# Complex Analysis and Applications

MAT 417

Instructor:

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Fall 1986

Homework and Tests

Benmei Chen

Exercises : Page 8–9

| NO: | Pair | Sum | Difference | Product | Quotient |
|---|---|---|---|---|---|
| 2. | $i, -i$ | $0$ | $2i$ | $1$ | $-1$ |
| 4. | $2-i, 3+i$ | $5$ | $-1-2i$ | $7-i$ | $\frac{5}{4}-\frac{5}{4}i$ |
| 6. | $2+i, 3-4i$ | $5-3i$ | $-1+5i$ | $10-5i$ | $\frac{2}{25}+\frac{11}{25}i$ |
| 8. | $5i, 2+i$ | $2+7i$ | $-2+4i$ | $-5+10i$ | $1+2i$ |
| 10. | $2+i, 2-i$ | $4$ | $2i$ | $5$ | $\frac{3}{5}+\frac{4}{5}i$ |
| 12. | $2+i, 2i$ | $2+3i$ | $2-i$ | $-2+4i$ | $\frac{1}{2}-i$ |

14. $(1-i)^3 = (1-i)(1-i)(1-i) = -2i\cdot(1-i) = -2-2i$

16. $i^2(1+i)^3 = -1\cdot(1+i)(1+i)(1+i) = -2i(1+i) = 2-2i$

18. $\frac{3+2i}{1+i} + \frac{5-2i}{-1+i} = \frac{(3+2i)(1-i)}{(1+i)(1-i)} + \frac{(5-2i)(-1-i)}{(-1+i)(-1-i)} = \frac{5}{2} - \frac{1}{2}i - \frac{7}{2} - \frac{3}{2}i = -1-2i$

20. $(1-i)(1-2i)(1-3i) = (-1-3i)(1-3i) = -10$

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24. SHOW that if $Im\, z > 0$, then $Im(1/z) < 0$

2

SOLUTION: Let $z = (z_1, z_2)$ and $z_2 = Im\, z > 0$

$$1/z = \frac{1}{z_1 + i z_2} = \frac{1}{z_1^2 + z_2^2}(z_1 - i z_2) = \frac{z_1}{z_1^2+z_2^2} - i\frac{z_2}{z_1^2+z_2^2}$$

$\therefore Im(1/z) = -z_2/(z_1^2+z_2^2) < 0$ (Because $z_2, z_2^2 > 0$; $z_1^2 \geq 0$) #

26. Prove $(z_1+z_2)^n = z_1^n + \binom{n}{1} z_1^{n-1} z_2 + \binom{n}{2} z_1^{n-2} z_2^2 + \cdots + z_2^n$ ... (1)

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SOLUTION: (1) FOR $n=1$,

The Left-hand side of the equation (1),

$$(z_1+z_2)^1 = z_1 + z_2$$

The right-hand side of the equation (1) equals to $z_1 + z_2$

So, for $n=1$, the equation (1) is true.

(2) Assume as $n = i-1$, the equation (1) is true, thus

$$(z_1+z_2)^{i-1} = z_1^{i-1} + \binom{i-1}{1} z_1^{i-2} z_2 + \binom{i-1}{2} z_1^{i-3} z_2^2 + \cdots + z_2^{i-1} \quad \cdots (2)$$

(3) FOR $n = i$, use the equation (2)

$$\begin{aligned}(z_1+z_2)^i &= (z_1+z_2)(z_1+z_2)^{i-1} \\ &= (z_1+z_2)\cdot\left[z_1^{i-1} + \binom{i-1}{1} z_1^{i-2} z_2 + \binom{i-1}{2} z_1^{i-3} z_2^2 + \cdots + z_2^{i-1}\right] \\ &= z_1^i + \left[1 + \binom{i-1}{1}\right] z^{i-1}\cdot z_2 + \left[\binom{i-1}{1} + \binom{i-1}{2}\right] z_1^{i-2}\cdot z_2^2 + \cdots + z_2^i\end{aligned}$$

WHERE

$$1 + \binom{i-1}{1} = 1 + \frac{(i-1)!}{1!(i-2)!} = \frac{(i-2)!\cdot i}{1!(i-2)!} = \frac{i!}{1!(i-1)!} = \binom{i}{1}$$

$$\binom{i-1}{1} + \binom{i-1}{2} = \frac{(i-1)!}{1!(i-2)!} + \frac{(i-1)!}{2!(i-3)!} = \frac{i!}{2!(i-2)!} = \binom{i}{2}$$

...

SO, $(z_1+z_2)^i = z_1^i + \binom{i}{1} z_1^{i-1}\cdot z_2 + \binom{i}{2} z_1^{i-2} z_2^2 + \cdots + z_2^i$

AND this is the proof of the truth of equation (1).

Note: The "i"s used in this problem are integers, not complex number $\sqrt{-1}$.

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2. $-i$

The absolute value : $|-i| = 1$ 2

The argument : $\arg(-i) = -\frac{\pi}{2} + 2k\pi$

And the polar representation : $-i = 1 \cdot [\cos(-\frac{\pi}{2} + 2k\pi) + i\sin(-\frac{\pi}{2} + 2k\pi)]$

4. $-3+4i$

The absolute value : $|-3+4i| = \sqrt{3^2+4^2} = \sqrt{25} = 5$ 2

The argument : $\arg(-3+4i) = \tan^{-1}(-\frac{4}{3}) + 2k\pi = -\tan^{-1}(\frac{4}{3}) + 2k\pi$

The polar representation : $-3+4i = 5 \cdot [\cos(-\tan^{-1}(\frac{4}{3}) + 2k\pi) + i\sin(-\tan^{-1}(\frac{4}{3}) + 2k\pi)]$

6. $5-12i$

The absolute value : $|5-12i| = \sqrt{5^2+12^2} = \sqrt{169} = 13$ 2

The argument : $\arg(5-12i) = \tan^{-1}(-\frac{12}{5}) + 2k\pi = -\tan^{-1}(\frac{12}{5}) + 2k\pi$

The polar representation : $5-12i = 13[\cos(\tan^{-1}(-\frac{12}{5}) + 2k\pi) + i\sin(\tan^{-1}(-\frac{12}{5}) + 2k\pi)]$

8. $2-i$

The absolute value : $|2-i| = \sqrt{2^2+1^2} = \sqrt{5}$ 2

The argument : $\arg(2-i) = \tan^{-1}(-\frac{1}{2}) + 2k\pi$

The polar representation : $2-i = \sqrt{5} \cdot [\cos(\tan^{-1}(-\frac{1}{2}) + 2k\pi) + i\sin(\tan^{-1}(-\frac{1}{2}) + 2k\pi)]$

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10. $(1+i)^{29}$

$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$ ; $Arg(1+i) = \frac{\pi}{4}$

$1+i = \sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$(1+i)^{29} = (\sqrt{2})^{29}\,(\cos\frac{29\pi}{4} + i\sin\frac{29\pi}{4})$ 2

$= 2^{29/2}\,(\cos(8\pi - \frac{3\pi}{4}) + i\sin(8\pi - \frac{3\pi}{4}))$

$= 2^{29/2}\,(\cos(\frac{3\pi}{4}) - i\sin\frac{3\pi}{4})$

$= -2^{29/2} \cdot \frac{1+i}{\sqrt{2}}$

$= -2^{14} - 2^{14}i = -16384 - 16384i$

12. $(-1-i)^{36}$

$|-1-i| = \sqrt{2}$ ; $Arg(-1-i) = -\frac{3\pi}{4}$

$-1-i = \sqrt{2}\,(\cos(-\frac{3\pi}{4}) + i\sin(-\frac{3\pi}{4}))$ 2

$(-1-i)^{36} = (\sqrt{2})^{36}\,[\cos(-\frac{108\pi}{4}) + i\sin(-\frac{108\pi}{4})]$

$= 2^{18}\,[\cos(-26\pi - \pi) + i\sin(-26\pi - \pi)]$

$= 2^{18}\,[\cos\pi \;- i\sin\pi] = -2^{18}$

14. $(\sqrt{3}+i)^{15}$

$|\sqrt{3}+i| = \sqrt{3+1} = 2$ ; $Arg(\sqrt{3}+i) = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\sqrt{3}+i = 2[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}]$

$(\sqrt{3}+i)^{15} = 2^{15}\,[\cos(\frac{15\pi}{6}) + i\sin(\frac{15\pi}{6})]$ 2

$= 2^{15}\,[\cos(2\pi + \frac{\pi}{2}) + i\sin(2\pi + \frac{\pi}{2})]$

$= 2^{15}\,[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}]$

$= 2^{15}i$

18. $z^2 = 2-i$

$$2-i = \sqrt{5}\,[\cos(-26.57^\circ) + i\sin(-26.57^\circ)]$$

$$\therefore\quad z_1 = \sqrt[4]{5}\,[\cos(-13.28^\circ) + i\sin(-13.28^\circ)]$$

2

$$z_2 = \sqrt[4]{5}\,[\cos(166.72^\circ) + i\sin(166.72^\circ)]$$

20. $z^3 = 2+i$

$$2+i = \sqrt{5}\,[\cos 26.57^\circ + i\sin 26.57^\circ]$$

$$\therefore\quad z_1 = \sqrt[6]{5}\,[\cos 8.86^\circ + i\sin 8.86^\circ]$$

2

$$z_2 = \sqrt[6]{5}\,[\cos 128.86^\circ + i\sin 128.86^\circ]$$

$$z_3 = \sqrt[6]{5}\,[\cos(-111.14^\circ) + i\sin(-111.14^\circ)]$$

28. Let $z = (a, b)$

$$\therefore\ |z| = \sqrt{a^2+b^2}\ ,\quad |\operatorname{Re} z| = |a| \quad \text{and} \quad |\operatorname{Im} z| = |b|$$

Because,

$$a^2+b^2 \leqslant a^2+b^2+2|a|\cdot|b| = (|a|+|b|)^2$$

$$\sqrt{a^2+b^2} \leqslant \sqrt{(|a|+|b|)^2} = |a|+|b|$$

2

Therefore, $|z| \leqslant |\operatorname{Re} z| + |\operatorname{Im} z|$ ...... (1)

Again. Because $(|a|-|b|)^2 = a^2 - 2|a|\cdot|b| + b^2 \geqslant 0$,

$$a^2+b^2 \geqslant 2|a|\cdot|b|$$

$$\therefore\ 2(a^2+b^2) \geqslant 2\cdot|a|\cdot|b| + a^2+b^2 = (|a|+|b|)^2$$

$$\sqrt{2}\cdot\sqrt{a^2+b^2} \geqslant |a|+|b|$$

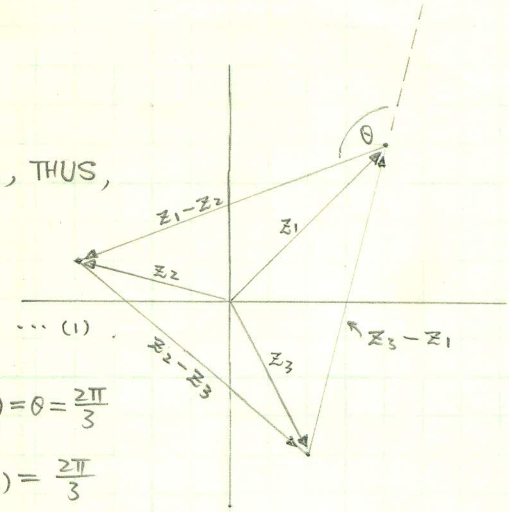
2

Thus, $\sqrt{2}\,|z| \geqslant |\operatorname{Re} z| + |\operatorname{Im} z|$ ...... (2)

From (1) and (2), we obtain

$$|z| \leqslant |\operatorname{Re} z| + |\operatorname{Im} z| \leqslant \sqrt{2}\cdot|z|$$

30. PROOF : See the figure on the right




FIRST, IF THE TRIANGLE IS EQUILATERAL, THUS,

$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$$

$$\therefore\ |z_1 - z_2|^2 = |z_2 - z_3| \cdot |z_3 - z_1| \quad \cdots (1)$$

And also $\quad \operatorname{Arg}(z_1 - z_2) - \operatorname{Arg}(z_3 - z_1) = \theta = \frac{2\pi}{3}$

$$\operatorname{Arg}(z_2 - z_3) - \operatorname{Arg}(z_1 - z_2) = \frac{2\pi}{3}$$

So, $\quad 2\operatorname{Arg}(z_1 - z_2) = \operatorname{Arg}(z_2 - z_3) + \operatorname{Arg}(z_3 - z_1) \quad \cdots (2)$

2

From the equations (1) and (2), we obtain

$$(z_1 - z_2)^2 = (z_2 - z_3) \cdot (z_3 - z_1)$$

Thus, $\quad z_1^2 - 2z_1z_2 + z_2^2 = z_2z_3 - z_1z_2 - z_3^2 + z_1z_3$

So, we get $\quad z_1^2 + z_2^2 + z_3^2 = z_1 \cdot z_2 + z_2 \cdot z_3 + z_3 \cdot z_1$

THEN, IF $\quad z_1^2 + z_2^2 + z_3^2 = z_1 \cdot z_2 + z_2 \cdot z_3 + z_3 \cdot z_1 \quad \cdots (3)$

We rewrite equation (3) as

$$(z_1 - z_2)^2 = (z_2 - z_3)(z_3 - z_1)$$

or, $\quad (z_2 - z_3)^2 = (z_3 - z_1)(z_1 - z_2)$

$$\therefore \quad |z_1 - z_2|^2 = |z_2 - z_3| \cdot |z_3 - z_1|$$

very good

2

$$|z_2 - z_3|^2 = |z_3 - z_1| \cdot |z_1 - z_2|$$

$$\frac{|z_1 - z_2|^2}{|z_2 - z_3|^2} = \frac{|z_2 - z_3| \cdot |z_3 - z_1|}{|z_3 - z_1| \cdot |z_1 - z_2|} = \frac{|z_2 - z_3|}{|z_1 - z_2|}$$

So, $\quad |z_1 - z_2|^3 = |z_2 - z_3|^3$

$$|z_1 - z_2| = |z_2 - z_3|$$

With the same method, we can obtain

$$|z_2 - z_3| = |z_3 - z_1|$$

SO, THE TRIANGLE IS EQUILATERAL.

34. PROOF: LET $z_1=(a_1, b_1)$ , $z_2=(a_2, b_2)$

$$z_1+z_2=(a_1+a_2, b_1+b_2)$$

$$z_1-z_2=(a_1-a_2, b_1-b_2)$$

$$|z_1+z_2|^2=(a_1+a_2)^2+(b_1+b_2)^2$$

$$=a_1^2+a_2^2+b_1^2+b_2^2+2a_1a_2+2b_1b_2$$

2

$$|z_1-z_2|^2=(a_1-a_2)^2+(b_1-b_2)^2$$

$$=a_1^2+a_2^2+b_1^2+b_2^2-2a_1a_2-2b_1b_2$$

$$\therefore |z_1+z_2|^2+|z_1-z_2|^2=2(a_1^2+a_2^2+b_1^2+b_2^2)$$

$$=2[(a_1^2+b_1^2)+(a_2^2+b_2^2)]$$

$$=2(|z_1|^2+|z_2|^2)$$

#

Mothed 2: $|z_1+z_2|^2+|z_1-z_2|^2$

$$=(z_1+z_2)(\overline{z_1+z_2})+(z_1-z_2)(\overline{z_1-z_2})$$

$$=(z_1+z_2)(\bar{z}_1+\bar{z}_2)+(z_1-z_2)(\bar{z}_1-\bar{z}_2)$$

$$=z_1\bar{z}_1+z_1\bar{z}_2+z_2\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_1-z_1\bar{z}_2-z_2\bar{z}_1+z_2\bar{z}_2$$

$$=2(z_1\bar{z}_1+z_2\bar{z}_2)=2(|z_1|^2+|z_2|^2)$$

1. Solve $z^2+2iz+i=0$

(a) $z^2+2iz+i = z^2+2iz+(i)^2-(i)^2+i$

$= (z+i)^2+i+1 = 0$

$(z+i)^2 = -i-1 = \sqrt{2}\,(\cos(-\frac{3\pi}{4}+2k\pi)+i\sin(-\frac{3\pi}{4}+2k\pi))$

$z_1+i = \sqrt[4]{2}\,[\cos(-\frac{3\pi}{8}+2k_1\pi)+i\sin(-\frac{3\pi}{8}+2k_1\pi)]$

$= 1.1892[0.3827 - i\,0.9239]$

$= 0.4551 - i\,1.0987$

$z_1 = (0.4551, -2.0987)$

$z_2+i = \sqrt[4]{2}\,[\cos(\frac{5\pi}{8}+2k_2\pi)+i\sin(\frac{5\pi}{8}+2k_2\pi)]$

$= 1.1892[-0.3827 + i\,0.9239]$

$= -0.4551 + i\,1.0987$

$z_2 = (-0.4551, 0.0987)$

answer correct but the problem said to use the quadratic formula

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(b) Let $z=(x,y)$

$(x,y)^2+(0,2)(x,y)+(0,1)=(0,0)$

$(x^2-y^2-2y,\ 2xy+2x+1) = (0,0)$

$\begin{cases} x^2-y^2-2y=0 \\ 2xy+2x+1=0 \end{cases}$ $\quad u(x,y) = x^2-y^2-2y$, $\quad v(x,y) = 2xy+2x+1$

$u_x(x,y) = 2x \qquad u_y(x,y) = -2y-2$

$v_x(x,y) = 2y+2 \qquad v_y(x,y) = 2x$

2

1. $(x_0, y_0) = (-\frac{1}{2}, 0)$

$\begin{cases} -0.25 = -1\cdot\Delta x - 2\Delta y \\ 0 = 2\Delta x - \Delta y \end{cases} \Rightarrow \begin{cases} \Delta x = \frac{1}{20} = 0.05 \\ \Delta y = \frac{1}{10} = 0.1 \end{cases}$

2. $(x_1, y_1) = (-0.45, 0.1)$

$\begin{cases} -0.0075 = -0.9\Delta x - 2.2\Delta y \\ -0.01 = 2.2\Delta x - 0.9\Delta y \end{cases} \Rightarrow \begin{cases} \Delta x = -0.0027 \\ \Delta y = 0.0045 \end{cases}$

1 — (b) (CONT.)

3. $(x_2, y_2) = (-0.4527, 0.1045)$

$$\begin{cases} 0.0150 = -0.9054\,\Delta x - 2.2090\,\Delta y \\ 0 = 2.2090\,\Delta x - 0.9054\,\Delta y \end{cases} \Rightarrow \begin{cases} \Delta x = -0.0024 \\ \Delta y = -0.0058 \end{cases}$$

$(x_3, y_3) = (-0.4551, 0.0987)$ ( same as $z_2$ in (a) )

2. Find $\pi$ from the equation

$$(5,1)^4 (1,-1) = (956, 4)$$

SOLUTION: $4 \arctan \frac{1}{5} - \arctan\left(\frac{4}{956}\right) = -\arctan(-1) = \frac{\pi}{4}$

$$\pi = 4\left[4 \arctan \tfrac{1}{5} - \arctan\left(\tfrac{1}{239}\right)\right]$$

2

4 only had to use 3 terms

$$\arctan\left(\tfrac{1}{5}\right) \doteq \left(\tfrac{1}{5}\right) - \left(\tfrac{1}{5}\right)^3/3 + \left(\tfrac{1}{5}\right)^5/5 - \left(\tfrac{1}{5}\right)^7/7 + \left(\tfrac{1}{5}\right)^9/9 - \left(\tfrac{1}{5}\right)^{11}/11$$

$$= 0.197395560$$

$$\arctan\left(\tfrac{1}{239}\right) = \left(\tfrac{1}{239}\right) - \left(\tfrac{1}{239}\right)^3/3 + \left(\tfrac{1}{239}\right)^5/5 - \left(\tfrac{1}{239}\right)^7/7 + \left(\tfrac{1}{239}\right)^9/9 - \left(\tfrac{1}{239}\right)^{11}/11$$

$$= 0.004184076$$

$$\therefore \pi = 4 \times (4 \times 0.197395560 - 0.004184076)$$

$$= 3.141592651$$

MAT 417 H.W. #5 Complex Var. Benmei Chen

P21, 32. PROVE THAT $\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$

PROOF: $p(n)$ is the statement $\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$

1. $p(1)$: $|z_1| \leq |z_1|$ is true.

2. Suppose $p(n)$ is true, thus

$$\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$$

2

3. $p(n+1)$:

$$\left|\sum_{k=1}^{n+1} z_k\right| = \left|\sum_{k=1}^{n} z_k + z_{n+1}\right|$$

$$\leq \left|\sum_{k=1}^{n} z_k\right| + |z_{n+1}| \qquad [\text{using } p(2)]$$

$$\leq \sum_{k=1}^{n} |z_k| + |z_{n+1}| \qquad [\text{using } p(n)]$$

$$= \sum_{k=1}^{n+1} |z_k|$$

SO, THE STATEMENT $\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$ IS TRUE.

P50, 21. $1 + \cos x + \cos 2x + \cdots + \cos nx$

23. $\sin x + \sin 2x + \cdots + \sin nx$

Solution: Let $z = \cos x + i \sin x$, so

$z^0 = 1$

$z^1 = \cos x + i \sin x$

$z^2 = \cos 2x + i \sin 2x$

$z^3 = \cos 3x + i \sin 3x$

$\vdots$

$z^n = \cos nx + i \sin nx$

$$(1 + \cos x + \cos 2x + \cdots + \cos nx) + i(\sin x + \sin 2x + \cdots + \sin nx)$$

$$= z^0 + z^1 + \cdots + z^n$$

(TO BE CONTD. NEXT PAGE)

MAT 417. H.W. #5 Complex Vax Benmei Chen

$P_{50}$, #21 'N' #23 (CONT.)

$$= 1 + z^1 + z^2 + \cdots + z^n$$

$$= (1 - z^{n+1}) / (1 - z)$$

$$= [1 - \cos(n+1)x - i\sin(n+1)x] / (1 - \cos x - i \sin x)$$

$$= [1 - \cos(n+1)x - i \cdot \sin(n+1)x] \cdot (1 - \cos x + i \sin x) / 2(1 - \cos x)$$

$$= (1 - \cos x - \cos(n+1)x + \cos nx) / 2(1 - \cos x)$$

$$+ i \cdot (\sin x + \sin nx - \sin(n+1)x) / 2(1 - \cos x)$$

SO,

21. $1 + \cos x + \cos 2x + \cdots + \cos nx$

$$= [1 - \cos x - \cos(n+1)x + \cos nx] / 2(1 - \cos x)$$

$$= \sin\frac{(n+1)x}{2} \cos\frac{nx}{2} / \sin\frac{x}{2}$$

this step is not very obvious

23. $\sin x + \sin 2x + \cdots + \sin nx$

$$= [\sin x + \sin nx - \sin(n+1)x] / 2(1 - \cos x)$$

$$= \sin\frac{(n+1)x}{2} \sin\frac{nx}{2} / \sin\frac{x}{2}$$

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show a bit more work please.

MAT 417 H.W. #6 | Complex Var. | Benmei Chen

In Exercises 1–10, classify the sets according to the terms open, closed, bounded, connected, and simply connected.

1. $|z+3| < 2$

2 open, bounded, connected and simply connected.

2. $|\mathrm{Re}\, z| < 1$

2 open, unbounded, connected, but not simply connected.

3. $|\mathrm{Im}\, z| > 1$

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2 open, unbounded, unconnected.

4. $0 < |z-1| \leq 1$

2 not open, not closed, bounded, connected, but not simply connected.

5. $|z| \leq \mathrm{Re}\, z + 2$

2 closed, unbounded and simply connected.

6. $|z-1| - |z+1| > 2$

2 open, unbounded and simply connected

7. $|z+1| + |z+i| \geq 2$

2 closed, unbounded and connected, but not simply connected.

8. $|z-1| < \mathrm{Im}\, z$

2 open, unbounded, and not connected.

9. $2\sqrt{2} < |z-1| + |z+1| < 3$

2 open, bounded, connected, but not simply connected.

10. $\left| |z-i| - |z+i| \right| < 1$

2 open, unbounded, connected, but not simply connected.

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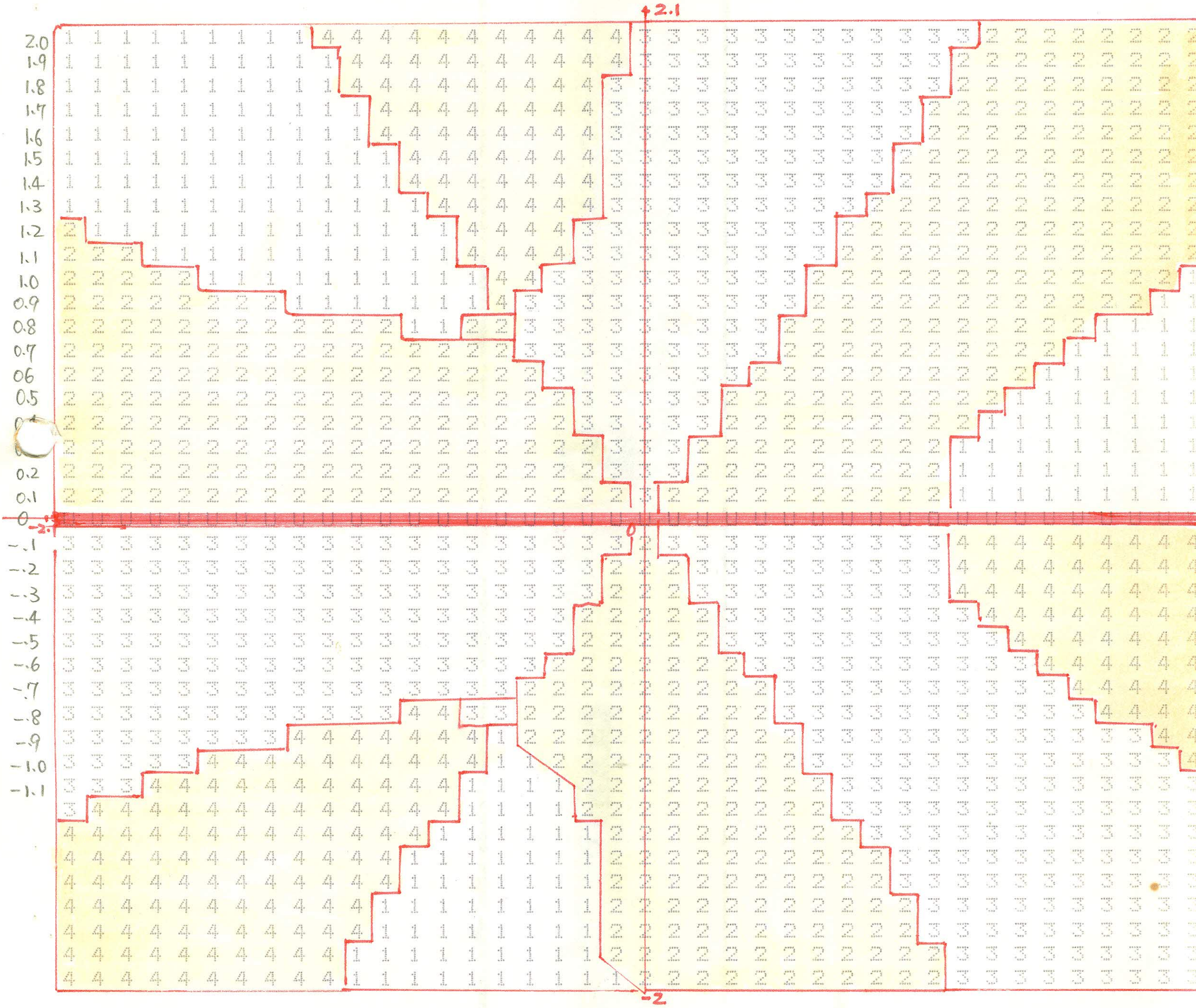
MAT 417 HOMEWORK #6 COMPLEX VAX. BENMEI CHEN

where is $z^2+2iz+i$ ?

F(z) = z*z*z -1 $= (x^3-3xy^2-1,\ 3x^2y-y^3)$

good




* "U"s represent that $f(z)=(s,0)$, $s\neq 0$ ; "0" represents $f(z)=(0,0)$.

MAT 417 H.W. #7 Complex Var. BENMEI CHEN

P39. 1. $f(z) = e^x(\cos y + i\sin y)$

SOLUTION: $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$

$$\frac{\partial u(x,y)}{\partial x} = e^x \cos y \qquad \frac{\partial v(x,y)}{\partial x} = e^x \sin y$$

2

$$\frac{\partial u(x,y)}{\partial y} = -e^x \sin y \qquad \frac{\partial v(x,y)}{\partial y} = e^x \cos y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin y$$

$f(z)$ satisfies the Cauchy-Riemann equations.

2. $f(z) = \cos x \cosh y - i \sin x \sinh y$

SOLUTION:

$$u(x,y) = \cos x \cdot \cosh y = \cos x \cdot (e^y + e^{-y})/2$$

$$v(x,y) = -\sin x \cdot \sinh y = \sin x \cdot (e^{-y} - e^y)/2$$

$$\frac{\partial u}{\partial x} = -\sin x \cdot (e^y + e^{-y})/2$$

2

$$\frac{\partial u}{\partial y} = \cos x \cdot (e^y - e^{-y})/2$$

$$\frac{\partial v}{\partial x} = \cos x \cdot (e^{-y} - e^y)/2 = -\cos \cdot (e^y - e^{-y})/2$$

$$\frac{\partial v}{\partial y} = \sin x \,(-e^{-y} - e^y)/2 = -\sin x\,(e^{-y} + e^y)/2$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

SO, $F(z)$ satisfies the Cauchy-Riemann equations.

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P39. 3. $f(z) = \sin x \cosh y + i \cos x \cdot \sinh y$

SOLUTION:

$$u(x,y) = \sin x \cdot \cosh y = \sin x \cdot (e^{y} + e^{-y})/2$$

$$v(x,y) = \cos x \cdot \sinh y = \cos x \cdot (e^{y} - e^{-y})/2$$

$$\frac{\partial u}{\partial x} = \cos x \cdot \cosh y$$

$$\frac{\partial u}{\partial y} = \sin x \; (e^{-y} - e^{-y})/2 = \sin x \sinh y$$

$$\frac{\partial v}{\partial x} = -\sin x \cdot \sinh y$$

$$\frac{\partial v}{\partial y} = \cos x \cdot (e^{y} + e^{-y})/2 = \cos x \cdot \cosh y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

SO, $f(z)$ stastifies the Cauchy-Riemann equations.

4. $f(z) = e^{x^2-y^2}(\cos 2xy + i \sin 2xy)$

SOLUTION:

$$u(x,y) = e^{x^2-y^2} \cos 2xy$$

$$v(x,y) = e^{x^2-y^2} \sin 2xy$$

$$\frac{\partial u}{\partial x} = e^{x^2-y^2} \cdot 2x \cos 2xy - e^{x^2-y^2} \cdot \sin 2xy \cdot 2y$$

$$\frac{\partial u}{\partial y} = e^{x^2-y^2} \cdot (-2y) \cos 2xy - e^{x^2-y^2} \cdot \sin 2xy \cdot 2x$$

$$\frac{\partial v}{\partial x} = e^{x^2-y^2} \, 2x \cdot \sin 2xy + e^{x^2-y^2} \cos 2xy \cdot 2y$$

$$\frac{\partial v}{\partial y} = e^{x^2-y^2} (-2y) \sin 2xy + e^{x^2-y^2} \cos 2xy \cdot 2x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

SO, $f(z)$ stastifies the Cauchy-Riemann equations.

IN EXERCISES 1 − 7 , EXPRESS EACH NUMBER IN THE FORM $x+iy$

2. $e^{(1+\pi i)/2} = e^{\frac{1}{2}} \cdot e^{i\frac{\pi}{2}} = e^{\frac{1}{2}} \cdot (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 0 + i \cdot e^{\frac{1}{2}}$ Z

4. $e^{(-1+\pi i)/4} = e^{-\frac{1}{4}} \cdot e^{i\frac{\pi}{4}} = e^{-\frac{1}{4}} \cdot (\cos\frac{\pi}{4} + i \cdot \sin\frac{\pi}{4}) = \frac{\sqrt{2}}{2} e^{-\frac{1}{4}} + i\frac{\sqrt{2}}{2} e^{-\frac{1}{4}}$ Z

6. $e^{-i\pi/2} = \cos(-\frac{\pi}{2}) + i \cdot \sin(-\frac{\pi}{2}) = 0 - i$ Z

IN EXERCISES 8−10 , FIND ALL THE COMPLEX NUMBERS $z$ THAT SATISFY THE GIVEN CONDITIONS.

8. $e^{2z} = -1$ , Let $z = x+iy$

$e^{2\cdot z} = e^{2(x+iy)} = e^{2x} \cdot (\cos 2y + i\sin 2y) = -1$

$|e^{2z}| = e^{2x} = |-1| = 1 \Rightarrow x = 0$

$\therefore \cos 2y + i \cdot \sin 2y = -1 + i \cdot 0$

$\Rightarrow \begin{cases} \cos 2y = -1 \\ \sin 2y = 0 \end{cases} \Rightarrow 2y = 2k\pi + \pi$ , $k = 0, \pm 1, \pm 2, \cdots$

$\therefore y = k\pi + \frac{\pi}{2}$ , $k = 0, \pm 1, \pm 2, \cdots$ Z

$z = (0, k\pi + \frac{\pi}{2})$ $k = 0, \pm 1, \pm 2, \cdots$

10. $e^{iz} = -1$ , Let $z = x+iy$

$e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}(\cos x + i\sin x) = -1$

$|e^{iz}| = e^{-y} = 1 \Rightarrow y = 0$ Z

$\cos x + i\sin x = -1 + i \cdot 0$

$\Rightarrow \begin{cases} \cos x = -1 \\ \sin x = 0 \end{cases} \Rightarrow x = (2k+1)\pi$ , $k = 0, \pm 1, \pm 2, \cdots$

$\therefore z = (2k\pi + \pi, 0)$ , $k = 0, \pm 1, \pm 2, \cdots$

12. SHOW THAT $\overline{(e^z)} = e^{\bar{z}}$ ; LET $z = x+iy$ Z

$\overline{(e^z)} = \overline{(e^{(x+iy)})} = \overline{e^x \cdot (\cos y + i\sin y)} = e^x(\cos y - i\sin y)$

$= e^x(\cos(-y) + i\sin(-y)) = e^x \cdot e^{-iy} = e^{(x-iy)} = e^{\bar{z}}$

Homework # 8 Complex Var. Benmei Chen

IN EXERCISES 13-20, CALCULATE EACH NUMBER USING De Moiver's THEOREM.

14. $(-1+i)^{17}$

$(-1+i) = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\cdot\sin\frac{3\pi}{4}\right)$

$(-1+i)^{17} = (\sqrt{2})^{17}\cdot\left(\cos\frac{3\times17}{4}\pi + i\sin\frac{3\times17}{4}\pi\right)$ 2

$= 256\sqrt{2}\left(\cos\left(12\pi+\frac{3\pi}{4}\right) + i\sin\left(12\pi+\frac{3\pi}{4}\right)\right)$

$= 256\sqrt{2}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -256 + i256$

16. $(2+2i)^{12}$ 2

$2+2i = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$(2+2i)^{12} = (2\sqrt{2})^{12}\left(\cos\frac{\pi}{4}\times12 + i\sin\frac{\pi}{4}\times12\right) = -2^{12}\cdot2^{6} = -2^{18}$

18. $(-\sqrt{3}+i)^{13}$

$(-\sqrt{3}+i) = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ 2

$(-\sqrt{3}+i)^{13} = 2^{13}\left(\cos\frac{5\pi}{6}\times13 + i\cdot\sin\frac{5\pi}{6}\times13\right)$

$= 2^{13}\left(\cos\left(10\pi+\frac{5\pi}{6}\right) + i\sin\left(10\pi+\frac{5\pi}{6}\right)\right)$

$= 2^{13}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2^{13}\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = -2^{12}\sqrt{3} + i2^{12}$

20. $\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^{19} = \left(\cos\left(-\frac{\pi}{4}\right), \sin\left(-\frac{\pi}{4}\right)\right)^{19}$

$= \left(\cos\left(-\frac{\pi}{4}\times19\right), \sin\left(-\frac{\pi}{4}\times19\right)\right)$ 2

$= \left(\cos\left(-4\pi-\frac{3\pi}{4}\right), \sin\left(-4\pi-\frac{3\pi}{4}\right)\right)$

$= \left(\cos\left(\frac{-3\pi}{4}\right), \sin\left(-\frac{3\pi}{4}\right)\right)$

$= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cdot i$

IN EXERCISES 1–8, EXPRESS EACH OF THE NUMBERS IN THE FORM $x+iy$.

2. $\cos(-i) = \frac{1}{2}(e^{i(-i)} + e^{-i(-i)}) = \frac{1}{2}(e + e^{-1}) + 0\cdot i$ 2

4. $\sinh \pi i = \frac{1}{2}(e^{\pi i} - e^{-\pi i}) = i \sin\pi = 0$ 2

6. $\tan 2i = \sin 2i / \cos 2i = (e^{i(2i)} - e^{-i(2i)}) / i(e^{i(2i)} + e^{-i(2i)})$ 2

$= (e^{-2} - e^{2}) / i(e^{-2} + e^{2})$

$= 0 - i \cdot \dfrac{1-e^4}{1+e^4}$

8. $\cosh(\pi i/4) = \frac{1}{2}(e^{i\cdot\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}) = \frac{1}{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$ 2

$= \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

IN EXERCISES 9–12, FIND ALL COMPLEX NUMBERS $z$ SUCH THAT THE GIVEN CONDITIONS ARE MET.

10. $\cos z = -i \sin z$, LET $z = x+iy$.

$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)})$

$= \frac{1}{2}(e^{(-y+ix)} + e^{(y-ix)})$

$= \frac{1}{2}[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)]$ 2

$= \frac{1}{2}[(e^{-y}+e^{y})\cos x + i\cdot(e^{-y}-e^{y})\sin x]$

$-i\sin z = -\frac{1}{2}[e^{iz} - e^{-iz}] = -\frac{1}{2}[e^{i(x+iy)} - e^{-i(x+iy)}]$

$= -\frac{1}{2}[e^{(-y+ix)} - e^{(y-ix)}]$

$= -\frac{1}{2}[(e^{-y}-e^{y})\cos x + i(e^{-y}+e^{y})\sin x]$

$\therefore \Rightarrow \begin{cases} (e^{-y}+e^{y})\cos x = (e^{y}-e^{-y})\cos x \\ (e^{-y}-e^{y})\sin x = +(e^{y}+e^{-y})\sin x \end{cases}$

$\Rightarrow \begin{cases} 2e^{-y}\cos x = 0 \\ 2e^{-y}\sin x = 0 \end{cases} \Rightarrow \begin{cases} \cos x = 0 \\ \sin x = 0 \end{cases}$ no exists

12. $\cosh z = i$, LET $z = x+iy$

$$\cosh z = \tfrac{1}{2}(e^{z}+e^{-z}) = \tfrac{1}{2}(e^{(x+iy)}+e^{(-x-iy)})$$

$$= \tfrac{1}{2}\left(e^{x}(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y)\right)$$

$$= \tfrac{1}{2}\left[(e^{x}+e^{-x})\cos y + i(e^{x}-e^{-x})\sin y\right] = i$$

$$\Rightarrow \begin{cases} \tfrac{1}{2}(e^{x}+e^{-x})\cos y = 0 \\ \tfrac{1}{2}(e^{x}-e^{-x})\sin y = 1 \end{cases} \Rightarrow \begin{cases} \cos y = 0 \\ (e^{x}-e^{-x})\sin y = 2 \end{cases}$$

$\cos y = 0 \Rightarrow y = 2k\pi \pm \frac{\pi}{2}$, $k = 0, \pm1, \pm2, \cdots$

$\Rightarrow \sin y = \pm 1$

$\therefore\ e^{x} - e^{-x} = \pm 2$

$(e^{x})^{2} - 1 = \pm 2e^{x}$

Let $u = e^{x}$

$\therefore\ u^{2} \pm 2u - 1 = 0$

$u_1 = 0.618 \Rightarrow x_1 = -0.481$

$u_2 = 1.618 \Rightarrow x_2 = 0.481$

$\therefore\ z_1 = (-0.481,\ 2k\pi + \frac{\pi}{2})$

$z_2 = (0.481,\ 2k\pi + \frac{\pi}{2})$, $k = 0, \pm1, \pm2, \cdots$

still, very good work!

14. SHOW THAT $\overline{\sin z} = \sin \bar{z}$

According to Homework #8. Problem 12. $\overline{(e^{z})} = e^{\bar{z}}$

$$\overline{\sin z} = \overline{\tfrac{1}{2i}(e^{iz} - e^{-iz})} = \overline{\tfrac{1}{2i}}\cdot(\overline{e^{iz}} - \overline{e^{-iz}})$$

$$= (\overline{-\tfrac{1}{2}i})\cdot(e^{\bar{i}\cdot\bar{z}} - e^{\overline{-i}\cdot\bar{z}})$$

$$= -\tfrac{1}{2i}\cdot(e^{-i\cdot\bar{z}} - e^{i\bar{z}}) = \tfrac{1}{2i}(e^{i\bar{z}} - e^{-i\bar{z}})$$

$$= \sin \bar{z}.$$

2

IN EXERCISES 16 – 21 , PROVE THE IDENTITIES.

16. $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \cdot \sin z_2$

$$\sin z_1 \cos z_2 \pm \cos z_1 \cdot \sin z_2 = \frac{1}{2i}(e^{iz_1} - e^{-iz_1}) \cdot \frac{1}{2}(e^{iz_2} + e^{-iz_2})$$

$$\pm \frac{1}{2}(e^{iz_1} + e^{-iz_1}) \cdot \frac{1}{2i}(e^{iz_2} - e^{-iz_2})$$

$$= \frac{1}{2i} \cdot \Big[ \frac{1}{2}\left(e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{-i(z_1-z_2)} - e^{-i(z_1+z_2)}\right)$$

$$\pm \frac{1}{2}\left(e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{-i(z_1-z_2)} - e^{-i(z_1+z_2)}\right)\Big]$$

$$\therefore \sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right] = \sin(z_1 + z_2)$$

$$\sin z_1 \cos z_2 - \cos z_2 \sin z_1 = \frac{1}{2i}\left[e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}\right] = \sin(z_1 - z_2)$$

SO, $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.

18. $\sin(-z) = -\sin z$ , $\cos(-z) = \cos z$

$$\sin(-z) = \frac{1}{2i}\left(e^{i(-z)} - e^{-i(-z)}\right) = \frac{1}{2i}\left(e^{-iz} - e^{iz}\right)$$

$$= -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\sin z$$

$$\cos(-z) = \frac{1}{2}\left(e^{i(-z)} + e^{-i(-z)}\right) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$= \cos z.$$

Homework # 10 | Complex Var. | Benmei Chen

IN EXERCISES 1-6, FIND ALL THE VALUES OF THE GIVEN EXPRESSIONS.

2. $\log(1+i) = \log|1+i| + i\cdot\arg(1+i)$ 2

$= \log\sqrt{2} + i\cdot(\frac{\pi}{4} + 2k\pi)$ , $k=0,\pm1,\pm2,\cdots$

2

4. $1^i = e^{i\log 1} = e^{i\cdot(\log 1 + i\cdot\arg(1))} = e^{i\cdot i\cdot(2k\pi)} = e^{-2k\pi}$ , $k=0,\pm1,\pm2,\cdots$

6. $(1+i)^{1+i} = e^{(1+i)\log(1+i)} = e^{(1+i)(\log\sqrt{2} + i(\frac{\pi}{4}+2k\pi))}$ $k=0,\pm1,\pm2,\cdots$

$= e^{\log\sqrt{2} - (\frac{\pi}{4}+2k\pi)}\cdot\cos(\log\sqrt{2}+\frac{\pi}{4}) + i\cdot e^{\log\sqrt{2}-(\frac{\pi}{4}+2k\pi)}\cdot\sin(\log\sqrt{2}+\frac{\pi}{4})$ 2

IN EXERCISES 7-10, FIND THE PRINCIPAL VALUES OF THE GIVEN EXPRESSIONS.

8. $\log(1+i)$ 2

$\mathrm{Log}(1+i) = \log|1+i| + i\,\mathrm{Arg}(1+i) = \log\sqrt{2} + i\frac{\pi}{4}$

10. $(1+i)^{1+i} = e^{(1+i)\mathrm{Log}(1+i)}$ 2

$= e^{(1+i)(\log\sqrt{2} + i\cdot\frac{\pi}{4})}$

$= e^{\log\sqrt{2}-\frac{\pi}{4}}\cdot\cos(\log\sqrt{2}+\frac{\pi}{4}) + i\cdot e^{\log\sqrt{2}-\frac{\pi}{4}}\cdot\sin(\log\sqrt{2}+\frac{\pi}{4})$

18 SHOW THAT $\mathrm{Log}(i^3) \neq 3\,\mathrm{Log}\,i$

$-\pi/2$ ~~2~~ 1

SHOW: $\mathrm{Log}(i^3) = \mathrm{Log}(-i) = \log|-i| + i(\frac{\pi}{2}) = i(\frac{\pi}{2})$

$3\cdot\mathrm{Log}\,i = 3\cdot(\log|i| + i\cdot\frac{\pi}{2}) = i\cdot\frac{3\pi}{2}$ Why?

$\therefore \mathrm{Log}(i^3) \neq 3\,\mathrm{Log}\,i$ $-\frac{\pi}{2}$ & $\frac{3\pi}{2}$ are not on the same branch

20. Is 1 raised to any power always equal to 1?

NO. 2

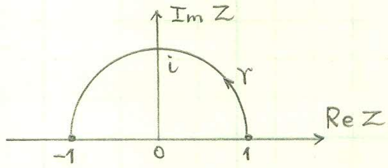
SEE PROBLEM # 4 , $1^i = e^{-2k\pi}$ , $k=0,\pm1,\pm2,\cdots$

WHEN $k=-2$ , $e^{-2k\pi} = e^{4\pi} = 286751.3148$.

MAT 417 H.W. # 12 Complex Vax. Benmei Chen

IN EXERCISES 2-5, DETERMINE PWS PARAMETRIZATIONS FOR THE INDICATED ARCS OR CURVES.

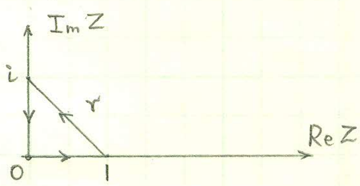
2. SEMICIRCLE FROM 1 to −1




$\gamma: z(t) = \cos t + i \sin t \qquad 0 \le t \le \pi$

2

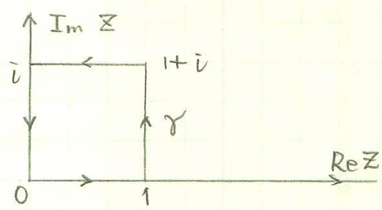
3. TRIANGLE




$$\gamma: \ z(t) = \begin{cases} i\cdot(1-t) & 0 \le t \le 1 \\ t-1 & 1 \le t \le 2 \\ (3-t)+i(t-2) & 2 \le t \le 3 \end{cases}$$

2

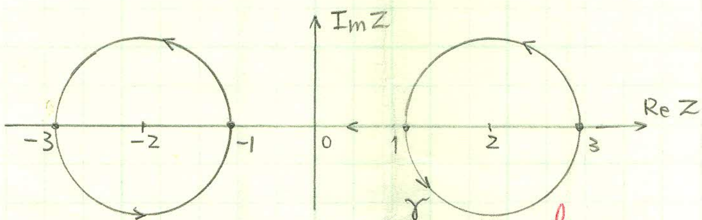
4. SQUARE




$$\gamma: z(t) = \begin{cases} i(1-t) & 0 \le t \le 1 \\ t-1 & 1 \le t \le 2 \\ 1+i(t-2) & 2 \le t \le 3 \\ 4-t+i & 3 \le t \le 4 \end{cases}$$

2

5. BARBELL BEGINNING AT 1



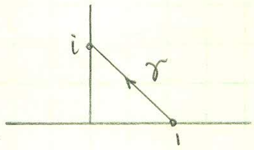

2

-ok

$$\gamma: \quad z(t) = \begin{cases} (\cos \pi t + 2) + i \sin \pi t & -1 \le t \le 1 \\ 2-t & 1 \le t \le 3 \\ \cos\pi(t-3) - 2 + i \sin \pi(t-3) & 3 \le t \le 5 \end{cases}$$

$\cancel{\cos \pi(t+1) - 2 + i \sin \pi(t+1)}$

10. EVALUATE $\int_\gamma y\,dz$, WHERE $\gamma$ is THE STRAIGHT LINE JOINING 1 TO $i$



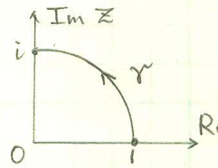

$\gamma$: $z(t) = (1-t) + i\cdot t \qquad 0 \le t \le 1$

$dz(t) = -1 + i$

2

$$\int_\gamma y\,dz = \int_0^1 t\cdot(-1+i)\,dt = -\tfrac{1}{2}t^2\Big|_0^1 + i\cdot\tfrac{1}{2}t^2\Big|_0^1 = -\tfrac{1}{2} + i\cdot\tfrac{1}{2}$$

11. EVALUATE $\int_\gamma y\,dz$, WHERE $\gamma$ is THE ARC IN THE FIRST QUADRANT ALONG $|z|=1$ JOINING 1 TO $i$.




$\gamma$: $z(t) = \cos t + i\sin t \qquad 0 \le t \le \frac{\pi}{2}$

$dz(t) = -\sin t + i\cos t$

$$\int_\gamma y\,dz = \int_0^{\frac{\pi}{2}} \sin t\,(-\sin t + i\cos t)\,dt$$

2

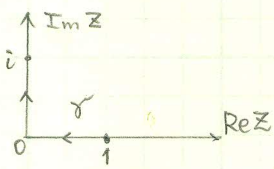
$$= \int_0^{\frac{\pi}{2}} -\sin^2 t\,dt + i\cdot\int_0^{\frac{\pi}{2}} \sin t\cdot\cos t\,dt$$

$$= -\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}} (1-\cos 2t)\,dt + i\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2t\,dt$$

$$= -\frac{1}{2}\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}i\left(-\frac{1}{2}\cos 2t\right)\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{4} + i\cdot\frac{1}{2}$$

12. EVALUATE $\int_\gamma y\,dz$, WHERE $\gamma$ is THE ARC ALONG THE COORDINATE AXES JOINING 1 TO $i$.




$$\gamma: \quad z(t) = \begin{cases} 1-t & 0 \le t \le 1 \\ i(t-1) & 1 \le t \le 2 \end{cases}$$

$$dz(t) = \begin{cases} -1 & 0 \le t \le 1 \\ i & 1 \le t \le 2 \end{cases}$$

2

$$\int_\gamma y\,dz = \int_{\gamma'} y\,dz + \int_{\gamma''} y\,dz$$

$$= \int_0^1 0\cdot(-1)\,dt + \int_1^2 (t-1)\cdot i\,dt$$

$$= i\int_1^2 (t-1)\,dt$$

$$= i\left(\frac{t^2}{2} - t\right)\Big|_1^2 = i\cdot\frac{1}{2}$$

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14. Evaluate the integral $\int (z-a)^n\,dz$, n: an integer, around the circle $|z-a|=R$.

$F(z)=\frac{1}{n+1}(z-a)^{n+1}$   $n\neq -1$

$F'(z)=(z-a)^n$

F(z) is an analytic function with a continuous derivative F'(z).

$\int_{|z-a|=R}(z-a)^n\,dz=0$   2

When $n=-1$,

$F(z)=\log(z-a)=\log|z-a|+i\arg(z-a)$   1

$\therefore \int_{|z-a|=R}(z-a)^{-1}\,dz=i\cdot 2k\pi$   $(k=0,\pm1,\pm2,\dots)$

$\int_{|z-a|=R}(z-a)^{-1}\,dz=2\pi i$

can't use anti-derivative log() as the derivative F(z) = log() is not defined everywhere on the arc |z-a|=R (not defined at the branch cut)

15. Evaluate $\int_\gamma e^z\,dz$, where $\gamma$ is the straight-line path joining 1 to $i$

Because $(e^z)'=e^z$,

$\int_\gamma e^z\,dz=-e^1+e^i=-e+(\cos 1+i\sin 1)$   2

$=(\cos 1-e)+i(\sin 1)$

MAT 417 H.W. #13 Complex Variable Benmei Chen

P86. USE GREEN'S THEOREM FOR Exercises 2-4, WHERE $A$ EQUALS THE AREA OF $G$ AND $\partial G$ IS THE BOUNDARY OF $G$.

2. SHOW THAT $\int_{\partial G} x\,dz = iA$

$f(z) = x$ ; $u(x,y) = x$, $v(x,y) = 0$, $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$

2

ALL ARE CONTINUOUS ON THE COMPLEX PLANE.

$$\int_{\partial G} x\,dz = -\iint_G \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_G \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

$$= i\iint_G dx\,dy = iA$$

3. SHOW THAT $\int_{\partial G} y\,dz = -A$

$u(x,y) = y$, $v(x,y) = 0$, $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 1$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$

ALL ARE CONTINUOUS ON THE COMPLEX PLANE.

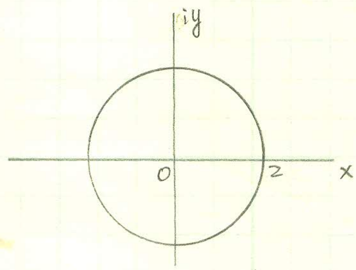
2

$$\int_{\partial G} y\,dz = -\iint_G \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_G \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

$$= -\iint_G (0+1)\,dx\,dy = -A$$

P96. 15. Without computing the integral, show that

$$\left|\int_{|z|=2} \frac{dz}{z^2+1}\right| \leq \frac{4\pi}{3}$$




2

$$|z^2+1| \leq |z^2|+1 \leq |z|^2+1 \leq 5$$

$$\therefore \left|\frac{1}{z^2+1}\right| \geq \frac{1}{5}$$

$$|dz| = 2\pi$$

$$\left|\int_{|z|=2} dz/(z^2+1)\right| \leq \int_{|z|=2} \left|\frac{1}{z^2+1}\right| \cdot |dz|$$

$$\leq \frac{1}{5}\cdot 2\pi = \frac{2\pi}{5} < \frac{4\pi}{3}$$

MAT 417 H.W. # | Complex Variables | Benmei Chen



IN EXERCISES 1–3, EVALUATE THE INTEGRAL

$$\int_\gamma \frac{dz}{(z-a)(z-b)}$$

BY DECOMPOSING THE INTEGRAND INTO PARTIAL FRACTIONS.

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{1}{a-b}\int_\gamma \frac{dz}{z-a} - \frac{1}{a-b}\int_\gamma \frac{dz}{z-b}$$

1. IF $a$ AND $b$ lie inside $\gamma$.

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{1}{a-b}\cdot 2\pi i - \frac{1}{a-b}\cdot 2\pi i = 0$$

2

2. IF $a$ lies inside and $b$ outside $\gamma$.

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{1}{a-b}\cdot 2\pi i$$

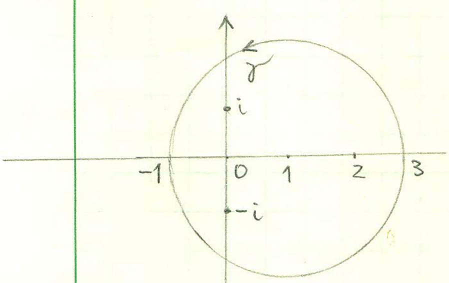
2

3. IF $b$ lies inside and $a$ outside $\gamma$.

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = 0 - \frac{2\pi i}{a-b} = \frac{2\pi i}{b-a}$$

2

LET $\gamma: z(t) = 2e^{it}+1,\ 0 \le t \le 2\pi$, EVALUATE THE INTEGRALS IN EX. 4–7.




4. $\int_\gamma \frac{e^z}{z}\,dz = 2\pi i\cdot e^0 = 2\pi i$

2

5. $\int_\gamma \frac{\cos z}{z-1}\,dz = \cos 1\cdot 2\pi i = 2\pi i\cdot\cos 1$

2

6. $\int_\gamma \frac{\sin z}{z^2+1}\,dz = \int_\gamma \frac{\sin z}{(z+i)(z-i)}\,dz$

oh OK sorry!

$$= \frac{1}{2i}\left[\int_\gamma \frac{\sin z}{z-i}\,dz - \int_\gamma \frac{\sin z}{z+i}\,dz\right]$$

$-\frac{1}{2i}$ or $\frac{i}{2}$

2

$$= \frac{2\pi i}{2i}\left[\sin i - \sin(-i)\right] = \quad 2\pi i\cdot\frac{e^{-1}-e^{1}}{i\cdot 2i} = \pi i(e-e^{-1})$$

$= 2\pi i\sinh 1$

$\frac{2\pi(i)(i)}{2} = -\pi$

7. $\int_\gamma \frac{\sin z}{z^2-z}\,dz = \int_\gamma \frac{\sin z}{z(z-1)}\,dz$

$$= \int_\gamma \frac{\sin z}{z-1}\,dz - \int_\gamma \frac{\sin z}{z}\,dz$$

$$= 2\pi i\cdot\sin 1 - 2\pi i\cdot\sin 0$$

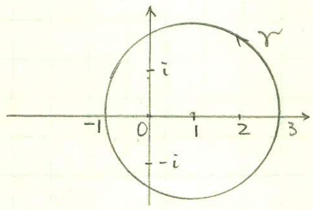
2

$$= 2\pi i\cdot\sin 1$$

MAT 417 H.W. # Complex Variables Benmei Chen

Let $\gamma: z(t) = 2e^{it}+1,\quad 0 \le t \le 2\pi$. Evaluate the integrals in Exercises 8 – 11

8. $$\int_\gamma \frac{e^z}{z^2}\,dz = \frac{2\pi i}{1!}(e^z)'\Big|_{z=0} = 2\pi i$$ 2



9. $$\int_\gamma \frac{\cos z}{(z-1)^2}\,dz = \frac{2\pi i}{1!}(\cos z)'\Big|_{z=1} = 2\pi i\cdot(-\sin z)\Big|_{z=1} = -2\pi\sin 1\cdot i$$ 2

* 10. $$\int_\gamma \frac{\sin z}{(z^2+1)^2}\,dz = \frac{1}{2}i\cdot\left[i\int_\gamma \frac{\sin z\,dz}{(z+i)^2} + i\int_\gamma \frac{\sin z}{(z-i)^2}dz + \int_\gamma \frac{\sin z}{z+i}dz - \int_\gamma \frac{\sin z}{z-i}dz\right]$$ 1

$$= \frac{1}{2}i\left[i\cdot 2\pi i\cdot\cos(-i) + i\cdot 2\pi i\cos i + 2\pi i\cdot\sin(-i) - 2\pi i\sin i\right] = \cancel{\frac{1}{2}}\pi i(\sinh 1 - \cosh 1)$$

$\frac{1}{4}$

11. $$\int_\gamma \frac{\sin z}{(z-1)^3}\,dz = \frac{2\pi i}{2!}(\sin z)''\Big|_{z=1} = \pi i\cdot(-\sin z)\Big|_{z=1} = -\pi i\cdot\sin 1$$ 2

18. Let $f(z)$ be analytic and bounded by $M$ in $|z| \le R$. Prove that

$$|f^{(n)}(z)| \le \frac{MRn!}{(R-|z|)^{n+1}},\qquad |z|<R$$

PROOF: By Cauchy's theorem for derivatives:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw$$

2

$$|f^{(n)}(z)| = \frac{n!}{2\pi}\left|\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw\right|$$

$$\le \frac{n!}{2\pi}\int_\gamma \frac{|f(w)|}{|(w-z)^{n+1}|}|dw|$$

$$= \frac{n!}{2\pi}\int_\gamma \frac{|f(w)|}{|w-z|^{n+1}}|dw|$$

$$|w-z| \ge \big||w|-|z|\big|$$

$$\therefore \frac{1}{|w-z|} \le \frac{1}{\big||w|-|z|\big|},\qquad \frac{1}{|w-z|^{n+1}} \le \frac{1}{\big||w|-|z|\big|^{n+1}}$$

$$\therefore |f^{n}(z)| \le \frac{n!}{2\pi}\cdot\frac{M\cdot 2\pi R}{(R-|z|)^{n+1}} = \frac{MRn!}{(R-|z|)^{n+1}}$$

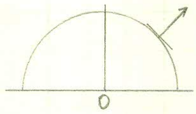
---

$$*.\quad \frac{i}{(z+i)^2} + \frac{i}{(z-i)^2} + \frac{1}{z+i} - \frac{1}{z-i} = \frac{i(z^2-2iz+1+z^2+2iz-1)-2i(z^2+1)}{(z^2+1)^2}$$

$$= \frac{2iz^2-2iz^2-2i}{(z^2+1)^2} = \frac{-2i}{(z^2+1)^2}$$

PROBLEM SET #6

(2) $C(t) = (\cos t, \sin t)$ $\quad 0 \le t \le \pi \quad$ $F = (x, y)$ $\quad$ Compute $\int_C F \cdot N ds$




$$N(t) = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$$

$$\int_C F \cdot N ds = \int_C (x, y)\left(\frac{dy}{ds}, -\frac{dx}{ds}\right) ds$$

$$= \int_C (x, y)(dy, -dx)$$

$$= \int_C -y dx + x dy$$

2

$$= \int_0^\pi -\sin t \cdot (-\sin t) dt + \cos t \cdot \cos t \, dt$$

$$= \int_0^\pi dt = \pi$$

(3) $C(t) = (\cos t, \sin t)$ $\quad 0 \le t \le \pi \quad$ $F = (-y, x)$ $\quad$ Compute $\int_C F \cdot N ds$

$$\int_C F \cdot N ds = \int_C (-y, x)(dy, -dx)$$

$$= \int_C -x\,dx - y\,dy$$

$$= -\frac{1}{2} \int_C dx^2 + dy^2$$

2

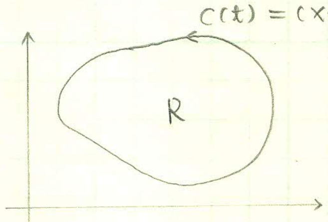
$$= -\frac{1}{2} \int_C d(x^2 + y^2)$$

$$= -\frac{1}{2} \int_C d(1) = 0$$

PROBLEM SET #7

(1) LET $F(x, y) = (P(x, y), Q(x, y))$

$C(t) = (x(t), y(t))$ $\quad$ $N(t) = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$




$$\int_C F \cdot N ds = \int_C (P, Q) \circ (dy, -dx)$$

2

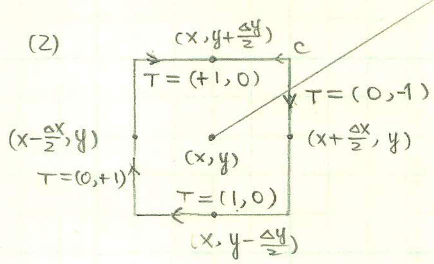
$$= \int_C P dy - Q dx$$

$$= \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx dy \quad \text{(Green's Thm)}$$

$$= \iint_R \operatorname{div} F \, dx dy$$

PROBLEM SET #7

(2)




$F = (P(x,y), Q(x,y))$

2

$$\int_C F\cdot T\,ds = -P(x, y-\tfrac{\Delta y}{2})\,\Delta x - Q(x+\tfrac{\Delta x}{2}, y)\Delta y + P(x, y+\tfrac{\Delta y}{2})\Delta x + Q(x-\tfrac{\Delta x}{2}, y)\Delta y$$

$$\text{Curl } F \triangleq \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{1}{\Delta x\cdot\Delta y}\int_C F\cdot T\,ds$$

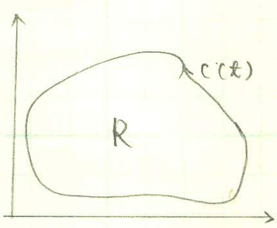
$$= \lim_{\substack{\Delta y\to 0\\ \Delta x\to 0}} \frac{-[P(x, y-\frac{\Delta y}{2}) - P(x, y+\Delta y/2)]\Delta x + [Q(x-\frac{\Delta x}{2}, y) - Q(x+\frac{\Delta x}{2}, y)]\Delta y}{\Delta x\Delta y}$$

$$= \lim_{\Delta y\to 0} -\frac{P(x, y-\frac{\Delta y}{2}) - P(x, y+\frac{\Delta y}{2})}{\Delta y} + \lim_{\Delta x\to 0}\frac{Q(x-\frac{\Delta x}{2}, y) - Q(x+\frac{\Delta x}{2}, y)}{\Delta x}$$

$$= -\frac{\partial P(x,y)}{\partial y} + \frac{\partial Q(x,y)}{\partial x}$$

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

(3)




$$\int_C F\cdot T\,ds$$

2

$$= \int_C (P, Q)(dx, dy)$$

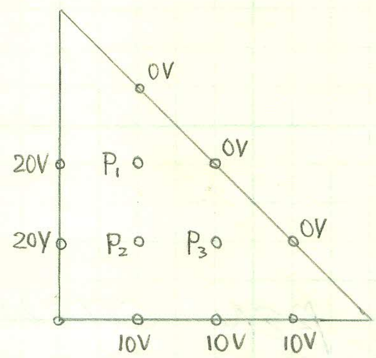
$$= \int_C P\,dx + Q\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad (\text{Green's Thm})$$

$$= \iint_R \text{Curl } F \; dx\,dy$$

MAT417 HW # Complex Variales Benmei Chen

(1)



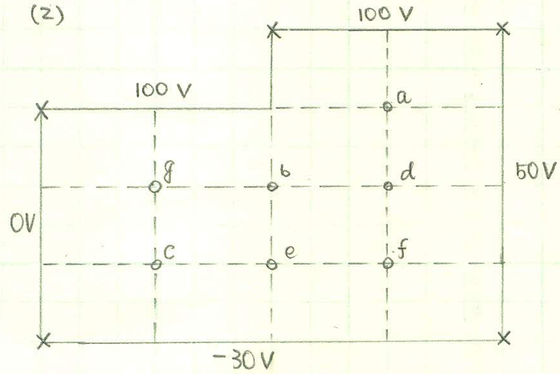

$$\begin{cases} P_1 = \frac{1}{4}(0+0+20+P_2) \\ P_2 = \frac{1}{4}(P_3+P_1+20+10) \\ P_3 = \frac{1}{4}(0+0+P_2+10) \end{cases}$$

$$\begin{cases} P_1 - 0.25P_2 + 0 \cdot P_3 = 5 \\ -P_1 + 4 \cdot P_2 - P_3 = 30 \\ 0P_1 - P_2 + 4P_3 = 10 \end{cases}$$

$$\begin{cases} P_1 = 7.6786 \text{ V} \\ P_2 = 10.7143 \text{ V} \\ P_3 = 5.1786 \text{ V} \end{cases}$$

2

(2)




$$\begin{cases} 4a = 100+100+d+50 \\ 4b = 100+g+e+d \\ 4c = g+0-30+e \\ 4d = a+b+f+50 \\ 4e = b+c-30+f \\ 4f = d+e-30+50 \\ 4g = 100+0+c+b \end{cases}$$

2

MAT 417 Complex Variable Benmei Chen

P112 . 2

SHOW $\int_0^{\pi/2} \cos^{2n}\theta\, d\theta = (2n)!\,\pi \,/\, 2\cdot(2^n n!)^2$

LET $f(z) = (z+1/z)^{2n}/z$ , $z = (\cos\theta, \sin\theta)$

$\therefore f(z)dz = (z+1/z)^{2n}/z = (z^2+1)^{2n}/z^{2n+1} = \cos^{2n}\theta\, d\theta / i$

$$\int_{|z|=1} f(z)dz = \int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}}\, dz = \int_{|z|=1} \frac{(z^2+1)^{2n}}{(z-0)^{2n+1}}\, dz$$

$$= 2\pi i \cdot g^{(2n)}(0) \,/\, (2n)!$$

WHERE $g(z) = (z^2+1)^{2n} \triangleq (h(z))^{2n}$

2

By USING THE EQUATION , IF $f(z) = x(y(z))$

$$f^{(2n)}(z) = x^{(2n)}(y(z))\cdot y'(z) + C_{2n}^{2}\, x^{(2n-1)}(y(z))\cdot y''(z) + C_{2n}^{3}\, x^{(2n-2)}(y(z))\, y^{(3)}(z)$$
$$+ \cdots + x'(y(z))\, y^{(2n)}(z)$$

IN THIS PROBLEM. $y(z) = z^2+1$

$\therefore$ $y'(z) = 2z$ , $y'(0) = 0$

$y''(z) = 2$

$y'''(z) = 0$ $\cdots$, $y^{(2n)}(z) = 0$

SO, WE HAVE $g^{(2n)}(0) = (2n!)^2/(2^n n!)^2$

$$\therefore \int_{|z|=1} f(z)dz = 2\pi i \cdot (2n)! \,/\, (2^n \cdot n!)^2$$

$$\int_0^{\pi/2} \cos^{2n}\theta\, d\theta = \frac{1}{4i}\int_{|z|=1} f(z)dz = \frac{(2n)!}{(2^n \cdot n!)^2}\cdot\frac{\pi}{2}$$

Ex. 2 , p. 83 ?

MAT417 H.W. # ~~LAST~~ 15 GRD Complex Vax. Benmei Chen.

Obtain the Maclaurin series given in Exercises 3-7

3. $\sin z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}$ , $|z| < \infty$

$(\sin z)' = \cos z \,|_{z=0} = 1$ $(\sin z)'' = -\sin z\,|_{z=0} = 0$

$(\sin z)''' = -\cos z\,|_{z=0} = -1$ $\sin^{(4)} z = \sin z\,|_{z=0} = 0$ 2

$$\sin^{(k)} z\Big|_{z=0} = \begin{cases} 1 & k = 4n+1 \\ 0 & k = 4n+2 \\ -1 & k = 4n+3 \\ 0 & k = 4n+4 \end{cases} \qquad n = 0, 1, 2, \cdots$$

$\therefore \sin z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}$ , $|z| < \infty$

4. $\cos z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$ , $|z| < \infty$

$\cos' z = -\sin z\,|_{z=0} = 0$ $\cos'' z = -\cos z\,|_{z=0} = -1$

$\cos''' z = \sin z\,|_{z=0} = 0$ $\cos^{(4)} z = \cos z\,|_{z=0} = 1$ 2

$$\cos^{(k)} z\Big|_{z=0} = \begin{cases} 0 & k = 4n+1 \\ -1 & k = 4n+2 \\ 0 & k = 4n+3 \\ 1 & k = 4n+4 \end{cases} \qquad n = 0, 1, 2, \cdots$$

$\therefore \cos z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$ $|z| < \infty$

5. $\sinh z = \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n-1)!}$ , $|z| < \infty$

$\sinh' z = \left(\frac{e^z - e^{-z}}{2}\right)' = \cosh z\,|_{z=0} = 1$ $\sinh'' z = \sinh z\,|_{z=0} = 0$

$$\therefore \sinh^{(k)} z\Big|_{z=0} = \begin{cases} 1 & k = 2n+1 \\ 0 & k = 2n \end{cases}$$

$\sinh = \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n-1)!}$ , $|z| < \infty$ 2

6. $\cosh z = \sum_{n=0}^{\infty} z^{2n}/(2n)!$ , $|z| < \infty$

$\cosh' z = \sinh z\,|_{z=0} = 0$ $\cosh'' z = \sinh' z = \cosh z\,|_{z=0} = 1$

$$\cosh^{(k)} z\Big|_{z=0} = \begin{cases} 0 & k = 2n \\ 1 & k = 2n+1 \end{cases}$$ 2

$\therefore \cosh z = \sum_{n=0}^{\infty} z^{2n}/(2n)!$ , $|z| < \infty$

MAT 417 H.W. #LAST GRD Complex Var. Benmei Chen

7. $\dfrac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}$ , $|z|<1$

USING THE RESULT IN EXPLE 2, WE HAVE

$$\frac{1}{1-\omega} = \sum_{n=0}^{\infty} \omega^n , \quad |\omega|<1$$

2

LET $\omega = z^2$ , SO

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n} , \quad |z|<1$$

9. $f(z) = \dfrac{1}{1-z}$ , $z_0 = i$

$$f(z) = \frac{1}{1-z_0-(z-z_0)} = \frac{1}{1-\left(\frac{z-i}{1-i}\right)} \cdot \frac{1}{1-i}$$

2

$$= \frac{1}{1-i} \cdot \left( 1 + \frac{z-i}{1-i} + \frac{(z-i)^2}{(1-i)^2} + \cdots \right)$$

$$\frac{|z-i|}{|1-i|} < 1 \quad \Rightarrow \quad |z-i| < \sqrt{2} .$$

10. $f(z) = \cos z$ , $z_0 = \frac{\pi}{2}$

$f'(z) = \cos' z = -\sin z \,\Big|_{z=\frac{\pi}{2}} = -1$ $\qquad$ $f''(z) = -\cos z \,\Big|_{z=\frac{\pi}{2}} = 0$

$f'''(z) = \sin z \,\Big|_{z=\frac{\pi}{2}} = 1$ $\qquad$ $f^{(4)}(z) = \cos z \,\Big|_{z=\frac{\pi}{2}} = 0$

$$f(z) = \cos z = -\left(z-\frac{\pi}{2}\right) + \frac{1}{3!}\left(z-\frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(z-\frac{\pi}{2}\right)^5 + \frac{1}{7!}\left(z-\frac{\pi}{2}\right)^7 - \cdots$$

$$\left|z-\frac{\pi}{2}\right| < \infty$$

2

FIND THE LAURENT SERIES OF THE FUNCTION $(z^2+z)^{-1}$ IN THE REGIONS GIVEN IN EXERCISES 1 – 3

1. $0 < |z| < 1$

$$f(z) = \frac{1}{z^2+z} = \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1} = \frac{1}{z} - \frac{1}{1-(-z)} \qquad |-z|<1$$

$$= \frac{1}{z} - \left[1 + (-z) + (-z)^2 + (-z)^3 + \cdots\right]$$

$$= \frac{1}{z} - 1 + z - z^2 + z^3 - z^4 + \cdots\cdots = \sum_{n=-1}^{\infty} (-1)^{n+1} \cdot z^n$$

2. $0 < |z-1| < 1$

$$f(z) = \frac{1}{z} - \frac{1}{1+z} = \frac{1}{1+(z-1)} - \frac{\frac{1}{2}}{1+\frac{z-1}{2}} \qquad |z-1|<1$$

$$= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

$$- \frac{1}{2}\left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2} - \frac{3}{4}(z-1) + \frac{7}{8}(z-1)^2 - \frac{15}{16}(z-1)^3 + \cdots = \sum_{n=0}^{\infty} \frac{2^{n+1}-1}{2^{n+1}} (-1)^n \cdot (z-1)^n$$

3. $1 < |z-1| < 2$

$$f(z) = \frac{1}{z} - \frac{1}{1+z} = \frac{1}{1-(-(z-1))} - \frac{1}{2-(-(z-1))}$$

$$= -\frac{1}{z-1}\left(\frac{1}{-1+\frac{-1}{z-1}}\right) - \frac{\frac{1}{2}}{1-\frac{[-(z-1)]}{2}}$$

$$= \frac{1}{z-1}\left[\frac{1}{1-\frac{-1}{z-1}}\right] - \frac{1}{2}\left[\frac{1}{1-\frac{[-(z-1)]}{2}}\right]$$

$$= \frac{1}{z-1}\left[1 + \left(\frac{-1}{z-1}\right) + \left(\frac{-1}{z-1}\right)^2 + \left(\frac{-1}{z-1}\right)^3 + \cdots\right] - \frac{1}{2}\left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{4} + \cdots\right]$$

$$= \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \cdots\cdots - \frac{1}{2} + \frac{1}{4}(z-1) - \frac{1}{8}(z-1)^2 + \frac{1}{16}(z-1)^3 + \cdots\cdots$$

Represent the function $(z^3 - z)^{-1}$ as a Laurent series in the regions given in Exercises 4 - 7

4. $0 < |z| < 1$

$$f(z) = \frac{1}{z^3 - z} = \frac{1}{z(z+1)(z-1)} = -\frac{1}{z} + \frac{1/2}{z+1} + \frac{1/2}{z-1}$$

$$= -\frac{1}{z} + \frac{1}{2}\,\frac{1}{1-(-z)} - \frac{1}{2}\,\frac{1}{1-z}$$

$$= -\frac{1}{z} + \frac{1}{2}\left[1 - z + z^2 - z^3 + \cdots\right] - \frac{1}{2}\left[1 + z + z^2 + z^3 + \cdots\right]$$

$$= -\frac{1}{z} - z - z^3 - z^3 - \cdots = \sum_{n=1}^{\infty} - z^{2n-1}$$

5. $1 < |z|$

$$f(z) = -\frac{1}{z} + \frac{1}{2}\cdot\frac{1}{z}\cdot\frac{1}{1+\frac{1}{z}} + \frac{1}{2}\cdot\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}$$

$$= -\frac{1}{z} + \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] + \frac{1}{2z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right]$$

$$= -\frac{1}{z} + \frac{1}{z} + \frac{1}{z^3} + \frac{1}{z^5} + \cdots$$

$$= \frac{1}{z^3} + \frac{1}{z^5} + \frac{1}{z^7} + \cdots = \sum_{n=1}^{\infty} z^{-2n-1}$$

6. $0 < |z-1| < 1$

$$f(z) = -\frac{1}{z-1+1} + \frac{1}{2}\cdot\frac{1}{z-1+2} + \frac{1}{2}\,\frac{1}{z-1}$$

$$= -\left[1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots\right] + \frac{1}{4}\left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \cdots\right] + \frac{1}{2}\,\frac{1}{z-1}$$

$$= \frac{1}{2}\cdot\frac{1}{z-1} - \frac{3}{4} + \frac{7}{8}(z-1) - \frac{15}{16}(z-1)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{2^{n+1}-1}{2^{n+1}} \cdot (z-1)^{n-1}$$

7. $1 < |z-1| < 2$

$$f(z) = \frac{1}{2}\cdot\frac{1}{z-1} + \frac{1}{4}\cdot\frac{1}{1-\frac{-(z-1)}{2}} - \frac{1}{z-1}\cdot\frac{1}{1-\frac{-1}{z-1}}$$

$$= \frac{1}{2}\cdot\frac{1}{z-1} + \frac{1}{4}\left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^5 + \cdots\right] - \frac{1}{z-1}\left[1 - \frac{1}{z-1} + \frac{1}{(z-1)^2} - \cdots\right]$$

$$= \frac{1}{4} - \frac{1}{8}(z-1) + \frac{1}{16}(z-1)^2 + \cdots - \frac{1}{2}\cdot\frac{1}{z-1} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^5} - \cdots$$