boxed{S_{\beta_2}^G = -\frac{3}{K+\frac{1}{\beta_2}}}$$

from the equation above, we see that

$S_{\beta_2}^G$ will be small if K is large or $\beta_2$ is small or both.

3.1 c)




$$G = \frac{y}{r} = \frac{-K^3}{1+\beta_3 K^3}$$

$$S^G_{\beta_3} = \frac{\beta_3}{G} \cdot \frac{K^3}{(1+\beta_3 K^3)^2} \cdot K^3$$

$$= \frac{\beta_3(1+\beta_3 K^3)}{-K^3} \cdot \frac{K^3 \cdot K^3}{(1+\beta_3 K^3)^2}$$

$$= -\frac{\beta_3 K^3}{1+\beta_3 K^3} = -\frac{1}{1+1/\beta_3 K^3}$$

$$\therefore \quad \boxed{S^G_{\beta_3} = -\frac{1}{1+1/\beta_3 K^3}}$$

from this equation, we see that

$S^G_{\beta_3}$ will be small if and only if $\beta_3 K^3$ is small

So, either $\beta_3$ and $K^3$ are small will be better.

Actually, we have already discussed the relative demands for precision in sensors ~~and actuators.~~ ?

$\beta_2, \beta_3$ corresponding to sensors

K's corresponding to actuators.

3.2 Compare the two structures shown in Fig. 3.22 with respect to sensitivity to changes in overall gain due to changes in amplifier gain. Use

$$S = \frac{d \ln F}{d \ln K} = \frac{K}{F} \cdot \frac{dF}{dK}$$

as the measure. Select $H_1$ and $H_2$ so that at the nominal $F_1 = F_2$

---

from (a), we have

$$F_1 = \frac{K^2}{(1+KH_1)^2} \cdot r \quad \cdots\cdots (1)$$

from (b), we have

$$F_2 = \frac{K^2}{1+K^2 H_2} r \quad \cdots (2)$$

$$F_1 = F_2 \Rightarrow (1+KH_1)^2 = 1+K^2 H_2$$

$$\Rightarrow 1 + 2KH_1 + K^2H_1^2 = 1 + K^2 H_2$$

$$\boxed{H_2 = \frac{2+KH_1}{K} \cdot H_1} \quad \cdots\cdots (3) \quad \text{(selected \& fixed)}$$

for $F_1$,

$$S = \frac{K}{F_1} \cdot \frac{2K\cancel{(1+KH_1)} - 2K^2\cancel{(1+KH_1)}H_1}{(1+KH_1)^3} \cdot r$$

$$= \frac{K\cancel{(1+KH_1)^2}}{K^2} \cdot \frac{2K \cdot r}{(1+KH_1)^{\cancel{3}}} = \frac{2 \cdot r}{1+KH_1} \quad \cdots (4)$$

for $F_2$,

$$S = \frac{K\cancel{(1+K^2H_2)}}{K^2} \cdot \frac{2K \cdot r}{(1+K^2H_2)^{2}} = \frac{2 \cdot r}{1+K^2H_2} \quad \cdots (5)$$

Subtitude (3) to the equation above, we have

$$S_K^{F_2} = \frac{2 \cdot r}{2KH_1 + K^2H_1^2 + 1} \quad \cdots\cdots (6)$$

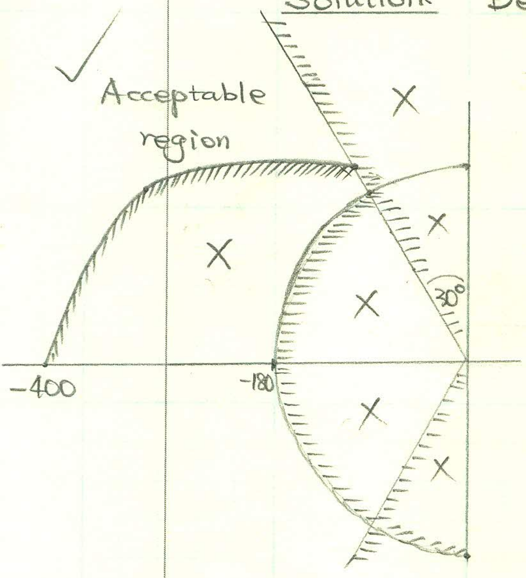
Compared (4) to (6), we see that $F_2$ has less sensitivity than $F_1$ for $K \gg 1$

3.3 Specifications: $t_r \leq 0.010$ s , $M_p \leq 17\%$

$e_{ss} \leq 0.005$ due to unit ramp.

---

Solution: Design equations: for second order system




$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$t_r \doteq 1.8/\omega_n$ , $e^{-\xi\omega_n t_s}$ = settled error

$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$ , $t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$

$t_r = 1.8/\omega_n \Rightarrow \omega_n \gtrsim 1.8/0.01 \gtrsim 180$

$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = 17\% \Rightarrow \xi \gtrsim 0.5$

$$E(s) = R(s) - G(s)\cdot R(s)$$

$$= R(s)\left[1 - G(s)\right]$$

$$= R(s)\cdot\frac{s^2 + 2\xi\omega_n\cdot s}{s^2 + 2\xi\omega_n\cdot s + \omega_n^2}$$

if $R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s\to 0} s\cdot E(s) = \lim_{s\to 0}\frac{1}{s}\cdot\frac{s^2 + 2\xi\omega_n\cdot s}{s^2 + 2\xi\omega_n\cdot s + \omega_n^2}$$

$$= \frac{2\xi}{\omega_n} \leq 0.005 \Rightarrow \omega_n \geq 400\xi$$

See the skecth above for (a)

(b) $y/r = G/(1+G) \Rightarrow$ we have unit negative feedback.

$$E_e = \left[1 - \frac{G(s)}{1+G(s)}\right]R(s) = \frac{1}{1+G(s)}\cdot R(s)$$

??

3.5 Givens: open loop transfer function:

$$G(s) = \frac{K}{s(s+2)}$$

$T_p = 1$ sec. and $M_p = 5\%$

(a) Find the closed-loop transfer function first:

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{K}{s^2+2s+K}$$

from the specification: $T_p = 1\,s \Rightarrow \omega_n\sqrt{1-\xi^2} = 3.1416$

$M_p = 5\% \Rightarrow \xi = 0.7 \quad \cdots (1)$

$\Rightarrow \omega_n = 4.4 \quad \cdots (2)$

In order for a system to meet (1) and (2), the characteatis Equation must be

$$s^2 + 2\xi\omega_n \cdot s + \omega_n^2 = s^2 + 6.16\,s + 19.36 = 0$$

Answer: there is no way to meet both spec. simultaneously.

(b) $T_p = 1(1+X) \Rightarrow \omega_n\sqrt{1-\xi^2} = \pi/(1+X)$

$M_p' = e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.05\,(1+X)$

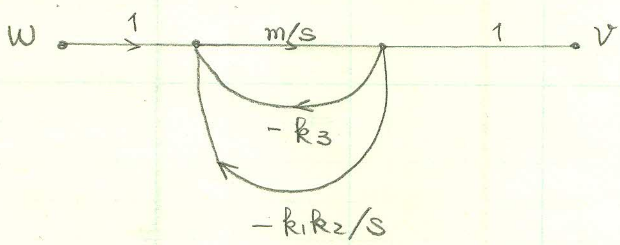
$$\Rightarrow \begin{cases} \xi\omega_n = 1 \quad (2\xi\omega_n = 2) \\ \omega_n\sqrt{1-\xi^2} = \pi/(1+X) \\ e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.05\,(1+X) \end{cases} \Rightarrow \begin{cases} \xi \doteq 0.573 \\ \omega_n \doteq 1.744 \\ X = 1.198 \end{cases}$$

$$\Rightarrow K = \omega_n^2 = 3.042$$

Specifications are relaxed by 120% ✓

3.6 Automobile control

(a) $v_e = 0$

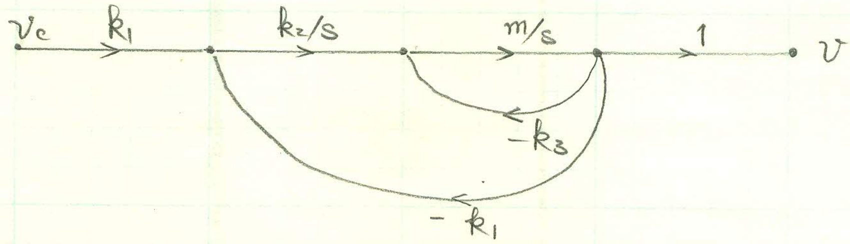



$$\frac{V(s)}{W(s)} = \frac{m/s}{1+(k_3+k_1k_2/s)\cdot m/s} = \frac{m\cdot s}{s^2+mk_3 s+k_1k_2 m}$$

(b) $$V_{ss} = \lim_{s\to 0} s\cdot\frac{m\cdot s}{s^2+mk_3\cdot s+k_1k_2 m}\cdot W(s)$$

for ramp : $$= \lim_{s\to 0}\frac{m}{s^2+mk_3 s+k_1k_2 m} = \frac{1}{k_1k_2}$$ (this concludes part d).

(c)




$$G_1 = k_1k_2 m/s^2$$

$$\Delta = 1-(-k_3m/s - k_1k_2m/s^2)$$

$$= 1+k_3m/s+k_1k_2m/s^2$$

$$T(s) = \frac{k_1k_2 m}{s^2+k_3m\cdot s+k_1k_2m}$$

$$E(s) = [1-T(s)]\cdot V_c(s) = \frac{s^2+k_3m\cdot s}{s^2+k_3m\cdot s+k_1k_2m}\cdot V_c(s)$$

This is type 1 system:

$$e_{ss} = \lim_{s\to 0}(sV_c(s))\cdot\frac{s^2+k_3m\cdot s}{s^2+k_3m\cdot s+k_1k_2m}$$

for ramp $sV_c(s) = \frac{1}{s}$ $\rightarrow$ type 1 and

$$e_{ss} = \frac{k_3m}{k_1k_2m} = \frac{k_3}{k_1k_2} = \frac{1}{k_v} \Rightarrow k_v = \frac{k_1k_2}{k_3}$$

(d) Due to disturbance $w$ = type 1 also and $e_{ss,w} = V_{ss} = \frac{1}{k_1k_2}$.

3.8

$$G(s) = \frac{y}{r} = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2}$$

a) $t_r \leq 0.1\,s$

$$t_r \simeq \frac{1.8}{\omega_n} \leq 0.1 \qquad \omega_n \geq 18 \quad \cdots \quad (1)$$

b) $M_p \leq 20\% = 0.2$

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \leq 0.2 \quad \Rightarrow \quad \zeta \geq 0.456 \quad \cdots \quad (2)$$

c) $t_s \leq 0.5\,s$ ($\pm 1\%$ Assumed)

$$t_s = \frac{4.6}{\sigma} = \frac{4.6}{\zeta\omega_n} \leq 0.5 \quad \Rightarrow \quad \zeta\omega_n \geq 9.2 \quad \cdots \quad (3)$$

d) $E(s) = R(s) - Y(s)$

$$= R(s) - G(s)R(s) = R(s)[1 - G(s)]$$

$$= R(s) \cdot \frac{s^2 + (a_1 - b_0)s + (a_2 - b_1)}{s^2 + a_1 s + a_2}$$

$$e_{ss} = \lim_{s\to 0} \cdot s \cdot \frac{c}{s} \cdot \frac{s^2 + (a_1 - b_0)s + (a_2 - b_1)}{s^2 + a_1 s + a_2} = 0 \quad \text{due to constant}$$

$$\Rightarrow \quad a_2 = b_1 \quad \cdots \quad (4)$$

e) due to ramp $R(s) = 0.1/s^2$

$$e_{ss} = \lim_{s\to 0} s \cdot \frac{0.1}{s^2} \cdot \frac{s^2 + (a_1 - b_0)s}{s^2 + a_1 s + a_2} = \frac{0.1(a_1 - b_0)}{a_2} \leq 0.001$$

$$\Rightarrow a_1 - b_0 \leq 0.01\, a_2 \quad \cdots \quad (5)$$

Now: from (3), we select $a_1 = 20$ $(\Rightarrow \zeta\omega_n = 10 > 9.2)$

from (2), we select $\zeta = 0.5 > 0.456 \Rightarrow \omega_n = 20 > 18$

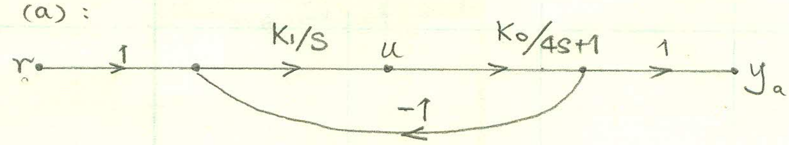
$\Rightarrow a_2 = \omega_n^2 = 400 \;.\; \Rightarrow b_1 = 400$

$\Rightarrow b_0 \geq 16$ then choose $b_0 = 16$.

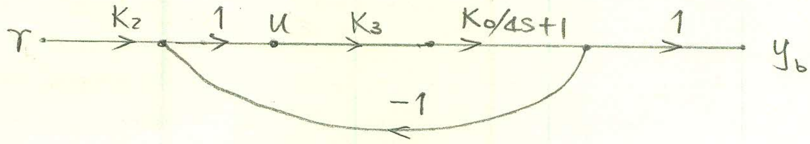
Thus, we have system

$$G(s) = \frac{y}{r} = \frac{16s + 400}{s^2 + 20s + 400}$$

3.9 system (a):




system (b):




(a) $K_0 = 1.0$

$$* \quad G_a(s) = \frac{y_a}{r} = \frac{K_1/s \cdot 1/(4s+1)}{1 + K_1/s \cdot 1/(4s+1)}$$

$$= \frac{K_1}{s(4s+1) + K_1} = \frac{0.25K_1}{s^2 + 0.25s + 0.25K_1}$$

$$** \quad G_b(s) = \frac{y_b}{r} = \frac{K_2K_3/(4s+1)}{1 + K_3/(4s+1)} = \frac{0.25K_2K_3}{s + 0.25(1+K_3)}$$

system a: $E_a(s) = R(s) - Y_a(s) = R(s)\,[1 - G_a(s)]$

$$= R(s) \cdot \frac{s^2 + 0.25s}{s^2 + 0.25s + 0.25K_1}$$

for step inpu $R(s) = \frac{1}{s}$

$$e_{ss,a} = \lim_{s \to 0} \frac{s^2 + 0.25s}{s^2 + 0.25s + 0.25K_1} = 0 \quad \text{for any } K_1 \neq 0$$

System b: $E_b(s) = R(s)\,[1 - G_b(s)]$

$$= R(s) \cdot \frac{s + 0.25(1 + k_3 - k_2k_3)}{s + 0.25(1 + k_3)}$$

$$e_{ss,b} = \lim_{s \to 0} \frac{s + 0.25(1 + k_3 - k_2k_3)}{s + 0.25(1 + k_3)} = \frac{1 + k_3 - k_2k_3}{1 + k_3} = 0$$

$$\Rightarrow \quad 1 + k_3 - k_2k_3 = 0 \quad \Rightarrow \quad K_3 = \frac{1}{K_2 - 1}$$

3.9 (a) (CONT.)

Due to the ramp input $R(s) = 1/s^2$

$$e_{ss,a} = \lim_{s\to 0} \frac{s + 0.25}{s^2 + 0.25s + 0.25K_1} = \frac{1}{K_1} = 1 \Rightarrow K_1 = 1$$

$$e_{ss,b} = \lim_{s\to 0} \cdot s \cdot \frac{1}{s^2} \cdot \frac{s}{s + 0.25(1+k_3)}$$

$$= \frac{1}{0.25(1+k_3)} = 1$$

$$\Rightarrow \quad 0.25(1+K_3) = 1$$

$$\Rightarrow K_3 = 3$$

$$\Rightarrow K_2 = \frac{4}{3}$$

Answer:

$$K_1 = 1$$
$$K_2 = \frac{4}{3}$$
$$K_3 = 3$$

b) from (*), we see that

$$G_a(s) = \frac{0.25 K_0 K_1}{s^2 + 0.25 s + 0.25 K_0 K_1}$$

(due to ramp) : $e_{ss,a} = \lim_{s} \frac{s + 0.25}{s^2 + 0.25 s + 0.25 K_0 K_1} = \frac{1}{K_0}$

$$K_v = K_0$$

So that system (a) is still a type I system whenever $K_0$ does not change to zero.

---

c) $G_b(s) = \frac{0.25 K_0 K_2 K_3}{s + 0.25(1 + K_0 K_3)} = \frac{K_0}{s + 0.25(1 + 3K_0)}$

from result in part (a), we see that $e_{ss}$ due to ramp input for system (b) is

$$e_{ss} = \lim_{s \to 0} s \cdot \frac{1}{s^2} \cdot \frac{s + 0.25(1 - K_0)}{s + 0.25(1 + 3K_0)}$$

$$= \lim_{s \to 0} \frac{s + 0.25(1 - K_0)}{s \cdot [s + 0.25(1 + 3K_0)]}$$

$e_{ss} = \infty$ if $K_0$ is no longer equal to 1. ⇒ no longer type I.

---

d) Many reasons. One of these reasons is as we have seen from part b) and c). for $K_0$ (in practice, which is difficult to be determined exactly), system (a) is much less sencilivity than system (b).

---

e) 1) $G_a(s) = \frac{0.25}{s^2 + 0.25 s + 0.25}$ ⇒ $\omega_n = 0.5$, $\xi = 0.25$

$t_s = \frac{4.6}{\xi \omega_n} = 36.8$ sec. ; $M_p = 44.43\%$ ; $t_r = 3.6$ sec.

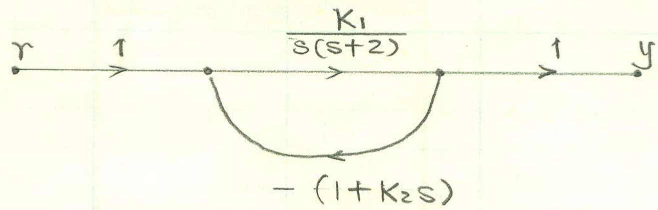
2) $Y_b(s) = \frac{1}{s+1} \cdot \frac{1}{s} = \frac{1}{s} - \frac{1}{s+1}$ ⇒ $y_b(t) = u(t) - e^{-t} u(t)$

$y_b(\infty) = 1$ ⇒ $\begin{cases} 1 - e^{-t_1} = 0.1 \Rightarrow t_1 = 0.105 \text{ sec.} \\ 1 - e^{-t_2} = 0.9 \Rightarrow t_2 = 2.303 \text{ sec.} \end{cases}$ ⇒ $t_r = 2.2$ sec.

$M_p = 0$ (never overshoot) ; $1 - e^{-t_s} = 0.99$ ⇒ $t_s = 4.6$ sec

In my opinion, system (b) is better.

3.11 system :




$$G(s) = \frac{y}{r} = \frac{K_1}{s^2 + (2+K_1K_2)s + K_1}$$

Specification: (1) steady-state error to a ramp input less than 10% of the input magnitude.

$$E(s) = R(s)[1 - G(s)]$$

$$= \frac{s^2 + (2+K_1K_2)s}{s^2 + (2+K_1K_2)s + K_1} \cdot \frac{1}{s^2}$$

$$e_{ss} = \lim_{s \to 0} s \cdot E(s) = \frac{2 + K_1K_2}{K_1} \le 0.1$$

$$0.1 K_1 - K_1K_2 \ge 2 \quad \cdots\cdots (1)$$

(2) $M_p \le 0.05 \Rightarrow \zeta \ge 0.69 \quad \cdots\cdots (2)$

(3) 2% $t_s \le 3$ sec. $\Rightarrow e^{-\zeta\omega_n t_s} \le 0.02$

$$t_s = \frac{3.912}{\zeta\omega_n} \le 3 \Rightarrow \zeta\omega_n \ge 1.30 \quad \cdots (3)$$

pick $\zeta = 0.7$, we have

$$\left\{\begin{array}{l} 1.4\,\omega_n = 2 + K_1K_2 \\ \omega_n \ge 1.857 \\ K_1 = \omega_n^2 \\ 0.1 K_1 - K_1K_2 \ge 2 \end{array}\right. \Rightarrow \left\{\begin{array}{l} K_1 \ge 196 \\ K_2 \le 0.0898 \end{array}\right.$$

Then pick $K_1 = 196$, $K_2 = 0.0898$

(a) $$G(s) = \frac{196}{s^2 + 19.6s + 196}$$

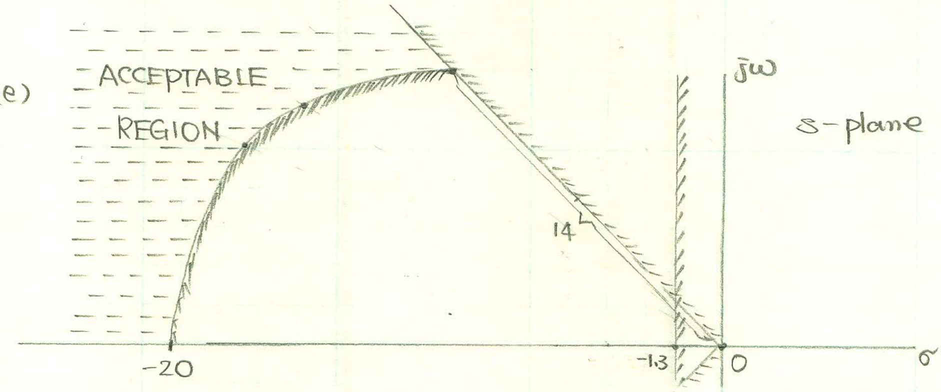
(b) $e_{ss} = \dfrac{2 + K_1 K_2}{K_1} = \dfrac{2 + 196 \times 0.0898}{196} = 10.00041\ \%$

(c) Specification (1) $\Rightarrow K_1 (0.1 - K_2) \geq 2 \Rightarrow K_1$ can be any value.
but $K_2$ must be small.

(d) Specification (3) $\Rightarrow \xi \omega_n \geq 1.30 \Leftrightarrow \sigma \geq 1.30$

That means that the real pont of closed-loops must be less than $-1.30$.

(e)




(f) $K_1 = 32 \Rightarrow 32(0.1 - K_2) = 2 \Rightarrow K_2 = 0.0375$

$K_1 < 196 \Rightarrow$ The closed-loop poles of such a system is not in acceptable region.

(g) From (a) we have $\omega_n = 14$, $\xi = 0.7$

$t_s \doteq 3.912 / \xi \omega_n = 0.4$ second.

3.16 Use the Routh criterion to determine if the closed-loop systems corresponding to the following open-loop transfer functions are stable.

a) $KG(s) = \dfrac{4(s+2)}{s(s^3+2s^2+3s+4)}$

$$CG(s) = \frac{G(s)}{1+G(s)} = \frac{4(s+2)}{s(s^3+2s^2+3s+4)+4(s+2)}$$

$$= \frac{4(s+2)}{s^4+2s^3+3s^2+8s+8}$$

$\therefore\ Q(s) = s^4+2s^3+3s^2+8s+8$

Routh Table

| | | | | |
|---|---|---|---|---|
| + | $s^4$ | 1 | 3 | 8 |
| + | $s^3$ | 2 | 8 | 0 |
| − | $s^2$ | −7 | 8 | |
| + | $s^1$ | 10.286 | 0 | |
| + | $s^0$ | 8 | | |

Two sign changes in Routh Table. Close-loop system unstable.

b) $KG(s) = \dfrac{2(s+4)}{s^2(s+1)}$

$$CG(s) = \frac{G(s)}{1+G(s)} = \frac{2(s+4)}{s^2(s+1)+2(s+4)} = \frac{2(s+4)}{s^3+s^2+2s+4}$$

$\therefore\ Q(s) = s^3+s^2+2s+4$

Routh Table:

| | | | |
|---|---|---|---|
| + | $s^3$ | 1 | 2 |
| + | $s^2$ | 1 | 4 |
| − | $s^1$ | −2 | 0 |
| + | $s^0$ | 4 | |

Two signs changed in Routh Table.

Close-loop system unstable.

c) $KG(s) = \dfrac{4(s^3+2s^2+s+1)}{s^2(s^3+2s^2-s-1)}$

$CG(s) = G(s)/[1+G(s)] = N(s)/[s^2(s^3+2s^2-s-1)+4(s^3+2s^2+s+1)]$

$Q(s) = s^5+2s^4+3s^3+7s^2+4s+1$

Two signs changed in the Table right.

The close-loop system unstable.

| | | | | |
|---|---|---|---|---|
| + | $s^5$ | 1 | 3 | 4 |
| + | $s^4$ | 2 | 7 | 1 |
| − | $s^3$ | −0.5 | 3.5 | |
| + | $s^2$ | 21 | 1 | |
| + | $s^1$ | 3.524 | 0 | |
| + | $s^0$ | 1 | | |

✓

3.17 Use Routh's stability criterion to determine how many roots with positive real parts each of the following equations has.

(a) $s^4 + 8s^3 + 32 \cdot s^2 + 80 \cdot s + 100 = 0.$

| | | | | |
|---|---|---|---|---|
| + | $s^4$ | 1 | 32 | 100 |
| + | $s^3$ | 8 | 80 | |
| + | $s^2$ | 22 | 100 | |
| + | $s^1$ | 43.64 | 0 | |
| + | $s^0$ | 100 | | |

No sign changes in Routh's table.

Thus, no roots on RHP.

(b) $s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0.$

| | | | | |
|---|---|---|---|---|
| + | $s^5$ | 1 | 30 | 344 |
| + | $s^4$ | 10 | 80 | 480 |
| + | $s^3$ | 22 | 296 | |
| − | $s^2$ | −54.55 | 480 | |
| + | $s^1$ | 489.56 | | |
| + | $s^0$ | 480 | | |

Two signs changed in R-table.

Thus, we have two roots in RHP.

(c) $s^4 + 2s^3 + 7s^2 - 2s + 8 = 0$

| | | | | |
|---|---|---|---|---|
| + | $s^4$ | 1 | 7 | 8 |
| + | $s^3$ | 2 | −2 | |
| + | $s^2$ | 8 | 8 | |
| − | $s^1$ | −4 | | |
| + | $s^0$ | 8 | | |

Two roots in RHP.

(d) $s^3 + s^2 + 20 \cdot s + 78 = 0.$

| | | | |
|---|---|---|---|
| + | $s^3$ | 1 | 20 |
| + | $s^2$ | 1 | 78 |
| − | $s^1$ | −58 | |
| + | $s^0$ | 78 | |

Two roots in RHP.

(e) $s^4 + 6s^2 + 25 = 0$

| | | | | |
|---|---|---|---|---|
| + | $s^4$ | 1 | 6 | 25 |
| + | $s^3$ | 0 4 | 0 12 | |
| + | $s^2$ | 3 | 25 | |
| − | $s^1$ | −21.3 | | |
| + | $s^0$ | 25 | | |

$A(s) = s^4 + 6s^2 + 25$

$A'(s) = 4s^3 + 12s$

Two roots in RHP.

✓

**3.18** For what range of K would all the roots of the following polynomial be in the left-half plane?

$$S^5 + 5S^4 + 10S^3 + 10S^2 + 5S + K = 0$$

**Solution:** Using Routh Table

| | | | |
|---|---|---|---|
| $S^5$ | 1 | 10 | 5 |
| $S^4$ | 5 | 10 | K |
| $S^3$ | 8 | $5-K/5$ | |
| $S^2$ | $\frac{1}{8}(55+K)$ | K | |
| $S^1$ | $(1375-350K-K^2)/5(55+K)$ | | |
| $S^0$ | K | | |

So, in order to have all the roots in LHP, we must choose

$$\begin{cases} \frac{1}{8}(55+K) > 0 \\ K > 0 \\ (1375-350K-K^2)/(275+5K) > 0 \end{cases}$$

$$\Rightarrow \begin{cases} K > -55 \\ K > 0 \\ 1375-350K-K^2 > 0 \end{cases}$$

OR. $K^2 + 350K - 1375 < 0$

$(K+175)^2 - 32000 < 0$

$(K+175)^2 < 32000$

Thus,

$$-178.88544 < K + 175 < 178.88544$$

Hence,

$$-353.88544 < K < 3.88544$$

Finally, we have

$$0 < K < 3.88544$$ Answer.

3.19 A typical transfer function for a tape-drive system would be (with time in milliseconds)

$$G(S) = \frac{K(S+4)}{S[(S+0.5)(S+1)(S^2+0.4S+4)]}$$

From Routh's criterion, what is the range of K for which this system is stable if the characteristic equation is $1+G(S)=0$?

Solution: Characterisitic Equation.

$$Q(S) = S[(S+0.5)(S+1)(S^2+0.4S+4)] + K(S+4)$$

$$= S^5 + 1.9S^4 + 5.1S^3 + 6.2S^2 + (2+K)S + 4K$$

Routh's Table:

| | | | |
|---|---|---|---|
| $S^5$ | 1 | 5.1 | 2+K |
| $S^4$ | 1.9 | 6.2 | 4K |
| $S^3$ | 1.83684 | 2−1.10526K | |
| $S^2$ | 4.13124+1.14326K | 4K | |
| $S^1$ | (6.53884−5.56103K−K²)/massy | | |
| $S^0$ | 4K | | |

Thus,

$$\begin{cases} 4K > 0 \\ 4.13124 + 1.14326K > 0 \\ 6.53884 - 5.56103K - K^2 > 0 \end{cases}$$

$$\begin{cases} K > 0 \\ K^2 + 5.56103K - 6.53884 < 0 \end{cases}$$

Hence

$$(K+2.78052)^2 - 14.27011 < 0$$

$$-3.77758 < K + 2.78052 < 3.77758$$

Thus, we have

$$\boxed{0 < K < 0.99706}$$

✓

3.19+

$$G_c(s) = \frac{K(s+4)}{s^5+1.9s^4+5.1s^3+6.2s^2+(2+K)s+4K}$$

(1) $K = 0.1$

$$G_c(s) = \frac{0.1s+0.4}{s^5+1.9s^4+5.1s^3+6.2s^2+2.1s+0.4}$$

(2) $K=0.3$

$$G_c(s) = \frac{0.3s+1.2}{s^5+1.9s^4+5.1s^3+6.2s^2+2.3s+1.2}$$

(3) $K = 0.5$

$$G_c(s) = \frac{0.5s+2}{s^5+1.9s^4+5.1s^3+6.2s^2+2.5s+2}$$

(4) $K=0.7$

$$G_c(s) = \frac{0.7s+2.8}{s^5+1.9s^4+5.1s^3+6.2s^2+2.7s+2.8}$$

The pole positions. (from MATLAB)

(1) $K = 0.1$

$P_{1,2} = -0.2079 \pm j1.98$ ; $P_{3,4} = -0.1951 \pm j0.2327$, $P_5 = -1.0940$

(2) $K = 0.3$

$P_{1,2} = -0.2252 \pm j1.9599$ ; $P_{3,4} = -0.1243 \pm j0.4912$, $P_5 = -1.2010$

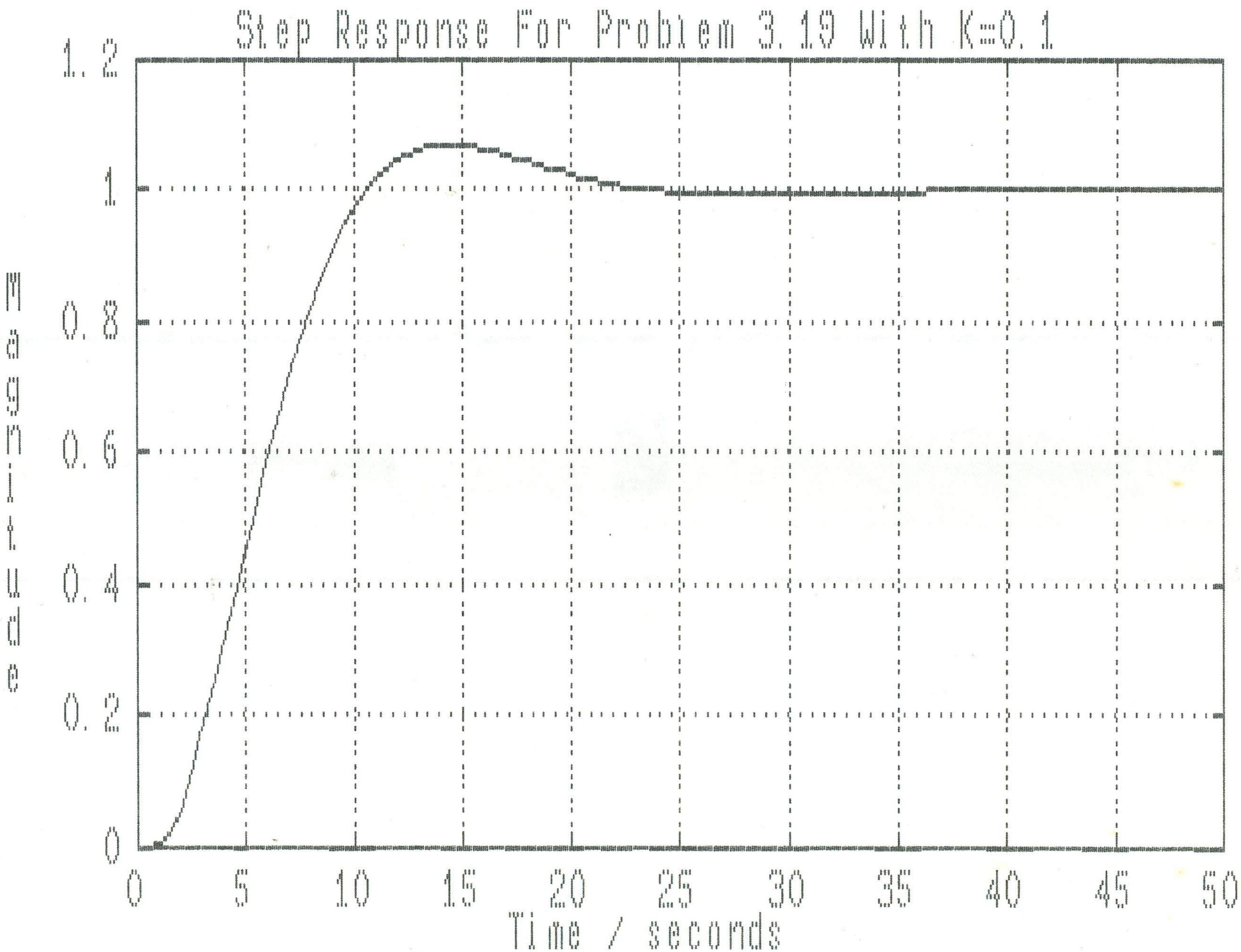
(3) $K = 0.5$

$P_{1,2} = -0.2450 \pm j1.9396$ ; $P_{3,4} = -0.0690 \pm j0.6377$, $P_5 = -1.2719$

(4) $K = 0.7$

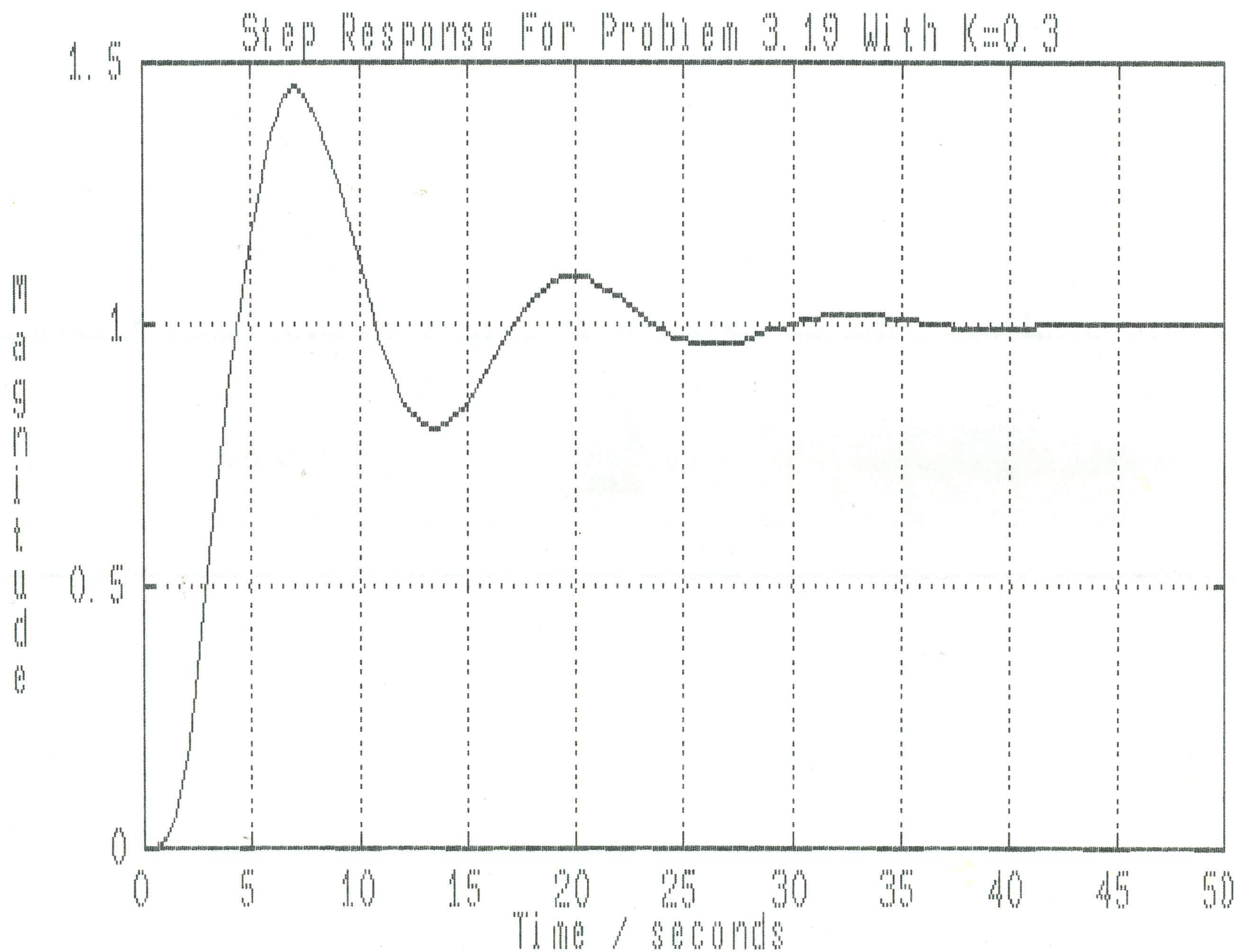
$P_{1,2} = -0.2677 \pm j1.9197$ ; $P_{3,4} = -0.0191 \pm j0.7493$, $P_5 = -1.3265$

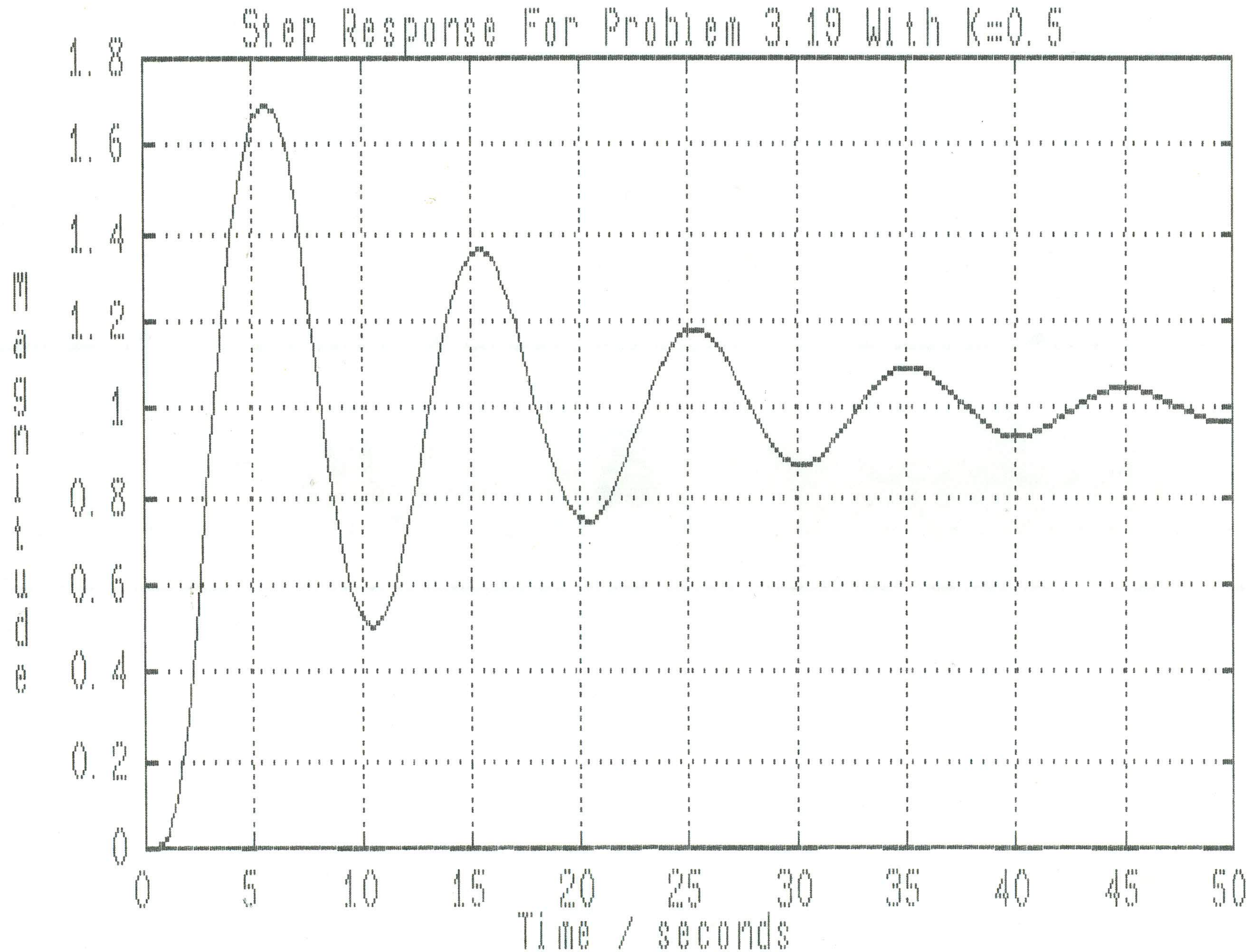
Step responses (see the plots).



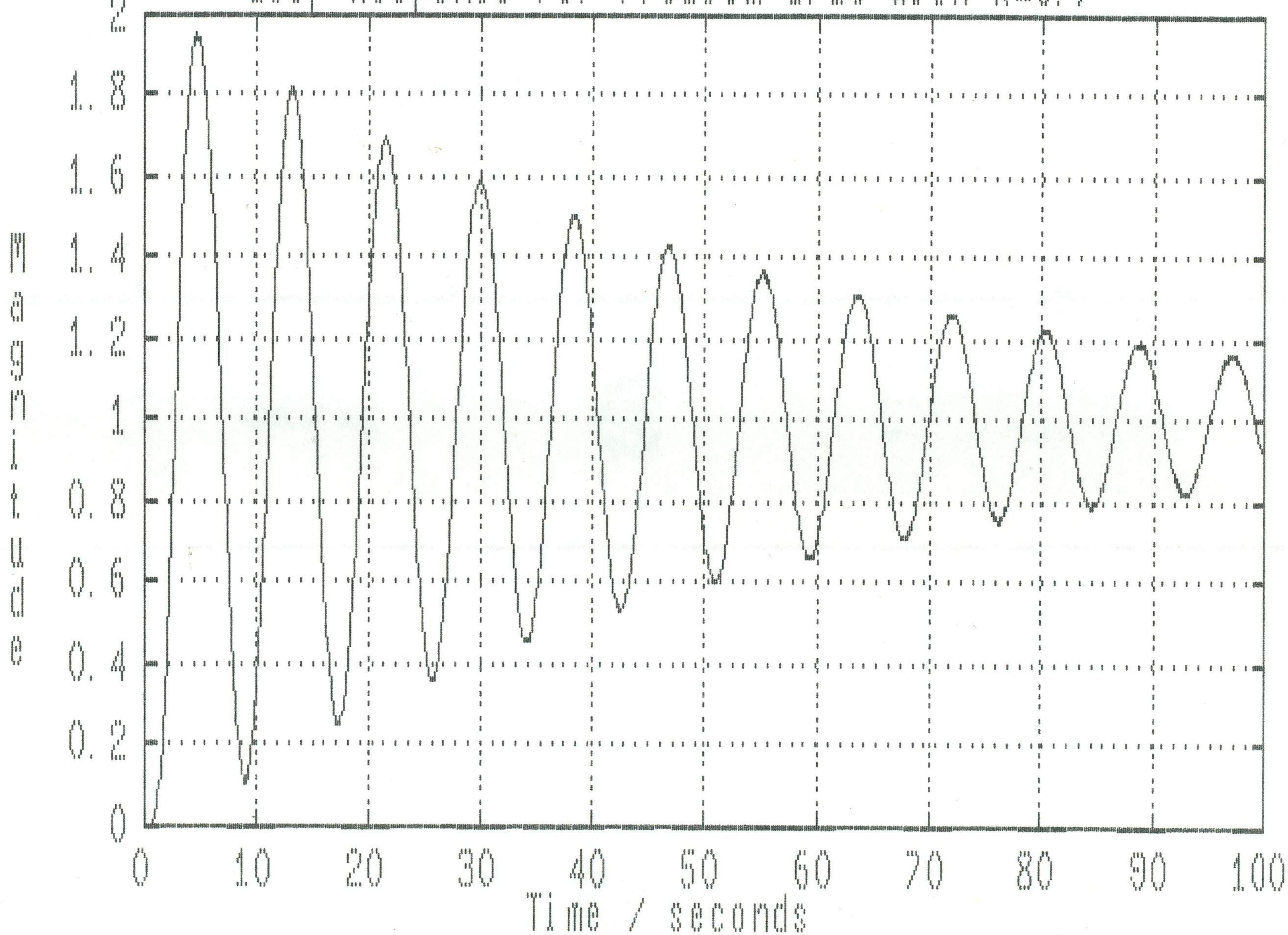
Step Response For Problem 3.19 With K=0.1
Magnitude
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30
35
40
45
50
Time / seconds


18


Step Response For Problem 3.19 With K=0.3
1.5
1
0.5
0
Magnitude
0
5
10
15
20
25
30
35
40
45
50
Time / seconds


Step Response For Problem 3.19 With K=0.5
Magnitude
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30
35
40
45
50
Time / seconds

Step Response For Problem 3.19 With K=0.7



## 3.19⁺ REMARKS

As we see the results obtained from MATLAB.

For the pole positions:

pole pair : $P_{1,2}$ move away slightly from jw axis when K increasing.

pole pair : $P_{3,4}$ move closer to jw axis when K is increased.

And the most important thing is the imagerary parts of poles $P_{3,4}$ increased when K is increased. This is why that we saw more damped in step responses for K=0.5 and 0.7.

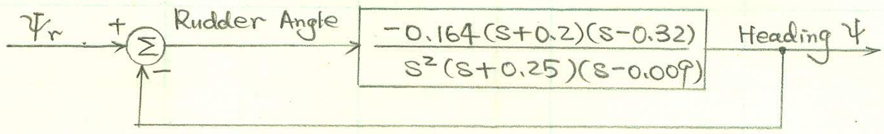
pole $P_5$ : Changed slightly, but inceasing the overshoots.

For the step responses.

From the plots we obtained from MATLAB. we see that percent overshoot is increased ripidly when K is increased. and the settling time is also, increased.

Anyway, the relationship between values K and the step responses can be seen clearly from the plots.

3.20 System:




(a)

$$\frac{\Psi}{\Psi_r}(\text{open}) = \frac{-0.164(s+0.2)(s-0.32)}{s^2(s+0.25)(s-0.009)}$$

$$= \frac{-0.164 s^2 + 0.0196 s + 0.010496}{s^4 + 0.241 s^3 - 0.00225 s^2}$$

Thus, we have

$$\psi^{(4)}(t) + 0.241\,\psi^{(3)}(t) - 0.00225\,\ddot{\psi}(t)$$

$$= -0.164\,\ddot{\psi}_r(t) + 0.0196\,\dot{\psi}_r(t) + 0.0105\,\psi_r(t)$$

(b)

$$G_c(s) = \frac{-0.164 s^2 + 0.0196 s + 0.0105}{s^4 + 0.241 s^3 - 0.00225 s^2 - (0.164 s^2 - 0.0196 s - 0.0105)K}$$

C. E. $Q(s) = s^4 + 0.241 s^3 - (0.164K + 0.00225) s^2$

$+ 0.0196 K s + 0.010496 K$

from the characteristic Equation above

| | | | |
|---|---|---|---|
| $s^4$ | 1 | $-0.00225 - 0.164K$ | $0.010496K$ |
| $s^3$ | 0.241 | $0.0196K$ | |
| $s^2$ | $-0.00225 - 0.24533K$ | | |
| $s^1$ | | | |
| $s^0$ | | | |

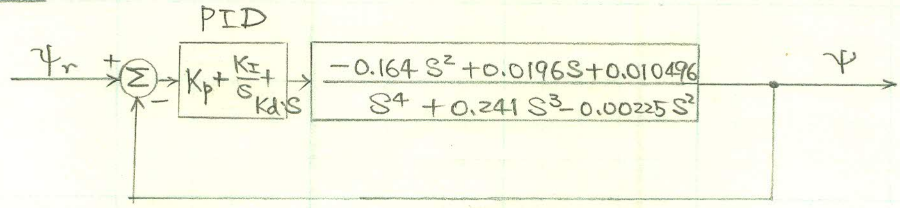
Note: $-0.00225 - 0.24533K < 0$ if $K > 0$

So the control system is unstable when $K = 1$

(c) No. As we have seen above. $-0.00225 - 0.24533K < 0$ for any $K > 0$.

(d) my seggestion control scheme is following:

3.20 (d)




$$G_c(S) = \frac{(K_d S^2 + K_p S + K_i)(-0.164 S^2 + 0.0196 S + 0.010496)}{\text{C.E.}}$$

C.E. $Q(s) = S^5 + 0.241 S^4 - 0.00225 S^3$

$$- 0.164 K_d S^4 + 0.0196 K_d S^3 + 0.010496 K_d S^2$$
$$- 0.164 K_p S^3 + 0.0196 K_p S^2 + 0.010496 K_p S$$
$$- 0.164 K_i S^2 + 0.0196 K_i S + 0.010496 K_i$$

$$= S^5 + (0.241 - 0.164 K_d) S^4 + (0.0196 K_d - 0.164 K_p - 0.00225) S^3$$
$$+ (0.010496 \cdot K_d + 0.0196 K_p - 0.164 K_i) S^2$$
$$+ (0.010496 K_p + 0.0196 K_i) S + 0.010496 K_i$$

This method is too massy. By chance, we try

$K_d = 1$, $K_p = 0.01$ $K_i = 0.05$

Subtitude these values, we have C.E.

$$Q(S) = S^5 + 0.077 S^4 + 0.01571 S^3 + 0.002492 S^2 + 0.00108496 S + 0.0005248$$

From MATLAB, we have closed-loop poles

$P_{1,2} = -0.0898 \pm j\, 0.2049$

$P_{3,4} = -0.1564 \pm j\, 0.1593$ $P_5 = -0.2103$

All the poles lie on the LHP, so the system is

Stable. ✓

A

**4.3** Set up the following characteristic equations in the form suited to the Evans root-locus method:

a) $s + (1/\tau) = 0$ versus $\tau$;

Set-up: $1 + \tau s = 0$

b) $s^2 + bs + b + 1 = 0$ versus $b$;

$$\Rightarrow \quad 1 + \frac{b(s+1)}{s^2+1} = 0$$

c) $(s+b)^3 + A(Ts+1) = 0$;

(1) versus $A$; $\quad 1 + A \cdot \dfrac{Ts+1}{(s+b)^3} = 0$

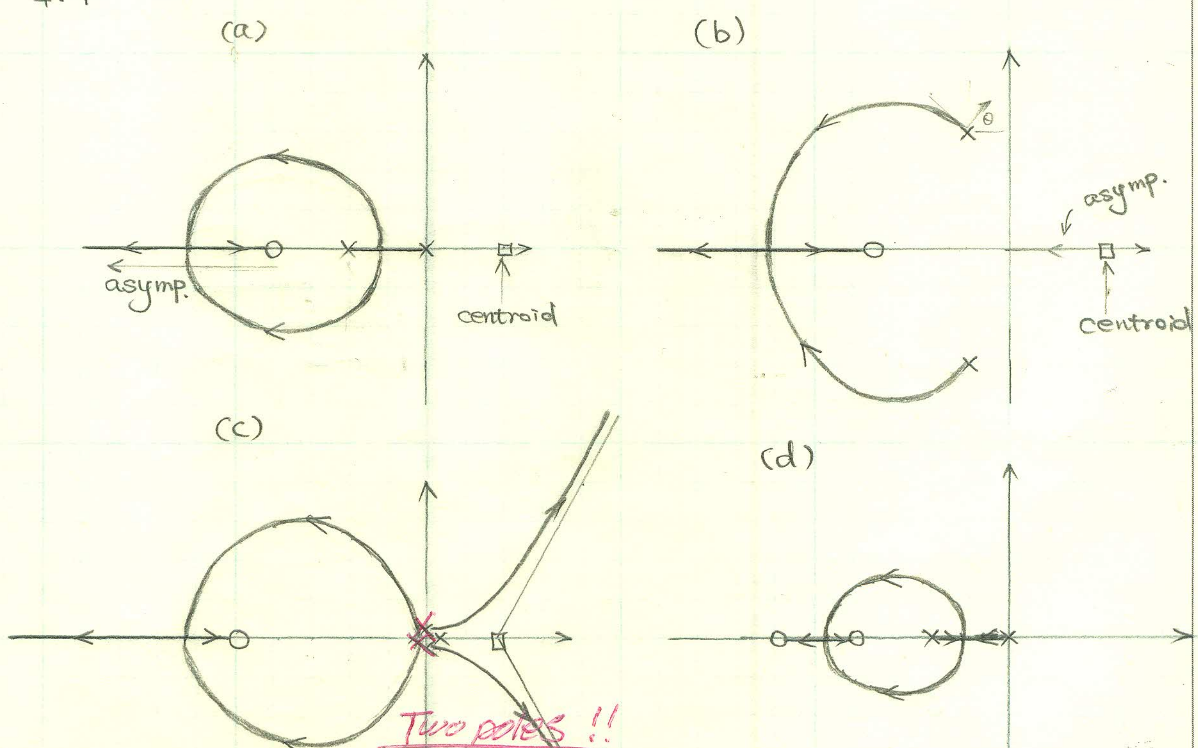
(2) versus $T$; $\quad 1 + T \cdot \dfrac{A \cdot s}{(s+b)^3 + A} = 0$

(3) No, because b is not linear in the C.E..

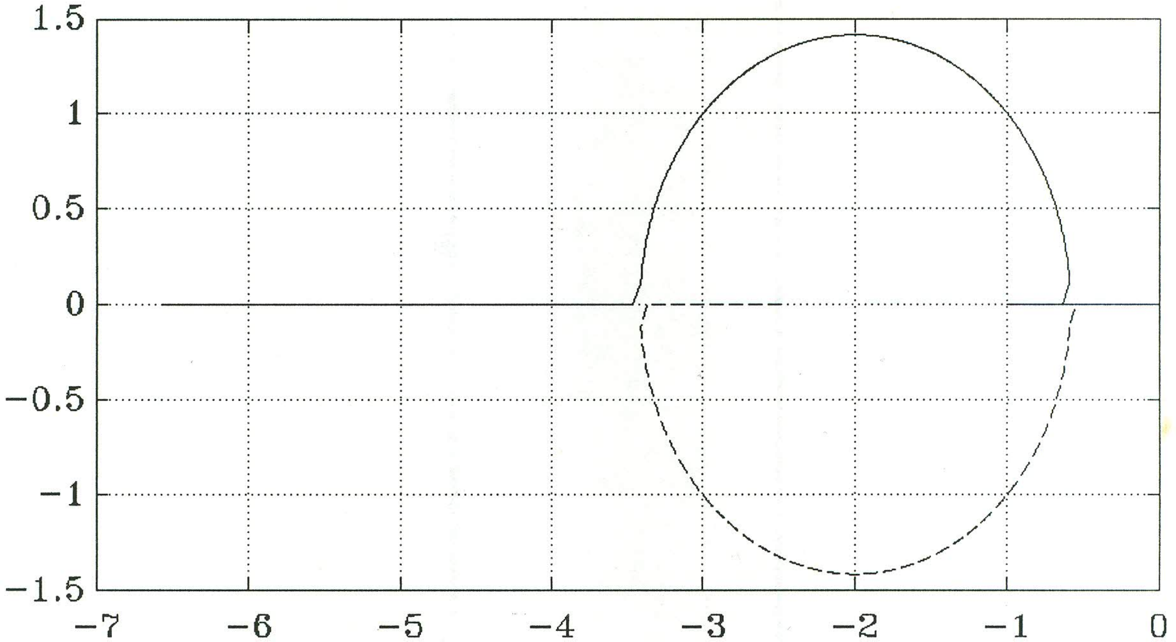
However, a locus can be drawn versus b for given of A and T. (pick a value from $0 \to \infty$ compute the roots and mark on s-plane)

---

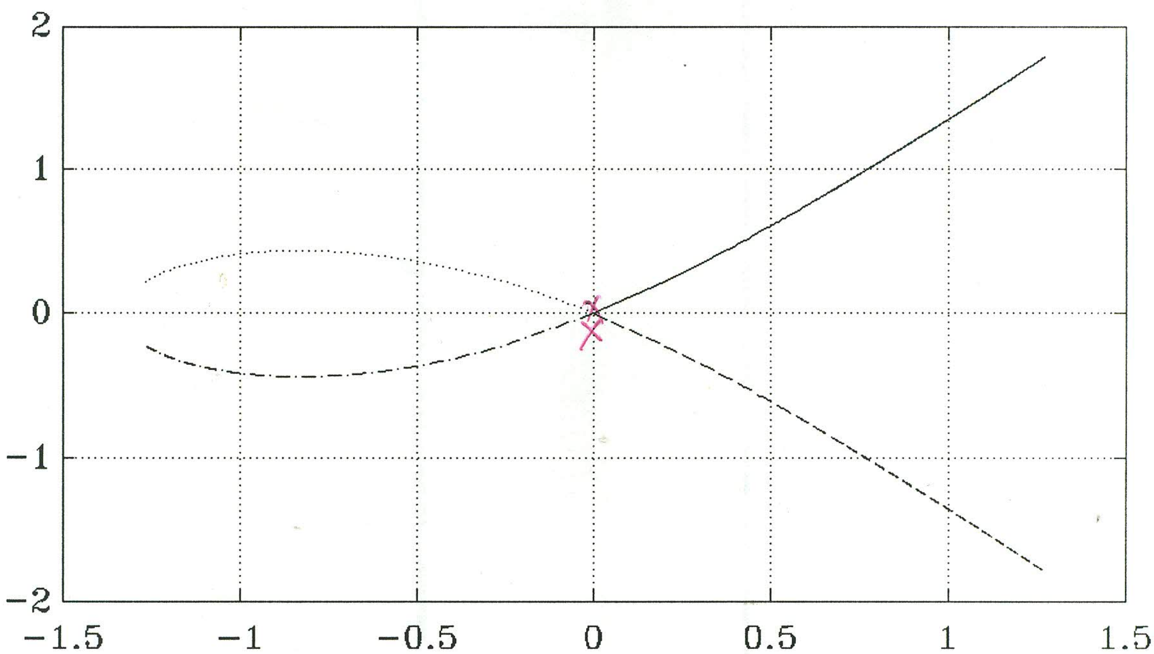
**4.4**




$n = m \Rightarrow$ no asymptote.



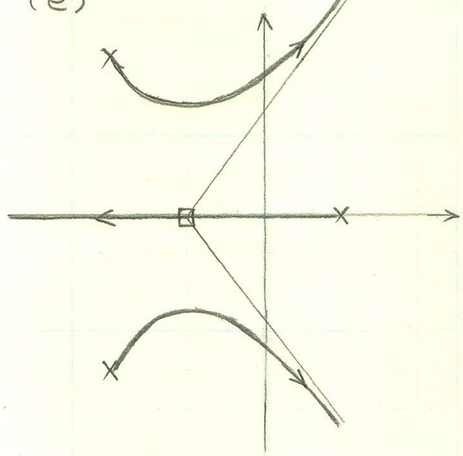

ROOT LOCUS FOR PROBLEM 4.4 a



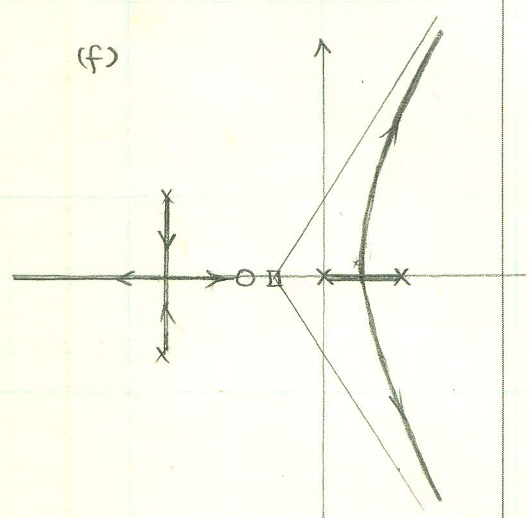

ROOT LOCUS FOR PROBLEM 4.4 c

4.4 (CONT.)

(e)



(f)



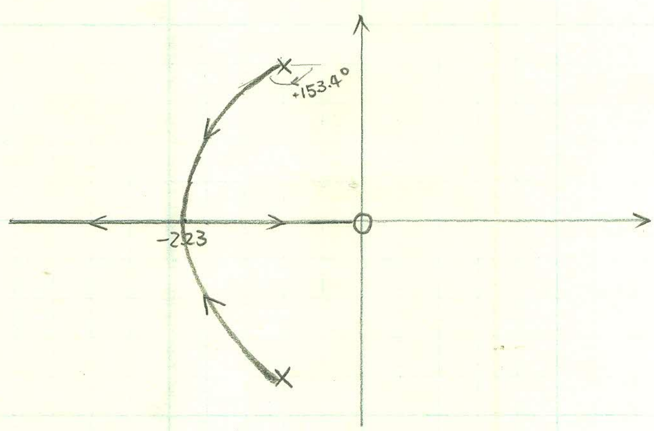
See the sketches from MATLAB for valification.

4.8 Sketh the root locus with respect to $\alpha$. Find step response for $\alpha = 0, 0.5, 2$

$$G_c = \frac{5}{s(s+2)} \Big/ \left[ 1 + (1+\alpha s) \cdot \frac{5}{s(s+2)} \right] = \frac{5}{(s^2+2s+5) + 5\alpha s}$$

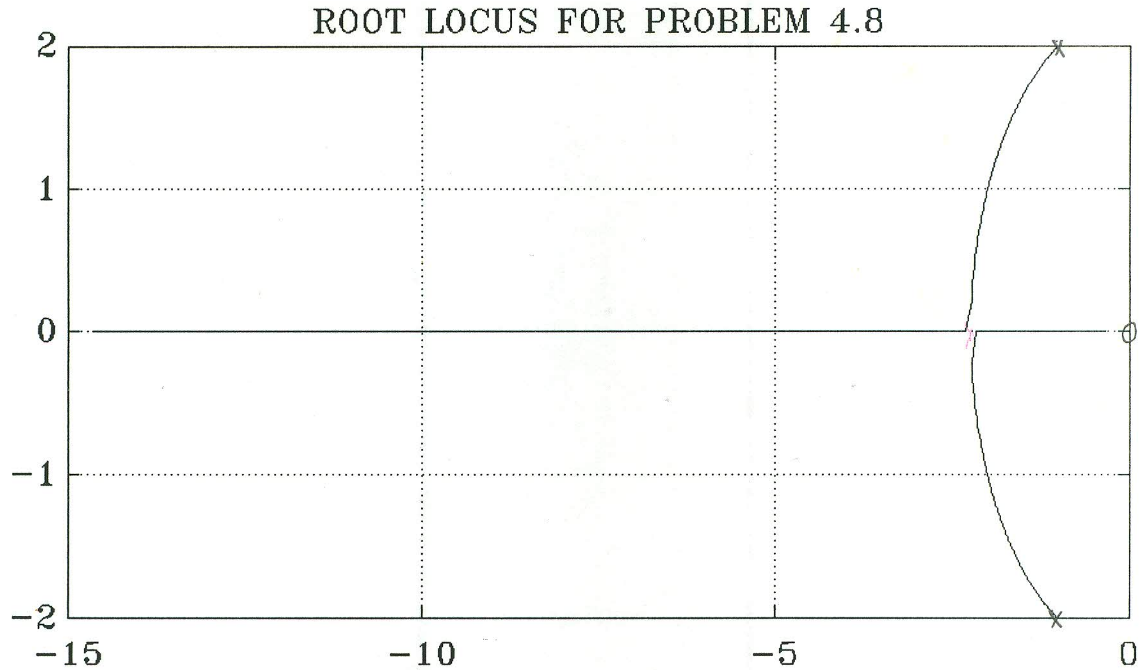
$$\Rightarrow \text{C.F.} = (s^2+2s+5) + 5\alpha \cdot s$$

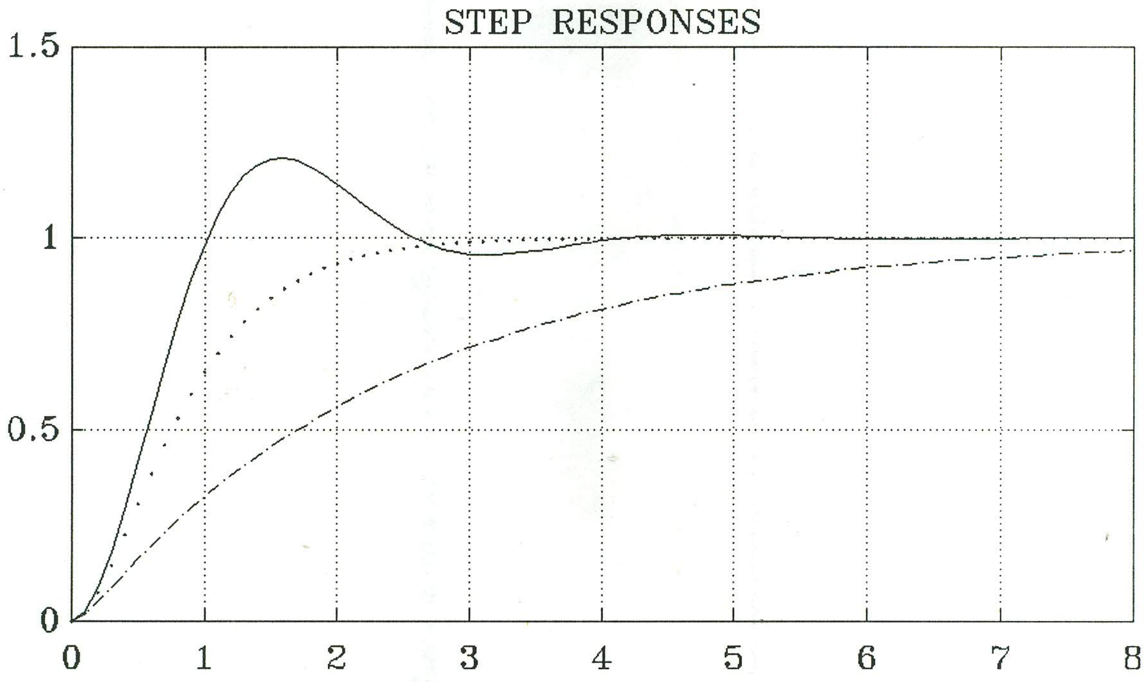
Set-up. $$1 + \alpha \cdot \frac{5s}{s^2+2s+5} = 0$$




See following pages for exact root locus from MATLAB

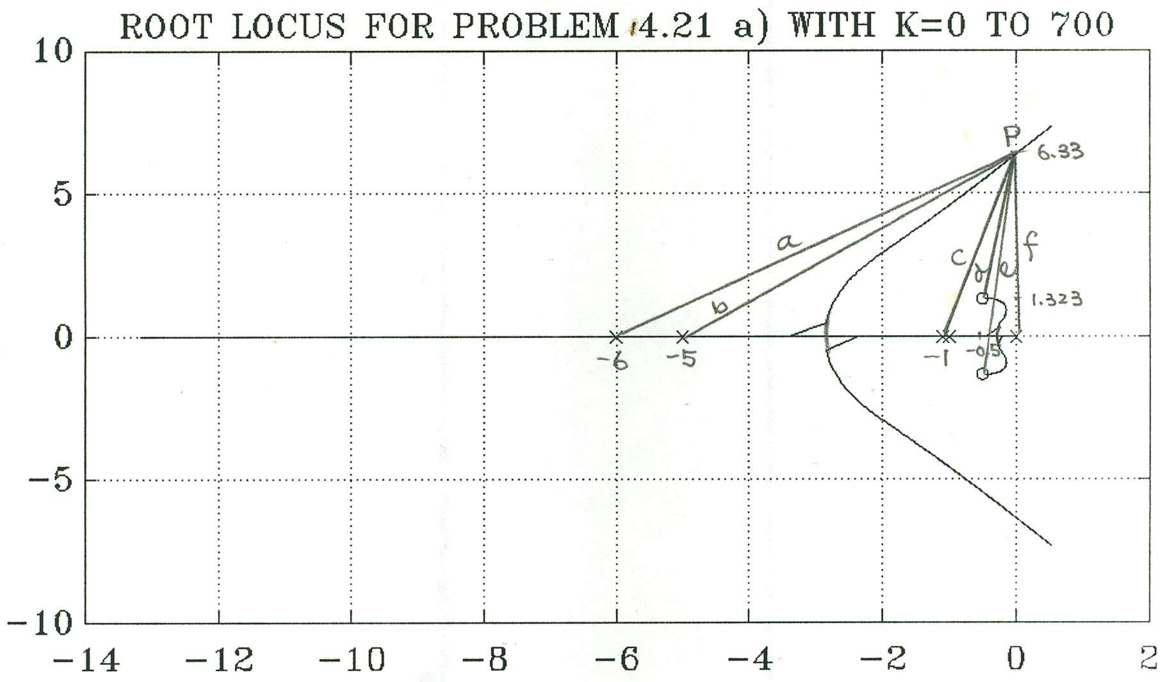
And step response for $\alpha = 0, 0.5, 2$.




Highest Mp for a=0. Mid. for a=.5 Lowest for a=2.5

PROBLEM 4.21 a)




Solution: from the root locus obtained from MATLAB, we can compute :

$a = \sqrt{6.33^2 + 6^2} = 8.72$ ; $b = \sqrt{6.33^2 + 5^2} = 8.066$

$C = \sqrt{6.33^2 + 1^2} = 6.41$ ; $d = \sqrt{0.5^2 + (6.33 - 1.323)^2} = 5.03$

$e = \sqrt{0.5^2 + (6.33 + 1.323)^2} = 7.67$ ; $f = 6.33$

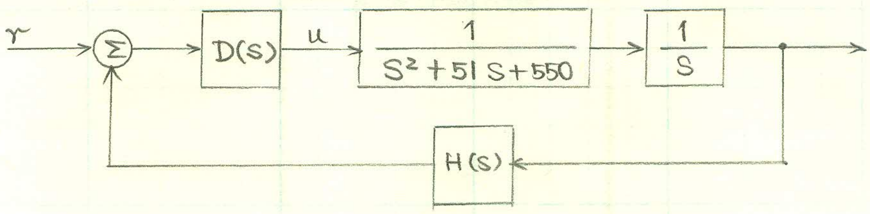
Then for the K at point P

$$K \doteq \frac{abc^2f}{de} = \frac{8.72 \times 8.066 \times (6.41)^2 \times 6.33}{5.03 \times 7.67} \approx 474.$$

The actual value of $K = 462$, computation error $= 2.3\%$

Thus we conclude system stable if $0 < K < 474$ $(1-5\%)$.

4.42



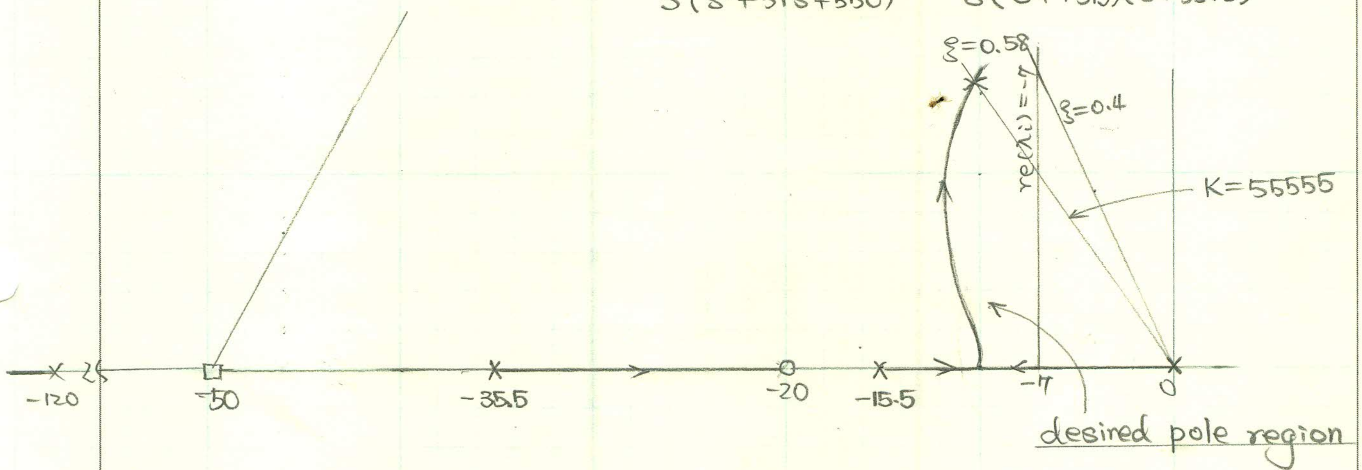

Case I: Lead Network

$$H(s)=1 \qquad D(s)=K\cdot\frac{s+z}{s+p} \quad \text{and} \quad \frac{p}{z}=6$$

spec. : $\xi \geq 0.4$ , $re(\lambda_i) \leq -7$ , $K_v \geq 16\frac{2}{3}$

Find : z and K.

$$G_c(S)=\frac{1}{S(S^2+51S+550)}=\frac{1}{S(S+15.5)(S+35.5)}$$




from $K_v=\lim_{s\to 0} s\cdot KG(s)\cdot\frac{z}{p}=\frac{K}{550\times 6}\geq 16\frac{2}{3} \Rightarrow \underline{K\geq 55000}$

If we place the zero between (−15.5, 0), we are not able to have two dominant roots nearest origin. So put $z=20, \Rightarrow p=120$

pick K = 55555 subtitute into C.E., we have

$$C.E.=S(S+51S+550)(S+120)+55555(S+20)$$
$$=S^4+171S^3+6670S^2+121555S+1111100$$

From MATLAB, we have poles at:

$P_1=-124.78$ , $P_2=-24.09$ , $\boxed{P_{3,4}=-11.07\pm j15.72}$

Check conditions: $\underline{Re(P_{3,4})=-11.07\leq -7}$ . $\underline{K_v=16.83\geq 16\frac{2}{3}}$

$\xi=11.07/\sqrt{11.07^2+15.72^2}=0.57576>0.4$ O.K.

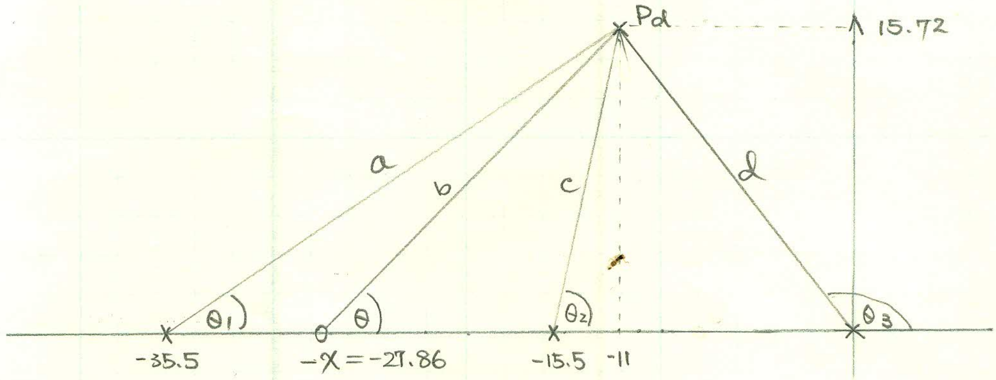
Case II. desired poles $P_{3,4} = -11.07 \pm 15.72$

$H(s) = 1 + K_T s$ and $D(s) = K$

$$T(s) = \frac{D(s) G_c(s)}{1 + H(s)D(s)G_c(s)}$$

$$C.E. = 1 + H(s)D(s)G_c(s) = 1 + (1 + K_T s) K \cdot \frac{1}{s(s^2+51s+550)}$$

$$= 1 + K \cdot K_T \cdot \frac{s + 1/K_T}{s(s+15.5)(s+35.5)}$$




Compute :

$\theta_1 = 32.71°$ ; $\theta_2 = 74.09°$ ; $\theta_3 = 125.03°$

$$\theta_1 + \theta_2 + \theta_3 - \theta = 32.71° + 74.09° + 125.03° - \theta$$

$$= 231.83° - \theta = -180° \Rightarrow \theta = 51.83°$$

$$\Rightarrow x = \frac{1}{K_T} = 27.8571 \Rightarrow \underline{K_T = 0.0359}$$

Compute :

$a = 29.09$ , $b = 23.03$ , $c = 16.35$ , $d = 19.20$

$$\Rightarrow K K_T = \frac{acd}{b} = \frac{29.09 \times 16.35 \times 19.20}{23.03} = 396.48.$$

$$\Rightarrow \underline{K = 396.48 / K_T = 11044.}$$

$$e(s) = (1 - T(s))R(s) = \frac{1 + D(s)G_c(s)(-1 + H(c))}{1 + H(s)D(s)G_c(s)} R(s)$$

$$= \frac{1 + K_T s K G_c(s)}{1 + (1 + K_T s) K G_c(s)} \cdot \frac{1}{s^2}$$

$$e_{ss} = \lim_{s \to 0} s \cdot e(s) = (1 + K_T K/550)/K/550 = 0.0857$$

$\Rightarrow \underline{K_v = 11.67}$ , Decreased in $K_v$ because we are comparing the input to output, as well as the rate of output changes.

4.42 Case III : Lag Network

(i) For the proportional control, we use

$$D(s) = K.$$

Then $$T_p(s) = \frac{K}{s^3 + 51s^2 + 550s + K}$$

$$E(s) = (1 - T_p(s)) \cdot R(s)$$

$$= \frac{(s^2 + 51s + 550)\cdot s}{s^3 + 51s^2 + 550s + K} \cdot R(s)$$

due to $R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s\to 0} s \cdot E(s) = \frac{550}{K} = \frac{1}{12} \quad \Rightarrow \quad K = 6600$$

Although it is useless in this problem, it is easy, however, to find the closed-loop poles for proportional control are

$$P_1 = -41.5915 \quad , \quad P_{23} = -4.7 \pm 11.69$$

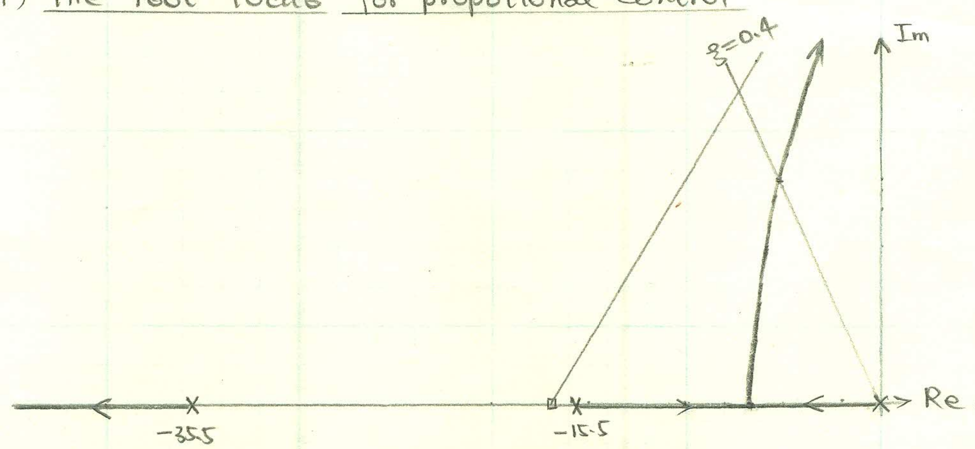
(ii) Adding a lag network, then

$$D(s) = K \cdot \frac{s+1}{s+p}$$

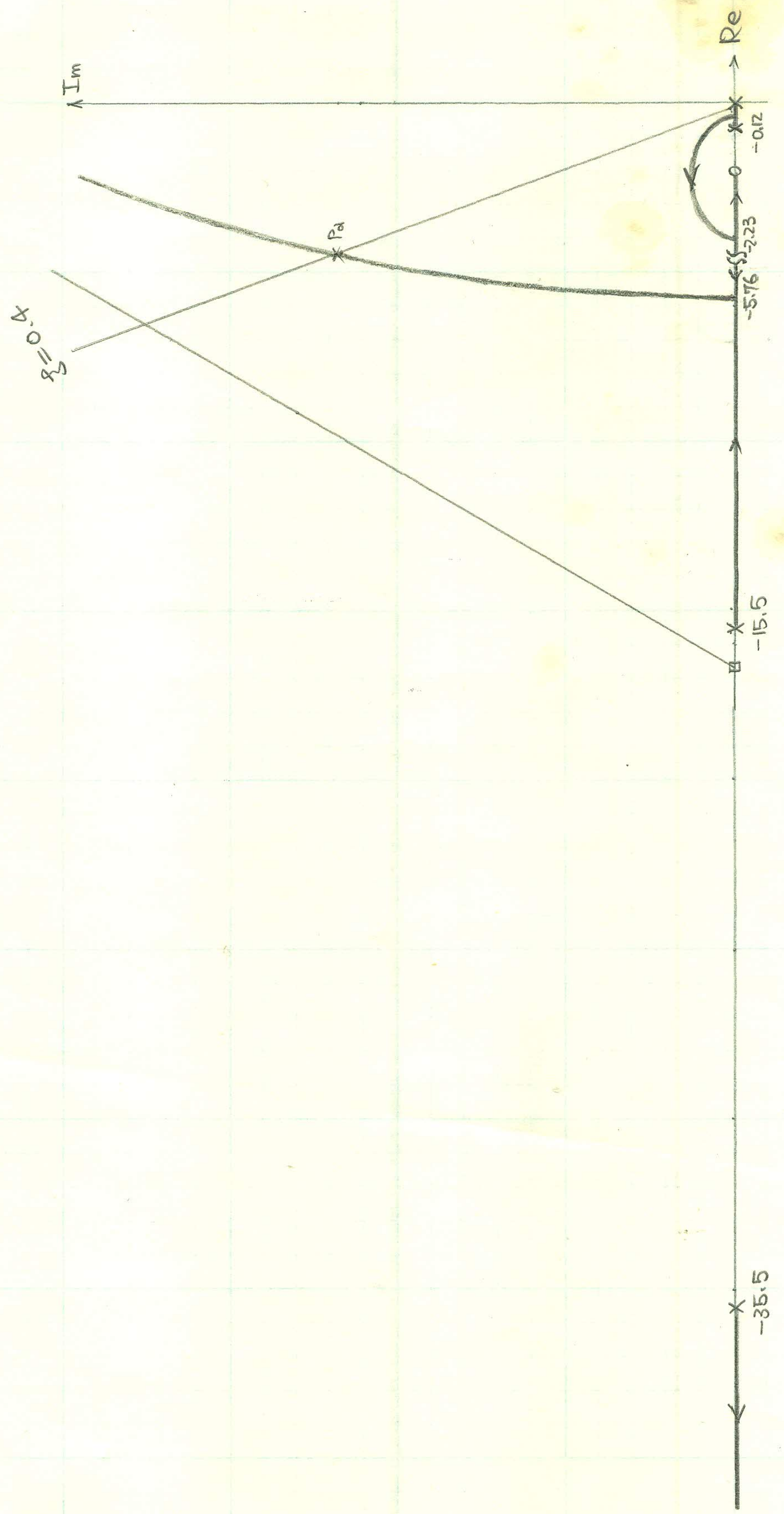
From Lecture, we know that

$$\frac{K_v \text{ compensated}}{K_v \text{ uncompensated}} = \frac{z}{p} = \frac{1}{p} = \frac{100}{12} \quad \Rightarrow \quad p = 0.12 .$$

(iii) The root locus for propotional control



The root-locus for compansated system.



A

**5.1** Draw the Bode plots for a normalized second-order system with $\xi = 0.5$ and an added zero. Do the plots lead one to expect extra transient overshoot? How?

$$G(s) = \frac{(S/z) + 1}{S^2 + S + 1}$$

Let $z = \alpha/2$ and plot for $\alpha = 0.1$, 1 and 10.

---

**Solution:** Refer to the plots given on next page. we can see clearly that Mr is decreasing when $\alpha$ is increased. Thus the overshoot is decresing when $\alpha$ is increased.

---

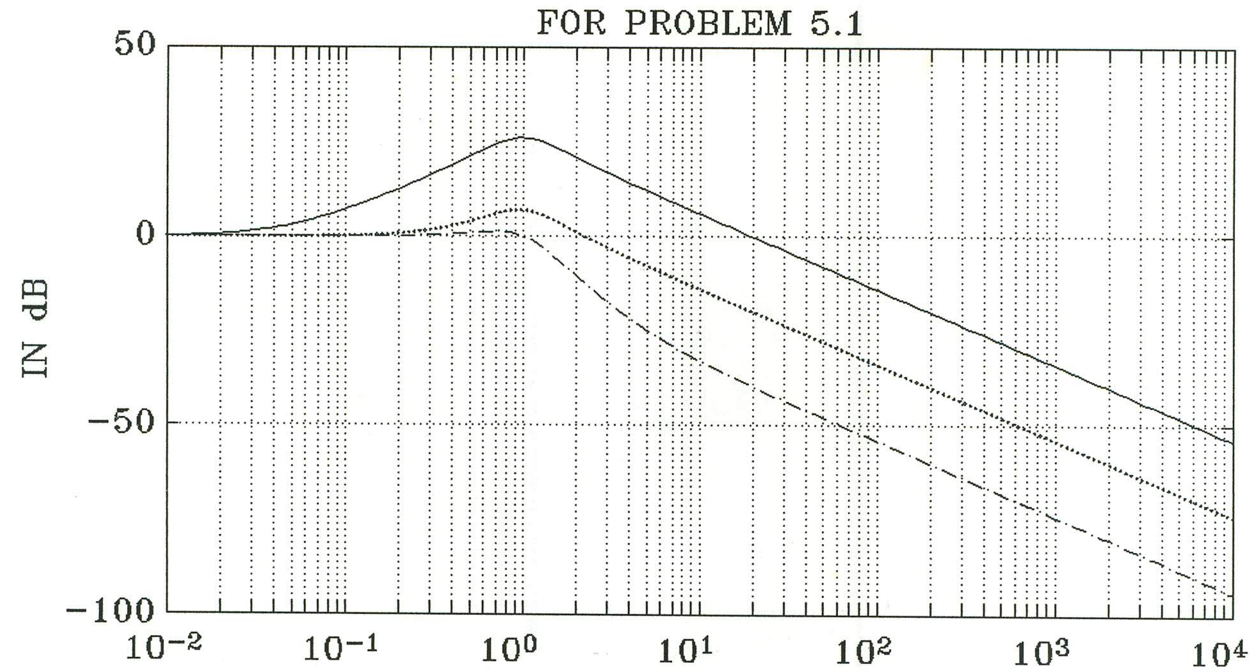
**5.2** Draw the Bode plot for a normalized second-order system with $\xi = 0.5$ and an added pole. Do the plots lead one to expect additional rise time? How?

$$G(s) = \frac{1}{[(S/p)+1](S^2 + S + 1)}$$

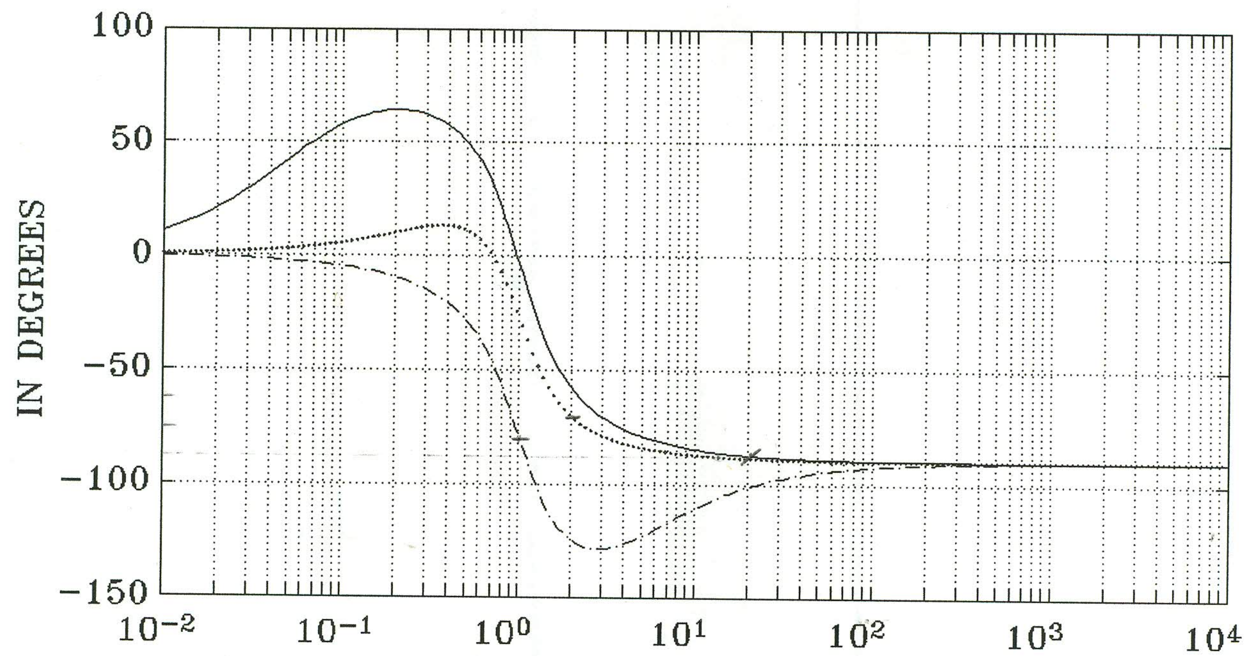
Let $p = \alpha/2$ and plot for $\alpha = 0.1$, 1 and 10.

---

**Solution:** Refer to the Bode's plots given on page 3. the Bandwidthes of the system is increasing when $\alpha$ is in~~de~~creasing. Hence, the rise time is decreasing.

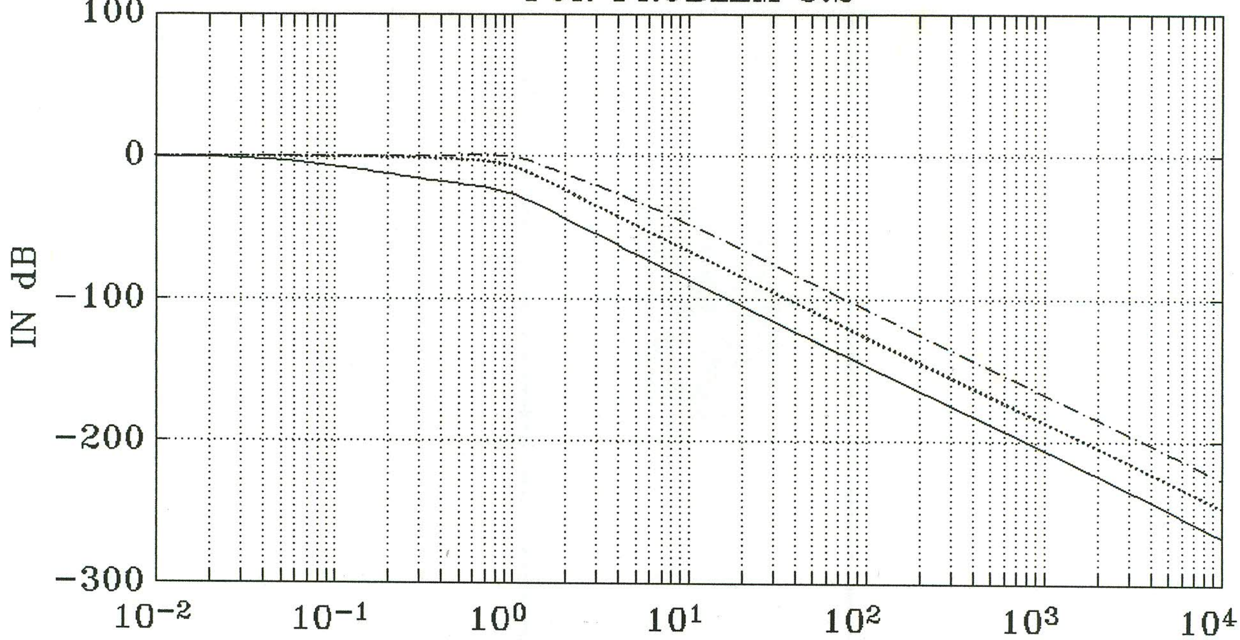



a=.1 with - & a=1 with . & a=10 with -.

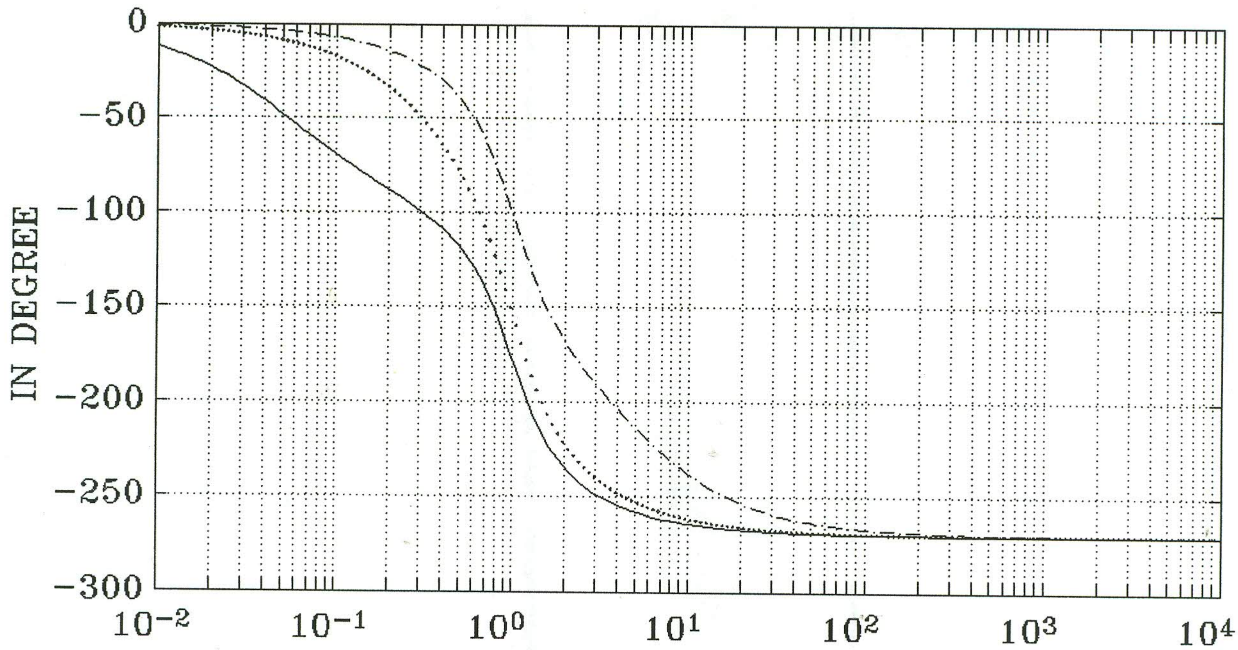



a=.1 with - & a=1 with . & a=10 with -.

FOR PROBLEM 5.2

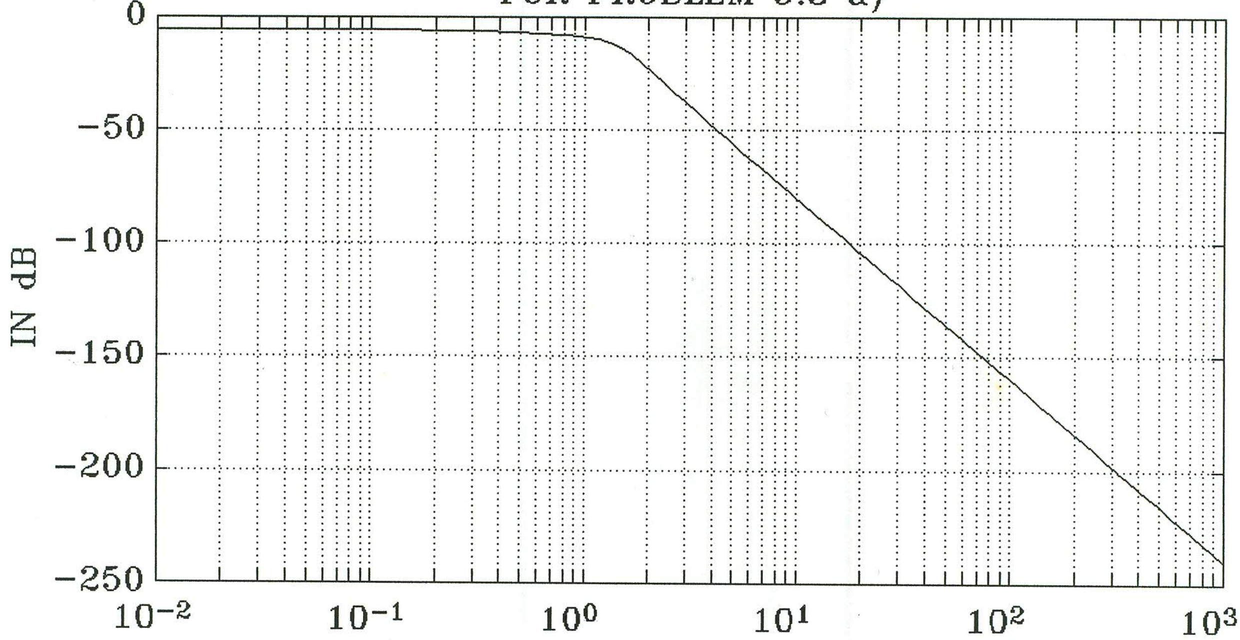



a=.1 with - & a=1 with . & a=10 with -.

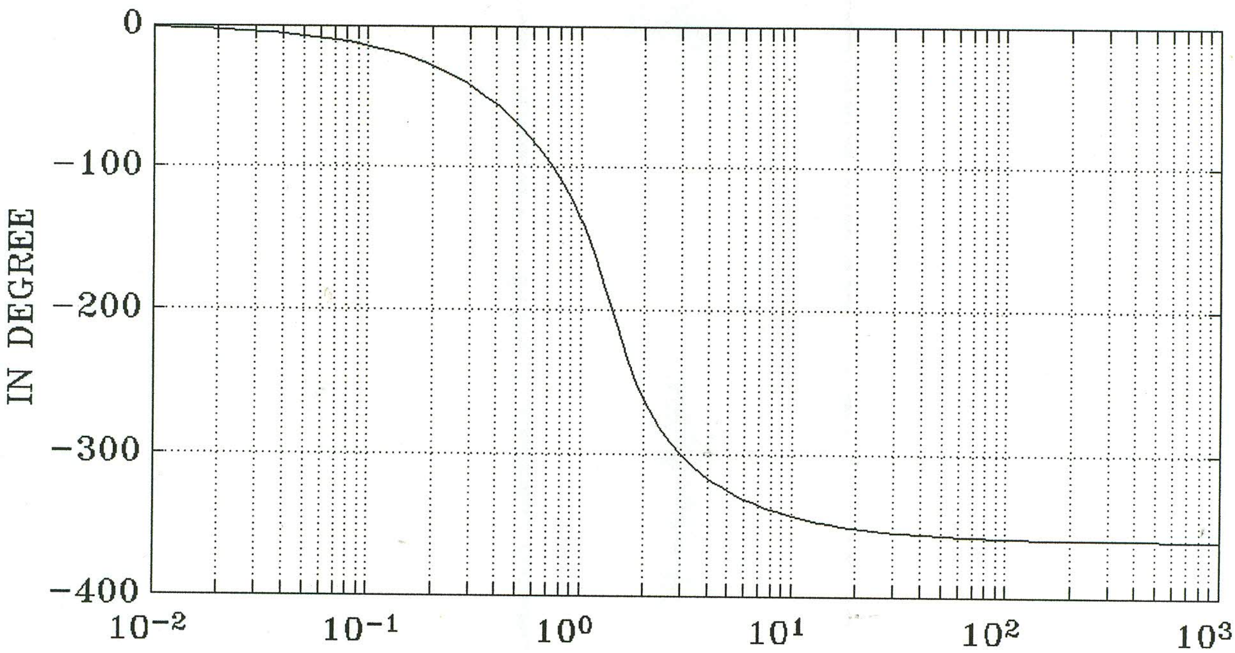



a=.1 with - & a=1 with . & a=10 with -.

FOR PROBLEM 5.3 a)

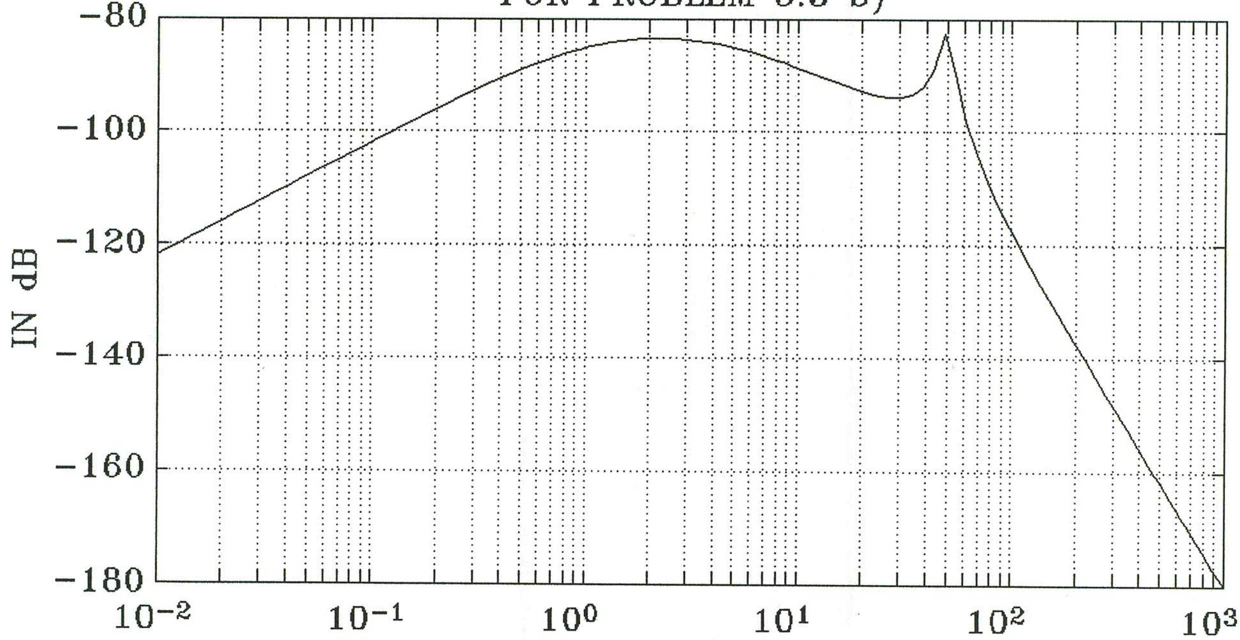



MAGNITUDE PLOT

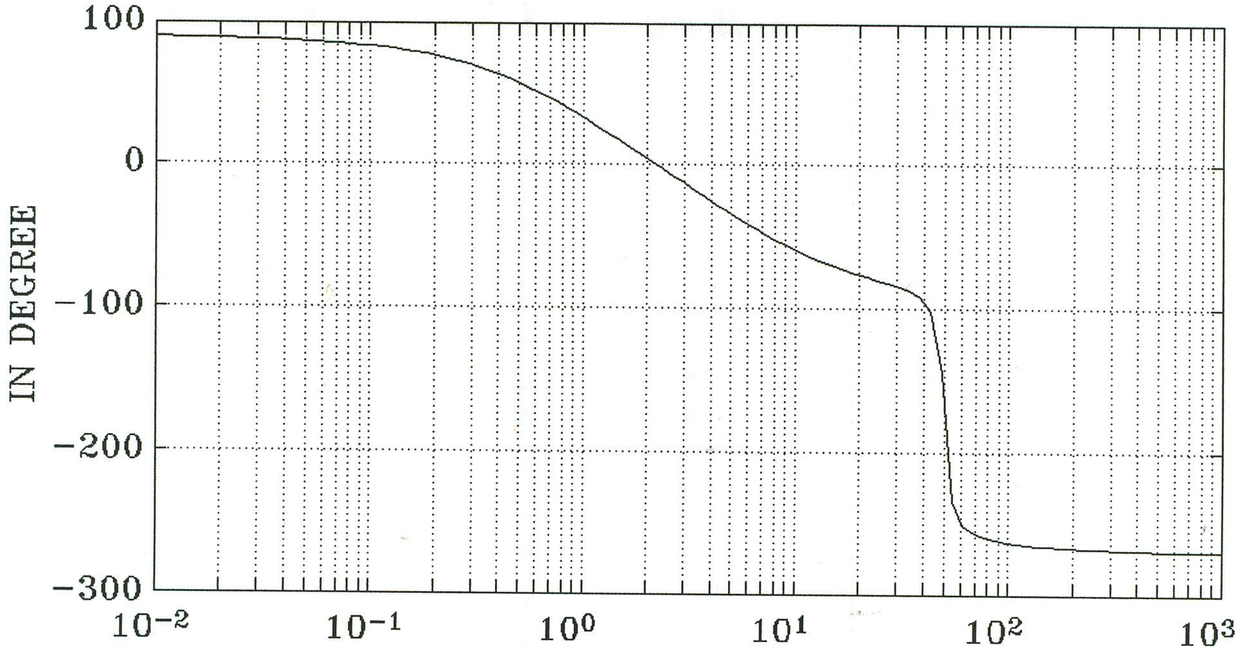



PHASE PLOT

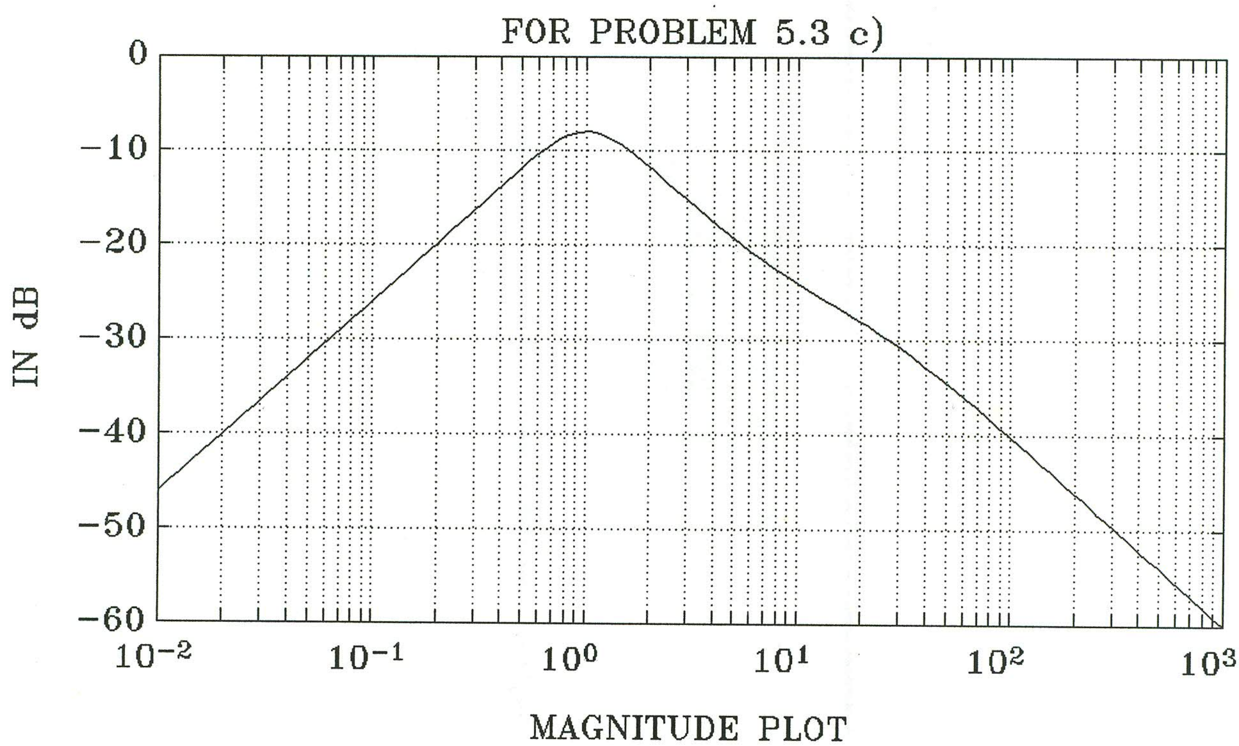
FOR PROBLEM 5.3 b)

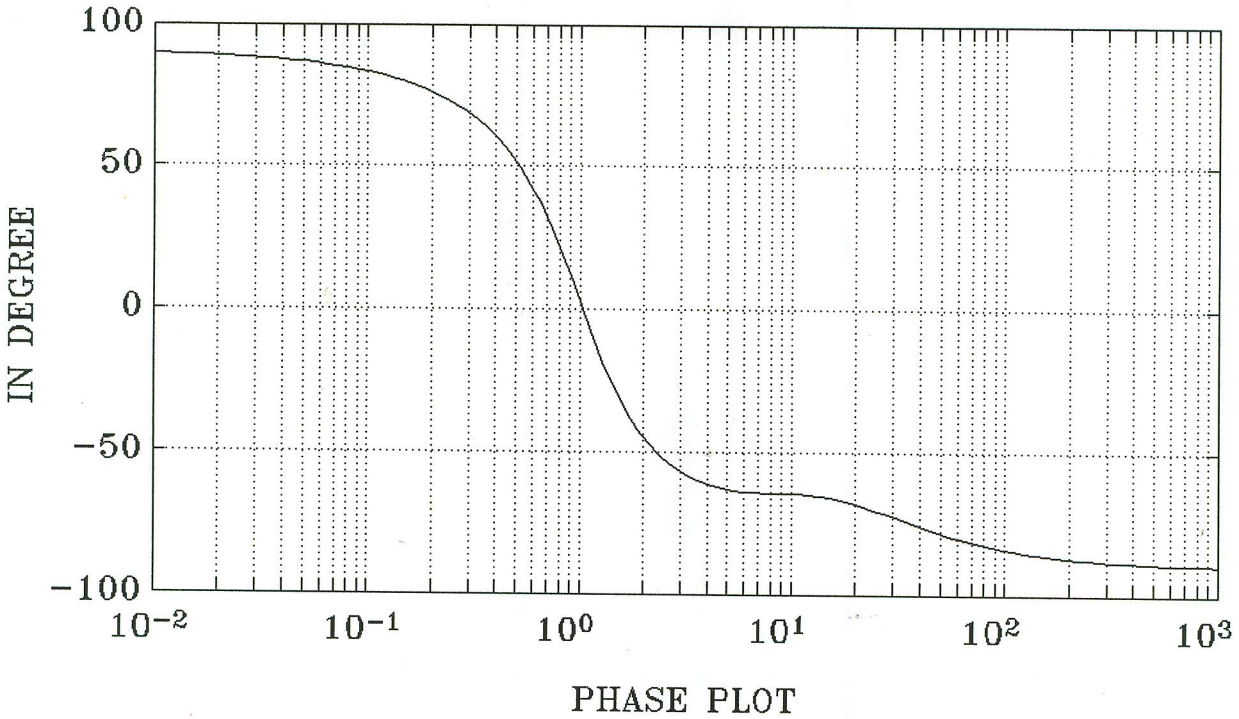



MAGNITUDE PLOT

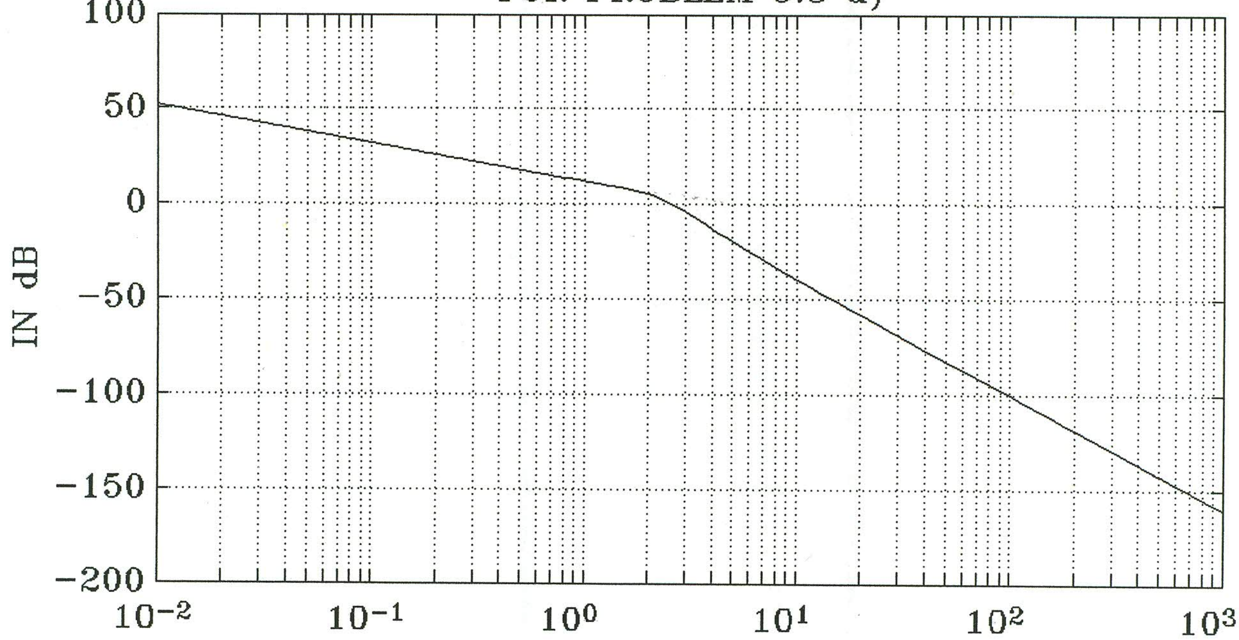



PHASE PLOT

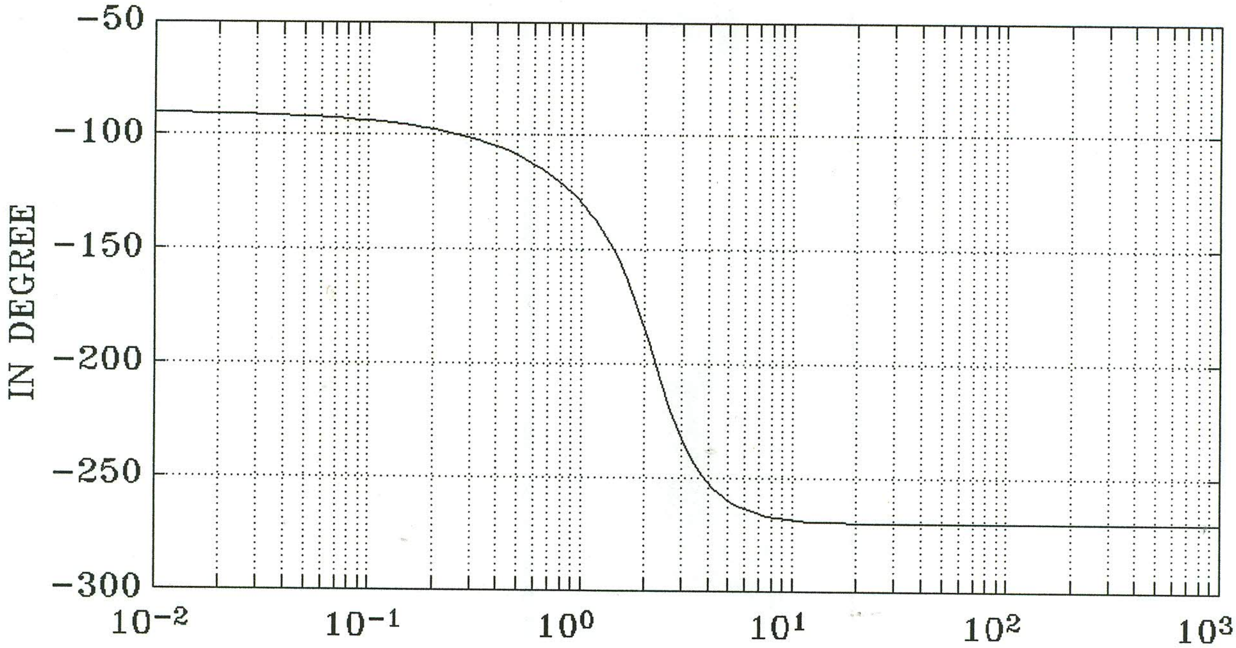

FOR PROBLEM 5.3 c)
0
−10
−20
−30
−40
−50
−60
IN dB
$10^{-2}$
$10^{-1}$
$10^{0}$
$10^{1}$
$10^{2}$
$10^{3}$
MAGNITUDE PLOT


100
50
0
−50
−100
IN DEGREE
$10^{-2}$
$10^{-1}$
$10^{0}$
$10^{1}$
$10^{2}$
$10^{3}$
PHASE PLOT

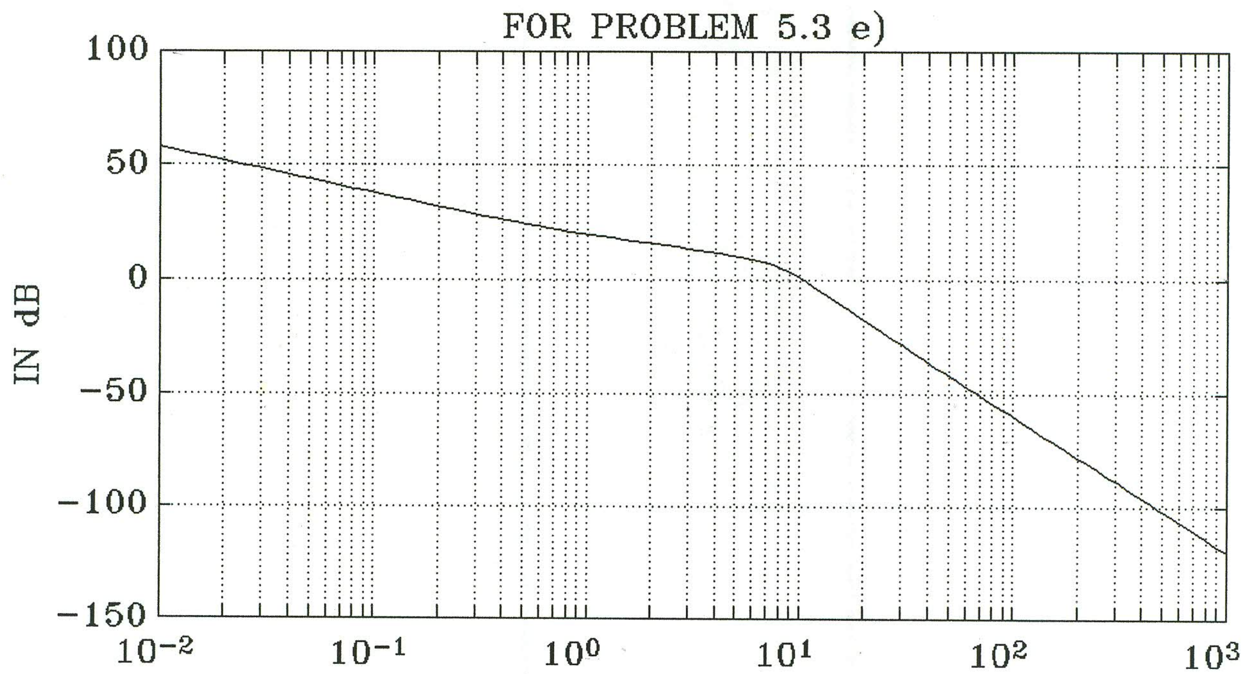
FOR PROBLEM 5.3 d)



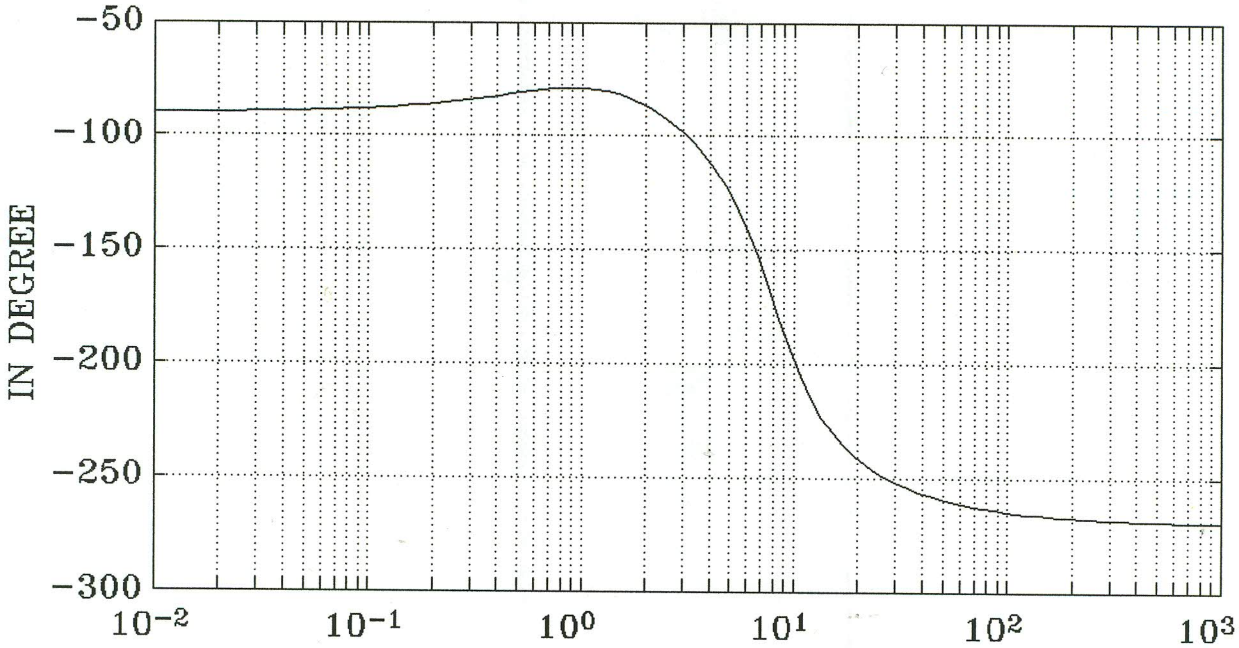

MAGNITUDE PLOT




PHASE PLOT

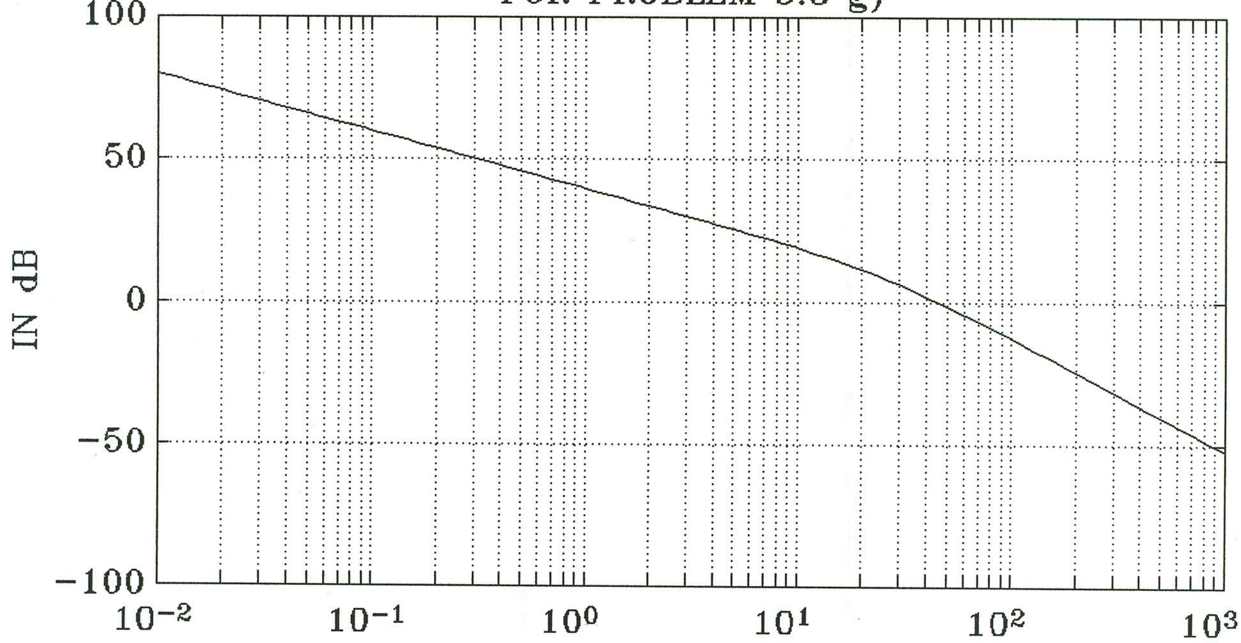


MAGNITUDE PLOT

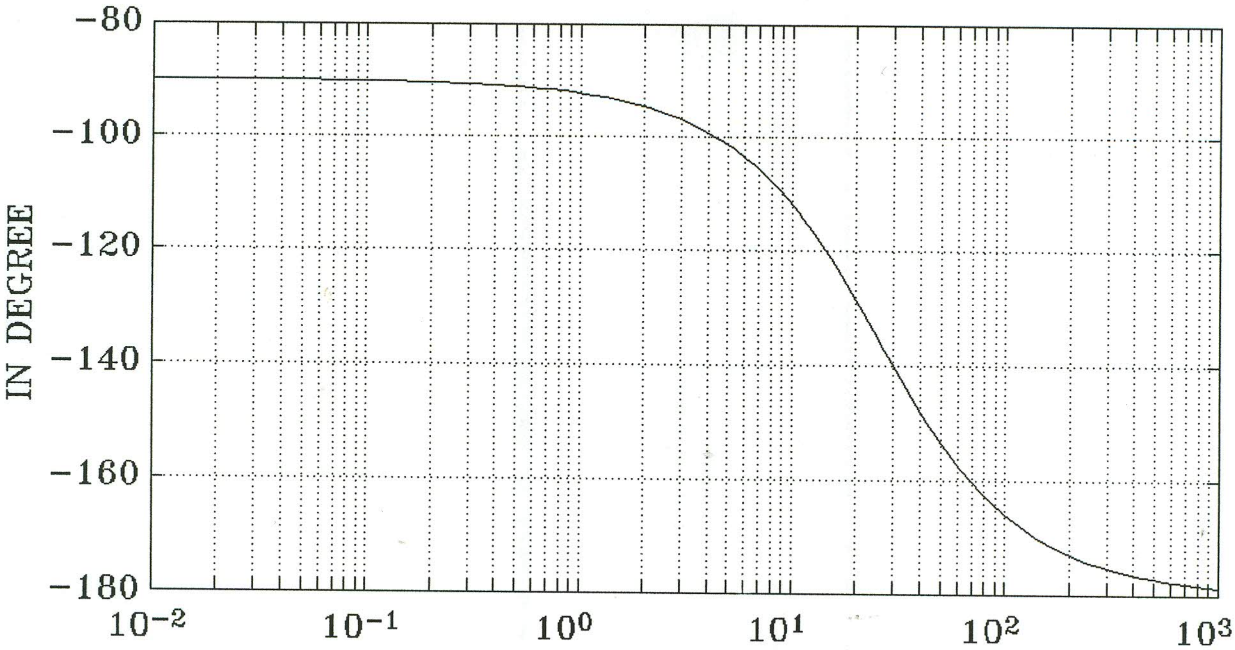


PHASE PLOT

FOR PROBLEM 5.3 g)

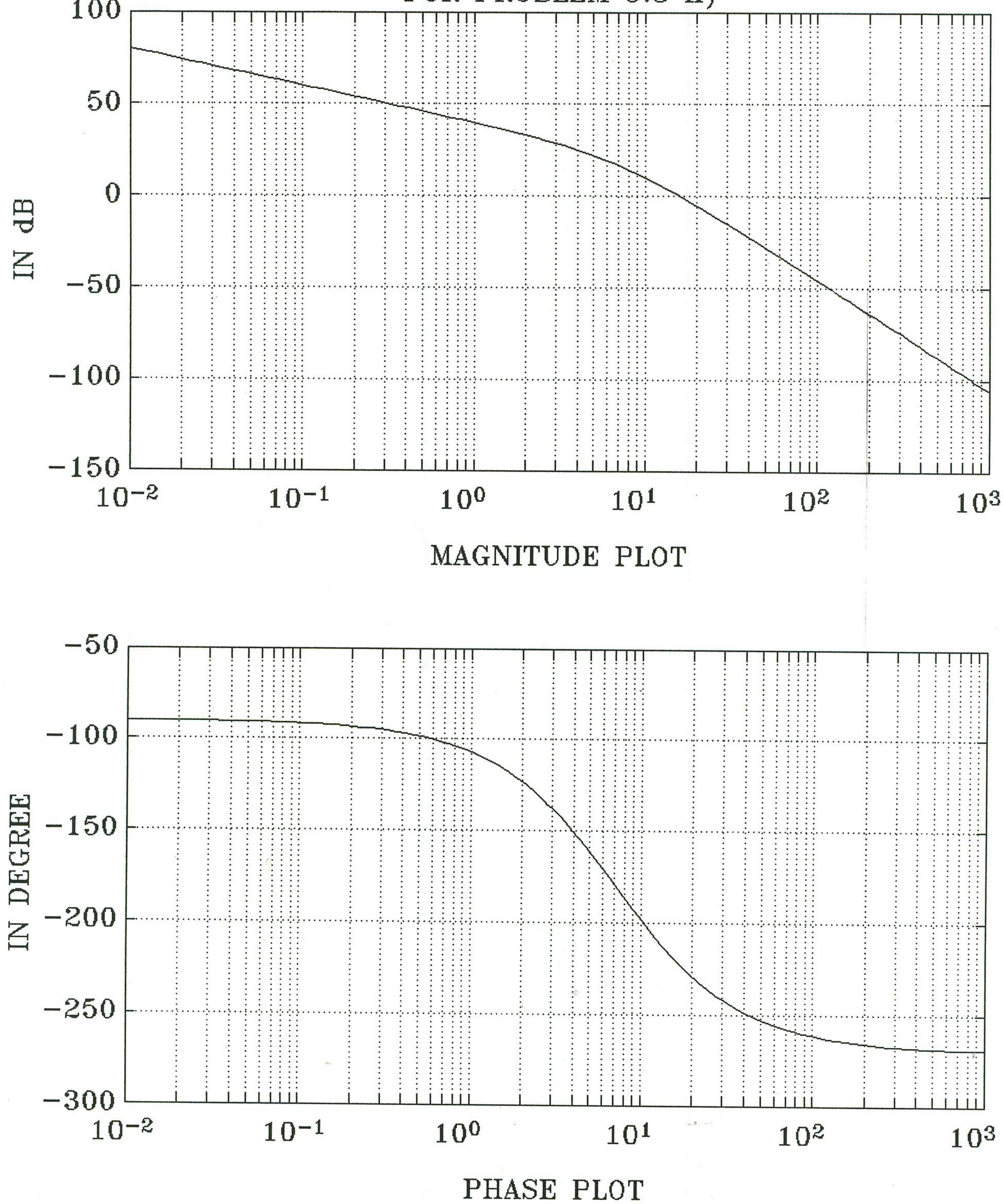



MAGNITUDE PLOT

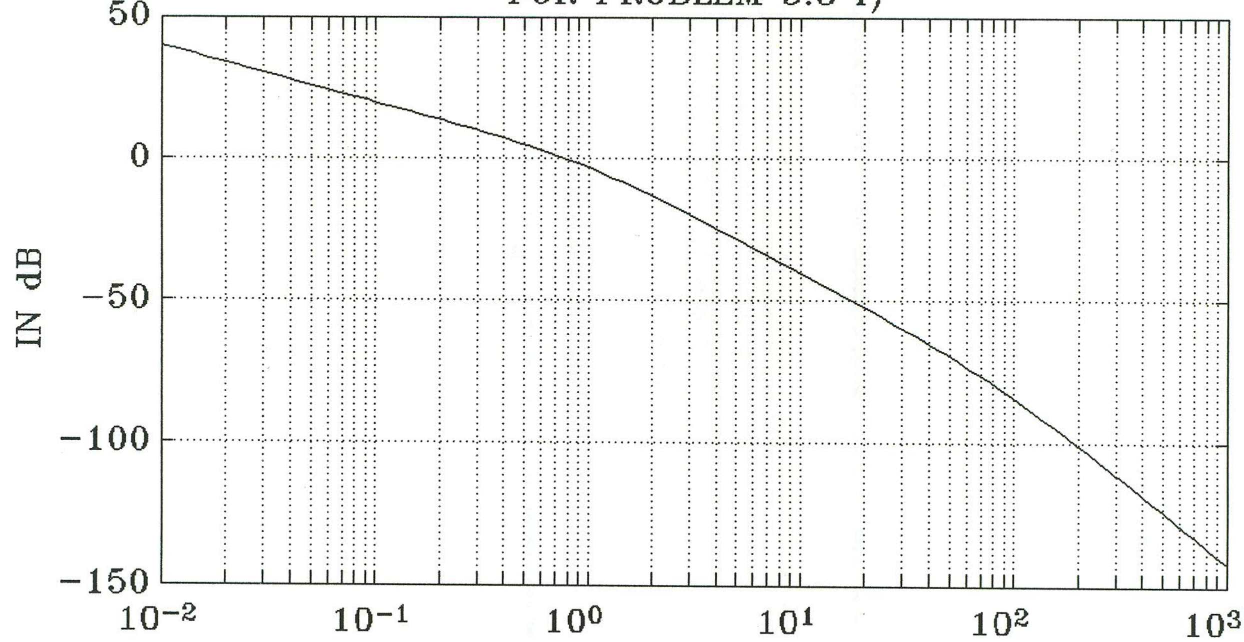



PHASE PLOT

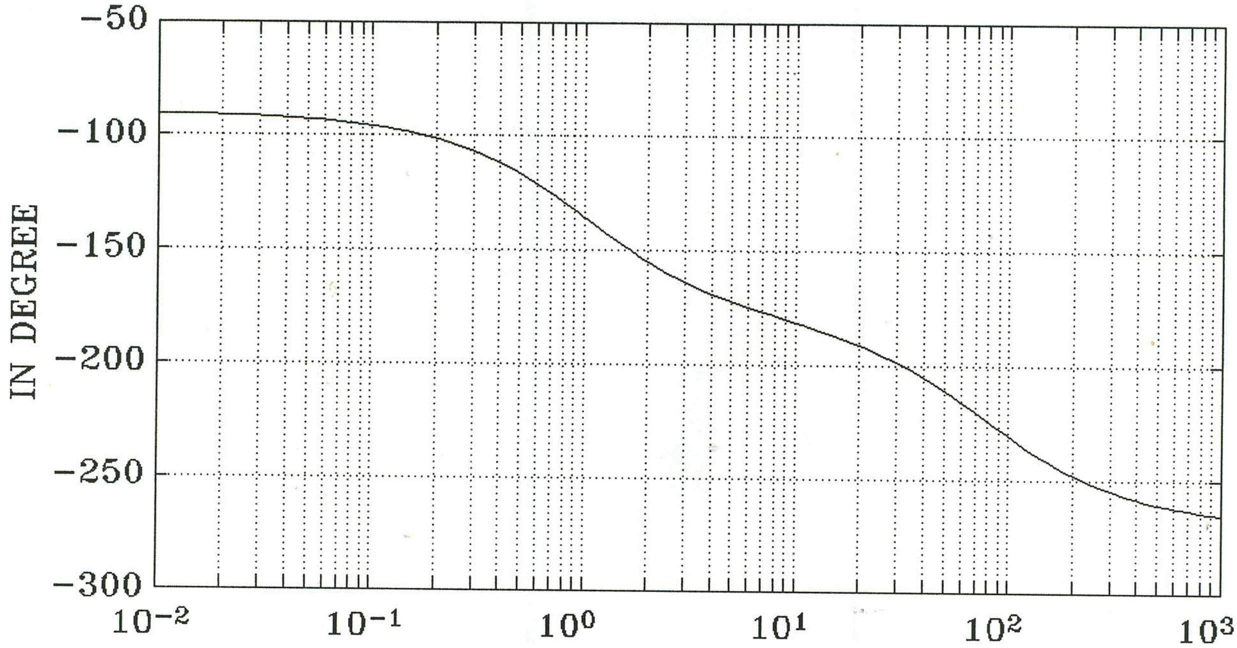
FOR PROBLEM 5.3 h)



FOR PROBLEM 5.3 i)

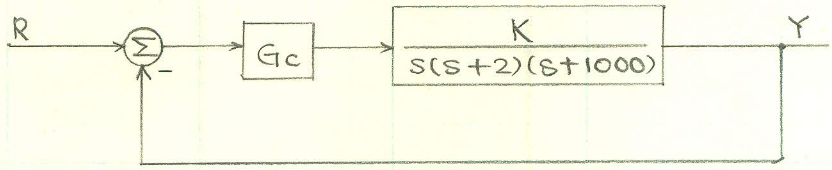



MAGNITUDE PLOT




PHASE PLOT

3. For the system given below :




(1) Design a lead compensator Gc and find K such that

$$\Phi_{pm} \geq 45^\circ$$

$$e_{ss}(\text{ramp}) \leq 0.02$$

(2) Obtain the step response of the uncompensated & Compensated Systems.

---

Solution : (1) Assuming we are going to design a unity DC gain Compencator, that is

$$\lim_{s \to 0} G_c = 1$$

$$K_v = \lim_{s \to 0} s \cdot G_c \cdot \frac{K}{s(s+2)(s+1000)} = \frac{K}{2000} = \frac{1}{0.02}$$

Thus we have $K = 100{,}000$.

from the Bode plots for the uncompensated given in next page we have $PM = 10^\circ$. So additional $PM = 45^\circ - 10^\circ = 35^\circ$

Looking carefully from the plot, we see that the slope at $\omega_1$ is $-2$ in magnitude plot. Hence, we need additional $PM = 45^\circ$. ??

$$\sin 45^\circ = \frac{1-\alpha}{1+\alpha} \qquad \alpha = 0.17157$$

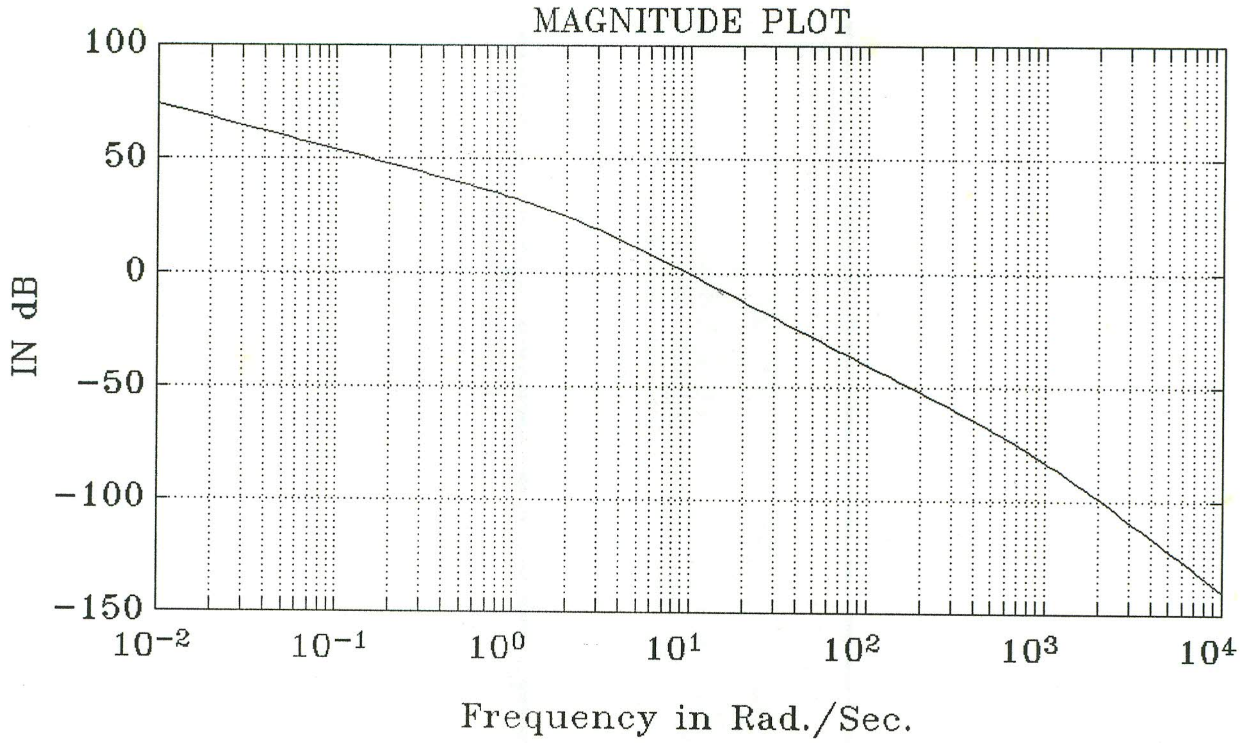
$$\omega_m = 10\,\text{rad/s} = \sqrt{\alpha}\,P \Rightarrow p = 24 \quad \& \; z = 4$$

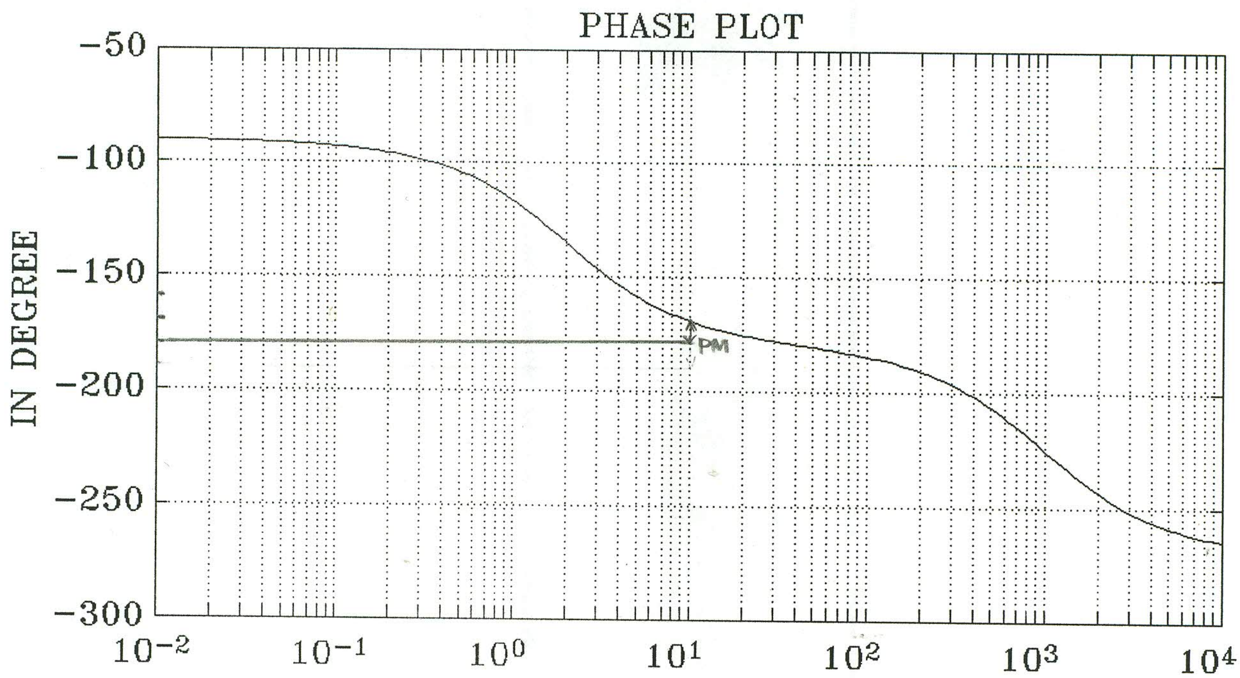
Thus, we have $\boxed{G_c = 6 \cdot \frac{s+4}{s+24}}$ (DC gain = 1)

Refer the Bode plots for compencated system for varifying.

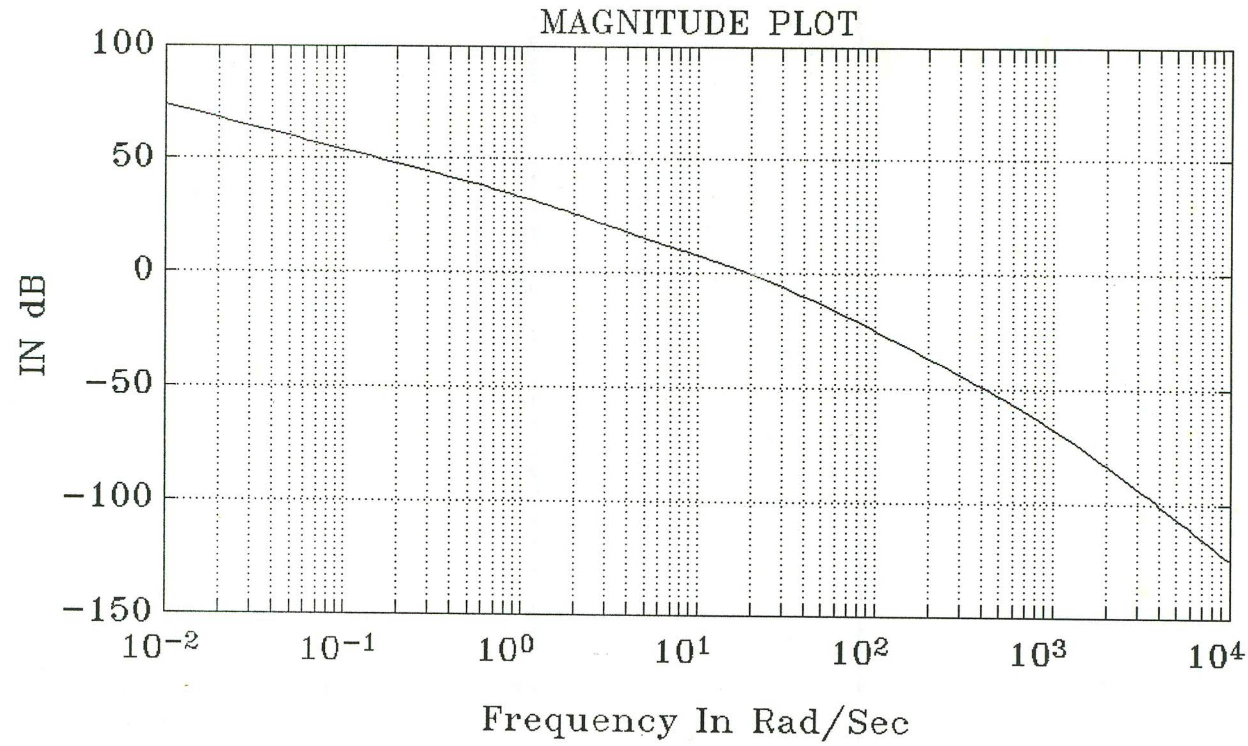
(2) Refer to step responses given on page 7.

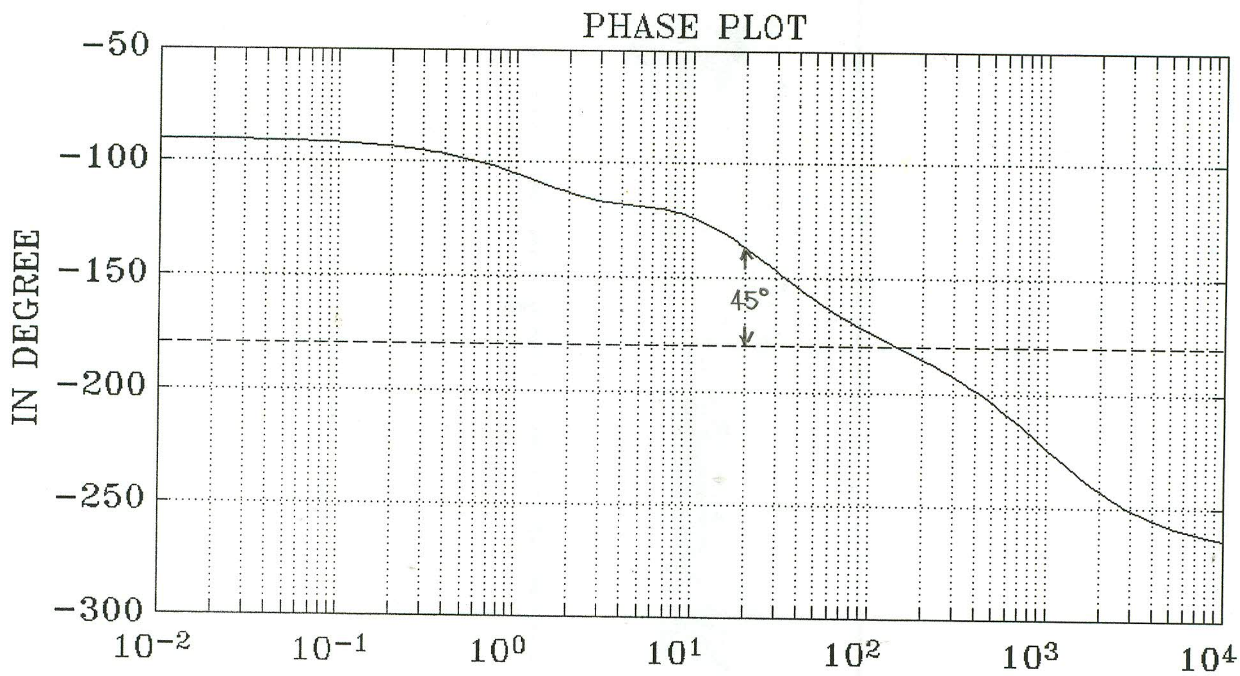
Q.E.D.



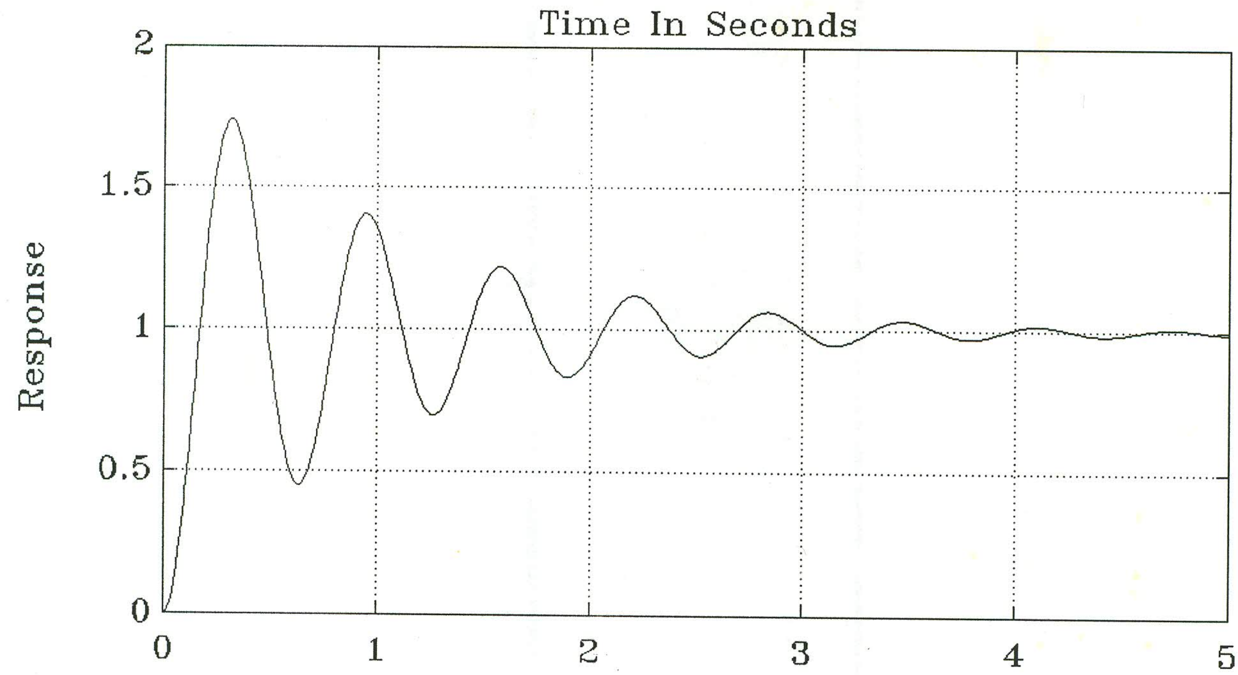




PROBLEM 3 : Bode Plots For The Uncompensated System

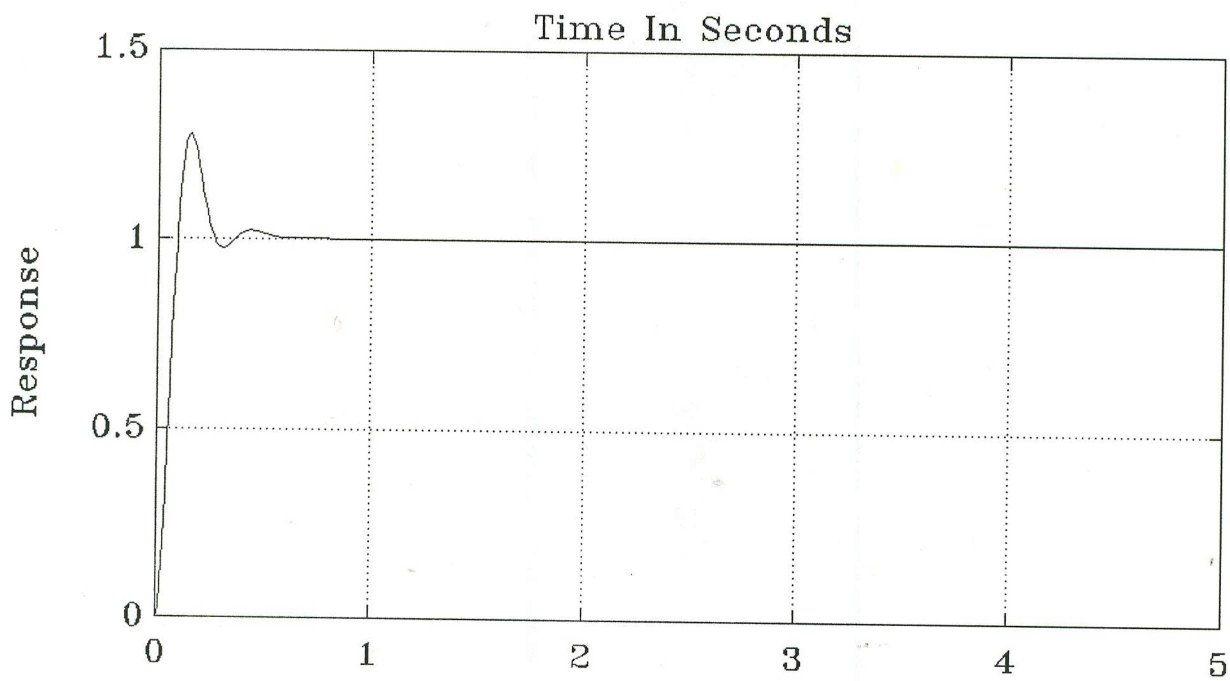



PROBLEM 3 : Bode Plots For The Compensated System

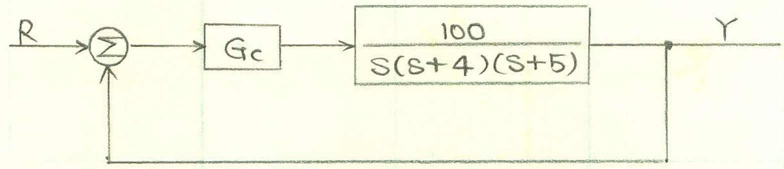



PROBLEM 3 : Step Response For Uncompensated System




PROBLEM 3 : Step Response For Compensated System

4. Considering the system




(i) <u>Obtain GM and PM of the uncompensated System.</u>

Refer to the plots given on next page, we have

$$GM = 4\,dB \quad , \quad PM = 20^\circ$$

(ii) <u>Design a lag compensator Gc. such that</u>

<u>$PM \geqq 45^\circ$ and $K_v = 5$</u>

$$K_{v,comp} = \lim_{s\to 0} G_c \cdot s \cdot \frac{100}{s(s+4)(s+5)} = 5 \cdot \lim_{s\to 0} G_c = 5$$

Thus, we must a compensator with unity dc gain

$$G_c = \frac{1+s\tau}{1+\alpha s\tau} \qquad \alpha > 1$$

$\omega_1 = 3.2$ rad/sec. additional $PM = 45^\circ - 20^\circ = 25^\circ \approx 30^\circ$

$A(\omega_1) = 16\,dB$

$$20 \log \frac{1}{\alpha} = -16 \Rightarrow \qquad \alpha = 6.3$$

$$z = \frac{\omega_1}{\alpha} = \frac{3.2}{10} \cong 0.32 \text{ and } \quad p = 0.05$$

Hence

$$\boxed{G_c = 0.15625 \frac{s+0.32}{s+0.05}}$$

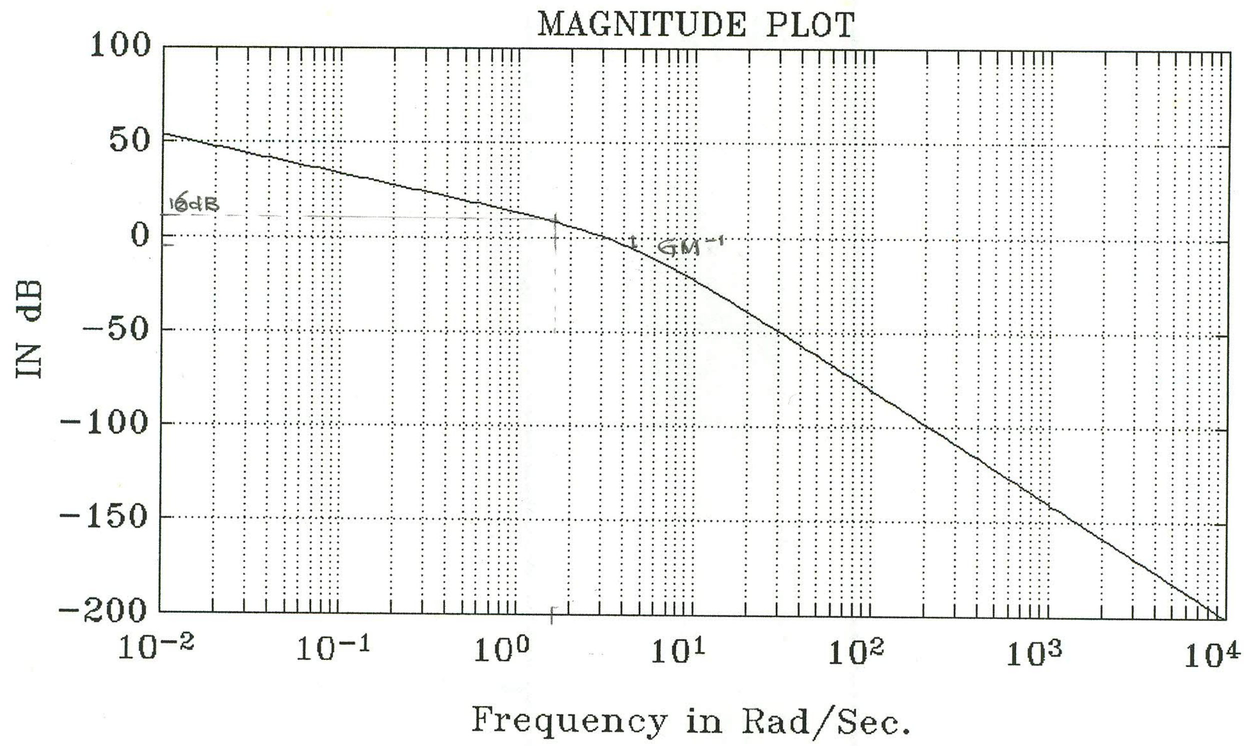
(iii) From the plots on next pages, we have for

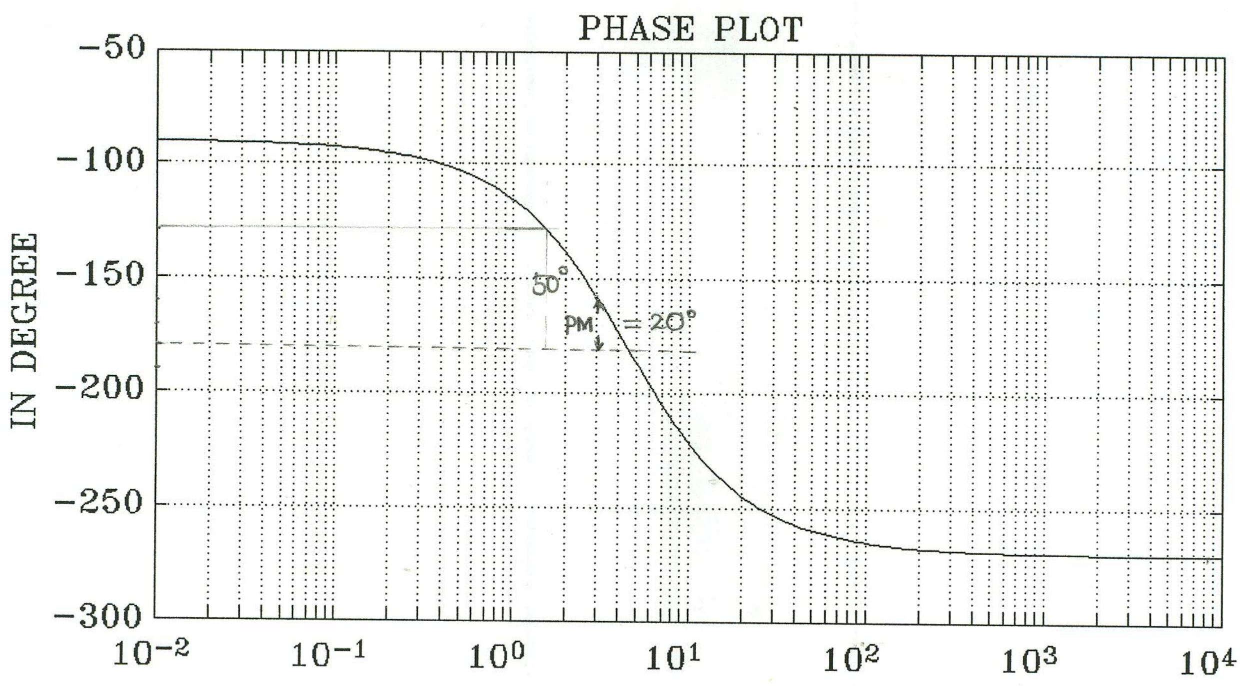
Uncompensated : $BW \cong 4.4$ rad/sec.

Compensated : $BW \cong 0.9$ rad/sec. $(PM) \approx 45^\circ$

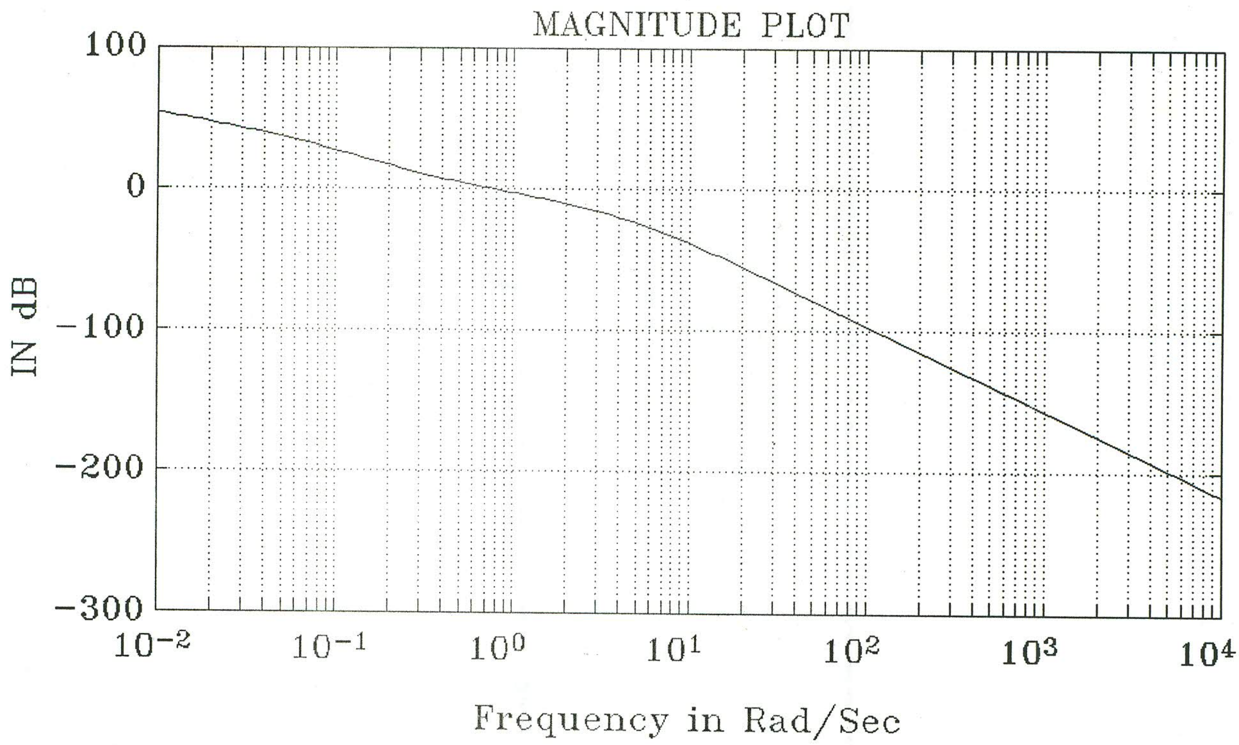
Also, the step response of the compenstated and uncompensated are given page 11.

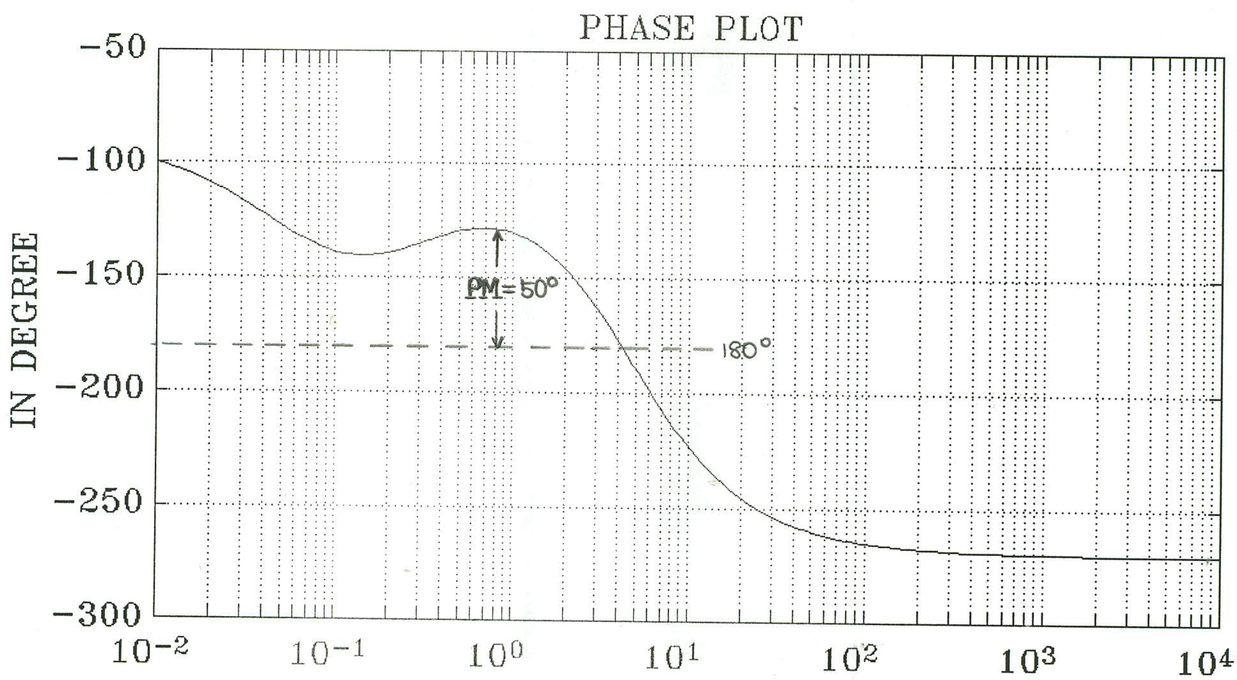
Q.E.D.



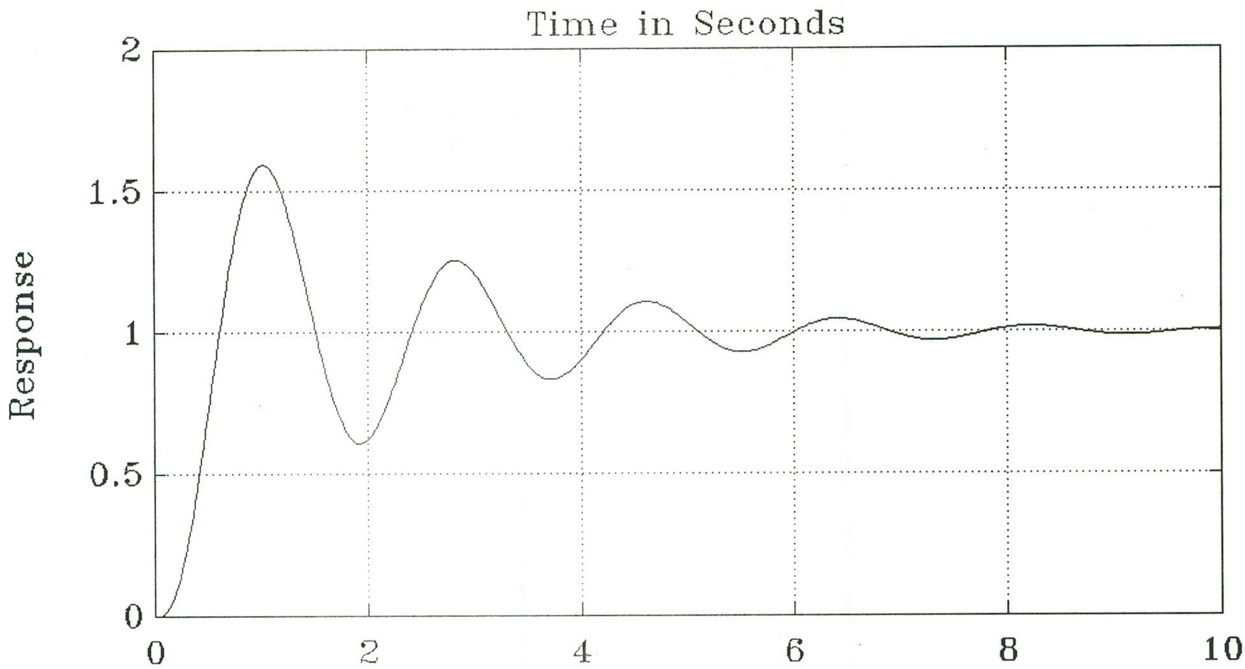




PROBLEM 4 : The Bode Plot For Uncompensated System

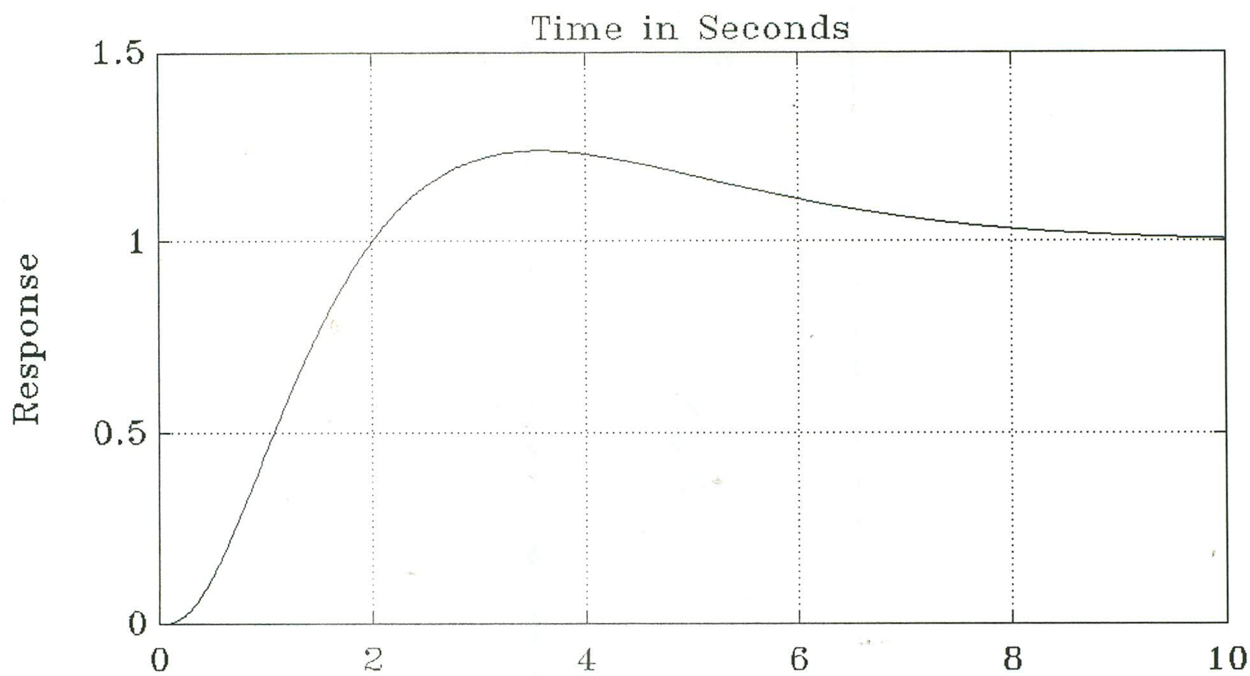



PROBLEM 4 : Bode Plots For The Compensated System




PROBLEM 4 : Step Response For Uncompensated System




PROBLEM 4 : Step Response For Compensated System

5. Using Root Locus Techniques. Design Lead and Lag Compensators for the open-loop system

$$G(s) = \frac{K}{s(s+10)^2}$$

such that $K_v = 20$ and $M_p \leq 5\%$ $(\xi \geq 0.7)$

(i) Obtain Root Locus & Step Response for Uncompensated System.

Refer to the root locus given on page 13.

We also notice that the closed-loop poles

$$P_{1,2} = -2.93 \pm j2.93$$

are on the R.L and $\xi = 0.7$ Line with $K = 242.64$

$$G(s) = \frac{242.64}{s^3 + 20s^2 + 100s}$$

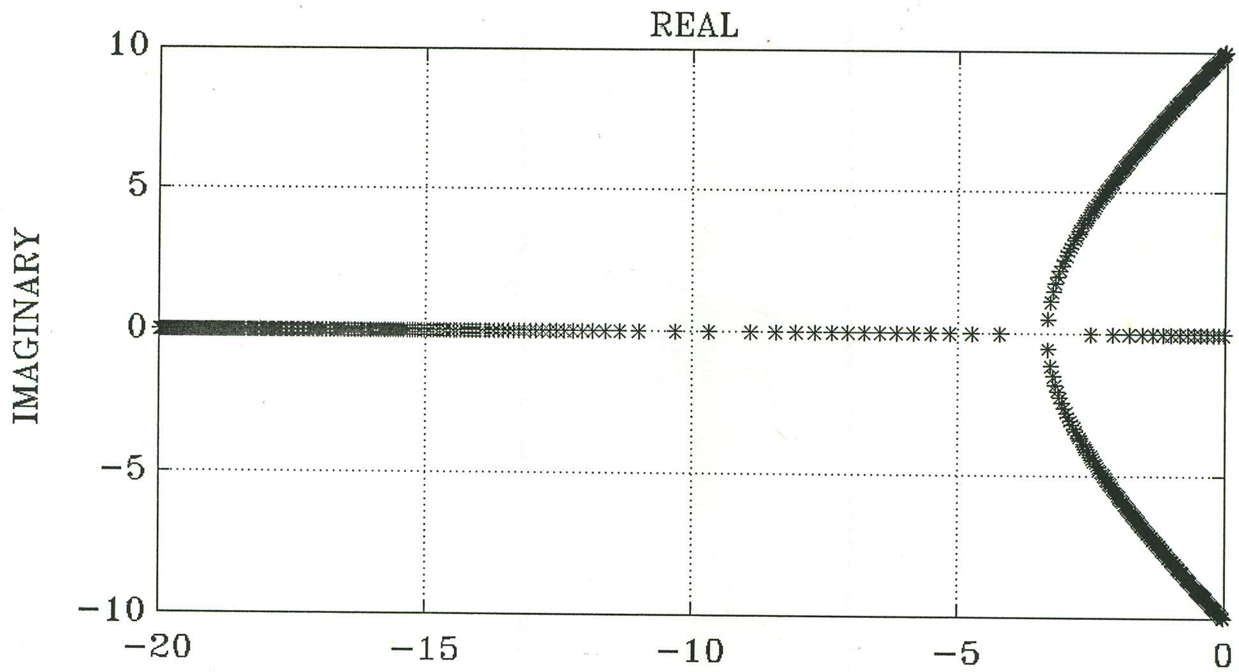
$$T(s) = \frac{242.64}{s^3 + 20s^2 + 100s + 242.64}$$

$$K_v = \lim_{s \to 0} sG(s) = \frac{242.64}{100} \cong 2.43 \ll 20$$

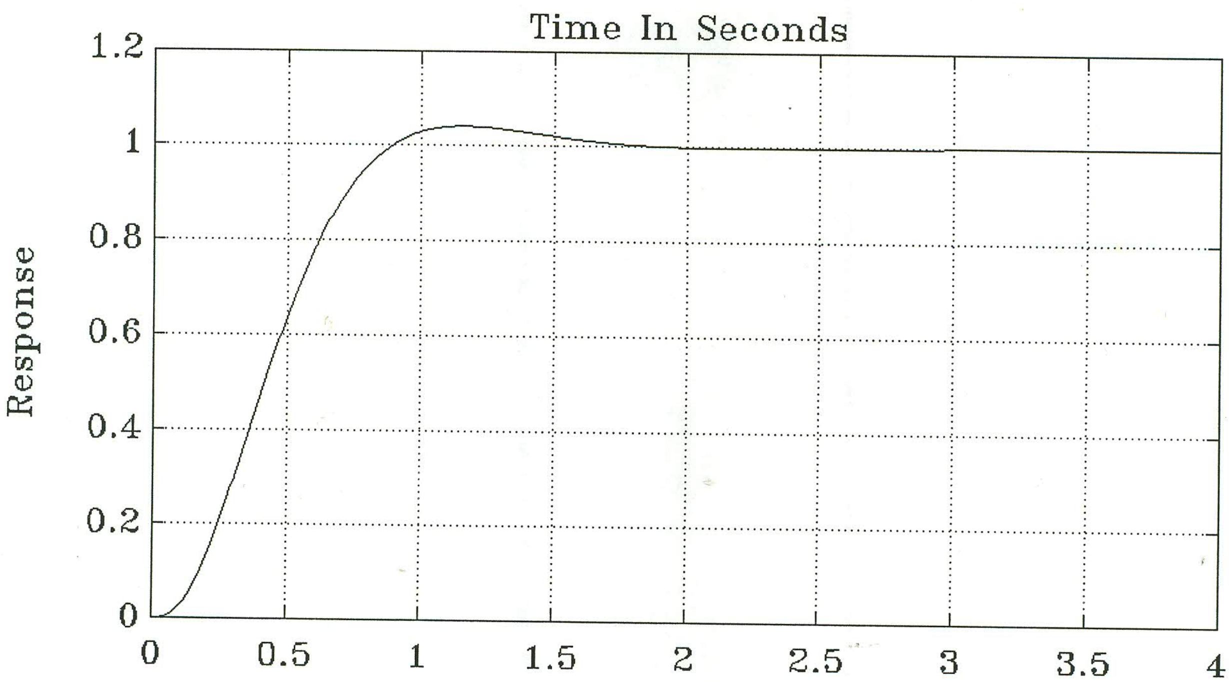
The step Response of Uncompensated System is also given on page 13.

---

for the step response, $e_{ss} = 0$ for this system.




PROBLEM 5 : Root Locus For The Uncompensated System



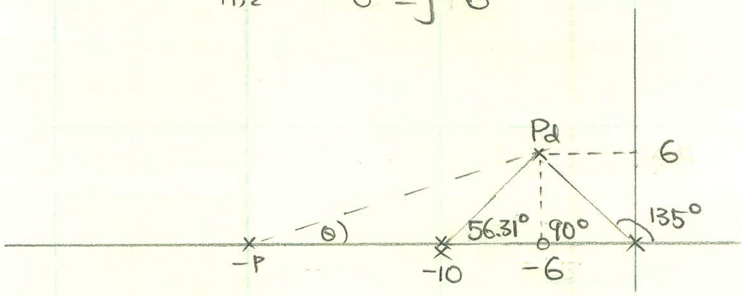

PROBLEM 5 : Step Response For The Uncompensated Sys

(ii) Obtain Root Locus and Step Response for Lead Compensated System.

From part (i), we see that $K_v \cong 2.43 \ll 20$.

Thus, we know we need a lag Compensator. Because Lead Compensator for this particular problem has noting to do with $K_v$. To Complete our problem, I design a lead comp. with $z = 6$. and closed-loop poles will be.

$$P_{1,2} = -6 \pm j\,6$$




$\theta = 22.38° \Rightarrow P = 20.57$

And the new $K = 1158.76$

$$G_{Lead}(s) = \frac{s+6}{s+20.57}$$

$$K_v = \lim_{s \to 0} \frac{s+6}{s+20.57} \cdot s \cdot G(s) = 3.38$$

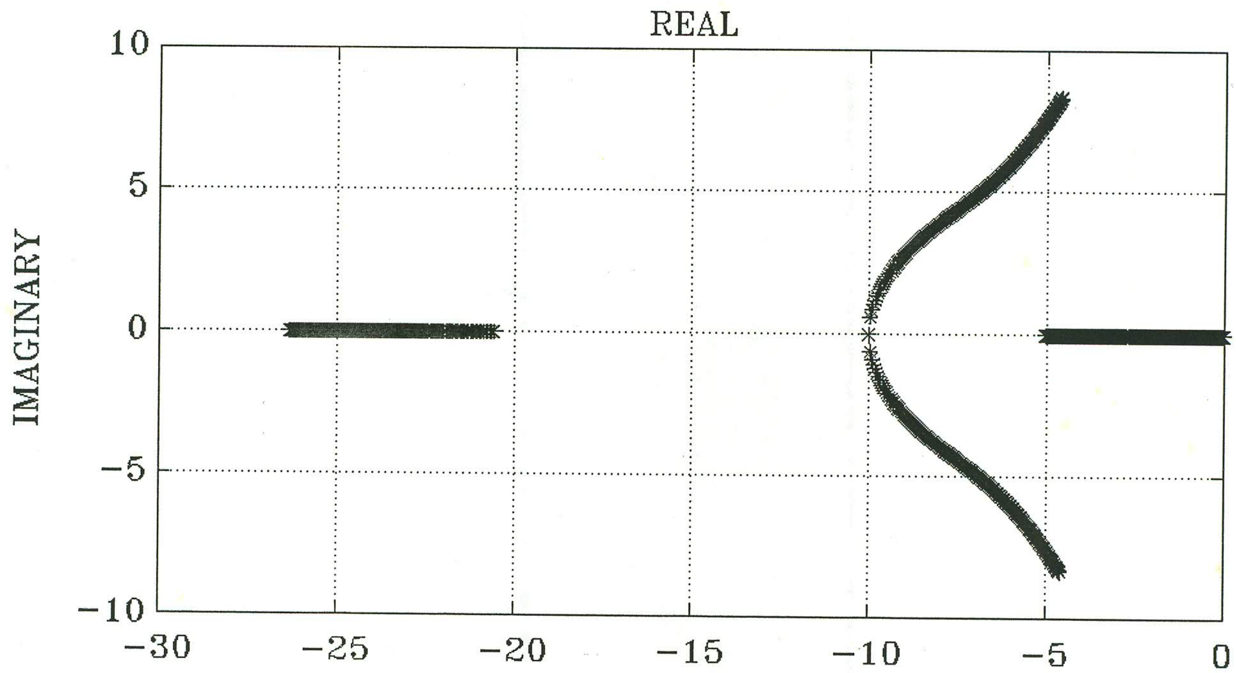
$K_v$ is still quite small compared to 20.

$$G_{Ld}(s) = \frac{1158.76(s+6)}{s^4 + 40.57s^3 + 511.4s^2 + 2057s}$$

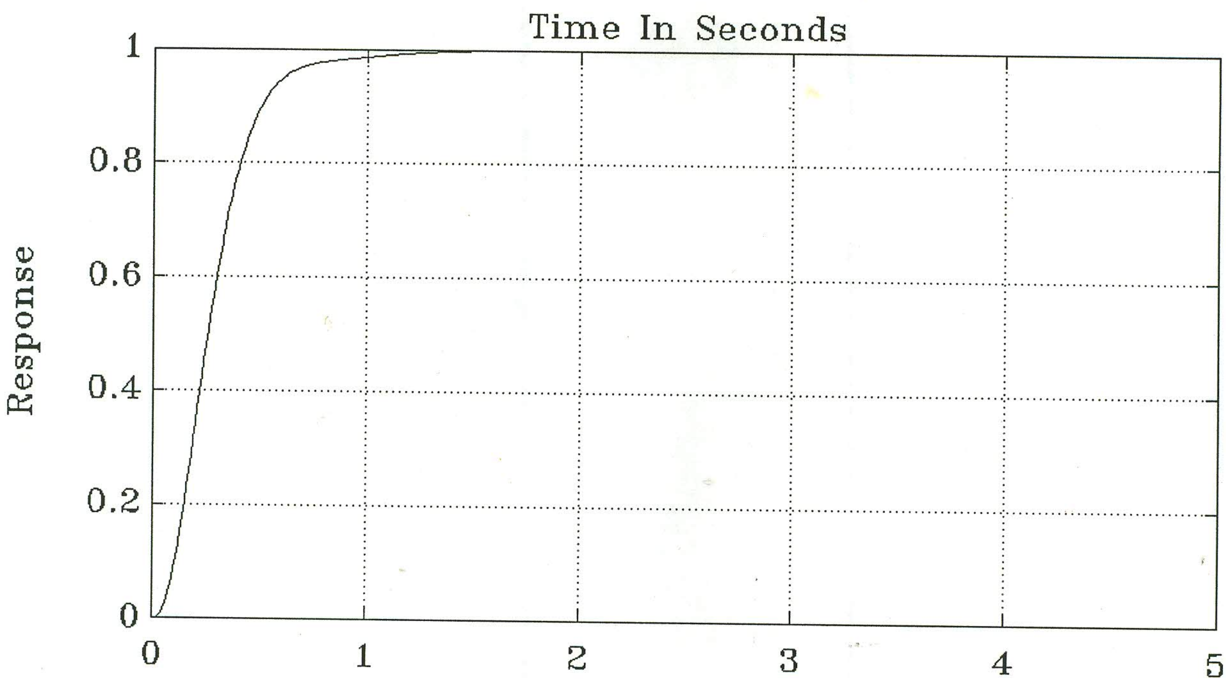
$$T_{Ld}(s) = \frac{1158.76(s+6)}{s^4 + 40.57s^3 + 511.4s^2 + 3215.76s + 6952.5}$$

---

The Root Locus and Step Response are given on page 15.




PROBLEM 5 : Root Locus For The Compensated System




PROBLEM 5 : Step Response For Lead Compensated

(iii) Obtain R.L. & Step Response for Lead & Lag Compensated System.

From Part (ii), $K_v = 3.38$.

Thus, we need a lag compensator with

$$\frac{z}{p} = \frac{20}{3.38} = 5.9 \simeq 6$$

Let $z = 0.4$. $p = 0.065$.

$$G_{Lag} = \frac{S+0.4}{S+0.065}$$

$$K_v = \lim_{S \to 0} \frac{S+0.4}{S+0.065} \times 3.38 = 20.8 \approx 20$$

Then

$$G_{compen}(S) = \frac{S+0.4}{S+0.065} \cdot \frac{1158.76(S+6)}{S^4+40.57S^3+511.4S^2+2057S}$$

$$= \frac{1158.76(S^2+6.4S+2.4)}{S^5+40.635S^4+514.037S^3+2090.24S^2+133.705S}$$

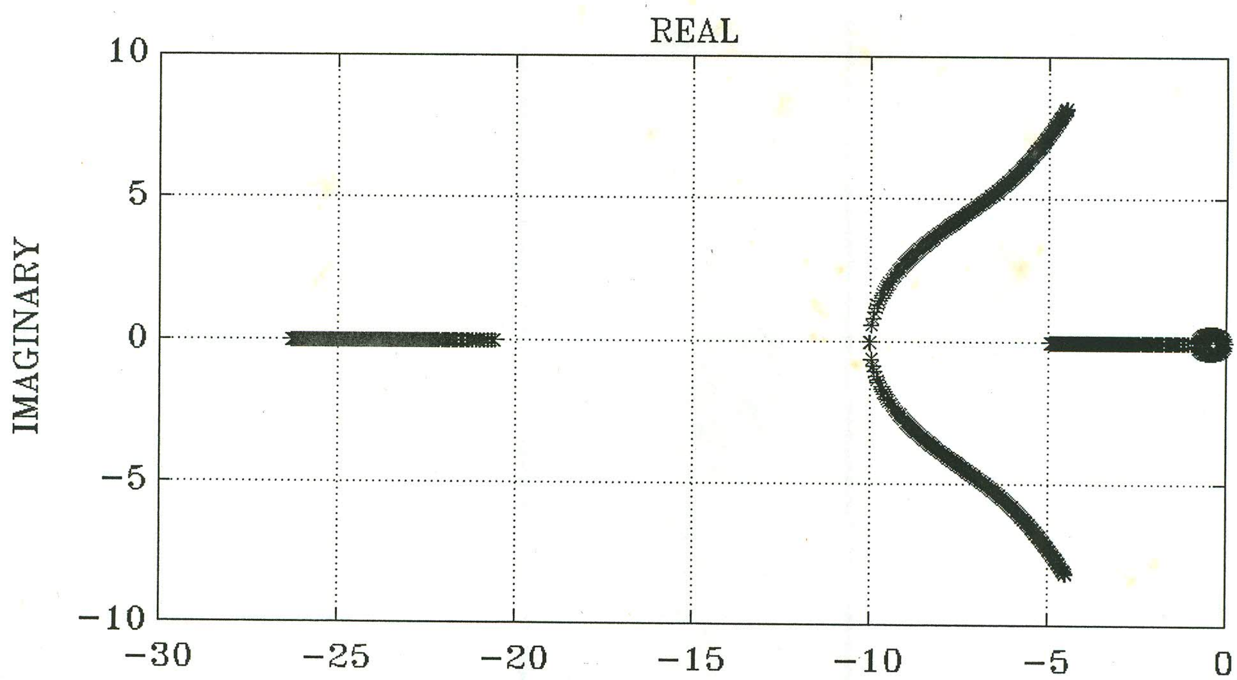
$T_{com,sys}(S) =$

$$\frac{1158.76(S^2+6.4S+2.4)}{S^5+40.635S^4+514.037S^3+3249S^2+7549.769S+2781.024}$$

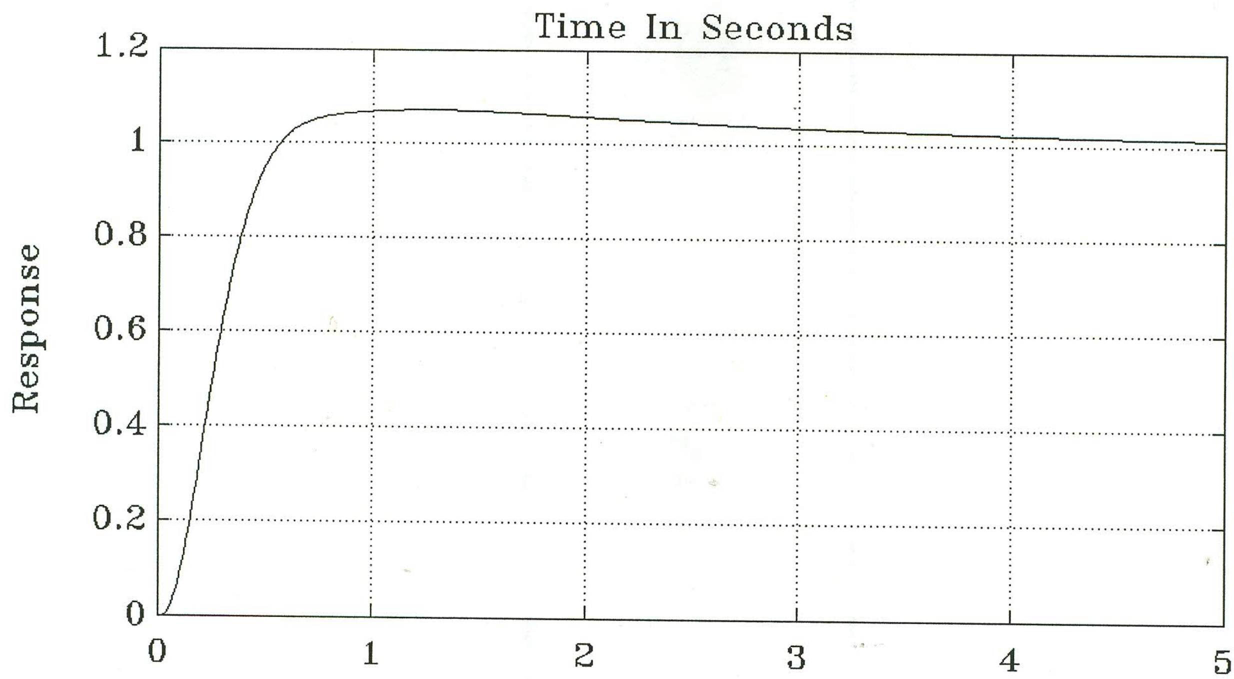
The Root locus and Step Response are given on next page

Q.E.D

✓




PROBLEM 5 : Root Locus For The Lead & Lag Compensated




PROBLEM 5 : Step Response For Lead & Lag Compensated

A

5.44 Two Nyquist diagrams for unity feedback systems that are open-loop stable are sketched in Fig. 5.83 and 5.84. The proposed operating gain is indicated as $K_0$ and arrows indicate increasing frequency. In each case, give your best estimates for :

a) Phase margin

b) Damping ratio to step response.

c) Closed-loop bandwidth.

d) Range of gain for stability (if any)

e) System type as 0, I or II.

---

Solution:

For the system given in Figure 5.83.

a) Phase Margin $\cong 20^\circ$

b) Damping ratio, $\zeta \cong PM/100 = 0.2$

c) $BW \cong 1.2$ rad/sec ( >1.0 rad/sec )

d) Range of gain $K_0 > K_1$

e) System is type II

For the system given in Figure 5.84

a) Phase Margin $\cong 72^\circ$

b) Damping ratio, $\zeta \cong 0.85$

c) $BW \cong 12$ rad/sec.

d) Range of gain $K_0 < K_1$

e) System is type I.

✓

6.2 For the redefinition of state variables described in section 6.2.2, derive Eq. (6.16). Note that, since $p = Tx$, ...

---

Since

$$p = \mathbf{T}x$$

Thus $\quad \dot{p} = \mathbf{T}\dot{x} \quad$ and $\quad \dot{x} = \mathbf{T}^{-1}\dot{p}, \quad x = \mathbf{T}^{-1}p$

Subtitute these into

$$\begin{cases} \dot{x} = \mathbf{F}x + \mathbf{G}u + \mathbf{G}_1 w \\ y = \mathbf{H}x + Ju \end{cases}$$

we have

$$\begin{cases} \mathbf{T}^{-1}\dot{p} = \mathbf{F}\cdot\mathbf{T}^{-1}\cdot p + \mathbf{G}u + \mathbf{G}_1 w \\ y = \mathbf{H}\cdot\mathbf{T}^{-1}p + J\cdot u \end{cases}$$

$$\Rightarrow \begin{cases} \dot{p} = (\mathbf{T}\mathbf{F}\mathbf{T}^{-1})\cdot p + (\mathbf{T}\mathbf{G})\cdot u + (\mathbf{T}\mathbf{G}_1) w \\ y = (\mathbf{H}\cdot\mathbf{T}^{-1})\cdot p + J\cdot u. \end{cases}$$

Thus, we obtain

$$\mathbf{F}' = \mathbf{T}\mathbf{F}\mathbf{T}^{-1}; \qquad \mathbf{G}' = \mathbf{T}\mathbf{G}$$

$$\mathbf{G}_1' = \mathbf{T}\mathbf{G}_1; \qquad \mathbf{H}' = \mathbf{H}\mathbf{T}^{-1}$$

$$J' = J$$

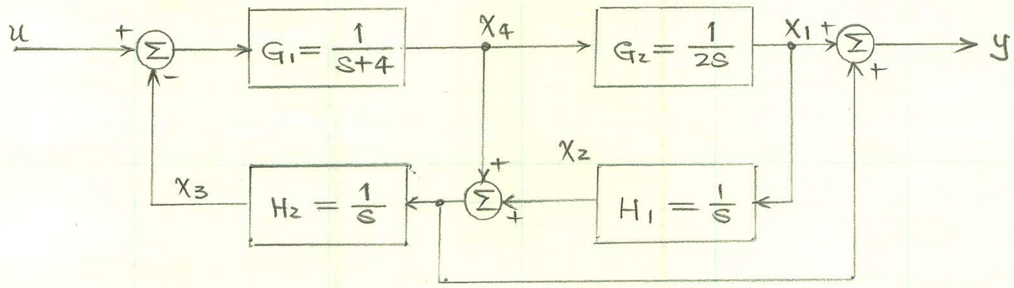
Q.E.D.

---

6.3 Consider the system in Fig. 6.62

a) Find the transfer function from u and y.

b) Write the state-variable equations using states indicated.




Solution:

$\dot{x}_2 = x_1$

$\dot{x}_3 = x_2 + x_4$

$x_1 = x_4 \cdot \frac{1}{2s} \Rightarrow x_4 = 2 \cdot s x_1 = 2 \cdot \dot{x}_1$

$\Rightarrow \dot{x}_1 = 0.5\, x_4$

$x_4 = (u - x_3) \cdot \frac{1}{s+4} \Rightarrow s x_4 + 4 x_4 = u - x_3$

$\Rightarrow \dot{x}_4 = -x_3 - 4x_4 + u$

$y = x_1 + x_2 + x_4$

b) State Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot u$$

$$y = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

a) From Matlab, we can obtain the transfer function

$$T(s) = \frac{s^3 + 0.5 s^2 + 0.5 s}{s^4 + 4 s^3 + s^2 + 0.5} \quad \text{(ss2tf)}$$

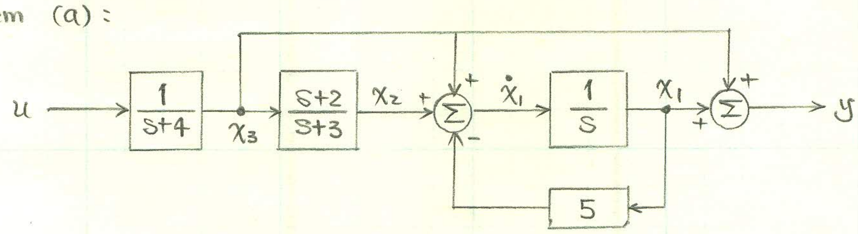
6.4 Using the indicated state variables in Fig. 6.63.

a) Find state variable equations.

b) Transfer function for each system.

---

System (a):




$$\dot{x}_1 = -5x_1 + x_2 + x_3$$

$$x_3 = u \cdot \left(\frac{1}{s+4}\right) \Rightarrow \quad s x_3 + 4x_3 = u \Rightarrow$$

$$\dot{x}_3 = -4x_3 + u.$$

$$x_2 = \frac{s+2}{s+3} x_3 \quad \Rightarrow \quad s x_2 + 3x_2 = s x_3 + 2x_3$$

$$\Rightarrow \quad \dot{x}_2 = -3x_2 + 2x_3 + \dot{x}_3 = -3x_2 + 2x_3 - 4x_3 + u$$

$$= -3x_2 - 2x_3 + u$$

$$y = x_1 + x_3.$$

Thus

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & -4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot u.$$

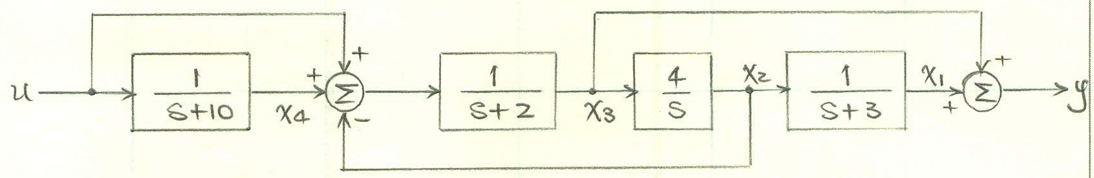
$$y = [1 \quad 0 \quad 1] \cdot \underline{x}$$

The transfer function of this system can be obtained easily from matlab.

$$T(s) = \frac{s^2 + 10s + 20}{s^3 + 12s^2 + 47s + 60} \qquad (\text{ss2tf})$$

6.4 (CONT.)

system (b)




$$X_1 = \frac{1}{s+3} X_2 \Rightarrow \quad sX_1 + 3X_1 = X_2$$

$$\dot{X}_1 = -3X_1 + X_2$$

$$X_2 = \frac{4}{s} X_3 \Rightarrow \quad \dot{X}_2 = 4X_3$$

$$X_3 = \frac{1}{s+2}(u + X_4 - X_2)$$

$$\dot{X}_3 = -X_2 - 2X_3 + X_4 + u$$

$$X_4 = \frac{1}{s+10} u \Rightarrow \quad \dot{X}_4 = -10X_4 + u$$

$$y = X_1 + X_3$$

State Equations:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & -10 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot u$$

$$y = [1 \quad 0 \quad 1 \quad 0] \cdot \underline{X}$$

Transfer function:

$$T(s) = \frac{s^3 + 14s^2 + 37s + 44}{s^4 + 15s^3 + 60s^2 + 112s + 120}$$

Q.E.D.

**6.6** Consider the system in Fig. 6.64

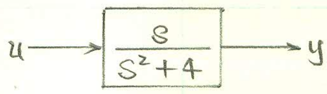
a) Write a set of equations that describes this system in the standard from

$\dot{x} = Fx + Gu$ and $y = Hx$.

b) Design a control law of the form $u = -[K_1\ K_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

that will place the closed-loop poles at $s = -2 \pm 2j$.

---




$$\frac{Y(s)}{U(s)} = \frac{X(s)}{U(s)} \cdot \frac{Y(s)}{X(s)} = \frac{s}{s^2+4} = \frac{1}{s^2+4} \cdot s$$

$$\Rightarrow \quad \frac{X(s)}{U(s)} = \frac{1}{s^2+4} \quad \Rightarrow \quad \ddot{x} + 4x = u \quad \text{and} \quad y = \dot{x}$$

Let $\quad x_1 = x \quad$ and $\quad x_2 = \dot{x}$

$$\Rightarrow \quad \dot{x}_1 = \dot{x} = x_2$$

$$\dot{x}_2 = \ddot{x} = -4x + u = -4x_1 + u \quad , \quad y = x_2 .$$

a) $$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u \quad , \quad y = [0 \quad 1] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) $$u = -[K_1, K_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow$$

$$F' = \begin{bmatrix} 0 & 1 \\ -4-K_1 & -K_2 \end{bmatrix}$$

$$|\lambda I - F'| = \begin{vmatrix} \lambda & -1 \\ 4+K_1 & \lambda + K_2 \end{vmatrix} = \lambda^2 + K_2\lambda + 4 + K_1$$

$$= (\lambda + 2 + j2)(\lambda + 2 - j2)$$

$$= \lambda^2 + 4\lambda + 8 .$$

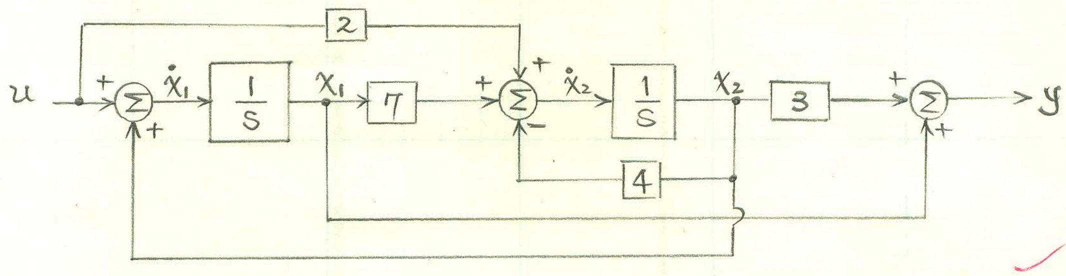
$$\Rightarrow \quad K_1 = 4 \quad , \quad K_2 = 4$$

Q.E.D.

6.7 Consider the plant described by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 7 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \;; \quad y = [1 \;\; 3]\, x$$

a) Draw the block diagram for the plant.




b) Transfer function

$$T(S) = [1 \;\; 3] \cdot \begin{bmatrix} S & -1 \\ -7 & S+4 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= [1 \;\; 3] \cdot \begin{bmatrix} S+4 & 1 \\ 7 & S \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Big/ (S^2+4S-7)$$

$$= \frac{7S + 27}{S^2 + 4S - 7}$$

c) Find characteristic equation if

(1) $u = -[K_1 \;\; K_2]\, X$

$$F' = \begin{bmatrix} 0 & 1 \\ 7 & -4 \end{bmatrix} - \begin{bmatrix} K_1 & K_2 \\ 2K_1 & 2K_2 \end{bmatrix} = \begin{bmatrix} -K_1 & 1-K_2 \\ 7-2K_1 & -4-2K_2 \end{bmatrix}$$

$$|SI - F'| = \begin{vmatrix} S+K_1 & K_2-1 \\ 2K_1-7 & S+4+2K_2 \end{vmatrix} =$$

$$= S^2 + (2K_2+K_1+4)S + 7K_2 + 6K_1 - 7$$

(2) $u = -Ky = -[K \;\; 3K]\, X$

Let $K_1 = K$, $K_2 = 3K$ subtitute into C.E. above

we have

$$C.E. = S^2 + (7K+4)S + (27K-7)$$

6.17 Given $\ddot{\theta} + \omega^2 \theta = 0$

a) $x_1 = \theta \quad , \quad x_2 = \dot{\theta}$

$$\Rightarrow \dot{x}_1 = \dot{\theta} = x_2$$

$$\dot{x}_2 = \ddot{\theta} = -\omega^2 \theta = -\omega^2 \cdot x_1$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad (u \equiv 0)$$

b) $\dot{\theta} = x_2 = [0 \quad 1] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad , \quad \omega = 5$ rad/sec.

$$|\lambda I - F + LH|$$

$$= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -25 & 0 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \cdot [0 \quad 1] \right|$$

$$= \begin{vmatrix} \lambda & -1 + L_1 \\ 25 & \lambda + L_2 \end{vmatrix} = \lambda^2 + L_2 \lambda + 25(1 - L_1) = \lambda^2 + 100\lambda + 200$$

$$\Rightarrow L_1 = -7 \quad , \quad L_2 = 100$$

Thus $L = \begin{bmatrix} -7 \\ 100 \end{bmatrix}$

Observer $\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ -25 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} -7 \\ 100 \end{bmatrix} \cdot (\dot{\theta} - [0 \quad 1] \hat{x})$

c)
$$\dot{\hat{x}} = \begin{bmatrix} 0 & 8 \\ -25 & -100 \end{bmatrix} \cdot \hat{x} + \begin{bmatrix} -7 \\ 100 \end{bmatrix} \cdot \dot{\theta}$$

$$\hat{\theta} = \hat{x}_1 = [1 \quad 0] \hat{x}.$$

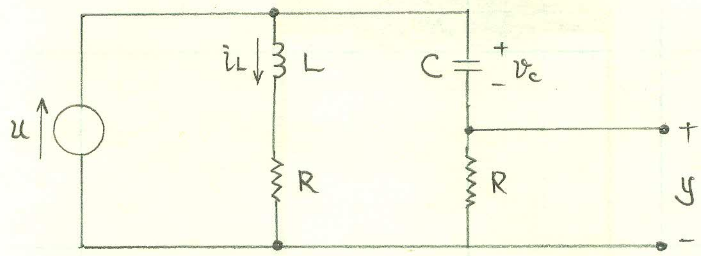
$$\frac{\hat{\theta}(s)}{\theta(s)} = \frac{-7s + 100}{s^2 + 100s + 200}$$

d)
$$|\lambda I - F + GK| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -25 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \quad K_2] \right|$$

$$= \begin{vmatrix} \lambda & -1 \\ 25 + K_1 & \lambda + K_2 \end{vmatrix} = \lambda^2 + K_2 \lambda + 25 + K_1 \equiv \lambda^2 + 8\lambda + 32$$

$$\Rightarrow K = [7 \quad 8]$$

6.21




$$i_c = C\cdot s\, V_c = C\cdot \dot{V}_c = u - i_L \Rightarrow \dot{V}_c = -\frac{1}{C} i_L + \frac{1}{C} u$$

$$V_L = L\cdot s\, i_L = L\dot{i}_L = V_c + i_c R - i_L R$$

$$= V_c + (u - i_L)R - i_L R$$

$$= V_c - 2R\cdot i_L + R\cdot u$$

$$\Rightarrow \dot{i}_L = \frac{1}{L} V_c - \frac{2R}{L} i_L + \frac{R}{L} u$$

$$y = i_c R = (u - i_L)R = -R i_L + R\cdot u$$

(a) State Equation :

$$\begin{bmatrix} \dot{i}_L \\ \dot{V}_c \end{bmatrix} = \begin{bmatrix} -\frac{2R}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} \cdot \begin{bmatrix} i_L \\ V_c \end{bmatrix} + \begin{bmatrix} \frac{R}{L} \\ \frac{1}{C} \end{bmatrix} \cdot u$$

$$y = \begin{bmatrix} -R & 0 \end{bmatrix} \begin{bmatrix} i_L \\ V_c \end{bmatrix} + R\cdot u.$$

(b) $\mathcal{C} = \begin{bmatrix} \frac{R}{L} & \frac{1}{LC} - \frac{2R^2}{L^2} \\ 1/C & -R/LC \end{bmatrix}$ , if $R^2 = L/C$

$$|\mathcal{C}| = -\frac{R^2}{L^2C} - \frac{1}{LC^2} + \frac{2R^2}{L^2C} = -\frac{L}{L^2C^2} - \frac{1}{LC^2} + \frac{2L}{L^2C^2} = 0$$

$\Rightarrow$ system is not complete controllable.

(c) $\mathcal{O} = \begin{bmatrix} -R & 0 \\ \frac{2R^2}{L} & -R/L \end{bmatrix}$ , $|\mathcal{O}| = \frac{R^2}{L} = \frac{L}{C\cdot L} = \frac{1}{C} \neq 0$

$\Rightarrow$ system is complete observable.

Q.E.D.