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MATH 417 HOLDEN Exam#1 10/1/86 Name: BENMEI CHEN

5 ① Write $(2+3i)(4+i)$ in the form $a+bi$ or (a,b).

$(2+3i)(4+i) = 8+2i+12i-3 = 5+14i = (5,14)$ ✓

3 ② Write $1+i\sqrt{3}$ in the form $r(\cos\theta, \sin\theta)$

unusual

OK

$1+i\sqrt{3} = 2\cdot\left(\frac{1}{2}+i\cdot\frac{\sqrt{3}}{2}\right) = 2\left(\cos\left(2k\pi+\frac{\pi}{3}\right), \sin\left(2k\pi+\frac{\pi}{3}\right)\right)$

$k = 0, \pm1, \pm2, \cdots$

6 ③ If $z = 2e^{i\frac{\pi}{4}}$ and $w = 3e^{i\frac{\pi}{3}}$ then what is:

a.) $\mathrm{Arg}(z^4 w) = -\frac{2\pi}{3}$ ✓

$z^4 w = (2e^{i\frac{\pi}{4}})^4 \cdot 3\cdot e^{i\frac{\pi}{3}} = 16\cdot e^{i\cdot\frac{\pi}{4}\cdot 4}\cdot 3e^{i\frac{\pi}{3}} = 48\cdot e^{i\cdot(\pi+\frac{\pi}{3})}$

b.) $|z^4 w| = 48$ ✓

5 ④ What is the real part of $\frac{1}{z}$ if $z = (x,y)$?

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{(x-iy)}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right)$$

$$\mathrm{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2}$$ ✓

5 ⑤ Prove: $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ for all complex numbers z_1 and z_2.

PROOF: LET $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

$z_1 \cdot z_2 = (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2)$

$\overline{z_1 \cdot z_2} = (x_1x_2 - y_1y_2,\ -x_1y_2 - y_1x_2)$

$\bar{z}_1 \cdot \bar{z}_2 = (x_1, -y_1)(x_2, -y_2) = (x_1x_2 - y_1y_2,\ -x_1y_2 - y_1x_2)$

$\therefore\ \overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ ✓

⑥ Prove: If $|z|=1$ then $\bar{z} = \frac{1}{z}$

5

PROOF: LET $z = e^{i\theta}$ (Because $|z|=1$)

$z = e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore \bar{z} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

$= \frac{1}{e^{i\theta}} = \frac{1}{z}$

⑦ Find all 3 cube roots of $(1, -\sqrt{3})$.

6

$(1, -\sqrt{3}) = 2\left(\cos(2k\pi - \frac{\pi}{3}),\ \sin(2k\pi - \frac{\pi}{3})\right)$

$z^3 = (1, -\sqrt{3}) = 2\left(\cos(2k\pi - \frac{\pi}{3}),\ \sin(2k\pi - \frac{\pi}{3})\right)$

$z = \sqrt[3]{2}\cdot\left(\cos\frac{2k\pi - \frac{\pi}{3}}{3},\ \sin\frac{2k\pi - \frac{\pi}{3}}{3}\right) \quad k = 0, 1, 2$

$\therefore z_1 = \sqrt[3]{2}\cdot\left(\cos(\frac{-\pi}{9}),\ +\sin(\frac{-\pi}{9})\right), \quad z_2 = \sqrt[3]{2}\left(\cos\frac{5\pi}{9},\ \sin\frac{5\pi}{9}\right)$

$z_3 = \sqrt[3]{2}\left(\cos\frac{11}{9}\pi,\ \sin\frac{11\pi}{9}\right) = \sqrt[3]{2}\left(\cos(\frac{-7\pi}{9}),\ +\sin(\frac{-7\pi}{9})\right)$

⑧ Write $(1+i)^{10}$ in the form $a+ib$ or (a,b)

5

$(1+i) = \sqrt{2}\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right) = \sqrt{2}\left(\cos\frac{\pi}{4},\ \sin\frac{\pi}{4}\right)$

$\therefore (1+i)^{10} = (\sqrt{2})^{10}\cdot\left(\cos\frac{10\pi}{4},\ \sin\frac{10\pi}{4}\right)$

$= 2^5\cdot\left(\cos(2\pi + \frac{\pi}{2}),\ \sin(2\pi + \frac{\pi}{2})\right)$

$= 32\ (0,\ 1) = (0,\ 32) = 0 + 32\cdot i$

⑨ Using DeMoivre's Theorem prove the following trigonometric identity:

5

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

Let $z = (\cos\theta,\ \sin\theta)$

$\therefore z^2 = (\cos 2\theta,\ \sin 2\theta), \quad z^3\ (\cos 3\theta,\ \sin 3\theta)$

$1 + z + z^2 = \frac{1 - z^3}{1 - z}$

$\therefore 1 - z^3 = (1 - z)(1 + z + z^2)$

$(1 - \cos 3\theta,\ -\sin 3\theta) = (1 - \cos\theta,\ -\sin\theta)(1 + \cos\theta + \cos 2\theta,\ \sin\theta + \sin 2\theta)$

$= \left((1 - \cos\theta)(1 + \cos\theta + \cos 2\theta) + \sin\theta(\sin\theta + \sin 2\theta),\ I_m\right)$

$\therefore 1 - \cos 3\theta = (1 - \cos\theta)(1 + \cos\theta + \cos 2\theta) + \sin^2\theta + \sin\theta\cdot 2\sin\theta\cos\theta$

* Go on last page.

(10.) $x^2 - y^2 - 1 = 0$ If your first guess is $(1,1)$ that is $x=1, y=1$
6 $xy - 2 = 0$ what is the next guess using Newtons method.

$u(x,y) = x^2 - y^2 - 1 \quad u_x(x,y) = 2x \quad , \; u_y(x,y) = -2y$
$V(x,y) = xy - 2 \quad V_x(x,y) = y \quad , \; V_y(x,y) = x$

$(x_0, y_0) = (1, 1)$

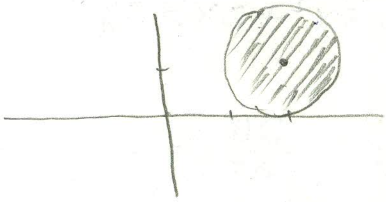
$-u(x_0,y_0) = u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y$
$-V(x_0,y_0) = V_x(x_0,y_0)\Delta x + V_y(x_0,y_0\,\Delta y$

$$\Rightarrow \begin{cases} 1 = 2\Delta x - 2\Delta y \\ 1 = \Delta x + \Delta y \end{cases} \Rightarrow \begin{cases} \Delta x = \frac{3}{4} \\ \Delta y = \frac{1}{4} \end{cases}$$

so, the next guess will be $(1+\frac{3}{4}, 1+\frac{1}{4}) = (\frac{7}{4}, \frac{5}{4})$

(11.) Make a sketch of the following sets and classify them
8 according to the terms open, bounded, connected, simply connected.

a.) $|z - (2,1)| \le 1$ b.) $|z + \bar{z}| > 1$



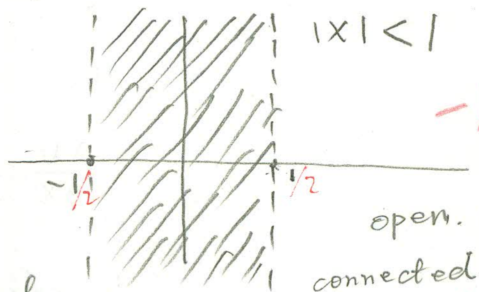
closed, bounded, connected.

According the defintion in text book,
? it is not simply connected.

-1

Let $z = (x, y)$
$|z + \bar{z}| = |(2x, 0)| = 2|x| > 1$

$|x| < 1$



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open, unbounded
connected and
~~not~~ simply
connected

(12.) State the Cauchy Riemann conditions in Cartesian and
6 polar form.

A function $f(z) = (u(x,y), V(x,y))$

The CR coditions are : $\frac{\partial u(x,y)}{\partial x} = \frac{\partial V(x,y)}{\partial y}$ and $\frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}$

if $x = r\cos\theta$, $y = r\sin\theta$, CR conditions are: $\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial V}{\partial \theta}$

$\frac{\partial V}{\partial r} = -\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}, \quad r \neq 0$

(13.) Define "f(z) is analytic at z_0"
5

Function $f(z)$ is said to be analytic at z_0, if and only if $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exists. In the other hand $f(z)$ has a derivative at z_0

← nbhd -2

one more

(14.) Prove that the following function is entire and find its derivative.

8

$$f(z) = (x^3+x^2-(3x+1)y^2,\ xy(3x+2)-y^3)$$

PROOF : $u(x,y) = x^3+x^2-(3x+1)y^2$

$V(x,y) = xy(3x+2)-y^3 = 3x^2y+2xy-y^3$

$$\frac{\partial u}{\partial x} = 3x^2+2x-3y^2$$

$$\frac{\partial u}{\partial y} = -2y(3x+1)$$

$$\frac{\partial V}{\partial x} = 6xy+2y = 2y(3x+1)$$

$$\frac{\partial V}{\partial y} = x(3x+2)-3y^2 = 3x^2+2x-3y^2$$

so, the CR conditions $\frac{\partial u}{\partial x} = \frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}$ hold.

And clearly, the first partials of $u(x,y)$ and $V(x,y)$ are continuous on the whole complex-plane $\mathbb{C}$. so, $f(z)$ is entire.

$$f'(z) = \left(\frac{\partial u}{\partial x}, \frac{\partial V}{\partial x}\right) = \left(3x^2+2x-3y^2,\ 2y(3x+1)\right)$$

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(9) $1-\cos 3\theta = 1+\cos\theta + \cos^2\theta - \sin^2\theta - \cos\theta - \cos^2\theta$

$- \cos\theta(\cos^2\theta - \sin^2\theta) + \sin^2\theta + 2\sin^2\theta \cos\theta$

$= 1-\cos^3\theta + \sin^2\theta\cos\theta + 2\sin^2\theta\cos\theta$

80 points

$= 1-\cos^3\theta + 3\cos\theta\sin^2\theta$

$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

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1 point

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① Evaluate the Following. Answer in the form $a+ib$ or (a,b)

25

a.) $e^{2+\frac{\pi}{3}i} = e^2 \cdot e^{i\frac{\pi}{3}} = e^2 (\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}) = \frac{1}{2}e^2 + i \cdot \frac{\sqrt{3}}{2} e^2$

b.) $\mathrm{Log}(1-i) = \log|1-i| + i \,\mathrm{Arg}(1-i) = \log\sqrt{2} - i\cdot\frac{\pi}{4}$

−3

c.) $i^{(1+i)} = e^{i \cdot \mathrm{Log}(1+i)} = e^{i(\log|1+i| + i\,\mathrm{Arg}(1+i))}$

$= e^{i(\log\sqrt{2} + i\frac{\pi}{4})} = e^{-\frac{\pi}{4} + i\log\sqrt{2}} = e^{-\frac{\pi}{4}}\cos(\log\sqrt{2}) + i\cdot e^{-\frac{\pi}{4}}\sin(\log\sqrt{2})$

d.) $\sin \pi i = \frac{1}{2i}(e^{i\pi i} - e^{-i\cdot\pi i}) = \frac{1}{2i}(e^{-\pi} - e^{\pi}) = i\cdot\frac{1}{2}(e^{\pi} - e^{-\pi})$

$= i \sinh \pi$

e.) $|e^{1+i}| = |e \cdot e^i| = |e|\cdot|e^i| = e$

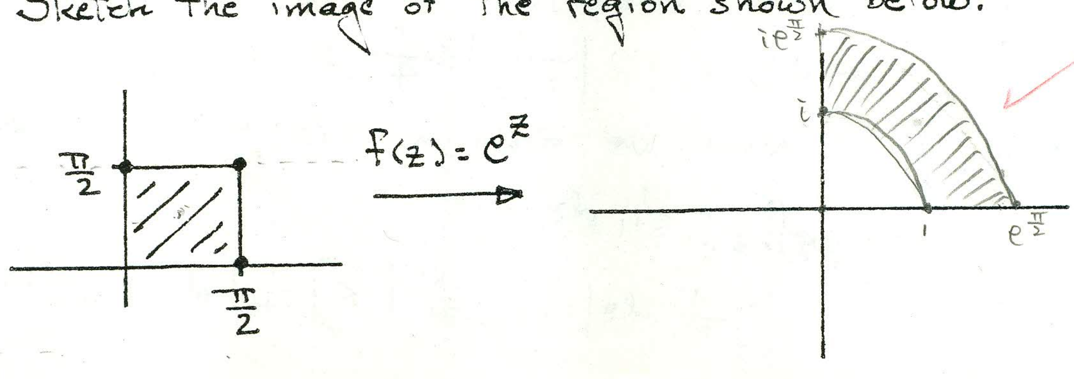
6 ② Prove $\overline{(e^z)} = e^{\bar{z}}$

Let $z = x + iy$

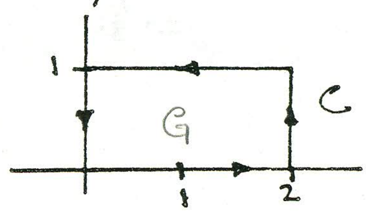
$\overline{e^z} = \overline{e^x \cdot e^{iy}} = \overline{e^x(\cos y + i \sin y)} = e^x(\cos y - i \sin y)$

$= e^x(\cos(-y) + i\sin(-y)) = e^{x-iy} = e^{\bar{z}}$

6 ③ Sketch the image of the region shown below.



6 ④ Using Green's Theorem, write $\int_C y\,dx - x\,dy$ as a double integral and evaluate it. $C(t)$ is the boundary of the rectangle shown at the right.



$$\int_C y\,dx - x\,dy = -\int_C x\,dy - y\,dx = -\iint_G (1+1)\,dx\,dy$$

$$= -2\cdot\iint_G dx\,dy = -4 \checkmark$$

⑤ State Cauchy's Theorem.

6

Let $f(z)$ be analytic on and in the pwc Jordan curve γ,

$f'(z)$ continuous

Then, $\int_\gamma f(z)\,dz = 0$

-1

⑥ Verify the following inequalities.

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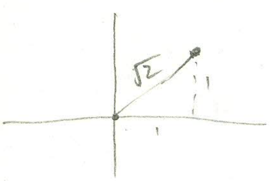
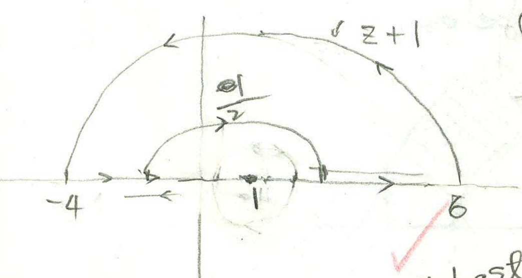
a.) $\left|\int_C z^2\,dz\right| \le 2\sqrt{2}$ $\quad C(t) = (t,t) \quad 0 \le t \le 1$



$|z^2| \le |z|^2 \le |\sqrt{2}|^2 = 2$; $|dz| = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore \left|\int_C z^2\,dz\right| \le \int_C |z^2|\cdot|dz| \le 2\cdot\sqrt{2}$. $\checkmark$

b.) $\left|\int_C \frac{1}{z+1}\,dz\right| \le \frac{\pi}{2}$ $\quad C(t) = 5(\cos t, \sin t) \quad 0 \le t \le \pi$



On the curve, the nearest point to orgiral is $= 4$, $\therefore \left|\frac{1}{z+1}\right| \le \frac{1}{4}$

Now, we apply a $C'(t) = 2(\cos t, \sin t)$ $0 \le t \le \pi$

Go ON Last page $\int_{C \mp C'} \frac{1}{z+1}\,dz = 0$

$$\left|\int_C \frac{1}{z+1}\,dz\right| = \left|\int_{C'} \frac{dz}{z+1}\right| \le \int_{C'} \left|\frac{1}{z+1}\right| dz \le \frac{1}{4}\cdot 2\pi = \frac{\pi}{2}$$

20 (7.) Evaluate the following integrals

a.) $\int_C \bar{z}\, dz$ $\quad C(t) = (1, t) \quad 1 \le t \le 2$

$c'(t) = (0, 1)$

$$\int_c \bar{z}\, dz = \int_1^2 (1, -t)(0, 1)\, dt = \int_1^2 t\, dt + i\int_1^2 1\, dt$$

$$= \tfrac{1}{2}t^2\Big|_1^2 + i = \tfrac{3}{2} + i$$

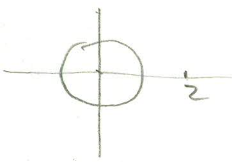
b.) $\int_C z^2\, dz$ $\quad C(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi$

$c(t) = e^{it} \qquad c'(t) = ie^{it}$

$$\int_c z^2 dz = \int_0^{2\pi} e^{i2t} \cdot ie^{it}\, dt = i\int_0^{2\pi} e^{i3t} dt = \tfrac{i}{3i} \cdot e^{i3t}\Big|_0^{2\pi}$$

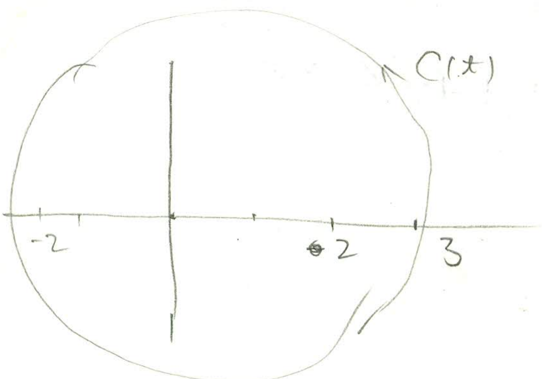
$$= \tfrac{1}{3}(\cos 6\pi + i\sin 6\pi) - \tfrac{1}{3} = 0$$

c.) $\int_C \frac{dz}{z-2}$ $\quad C(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi$



$$\int_c \frac{dz}{z-2} = 0$$

d.) $\int_C \frac{4}{z} + \frac{1}{z-2}\, dz$ $\quad C(t) = 3(\cos t, \sin t) \quad 0 \le t \le 2\pi$



$$\int_c \frac{4}{z} + \frac{1}{z-2} dz = 4\int_c \frac{1}{z} dz + \int_c \frac{1}{z-2} dz$$

$$= 4 \cdot 2\pi i + 2\pi i$$

$$= 10\pi \cdot i$$

$$\int_C \frac{dz}{z+1} = \int_0^{\pi} \frac{-5\sin t + i5\cos t}{(5\cos t+1)+i5\sin t}\,dt$$

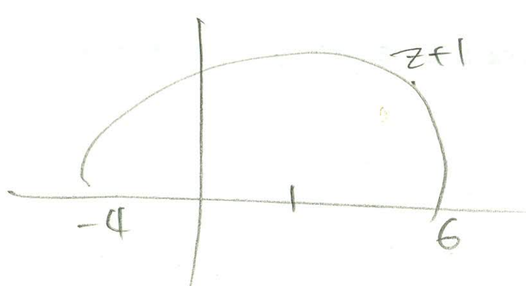
$$= \int_0^{\pi} \frac{\cancel{-25\cos t\sin t} - 5\sin t + i25\sin^2 t + i25\cos^2 t + i5\cos t + \cancel{25\cos t\sin t}}{25\cos^2 t + 10\cos t + 1 + 25\sin^2 t}\,dt$$

$$= \int_0^{\pi} \frac{-5\sin t + i(25+5\cos t)}{26+10\cos t}\,dt$$

$$= \int_0^{\pi} \frac{5\,d\cos t}{26+10\cos t} + i\int_0^{\pi} \frac{25+5\cos t}{26+10\cos t}\,dt$$

$$=$$

6. b) part. the nearest point of $z+1$ to the orginal is -4



$\therefore \left|\frac{1}{z+1}\right| \le \frac{1}{4}$, $|dz| = 5\pi$

$$\left|\int_C \frac{dz}{z+1}\right| \le \int_C \left|\frac{1}{z+1}\right| |dz| \le \frac{5\pi}{4}$$

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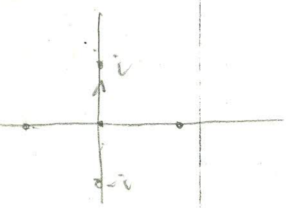
① Evaluate the following integrals.

a.) $\int_C z^4 dz$ $\quad C(t) = (0, t) \quad -1 \le t \le +1$



$$\int_C z^4 dz = \int_{-1}^{1} (0, t)^4 \cdot (0, 1)\, dt$$

$$= i\int_{-1}^{1} t^4\, dt = \frac{i}{5} t^5 \Big|_{-1}^{1} = \frac{i}{5} + \frac{i}{5} = \frac{2}{5} i$$

b.) $\int_C \frac{z+1}{z(z-2)} dz$ $\quad C(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi$

$$\int_C \frac{z+1}{z(z-2)} dz = \int_C \frac{\frac{z+1}{z-2}}{z-0} dz = 2\pi i \cdot \frac{0+1}{0-2} = -\pi i$$

c.) $\int_C \frac{\cos z}{z^5} dz$ $\quad C(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi$

$$\int_C \frac{\cos z}{z^5} dz = \frac{2\pi i}{4!} (\cos^{(4)} z)\Big|_{z=0} = \frac{2\pi i}{4\times3\times2\times1} \cos 0 = \frac{\pi}{12} i$$

$\cos' z = -\sin z$, $\quad (\cos z)'' = -\cos z$, $\quad \cos^{(3)} z = \sin z$, $\quad \cos^{(4)} z = \cos z$

② Fluid is flowing in the xy-plane and its velocity at each point in the plane is given by $F(x,y) = (x^2, y^2)$.
$C(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi$

a.) Find the circulation around C.

$F(x,y) = (x^2, y^2) \qquad P(x,y) = x^2, \quad Q(x,y) = y^2$

$c(t) = (\cos t, \sin t) \qquad T = (\frac{dx}{ds}, \frac{dy}{ds})$

$$\int_C F \cdot T\, ds = \int_C (P, Q) \circ (\frac{dx}{ds}, \frac{dy}{ds}) ds = \int_C P dx + Q dy.$$

$$= \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dx dy = \iint_R (0 - 0)\, dx dy = 0$$

$$\left(\text{curl } F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \right)$$

(2) b.) Find the divergence at $(\frac{1}{2}, -\frac{1}{2})$

$$\text{Div } F = \frac{\partial Q}{\partial y} + \frac{\partial P}{\partial x} = 2(x+y) = 0$$

$$\int_c F \cdot N\,ds = \int_c (P, Q)(dy, -dx) = \int_c P\,dy - Q\,dx = \iint_R \left(\frac{\partial Q}{\partial y} + \frac{\partial P}{\partial x}\right) dx\,dy.$$

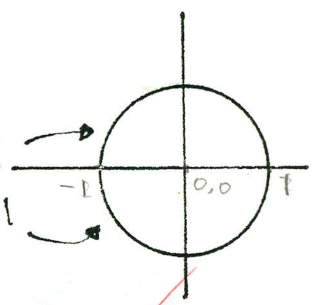
(3) $F(\xi) = \int_0^{\xi} z^3\,dz$ Evaluate $F''(i)$

$F'(z) = z^3$

$F''(z) = 3z^2$

$F''(i) = 3 \cdot i^2 = -3.$

(4) $f(z)$ is analytic for $|z| < 2$.
On the upper half of the unit circle $f(z) = i$.
On the lower half of the unit circle $f(z) = 1$



a.) Evaluate $f(0,0)$. Go on last page!

$$\cancel{f(0,0) = \frac{1}{2\pi}\left(\int_0^{\pi} i \cdot (-\sin t, \cos t)\,dt + \int_{\pi}^{2\pi} (-\sin t, \cos t)\,dt\right)}$$

$$\cancel{= \frac{1}{2\pi}\left(i \cdot \left[-\int_0^{\pi} \sin t\,dt + i\int_0^{\pi} \cos t\,dt\right] + \int_{\pi}^{2\pi} -\sin t\,dt + \int_{\pi}^{2\pi} \cos t\,dt\right) = \frac{1}{\pi}(1-i)}$$

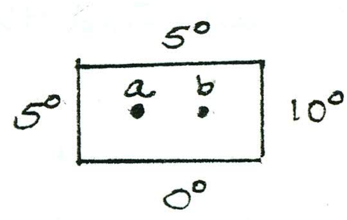
b.) Estimate $|f^{(3)}(0,0)|$

By maximum principle $|f(z)| \leq 1 = M$

By Cauchy Inequality: $|f^{(3)}(0,0)| \leq \frac{M\,n!}{r^n} = \frac{1 \cdot 3!}{1^3} = \frac{3 \times 2}{1} = 6$

OR $|f^{(3)}(0,0)| \leq \frac{M\,n!}{r^n} = \frac{3}{4}$?

(5) What is the steady state temperature (approximately) at the points a and b in the diagram.



3.667
14.668

$a = \frac{1}{4}(5+5+0+b) = \frac{5}{2} + \frac{b}{4}$ $\qquad 4a - b = 10$

$b = \frac{1}{4}(5+10+0+a) = \frac{15}{4} + \frac{a}{4}$ $\qquad 4b - a = 15$

3.66
55
45
100
90

$\begin{cases} 16a - 4b = 40 \\ 4b - a = 15 \end{cases}$ $\qquad 15a = 55 \Rightarrow a = \frac{55}{15} = 3.667°$

$4 \cdot \frac{55}{15} - 10$

$b = 4 \times \frac{55}{15} - 10 = 4.668°$

⑥ State the Maximum Modulus Theorem

Let the function $f(z)$ be analytic in a bounded region G and be continuous on ∂G. Then $|f(z)|$ attains its maximun on the boundary of G ✓

⑦ Evaluate $\int_0^{2\pi} \frac{1}{5+4\cos\theta}\, d\theta$

$$z = (\cos\theta, \sin\theta) = e^{i\theta}$$

$$\bar{z} = e^{-i\theta} = \frac{1}{e^{i\theta}} = \frac{1}{z} = (\cos\theta, -\sin\theta)$$

$2z^2+5z+2=0$

$$z + \frac{1}{z} = (2\cos\theta, 0)$$

$$dz = ie^{i\theta}\, d\theta \qquad d\theta = -ie^{-i\theta}\, dz$$

$25-16$
9

$$\frac{1}{5+4\cos\theta}\, d\theta = \frac{1}{5+2(z+\frac{1}{z})}\cdot(-i)\cdot\frac{1}{z}\, dz$$

$z = \frac{-5\pm3}{4}$

$$= -i\cdot\frac{1}{5z+2(z^2+1)}\, dz$$

$(z+1)(z+$

$$= -i\cdot\frac{dz}{2(z+2)(z+\frac{1}{2})}$$

$z^2+\frac{1}{2}z+2z+1$

$\frac{5}{2}z$

$$\int_0^{2\pi}\frac{1}{5+4\cos\theta}\, d\theta = -i\cdot\int_{|z|<1}\frac{dz}{2(z+2)(z+\frac{1}{2})}$$

$$= -i\int\frac{\frac{1}{2(z+2)}}{z+\frac{1}{2}}\, dz$$

$\frac{3}{2}\times 2$

$$= -i\cdot 2\pi i\cdot\frac{1}{2(-\frac{1}{2}+2)} = \frac{2\pi}{3}$$ ✓

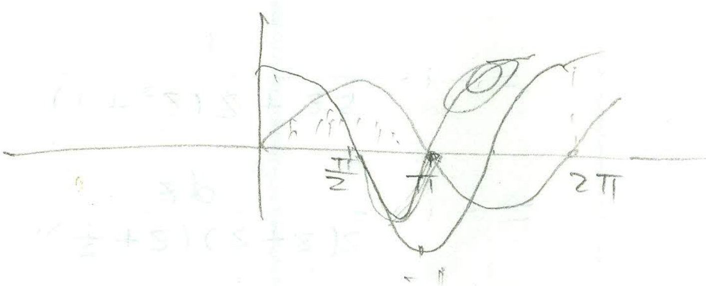
① (a) By Gauss Mean Value Thm:

$$f(0,0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\,d\theta \qquad \text{Let } r=1<2$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,d\theta$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} i\,d\theta + \int_{\pi}^{2\pi} 1\,d\theta\right] = \frac{1}{2\pi}[\pi i + \pi] = \frac{1}{2}[1+i]$$

$(0,t)(0,t) = (0,-t)(0,-t)$
$= (-t^2, 0)(-t^2, 0)$
$= (t^4, 0)$
$= t^4$



$\cos t \Big|_0^{\pi}$

$2\,($ $\quad -1-1 \quad$ $\cos t \Big|_0^{\pi}$ $\quad \cos t \Big|_{\pi}^{2\pi}$

$-2i \quad -\cos t \Big|_0^{\pi} \quad -1-1 \quad -1-$

$1+1$

$-1-1 \quad +2 \quad 2$

$-2i$

cos

80

MATH 417 - Holden EXAM#4 12/10/86 Name: BENMEI CHEN

8 (1.) Give the Taylor series expansion of $\frac{1}{z}$ about $z_0 = i$.

Show the first four terms and give the radius of convergence.

$$\frac{1}{z} = \frac{1}{z-i+i} = \frac{1}{i}\cdot\frac{1}{1-\frac{i-z}{i}} = \frac{1}{i}\left(1+(i-z)+(i-z)^2+\cdots\right)$$

$$= \frac{1}{i} - \frac{1}{i}(z-i) + \frac{1}{i}(z-i)^2 - \frac{1}{i}(z-i)^3 + \cdots$$

$$\frac{|i-z|}{|i|} < 1, \quad |z-i| \cancel{<} 1 \leftarrow R$$

OK 1

(2.) Give the Laurent series expansion for the following:

8 a.) $\sin\left(\frac{1}{z}\right)$ $0<|z|$ Show at least 3 non-zero terms

$$\sin\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right) - \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \frac{1}{5!}\left(\frac{1}{z}\right)^5 - \frac{1}{7!}\left(\frac{1}{z}\right)^7 + \cdots$$

$$= \frac{1}{z} - \frac{1}{3!}\cdot\frac{1}{z^3} + \frac{1}{5!}\cdot\frac{1}{z^5} - \frac{1}{7!}\cdot\frac{1}{z^7} + \cdots$$

8 b.) $\frac{1}{z}$ $0<|z-1|<1$ Show at least 4 non-zero terms

$$\frac{1}{z} = \frac{1}{z-1+1} = \frac{1}{1-(-(z-1))} = 1-(z-1)+(z-1)^2-(z-1)^3+\cdots$$

8. (3.) $f(z)$ is analytic for $0<|z|<1$. $f(z) = \sum_{-\infty}^{\infty} a_n z^n$

Give a formula for a_n. Describe or define all symbols.

$$a_n = \frac{1}{2\pi i}\int_{|\xi|=\rho} \frac{f(\xi)\,d\xi}{\xi^{n+1}}$$

WHERE $0<\rho<1$, AND z is ON THE curve $|\xi|=\rho$.

MATH 417 EXAM#4 p.2

④ Evaluate the following residues

*✓ 8 a.) $f(z) = \dfrac{2z+3}{e^z - 1}$ $\quad \text{Res}(f,0) =$

$$\text{Res}(f,0) = \lim_{z\to 0} \frac{2z+3}{(e^z-1)'} = \lim_{z\to 0} \frac{2z+3}{e^z} = 3$$ ✓

8 b.) $f(z) = z e^{1/z}$ $\quad \text{Res}(f,0) =$

$$= z\left(1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \cdots\right)$$

$$= z + 1 + \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \cdots$$

$$\text{Res}(f,0) = a_{-1} = \frac{1}{2!} = \frac{1}{2}$$ ✓

*✓ 8 c.) $f(z) = \dfrac{z^3+1}{(z+1)^3}$ $\quad \text{Res}(f,-1) =$

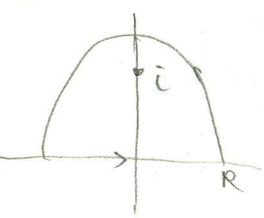
$$\lim_{z\to -1} = \lim_{z\to -1} \frac{d^2}{dz^2} (z+1)^3 \cdot \frac{z^3+1}{(z+1)^3} = \lim_{z\to -1} \frac{d^2}{dz^2}(z^3+1)$$

$$= \lim_{z\to -1} 6z = -6 = 2! \cdot a_{-1}$$

$$\Rightarrow \text{Res}(f,-1) = a_{-1} = \frac{-6}{2!} = -3$$ ✓

⑤ Evaluate the following integrals by using residues.

*✓ 8 a.) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx = 2\pi i \,\text{Res}(f,i) = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}$ ✓



$$f(z) = \frac{1}{(z^2+1)^2} \qquad \text{Res}(f,i) = \lim_{z\to i} \frac{d}{dz} (z-i)^2 \cdot \frac{1}{(z^2+1)^2}$$

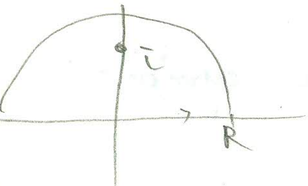
$$= \lim_{z\to i} \left[\frac{1}{(z+i)^2}\right]' = \lim_{z\to i} \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{(2i)^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

* 8 b.) $\displaystyle\int_{-\infty}^{\infty} (\sin x)\frac{(x+1)}{x^2+1}\,dx = \text{Im}\{2\pi i \,\text{Res}(f,i)\} = \text{Im}\{\pi e^{-1}(1+i)\} = \pi e^{-1}$

$= \pi/e$ ✓

WHERE $f(z) = \dfrac{e^{iz}(z+1)}{z^2+1}$



$$\text{Res}(f,i) = \lim_{z\to i} (z-i)\cdot \frac{e^{iz}(z+1)}{z^2+1}$$

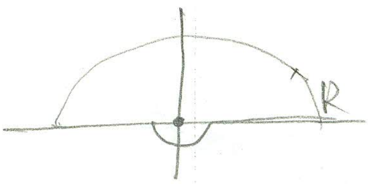
$$= \lim_{z\to i} \frac{e^{iz}(z+1)}{z+i}$$

$$= \frac{e^{-1}\cdot(1+i)}{2i}$$

MATH 417 EXAM#4 P3

5.) continued

8 c.) $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \mathrm{Im}\{\pi i \,\mathrm{Res}(f,0)\} = \mathrm{Im}\{\pi i \cdot 1\} = \pi$ ✓



WHERE $f(z) = \frac{e^{iz}}{z}$

$\mathrm{Res}(f,0) = \lim_{z\to 0} z \cdot \frac{e^{iz}}{z} = 1$ ✓

$\frac{2z+3}{\ldots z + \frac{1}{2!} z^2 \ldots}$

$\frac{3}{\ldots}$

$$\frac{z^3+1}{(z} = (z-1)^3 + 3(z-1)^2 + 4(z-1) + 3$$

$$z^3+1 = (z-1)^3 + 3z^2 - 2z + 2 = (z-1)^3 + 3(z-1)^2 + 6z - 3 - 2z + 2$$

$$+ 4z - 1$$

$$+ 4(z-1) + 3$$

$$(z-1)^3 = z^3 + 3z^2 + 3z + 1$$

$$(z^2 - 2z + 1)(z-1)$$

$$z^3 - 2z^2 + z - z^2 + 2z - 1$$

$$z^3 - 3z^2 + 2z - 1$$

$$3z^2 - 6z + 3$$

$$z^3+1 = (z+1)^3 - 3z^2 - 3z = (z+1)^2 - \frac{3(z+1)^2}{(z+1)^3} + 3(z+1)$$

$$-\cancel{0}3(z+1)^2$$

$$= \quad = -3z^2 - 6z - 3$$

$$+6z + 3 - 3z$$

$$3z + 3$$

$$3(z+1)$$

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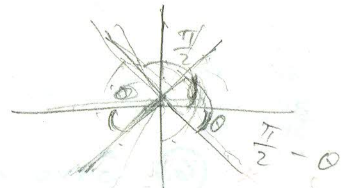
1 point



$-\pi+\theta = \frac{\pi}{2}$

6 (1.) Let $z = x+iy$. Define the following

a.) $\bar{z} = x - iy$

b.) $|z| = \sqrt{x^2+y^2}$

c.) $\operatorname{Arg} z = \begin{cases} \arctan\left(\frac{y}{x}\right) & x \geq 0 \\ \pi + \arctan\left(\frac{y}{x}\right) & x<0 \quad y \geq 0 \\ -\pi + \arctan\left(\frac{y}{x}\right) & x<0 \quad y<0 \end{cases}$

$-\frac{\pi}{2}-\text{arc}$

$-\pi + \text{arctg}\frac{y}{x}$

$\pi + \text{arctg}\frac{y}{x}$

2 (2.) Write $\frac{2+3i}{3+4i}$ in the form $a+bi$

$$\frac{2+3i}{3+4i} = \frac{2+3i}{3+4i}\cdot\frac{3-4i}{3-4i} = \frac{6+12+9i-8i}{9+16} = \frac{18}{25}+\frac{1}{25}i$$

4 (3.) Give all values of $(1+i)^{1/5}$. -1

$k=-1$ $k=-2$

$$(1+i) = \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$(1+i)^{1/5} = (\sqrt{2})^{\frac{1}{5}}\left(\cos\frac{2k\pi+\frac{\pi}{4}}{5}+i\sin\frac{2k\pi+\frac{\pi}{4}}{5}\right) \qquad k=0,1,2,3,4$$

$z_1 = \sqrt[10]{2}\left(\cos\frac{\pi}{20}, \sin\frac{\pi}{20}\right)$, $z_2 = \sqrt[10]{2}\left(\cos\frac{9\pi}{20}, \sin\frac{9\pi}{20}\right)$, $z_3 = \sqrt[10]{2}\left(\cos\frac{17\pi}{20}, \sin\frac{17}{20}\pi\right)$

$z_3 = \sqrt[10]{2}\left(\cos\frac{-7\pi}{20}, \sin\frac{-7\pi}{20}\right)$, $z_5 = \sqrt[10]{2}\left(\cos\frac{-15\pi}{20}, \sin\frac{-15\pi}{20}\right)$

12 (4.) Write the following complex numbers in the form $a+bi$

a.) $\operatorname{Log}(1+2i) = \log|1+2i| + i\operatorname{Arg}(1+2i) = \log\sqrt{5} + i\cdot(\tan^{-1}2)$

b.) $e^{\frac{\pi}{4}+\frac{\pi}{4}i} = e^{\frac{\pi}{4}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}e^{\frac{\pi}{4}} + \frac{\sqrt{2}}{2}e^{\frac{\pi}{4}}i$

c.) $\sin\left(\frac{\pi}{2}i\right) = \frac{1}{2i}\left(e^{i\cdot\frac{\pi}{2}i} - e^{-i\frac{\pi}{2}i}\right) = \frac{1}{2i}\left(e^{-\frac{\pi}{2}} - e^{\frac{\pi}{2}}\right) = \frac{1}{2}\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)i$

$= i\sinh\frac{\pi}{2}$

d.) $i^i = e^{i\operatorname{Log} i} = e^{i\left(\log|i| + i\cdot\frac{\pi}{2}\right)} = e^{i\left(i\cdot\frac{\pi}{2}\right)} = e^{-\frac{\pi}{2}}$

(5.) How can you determine whether or not $f(z)$ is analytic?

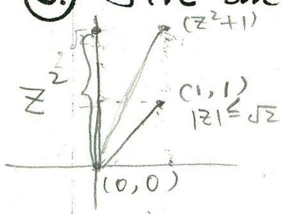
2

Let $f(z) = (u(x,y), v(x,y))$, check if $f(z)$ satisfies Cauchy-R

$\boxed{\begin{matrix} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y} \end{matrix}}$ $\Rightarrow$ conditions, and then check if the first portral of u, v is continous.

* (6.) Give an upper bound for $\left| \int_C z^2+1 \, dz \right|$ on $C(t) = (t,t)$ $0 \le t \le 1$

4



z^2+1

$\sqrt{5}$

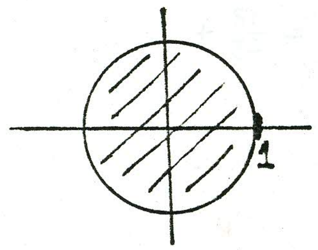
$$\left| \int_C z^2+1 \, dz \right| \le \int_C |z^2+1||dz| \le \int_C (|z^2|+1)|dz|$$

$$= \int_C (|z|^2+1) \cdot \sqrt{2} \le (2+1) \cdot \sqrt{2} = 3\sqrt{2}$$

$|z^2+1| \le \sqrt{2^2+1^2} = \sqrt{5}$ $\Rightarrow \left| \int_C z^2+1 dz \right| \le \sqrt{5} \cdot \sqrt{2} = \sqrt{10}$ too!!

(7.) Sketch the image of the region shown below.

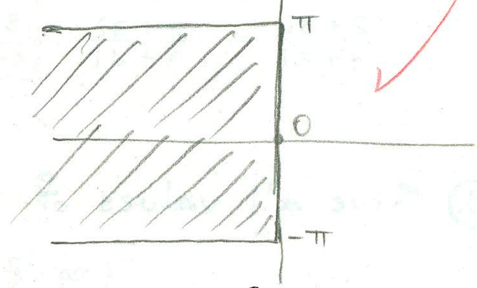
3



$\xrightarrow{f(z) = \text{Log}\, z}$

$\text{Log}\, z = \log|z| + i \text{Arg}\, z$

$-\pi < \text{Arg}\, z \le \pi$



(8.) The conclusion of the Cauchy Theorem is $\int_C f(z) dz = 0$.

4 What is the hypothesis? (i.e. what is the significance of the symbols "f" and "C")

$f(z)$ must be analytic on and inside C and $f'(z)$ must be continuous inside C; and C is the pws Jordan curve.

(9.) What is the Gauss Mean Value Theorem?

4

Let the function $f(z)$ be analytic in $|z - \xi| < R$, Then

$$f(\xi) = \frac{1}{2\pi} \int_0^{2\pi} f(\xi + re^{i\theta}) d\theta, \quad \text{WHERE} \quad 0 < r < R$$

(10) $f(z)$ is analytic inside and on the simple closed curve C.

4 z_0 is an interior point to C. Give the Cauchy integral Formula for evaluating $f'''(z_0)$.

$$f'''(z_0) = \frac{3!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^4} dz = \frac{3}{\pi i} \int_C \frac{f(z)}{(z-z_0)^4} dz$$

$3z \quad z \quad \frac{1}{z-1} - \frac{1}{z} = \frac{z-z+1}{z(z-1)}$

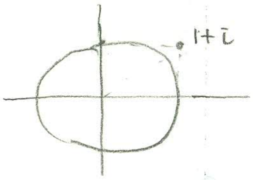
$(3z+1)(z+3)$

(11) Evaluate the following integrals

20

* a.) $\int_C \frac{1}{z-(1+i)}\,dz \qquad C(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi$

$\frac{9}{3} - \frac{1}{3}$

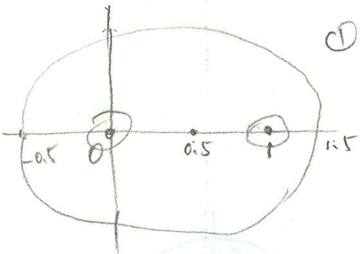


$= 0$

$\frac{8}{3}$

$\frac{3}{8} \times \frac{4\pi}{2\sqrt{3}}$

* b.) $\int_C \frac{1}{z(z-1)}\,dz \qquad C(t) = \frac{1}{2} + (\cos t, \sin t) \quad 0 \le t \le 2\pi$



$$\int_C \frac{1}{z(z-1)}\,dz = \int_C \frac{dz}{z-1} - \int_C \frac{dz}{z}$$

$$= 2\pi i - 2\pi i$$

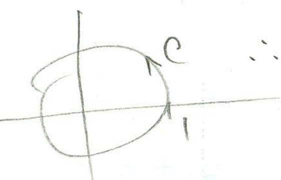
$$= 0$$

* c.) $\int_0^{2\pi} \frac{1}{3\cos\theta + 5}\,d\theta$

LET $z = (\cos\theta, \sin\theta) = e^{i\theta} \qquad \frac{1}{z} = e^{-i\theta} = (\cos\theta, -\sin\theta)$

$2\cos\theta = z + \frac{1}{z} \Rightarrow \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$dz = ie^{i\theta}\,d\theta \qquad d\theta = \frac{1}{i}e^{-i\theta}\,dz = \frac{1}{i}\cdot\frac{1}{z}\,dz$



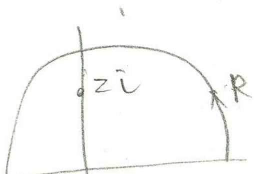
$$\therefore \int_0^{2\pi} \frac{1}{3\cos\theta + 5}\,d\theta = \int_C \frac{1}{3 \times \frac{1}{2}\left(z + \frac{1}{z}\right) + 5} \cdot \frac{1}{iz}\,dz$$

$$= \frac{1}{i}\int_C \frac{2}{3z^2 + 10z + 3}\,dz = \frac{1}{i}\int_C \frac{2\,dz}{(z+3)(3z+1)} = \frac{2}{3i}\int_C \frac{\frac{1}{z+3}}{z+\frac{1}{3}}\,dz$$

$$= \frac{2}{3i} \cdot 2\pi i \cdot \frac{1}{-\frac{1}{3}+3} = \frac{\pi}{2}$$

* d.) $\int_0^{\infty} \frac{1}{(x^2+4)^2}\,dx$

$$\int_0^{\infty} \frac{1}{(x^2+4)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,dx = \pi i \operatorname{Res}(f, 2i) = \pi i \cdot \frac{1}{32i} = \frac{\pi}{32}$$



$$f(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$$

$$\operatorname{Res}(f, 2i) = \lim_{z\to 2i} \frac{d}{dz}(z-2i)^2 \cdot \frac{1}{(z+2i)^2(z-2i)^2} = \lim_{z\to 2i} \frac{-2}{(z+2i)^3}$$

$$= \frac{-2}{(4i)^3} = \frac{1}{32i}$$

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4 (12.) $1+i$ is a pole of order 4 of $f(z)$. Give a formula for evaluating $\operatorname{Res}(f, 1+i)$.

$$\operatorname{Res}(f, 1+i) = \frac{1}{3!} \lim_{z \to 1+i} \frac{d^3}{dz^3} (z-1-i)^4 f(z)$$

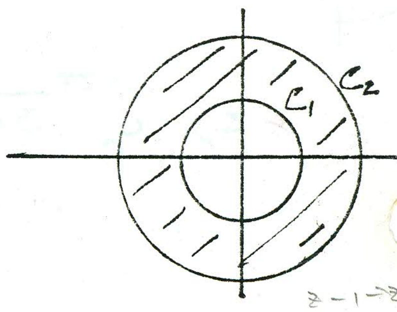
4 (13.) $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$ ← Taylor Series

Give a formula for a_n.

$$a_n = \frac{1}{n!} f^{(n)}(0) \qquad n = 0, 1, 2, \ldots$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}}\, dz$$

*6 (14.) $C_1: |z| = 1$ $C_2: |z| = 2$ $f(z) = \dfrac{1}{(z-1)(z-2)}$



Give the Laurent series for $f(z)$ in the region $1 < z < 2$.

$z-1-z+2$

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{-1}{2} \cdot \frac{1}{1-\frac{z}{2}} - \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}}$$

$$= -\frac{1}{2}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)$$

$$= -\frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \cdots - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots$$

80 points total.

Enjoy the holidays!