1.1 In straight poker, five cards are dealt to each player from a deck of ordinary playing cards. What is the probability that a player will be dealt a flush.

SOLUTION: Surppose that the ordinary playing poker has 54 cards.

i) THE TOTAL OUTCOMES $= \binom{n}{r} = \binom{54}{5} = \frac{54!}{5!\,(54-5)!} = 3162510$

THE TOTAL OUTCOME NUMBER FOR FIVE CARDS ALL OF ONE SUIT:

$$= 4 \times \binom{13}{5} = \frac{4 \times 13!}{5!\,(13-5)!} = 1287 \times 4 = 5148.$$

THE PROBABILITY THAT A PLAYER IS DEALT A FLUSH $= \frac{5148}{3162510} = 0.001628$

ii) IF THE TOTAL CARDS NUMBER IS 52.

THE PROBABILITY $= 0.001981$ ✓

1.3 An urn contains two white, one black, and four red balls. Three balls are drawn out simultaneously.

(a) How many elements are in the sample space for this experiment?

(b) What is the probability that the produce one white and two red balls?

SOLUTION:

(a) TOTAL ELEMENTS $= \binom{7}{3} = \frac{7!}{3!(7-3)!} = 35$ ✓

(b) TOTAL ELEMENTS FOR "ONE WHITE AND TWO RED BALLS"

$$= \binom{2}{1} \cdot \binom{4}{2} = \frac{2!}{1!(2-1)!} \cdot \frac{4!}{2!(4-2)!} = 12$$

THE PROBABILITY OF " " $= 12/35 = 0.342857$ ✓

1.4 Five dice are rolled simultaneously

(a) Describle the sample space for this experiment.

(b) What is the probability that exactly one 6 will be rolled? 0.4

(c) What is the probability that at least one 6 will be rolled? 0.6

SOLUTION:

(a) TOTAL ELEMENTS $= 6^5 = 7776$ ✓

(b) $P_{\text{one } 6} = \cancel{P\epsilon}\ 5 \times 5^4 / 6^5 =$ 0.401878

(c) $P_{\text{at least one } 6} = 5 \times 6^4 / 6^5 =$ 0.66667

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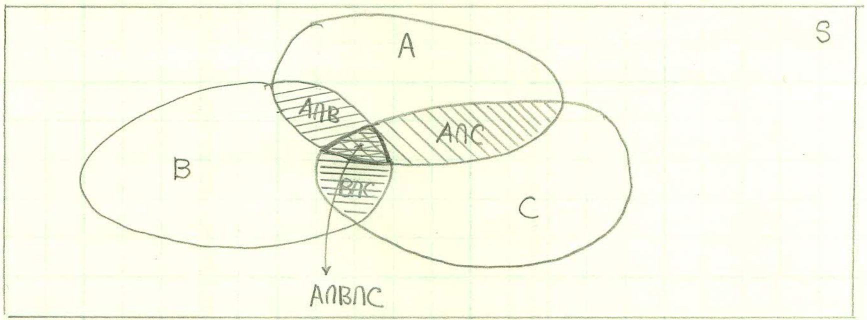
1.5 Sketch a Venn diagram for three events A, B, and C. Use the diagram to justify the following relationship:

SOLUTION: $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$



1.9 Assume equal likelihood for the birth of a boys and girls. What is the probability that a four-children family chosen at random will have two boys and two girls, irrespective of the order of birth.

SOLUTION: USING "1" FOR BOY. "0" FOR GIRL, SO THE TOTAL OUTCOMES

0000	0001	0010	0011
0100	0101	0110	0111
1000	1001	1010	1011
1100	1101	1110	1111

$$P_{2boys\,2girls} = \frac{6}{16} = \frac{3}{8}$$

1.10. Consider a sequence of random binary digits, zeros and ones. Each digit may be having a probability of $\frac{1}{2}$. For a six-digit sequence, what is the probability of having:

(a) Exactly 3 "0"s and 3 "1"s arranged in any order?

(b) Exactly 4 "0" and 2 "1" " " ?

(c) " 5 "0" and 1 "1" " " " ?

(d) Exactly 6 "0"s ?

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SOLUTION:

000000	000001	000010	000011	000100	000101	000110	000111
001000	001001	001010	001011	001100	001101	001110	001111
010000	010001	010010	010011	010100	010101	010110	010111
011000	011001	011010	011011	011100	011101	011110	011111

TOTAL OUTCOMES $= 2^6 = 64$ ✓

a) $P_{3 \text{ and } 3} = \binom{6}{3} / 64 = 20/64 = 5/16 = 0.312500$ ✓

b) $P_{4 \text{ and } 2} = \binom{6}{4} / 64 = 15/64 = 0.234375$ ✓

c) $P_{5 \text{ and } 1} = \binom{6}{5} / 64 = 6/64 = 3/32 = 0.093750$ ✓

d $P_{6 \text{ and } 0} = \binom{6}{6} / 64 = 1/64 = 0.015625$ ✓

1.11 *A certain binary message is n bits in length. If the probability of making an error in the transmission of a single bit is p. and if the error probability does not depend on the outcome of any previous Xmisions. Show that the Prob. of occurence of exactly k bits errors in a message is*

$$P(k \text{ errors}) = \binom{n}{k} p^k (1-p)^{n-k}$$

SOLUTION: consider the following sample space.

ALL bi-message	00···00	00···01	00···10	····	11···10	11···11
ERROR p	p p					
NO-ERROR 1−p						

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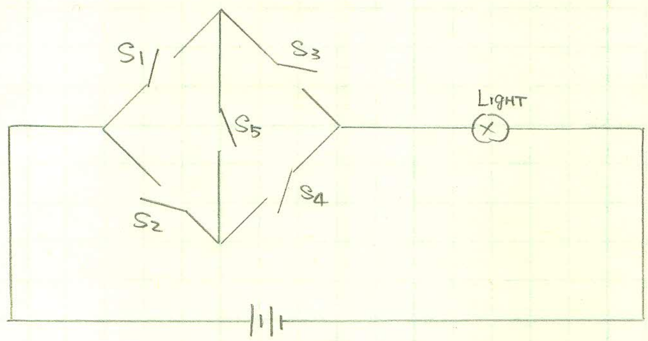
1.12



S1	S2	S3	S4	S5		
0	0	0	0	1	OFF	
0	0	0	1	0	OFF	
0	0	0	1	1	OFF	
0	0	1	0	0	OFF	
0	0	1	0	1	OFF	
0	0	1	1	0	OFF	
0	0	1	1	1	OFF	
0	1	0	0	0	OFF	
0	1	0	0	1	OFF	
0	1	0	1	0	ON	
0	1	0	1	1	ON	S5
0	1	1	0	0	OFF	
0	1	1	0	1	ON	S5
0	1	1	1	0	ON	
0	1	1	1	1	ON	S5
1	0	0	0	0	OFF	
1	0	0	0	1	OFF	
1	0	0	1	0	OFF	
1	0	0	1	1	ON	S5
1	0	1	0	0	ON	
1	0	1	0	1	ON	S5
1	0	1	1	0	ON	
1	0	1	1	1	ON	S5
1	1	0	0	0	OFF	
1	1	0	0	1	OFF	
1	1	0	1	0	ON	
1	1	0	1	1	ON	S5
1	1	1	0	0	ON	
1	1	1	0	1	ON	S5
1	1	1	1	0	ON	
1	1	1	1	1	ON	S5

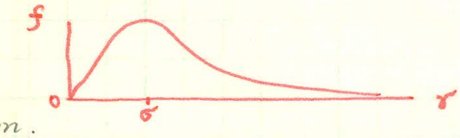
"0" FOR "OFF" AND "1" FOR "ON"

$P_{\text{"ON"}}$ — $16/32 = 0.5$ ✓

$P(S5/\text{"ON"}) = 9/16 = 0.5625$ ✓

1.27 The Rayleigh probability density function is defined as

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$



where σ^2 is a parameter of the distribution.

(a) Find the mean and variance of a Rayleigh random variable R.

$$E[R] = \int_{-\infty}^{\infty} r f_R(r)\,dr = \int_{-\infty}^{\infty} \frac{r^2}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\,dr$$

$$= 2\sqrt{2}\sigma \cdot \int_{-\infty}^{\infty} \frac{r^2}{2\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}}\, d\left(\frac{r}{\sqrt{2}\sigma}\right)$$

$$= 2\sqrt{2}\sigma \cdot \int_{\cancel{-\infty}\ 0}^{\infty} t^2 e^{-t^2}\,dt = \frac{\sqrt{2\pi}}{2}\sigma \checkmark$$

$$E[(R-E[R])^2] = E[(R-\sqrt{2\pi}\sigma)^2] = E[R^2] - (\sqrt{2\pi}\sigma)^2$$

$$= \int_{-\infty}^{\infty} r^2 \cdot \frac{r}{\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}}\,dr - 2\pi\sigma^2$$

$$= \int_{-\infty}^{\infty} 2\sigma^2 \frac{r^2}{2\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}}\, d\left(\frac{r^2}{2\sigma^2}\right) - 2\pi\sigma^2$$

$$= 2\sigma^2 \cdot \int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt^2 - 2\pi\sigma^2$$

$$= \cancel{4}\,2\sigma^2 \int_0^{\infty} t^2 e^{-t^2}\,dt^2 - \frac{\cancel{2}\pi\sigma^2}{2} = 4\sigma^2 \cdot \left(e^{-t^2}(-t^2-1)\right)\Big|_0^{\infty} - 2\pi\sigma^2$$

$$= \cancel{4\sigma^2 - 2\pi\sigma^2}\quad \left(2 - \frac{\pi}{2}\right)\sigma^2 \checkmark$$

(b) Find the mode of R.

$$\frac{df_R(r)}{dr} = \frac{1}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} + \frac{r^2}{\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}} \cdot \left(-\frac{1}{2\sigma^2}\right) \cdot 2r$$

$$= \frac{1}{\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}} \cdot \left[1 - \frac{r^3}{\sigma^2}\right] = 0$$

THUS, $r^2 = \sigma^2$

SO, THE MODE OF $R = \cancel{\sqrt[3]{\sigma^2}}\ \sigma$

$$** \quad \int_{-\infty}^{\infty} f_R(r)\,dr = \int_{-\infty}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\,dr = \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}}\, d\left(\frac{r^2}{2\sigma^2}\right)$$

$$= \int_{-\infty}^{0} e^{-\frac{r^2}{2\sigma^2}}\, d\left(\frac{r^2}{2\sigma^2}\right) + \int_0^{\infty} e^{-\frac{r^2}{2\sigma^2}}\, d\left(\frac{r^2}{2\sigma^2}\right) = \int_{\infty}^{0} e^{-t}\,dt + \int_0^{\infty} e^{-t}\,dt = 0$$

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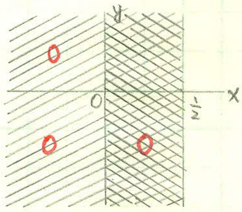
1.33 Random variable X and Y have a joint probability density function

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)}, & x \geq 0 \text{ and } y \geq 0 \\ 0, & \text{OTHERWISE} \end{cases}$$

(a) Find $P(X \leq \frac{1}{2})$



$$P(X \leq \tfrac{1}{2}) = \int_{-\infty}^{\frac{1}{2}} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$$

$$= \int_0^{\frac{1}{2}} \int_0^{\infty} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^{\frac{1}{2}} e^{-x}\,dx \cdot \int_0^{\infty} e^{-y}\,dy = +e^{-x}\Big|_0^{\frac{1}{2}} \cdot e^{-y}\Big|_0^{\infty}$$

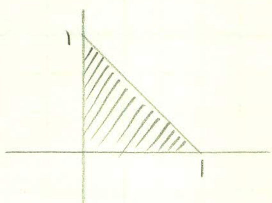
$$= (1/\sqrt{e} - 1)(-1) = 1 - \frac{1}{\sqrt{e}} = 0.39347$$

(b) FIND $P[(X+Y) \leq 1]$

FROM THE NOTES, WE HAVE, $Z = X + Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{xy}(z-y, y)\,dy$$

$$= \int_0^{\infty} e^{-(z-y+y)}\,dy \qquad \text{WHY ?}$$



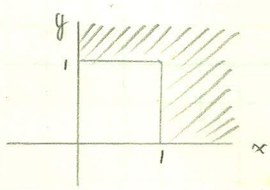
$$P[(X+Y) \leq 1] = \int_{-\infty}^{\infty} \int_{-\infty}^{1-y} f_{XY}(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^1 e^{-y} \int_0^{1-y} e^{-x}\,dx\,dy$$

$$= 1 - 2e^{-1} = 0.26424$$

(c) FIND $P[(X \text{ OR } Y) \geq 1]$

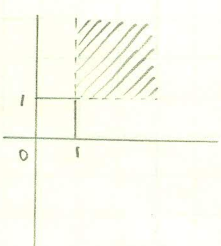


$$P[(X \text{ OR } Y) \geq 1] = 1 - P[(X \text{ AND } Y) < 1]$$

$$= 1 - \int_0^1 \int_0^1 e^{-(x+y)}\,dx\,dy$$

$$= 1 - (1 - e^{-1})^2 = 0.60042$$

(d) FIND $P[(X \text{ and } Y) \geq 1]$



$$P[(X \text{ and } Y) \geq 1] = \int_1^{\infty} \int_1^{\infty} e^{-(x+y)}\,dx\,dy$$

$$= \left(\int_1^{\infty} e^{-x}\,dx\right)^2$$

$$= e^{-2} = 0.13534$$

1-34 Are the random variables of Problem 1.33 statistically independent?

YES. Let $f_X(x) = e^{-x}$ FOR $x \geq 0$ AND $=0$ OTHERWISE

$f_Y(y) = e^{-y}$ FOR $y \geq 0$ AND $=0$ OTHERWISE

$f_X(x) \cdot f_Y(y) = f_{XY}(x,y)$ INDEPENDENT.

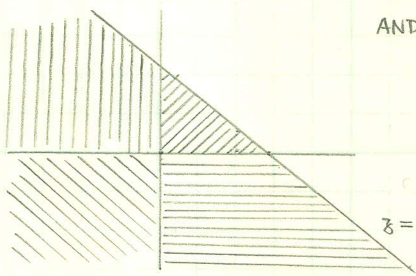
1.35. Random variables X and Y are satistically independent and their respective probability density function are

$$f_X(x) = \tfrac{1}{2} e^{-|x|} \quad ; \quad f_Y(y) = e^{-2|y|}$$

Find the probability density function associated with $X+Y$.

FROM THE NOTES: LET $Z = X+Y$, $z = x+y$

IN CASE I, AND II $P[Z < z \mid z \geq 0]$



$$= \int_{-\infty}^{0}\int_{-\infty}^{0} f_X(x) f_Y(y)\,dx\,dy + \int_{-\infty}^{0}\int_{0}^{z-y} f_X(x) f_Y(y)\,dx\,dy$$

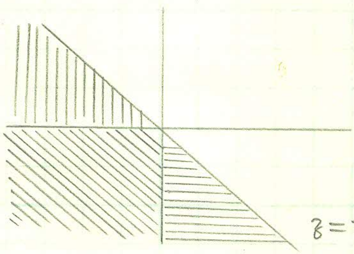
$$+ \int_{0}^{z-x}\int_{-\infty}^{0} f_X(x) f_Y(y)\,dx\,dy + \int_{0}^{z}\int_{0}^{z-y} f_X(x) f_Y(y)\,dx\,dy$$

$z = x+y > 0$

$$= \int_{-\infty}^{0} \tfrac{1}{2} e^{x}dx \int_{-\infty}^{0} e^{2y}dy + \int_{-\infty}^{0} e^{2y} \cdot \int_{0}^{z-y} \tfrac{1}{2} e^{-x} dx\, dy$$

$$+ \int_{-\infty}^{0} \tfrac{1}{2} e^{x} \int_{0}^{z-x} e^{-2y}\,dy\,dx + \int_{0}^{z} e^{-2y} \int_{0}^{z-y} \tfrac{1}{2} e^{-x} dx \cdot dy$$

CASE I



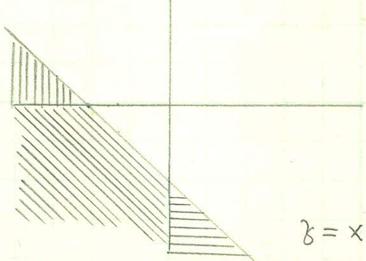
$$= \frac{1}{4} + \frac{1}{4} - \frac{e^{-z}}{3} + \frac{1}{4} - \frac{e^{-2z}}{12} + \frac{1}{4} + \frac{1}{4} e^{-2z} - \frac{1}{4} e^{-z}$$

$$= 1 + \frac{1}{6} e^{-2z} - \frac{7}{12} e^{-z}$$

$z = x+y = 0$

CASE II

IN CASE III, $P[Z < z \mid z < 0]$



$$= \int_{-\infty}^{z}\int_{0}^{z-x} \tfrac{1}{2} e^{x} \cdot e^{-2y}\,dy\,dx + \int_{-\infty}^{z}\int_{0}^{z-y} \tfrac{1}{2} e^{-x} e^{2y}\,dx\,dy$$

$$+ \int_{-\infty}^{0}\int_{-\infty}^{0} \tfrac{1}{2} e^{+x} \cdot e^{2y}\,dx\,dy - \int_{z}^{0}\int_{z-y}^{0} \tfrac{1}{2} e^{x} \cdot e^{2y}\,dx\,dy$$

$z = x+y < 0$

$$= \frac{1}{4} + \frac{1}{6} e^{z} + \frac{1}{12} e^{2z} - \frac{1}{4} + \frac{1}{4} e^{2z} + \frac{1}{2} e^{z} - \frac{1}{2} e^{2z}$$

$$= -\frac{1}{6} e^{2z} + \frac{7}{12} e^{z}$$

CASE III

$f_Z(z) = \frac{7}{12} e^{-z} - \frac{1}{3} e^{-2z}$ FOR $z \geq 0$

$f_Z(z) = \frac{7}{12} e^{z} - \frac{1}{3} e^{2z}$ FOR $z < 0$

SO, $f_Z(z) = \frac{7}{12} e^{-|z|} - \frac{1}{3} e^{-2|z|}$ $\frac{2}{3}$

check

1.37 X and Y are independent, zero-mean random variables with variance σ_x^2 and σ_y^2

Another set of random variables U and V are related to X and Y through the equations

$$u = 2x + y \quad , \quad v = x - y$$

Find the correlation coefficient of U and V. Let $\sigma_X^2 = \sigma_Y^2 = \sigma^2$

$$\rho = \frac{\sigma_{UV}^2}{\sigma_U \cdot \sigma_V} = \frac{E[(U-E[U])(V-E[V])]}{\sqrt{E[U^2]} \cdot \sqrt{E[V^2]}}$$

$$E[U] = E[2x+y] = 2E[X] + E[Y] = 0 \quad ; \quad E[V] = 0$$

$$E[U^2] = E[(2X+Y)^2] = E[4X^2 + 4XY + Y^2]$$

$$= 4E[X^2] + 4E[X]E[Y] + E[Y^2] = 4\sigma_x^2 + \sigma_y^2 = 5\sigma^2$$

$$E[V^2] = E[(X-Y)^2] = E[X^2 - 2XY + Y^2] = E[X^2] + E[Y^2] = 2\sigma^2$$

$$E[(U-E[U])\cdot(V-E[V])] = E[U\cdot V] = E[(2X+Y)\cdot(X-Y)]$$

$$= E[2X^2 - XY - Y^2] = 2E[X^2] - E[Y^2] = \sigma^2$$

$$\rho = \frac{\sigma^2}{\sqrt{5\sigma^2 \cdot 2\sigma^2}} = \frac{1}{\sqrt{10}} = 0.31623$$

1.38 The vector Gaussian random variable

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is completely described by its mean and covariance matrix. In this case they are

$$m_X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$C_X = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$$

Now consider another vector random variable Y that is related to X by the equation

$$y = Ax + b$$

where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} , \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find the mean and covariance matrix for Y.

1.38 (CONT.)

$$E(Y) = E[AX+b] = A \cdot E[X] + b$$

$$= A \cdot m_x + b$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$E[(Y-\bar{Y})(Y-\bar{Y})^T]$$

$$= E\left[\begin{bmatrix} Y_1 - 5 \\ Y_2 - 0 \end{bmatrix} \cdot [Y_1 - 5, Y_2] \right]$$

$$= E\left[\begin{bmatrix} (Y_1-5)^2 & (Y_1-5)Y_2 \\ Y_2(Y_1-5) & Y_2^2 \end{bmatrix} \right]$$

$$= \begin{bmatrix} E[Y_1^2] - 25 & E[Y_1 \cdot Y_2] \\ E[Y_1 \cdot Y_2] & E[Y_2^2] \end{bmatrix}$$

$$C_Y = E[(Y-\bar{Y})(Y-\bar{Y})^T]$$

$$= E[(Y-\bar{Y})(Y^T - \bar{Y}^T)]$$

$$= E[YY^T - \bar{Y}Y^T - Y\bar{Y}^T + \bar{Y}\bar{Y}^T]$$

$$= E[(AX+b)(X^TA^T + b^T) - (A\bar{X}+b)(X^TA^T+b^T)$$

$$- (AX+b)(\bar{X}^TA^T + b^T) + (A\bar{X}+b)(\bar{X}^TA^T + b^T)]$$

$$= E[AXX^TA^T + \cancel{AXb^T} + \cancel{bX^TA^T} + \cancel{bb^T} - A\bar{X}X^TA^T - \cancel{A\bar{X}b^T} - \cancel{bX^TA^T} - \cancel{bb^T}$$

$$- AX\bar{X}^TA^T - \cancel{AXb^T} - \cancel{b\bar{X}^TA^T} - \cancel{bb^T} + A\bar{X}\bar{X}^TA^T + \cancel{A\bar{X}b^T} + \cancel{b\bar{X}^TA^T} + \cancel{bb^T}$$

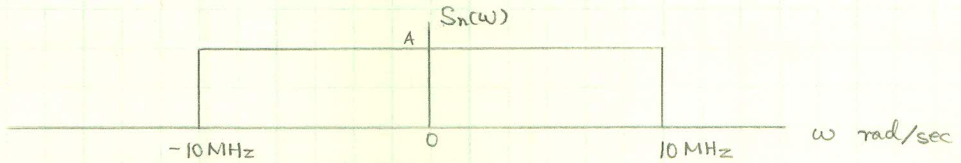
$$= E[AXX^TA^T - A\bar{X}X^TA^T - AX\bar{X}^TA^T + A\bar{X}\bar{X}^TA^T]$$

$$= E[A(X-\bar{X})(X-\bar{X})^TA^T]$$

$$= AC_XA^T = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \cancel{15}\,21 & 6 \\ 6 & 3 \end{bmatrix}$$

2.1 *Noise measurements at the output of a certain amplifier (with its input shorted) indicate that the rms output voltage due to internal noise is 100 μV. If we assume that the frequency spectrum of the noise is flat from 0 to 10 MHz and zero above 10 MHz, find:*

(a) *The spectral density function for the noise.*



According to the Parseval's theorem.

$$\int_{-\infty}^{\infty} |n(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |N(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |N(f)|^2 df$$

FROM $P = \overline{n^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)\, d\omega = \int_{-\infty}^{\infty} S_n(f)\, df = A \cdot 20\times10^6 = (100\times10^{-6})^2$

$A = 5\times10^{-16}\ V^2 \cdot SEC.$

(b) *The autocorrelation function for the noise.*

$$R_n(\tau) = \mathcal{F}^{-1}\{S_n(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)\cdot e^{j\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-2\pi\times10^7}^{2\pi\times10^7} 5\times10^{-16} e^{j\omega t}\, d\omega$$

$$= \frac{5\times10^{-16}}{j\,t\cdot 2\pi}\cdot e^{j\omega t}\Big|_{-2\pi\times10^7}^{2\pi\times10^7}$$

$$= \frac{5\times10^{-16}}{t\cdot\pi\cdot 2j}\cdot\left[e^{j2\pi\times10^7\tau} - e^{-j2\pi\times10^7\tau}\right]$$

$$= \frac{5\times10^{-16}}{\tau\,\pi}\cdot \sin(2\pi\times10^7\tau)$$

$$= 10^{-8}\cdot\frac{\sin(2\pi\times10^7\tau)}{2\pi\times10^7\tau}$$

$$= 10^{-8}\ Sa(2\pi\times10^7\tau)$$

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2-4 *Find the autocorrelation function corresponding to the spectral density function*

$$S(j\omega) = \delta(\omega) + \tfrac{1}{2}\delta(\omega-\omega_0) + \tfrac{1}{2}\delta(\omega+\omega_0) + e^{-2|\omega|}$$

Because the autocorrelation function corresponding to the spectral density function is

$$R(\tau) = \mathcal{F}^{-1}\{S(j\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [\delta(\omega) + \tfrac{1}{2}\delta(\omega-\omega_0) + \tfrac{1}{2}\delta(\omega+\omega_0) + e^{-2|\omega|}]\, e^{j\omega\tau}\, d\omega$$

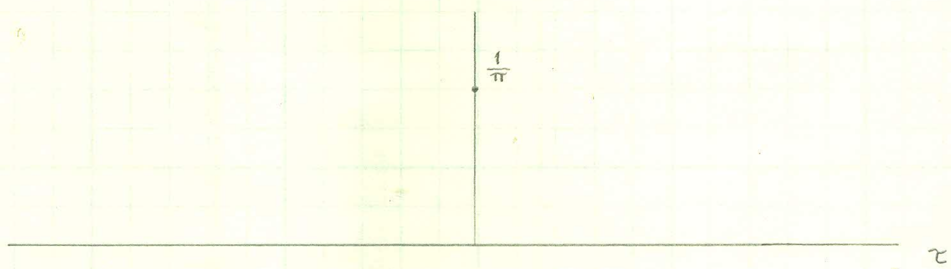
$$= \frac{1}{2\pi}\left[e^{j0\tau} + \frac{1}{2}e^{j\omega_0\tau} + \frac{1}{2}e^{-j\omega_0\tau} + \int_{-\infty}^{\infty} e^{-2|\omega|+j\omega\tau}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[1 + \cos(\omega_0\tau) + \int_{-\infty}^{0} e^{2\omega+j\omega\tau}\, d\omega + \int_{0}^{\infty} e^{-2\omega+j\omega\tau}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[1 + \cos\omega_0\tau + \frac{1}{2+j\tau}\, e^{(2+j\tau)\omega}\Big|_{-\infty}^{0} + \frac{1}{-2+j\tau}\, e^{(-2+j\tau)\omega}\Big|_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[1 + \cos\omega_0\tau + \frac{1}{2+j\tau} + \frac{1}{2-j\tau}\right]$$

$$= \frac{1}{2\pi}\left[1 + \cos\omega_0\tau + \frac{4}{4+\tau^2}\right]$$

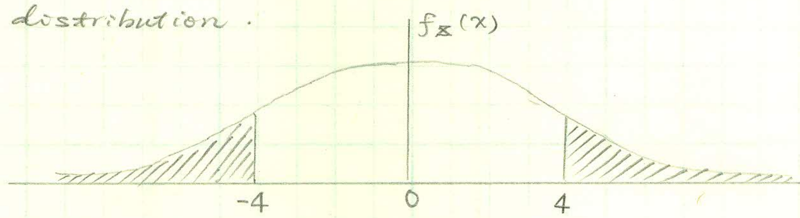


2.5 A stationary Gaussian random process $X(t)$ has an autocorrelation function of the form

$$R_X(\tau) = 4e^{-|\tau|}$$

What fraction of the time will $|X(t)|$ exceed four units?

Because the random process $X(t)$ is a stationary one and has Gaussian distribution.



$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\frac{x^2}{\sigma^2}}$$

$$E[X(t)] = 0 \qquad E[X^2(t)] = \sigma^2$$

$$E[X^2(t)] = \sigma^2 = R_X(0) = 4e^{-|0|} = 4$$

$$\therefore \quad \sigma = 2$$

$$f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{8}x^2}$$

$$P[\,|x| > 4\,] = 2\cdot\int_4^\infty f_X(x)\,dx$$

$$= 2\cdot\int_4^\infty \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{8}x^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_4^\infty e^{-\frac{1}{8}x^2}\,dx$$

LET $v = x - 4$ $\quad = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{8}(v+4)^2}\,dv$

$$= 0.0455$$

SO, The random process has 4.55% Chance for $|X(t)| > 4$.

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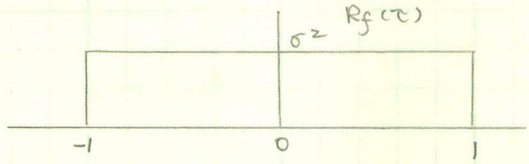
2.8 It's suggested that a certain real process has an autocorrelation function as shown in the figure. Is this possible? Justify your answer.



Answer: It is impossible. Because the power spectral density of the process

$$S_f(\omega) = \mathcal{F}\{R_f(\tau)\}$$

$$= \int_{-\infty}^{\infty} R_f(\tau) e^{-j\omega\tau} d\tau$$

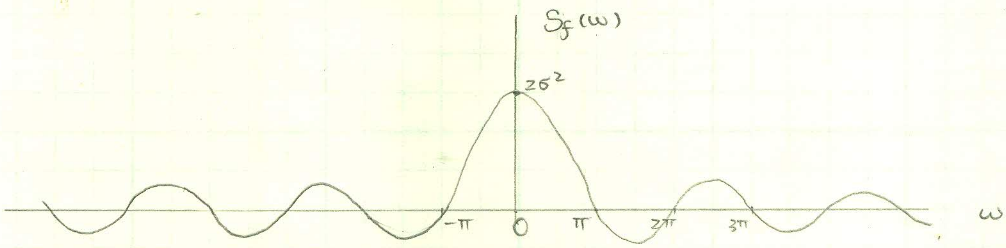
$$= \int_{-1}^{1} \sigma^2 e^{-j\omega\tau} d\tau$$

$$= \frac{\sigma^2}{-j\omega} e^{-j\omega\tau}\Big|_{-1}^{1}$$

$$= \frac{\sigma^2}{+j\omega} \cdot [e^{j\omega} - e^{-j\omega}]$$

$$= \frac{2\sigma^2}{\omega} \cdot \sin(\omega)$$

$$= 2\sigma^2 \, Sa(\omega)$$



A real process can not have a frequency at ~~infinite~~ and negative power.

2.9 Consider the random process $X(t) = 2\sin\omega t$ where ω is a random variable with uniform distribution between $\omega=2$ and $\omega=6$. Is the process (a) Stationary, (b) ergodic, and (c) deterministic or nondeterministic?

$$E[X(t)] = \int_{-\infty}^{\infty} x(t)\, f_X(x,t)\, dx$$

$$= \int_{-\infty}^{\infty} 2\sin\omega t \cdot f_X(\omega)\, d\omega$$

$$= \int_2^6 2\sin\omega t \cdot \frac{1}{6-2}\, d\omega$$

$$= \frac{1}{2}\int_2^6 \sin\omega t\, d\omega$$

$$= -\frac{1}{2t}\cos\omega t\Big|_2^6 = \frac{1}{2t}[\cos 2t - \cos 6t]$$

$$E[X(t_1)X(t_2)] = \int_2^6 2\sin\omega t_1 \cdot 2\sin\omega t_2 \cdot \frac{1}{6-2}\, d\omega$$

$$= \int_2^6 \sin\omega t_1 \cdot \sin\omega t_2\, d\omega$$

$$= \int_2^6 \frac{1}{2}[\cos\omega(t_1-t_2) - \cos\omega(t_1+t_2)]\, d\omega$$

$$= \frac{1}{2(t_1-t_2)}\cos\omega(t_1-t_2)\Big|_2^6 - \frac{1}{2(t_1+t_2)}\cos\omega(t_1+t_2)\Big|_2^6$$

$$= \frac{1}{2(t_1-t_2)}[\cos 6(t_1-t_2) - \cos 2(t_1-t_2)]$$

$$- \frac{1}{2(t_1+t_2)}[\cos 6(t_1+t_2) - \cos 2(t_1+t_2)]$$

So, the random process $X(t)$ is not stationary and ergodic.

But maybe deterministic.

2.16 A stationary random process $X(t)$ has a spectral density function of the form

$$S_X(\omega) = \frac{6\omega^2+12}{(\omega^2+4)(\omega^2+1)}$$

What is the mean square value of $X(t)$?

$$S_X(\omega) = \frac{6\omega^2+12}{(\omega^2+4)(\omega^2+1)}$$

$$= \frac{4}{\omega^2+4} + \frac{2}{\omega^2+1}$$

$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(\omega)\}$$

$$= \mathcal{F}^{-1}\left\{\frac{4}{\omega^2+4} + \frac{2}{\omega^2+1}\right\}$$

$$= e^{-2|\tau|} + e^{-|\tau|}$$

$$E[X^2(t)] = R_X(0) = e^{-2\cdot 0} + e^{-0} = 2$$

2.17. The stationary process $X(t)$ has an autocorrelation function of the form

$$R_X(\tau) = \sigma^2 e^{-\beta|\tau|}$$

Another process $Y(t)$ is related to $X(t)$ by the deterministic equation

$$Y(t) = aX(t) + b$$

where a and b are known constants.

(a) What is the autocorrelation function for $Y(t)$?

$$E[Y(t_1)Y(t_2)] = E[\{aX(t_1)+b\}\cdot\{aX(t_2)+b\}]$$

$$= E[a^2X(t_1)X(t_2) + abX(t_1) + abX(t_2) + b^2]$$

$$= a^2E[X(t_1)X(t_2)] + 2ab\,E[X(t)] + b^2$$

(Because $R_X(\tau) \to 0$ as $\tau \to \infty$. $\therefore E[X(t)] = 0$)

$$= a^2E[X(t_1)X(t_2)] + b^2$$

$$\therefore R_Y(\tau) = a^2R_X(\tau) + b^2 = a^2\sigma^2 e^{-\beta|\tau|} + b^2$$

(b) What is the crosscorrelation function $R_{XY}(\tau)$?

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E[X(t)\cdot[aX(t+\tau)+b]]$$

$$= aE[X(t)X(t+\tau)] + bE[X(t)] = aR_X(\tau) = a\sigma^2 e^{-\beta|\tau|}$$

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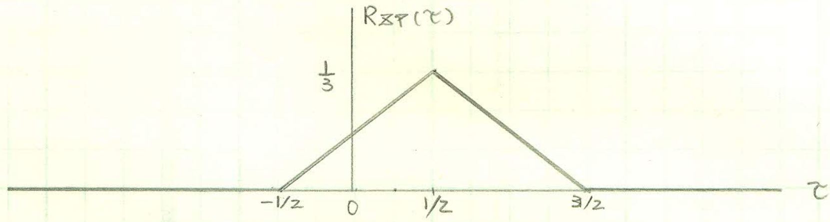
2.18 The crosscorrelation function $R_{xy}(\tau)$ for Example 2.8 is sketched below. What is the corresponding cross spectral density function?



$$S_{xy}(\omega) = \mathcal{F}\{R_{xy}(\tau)\}$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau)\cdot e^{-j\omega\tau}\,d\tau$$

$$= \int_{-\frac{1}{2}}^{1/2} \left(\frac{\tau}{3}+\frac{1}{6}\right)e^{-j\omega\tau}\,d\tau + \int_{1/2}^{3/2}\left(-\frac{\tau}{3}+\frac{1}{2}\right)e^{-j\omega\tau}\,d\tau$$

$$= \frac{1}{3(-j\omega)^2}\cdot e^{-j\omega\tau}\cdot(-j\omega\tau-1)\Big|_{-1/2}^{1/2} + \frac{1}{6}\cdot\frac{1}{-j\omega}e^{-j\omega\tau}\Big|_{-1/2}^{1/2}$$

$$- \frac{1}{3(-j\omega)^2}e^{-j\omega\tau}(-j\omega\tau-1)\Big|_{1/2}^{3/2} + \frac{1}{2}\cdot\frac{1}{-j\omega}e^{j\omega\tau}\Big|_{1/2}^{3/2}$$

$$= -\frac{1}{3\omega^2}e^{-j0.5\omega}(-j0.5\omega-1) + \frac{1}{3\omega^2}e^{j0.5\omega}(j0.5\omega-1)$$

$$-\frac{1}{j6\omega}e^{-j0.5\omega} + \frac{1}{j6\omega}e^{j0.5\omega}$$

$$+\frac{1}{3\omega^2}e^{-j1.5\omega}(-j1.5\omega-1) - \frac{1}{3\omega^2}e^{-j0.5\omega}(-j0.5\omega-1)$$

$$-\frac{1}{j2\omega}e^{-j1.5\omega} + \frac{1}{j2\omega}e^{-j0.5\omega}$$

$$= e^{-j0.5\omega}\left[\frac{1+j0.5\omega}{3\omega^2} + \frac{j}{6\omega} + \frac{1+j0.5\omega}{3\omega^2} - \frac{j}{2\omega}\right]$$

$$+ e^{j0.5\omega}\left[\frac{-1+j0.5\omega}{3\omega^2} - \frac{j}{6\omega}\right]$$

$$+ e^{j1.5\omega}\left[\frac{j}{2\omega} - \frac{1+j1.5\omega}{3\omega^2}\right]$$

$$= e^{-j0.5\omega}\cdot\frac{1}{3}\cdot\frac{1}{2}\left[Sa(\omega/4)\right]^2$$

$$= \frac{1}{6}e^{-j0.5\omega}\cdot\left[Sa(\omega/4)\right]^2$$

2.20 Two deterministic random processes are defined by

$$X(t) = A \sin(\omega t + \theta)$$

$$Y(t) = B \sin(\omega t + \theta)$$

where θ is a random variable with uniform distribution between 0 and 2π, and ω is a known constant. The A and B coefficients are both normal random variable $N(0, \sigma^2)$ and are correlated with a correlation coefficient ρ. What is the crosscorrelation function $R_{XY}(\tau)$?

(Assume A and B are independent of θ)

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

$$= E[\,[A\sin(\omega t+\theta)]\cdot[B\sin(\omega t+\omega\tau+\theta)]\,]$$

$$= E[AB]\cdot E[\sin(\omega t+\theta)\cdot\sin(\omega t+\omega\tau+\theta)]$$

$$= \rho\sigma^2[\int_0^{2\pi}[\sin(\omega t+\theta)\cdot\sin(\omega t+\omega\tau+\theta)]\cdot\frac{1}{2\pi}\,d\theta]$$

$$= \frac{\rho\sigma^2}{2\pi}[\int_0^{2\pi}\frac{1}{2}(\cos\omega\tau+\cos(2\omega t+\omega\tau+2\theta))\,d\theta]$$

$$= \frac{\rho\sigma^2}{\cancel{4\pi}\ 2}\cos\omega\tau$$

Because correlation coefficient $\rho \triangleq \sigma_{xy}^2/\sigma_x\cdot\sigma_y$.

$$\sigma_{xy}^2 = E[XY] - m_x m_y = E[XY] = \rho\,\sigma_x\cdot\sigma_y = \rho\sigma^2.$$

2.22 Let the process $Z(t)$ be the product of two independent stationary processes $X(t)$ and $Y(t)$. Show that the spectral density function for $Z(t)$ is given by (in the s domain) [Hint: First show that $R_Z(\tau) = R_X(\tau) R_Y(\tau)$]

$$S_Z(s) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} S_X(w) S_Y(s-w)\,dw$$

Because $Z(t) = X(t)Y(t)$,

$$R_Z(\tau) = E[Z(t)Z(t+\tau)]$$

$$= E[X(t)Y(t)X(t+\tau)Y(t+\tau)]$$

$$= E[X(t)X(t+\tau)\cdot Y(t)Y(t+\tau)]$$

$$= E[X(t)X(t+\tau)]\cdot E[Y(t)Y(t+\tau)] \quad (X(t) \text{ and } Y(t) \text{ indep.})$$

$$= R_X(\tau)\cdot R_Y(\tau)$$

$$S_Z(s) = \mathcal{L}\{R_Z(\tau)\}$$

$$= \mathcal{L}\{R_X(\tau)\cdot R_Y(\tau)\}$$

$$= \int_{-j\infty}^{j\infty} R_X(\tau)R_Y(\tau)e^{-s\tau}\,d\tau$$

$$= \int_{-j\infty}^{j\infty} R_Y(\tau)\cdot\left[\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} S_X(w)e^{w\tau}dw\right]e^{-s\tau}\,d\tau$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} S_X(w)\cdot\left[\int_{-j\infty}^{j\infty} R_Y(\tau)e^{-(s-w)\tau}\,d\tau\right]dw$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} S_X(w)\, S_Y(s-w)\,dw$$

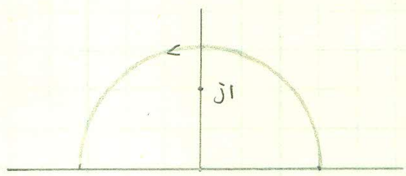
2.23. The spectral density function for the stationary process $X(t)$ is

$$S_X(j\omega) = \frac{1}{(1+\omega^2)^2}$$

Find the autocorrelation function for $X(t)$.

$$R_X(\tau) = \mathfrak{F}^{-1}\{S_X(j\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega)e^{j\omega\tau}\,d\omega$$



$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} e^{j\omega\tau}\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\frac{1}{(1+\omega^2)^2}\cos\omega\tau\,d\omega + j\cdot\int_{-\infty}^{\infty}\frac{1}{(1+\omega^2)^2}\sin\omega\tau\,d\omega\right]$$

$$= \frac{1}{2\pi}\operatorname{Re}\left\{2\pi i \sum_{y>0} \operatorname{Res}\frac{e^{z\tau}}{(1+z^2)^2}\right\}$$

$$+ \frac{1}{2\pi j}\operatorname{Im}\left\{2\pi i \sum_{y>0} \operatorname{Res}\frac{e^{z\tau}}{(1+z^2)^2}\right\}$$

$$= \frac{1}{2\pi}\operatorname{Re}\left\{2\pi i\cdot\frac{d}{dz}\cdot\frac{e^{z\tau}}{(z+j)^2}\Big|_{z=j}\right\}$$

$$+ \frac{1}{2\pi j}\operatorname{Im}\left\{2\pi j\cdot\frac{d}{dz}\cdot\frac{e^{z\tau}}{(z+j)^2}\Big|_{z=j}\right\}$$

$$= \frac{1}{2\pi}\operatorname{Re}\left\{2\pi j\cdot\frac{\tau e^{z\tau}(z+j)^2 - 2(z+j)e^{jz\tau}}{(z+j)^4}\Big|_{z=j}\right\}$$

$$+ \frac{1}{2\pi j}\operatorname{Im}\left\{2\pi j\cdot\frac{\tau e^{jz\tau}(z+j)^2 - 2(z+j)e^{jz\tau}}{(z+j)^4}\Big|_{z=j}\right\}$$

$$= \frac{1}{2\pi}\operatorname{Re}\left\{2\pi j\cdot\frac{-4\tau e^{-\tau} - 4je^{-\tau}}{16}\right\}$$

$$+ \frac{1}{2\pi j}\operatorname{Im}\left\{2\pi j\cdot\frac{-4\tau\cos\tau - j4\tau\sin\tau - j4\tau\cos\tau + 4\tau\sin\tau}{16}\right\}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{16}\cdot 2\pi\cdot 4\tau(\sin\tau + \cos\tau)$$

$$+ \frac{1}{2\pi j}\cdot\frac{1}{16}\cdot 2\pi\cdot 4\tau(\sin\tau - \cos\tau)$$

$$= \frac{1}{4}e^{-\tau} - \frac{\tau}{4}e^{-\tau}$$

$=$

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2.26 We wish to determine the autocorrelation function a random signal empirically from a single time record. Let us say we have good reason to believe the process is ergodic and at least approximately Gaussian and furthermore, that the ACF of the process decays exponentially with a time constant no greater than 10 sec. Estimate the record length needed to achieve 5 percent accuracy in the determination of the autocorrelation function).

FROM THE NOTES, WE HAVE

$$VAR[V_x(\tau)] \leq \frac{4}{T}\int_0^\infty R_x^2(\tau)\,d\tau$$

T -- THE LENGTH OF RECORD

$R_x(\tau) = \sigma^2 \cdot e^{-\beta|\tau|}$, ACF $\beta = 0.1$

$V_x(\tau)$ -- COMPUTED ACF

$$VAR[V_x(\tau)] \leq \frac{4}{T}\int_0^\infty R_x^2(\tau)\,d\tau = \frac{4}{T}\sigma^4 \int_0^\infty e^{-2\beta|\tau|}\,d\tau$$

$$= \frac{4}{T}\sigma^4 \cdot -\frac{1}{2\beta} e^{-2\beta|\tau|}\Big|_0^\infty$$

$$= \frac{2\sigma^4}{T\beta}$$

According to the defination of '$V_x(\tau)$ IS 10% ACCURATE' on Page 15,

$$\frac{\sqrt{VAR[V_x(\tau)]}}{\sigma^2} \leq \sqrt{\frac{2}{T\beta}} = 0.05 = 5\%$$

$$T = \frac{2}{0.1 \times 0.05^2} = 8000 \text{ SECONDS}$$

2.27 In Problem 2.26 the signal is not know to be truly bandlimited. but it is reasonable to assume that essentially all the signal energy will lie between 0.1 Hz. Let us assume 0.1 Hz to be the signal bandwidth, and let us say we wish to sample the signal at the Nyquist rate.

(a) How many discrete samples would be required to describe the record length of Problem 2.26?

According the Nyquist rate. sample period must be

$$T' = \frac{1}{2B} = \frac{1}{2\times 0.1} = 5 \text{ SEC.}$$

So, the sample-points $= 8000/5 = 1600$

(b) Suppose that we wish to compute the discrete Fourier transform of the finite-time signal using the Fast Fourier Transform. This requires that the number of samples N be an integer power of 2. What should N be in this case?

$$N = 2^m > 1600$$

$$m \log 2 > \log 1600$$

$$m > \log 1600 / \log 2 = 10.6439 \quad , \text{ SO } \quad m = 11$$

$$N = 2^{11} = 2048 \quad \text{points.}$$

(c) THE value of N in part (b) should work out to be greater than that found in part (a). In order to achive the approciate N for the FFT algorithm, would we be better off to increase the sampling rate for the length of time computed in Problem 2.26, or should we keep the sampling rate at .2 Hz and increase the time length of the record accordingly? Presumably, the computational effort would be the same.

INCREASING THE TIME LENGTH TO $5 \times 2048 = 10240$ SEC.

VAR ACCURATE % $\leq \sqrt{\frac{2}{T\beta}} = 0.1953\,\%$

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1. Verify Equations (4.12), (4.13), (4.15), (4.16), (4.22) and (4.23)

4.2.1 Bernoulli Process

$$P\{X_n = 1\} = p \quad , \quad \text{so,} \quad P\{X_n = 0\} = 1-p$$

Equation (4.12): $m_X(n) = E[X_n] = 1\cdot p + 0\cdot(1-p) = p$ QED

Equation (4.13): $K_X(n_1,n_2) = R_X(n_1,n_2) - m_X(n_1)\,m_X(n_2)$

$$R_X(n_1,n_2) = E[X_{n_1}\cdot X_{n_2}]$$

$$= 1\cdot1\cdot p\cdot p + 0\cdot1\cdot(1-p)\cdot p + 1\cdot0\cdot p\cdot(1-p) + 0\cdot0\cdot(1-p)^2$$

$$= p^2 \quad \text{FOR} \quad n_1 - n_2 \neq 0$$

IF $n_1 - n_2 = 0$, THUS $n_1 = n_2$

$$R_X(n_1,n_2) = R_X(n_1,n_1) = E[X_{n_1}^2] = [1\times p - 0\cdot(1-p)] = p$$

$$K_X(n_1,n_2) = R_X(n_1,n_2) - p^2 = p^2 - p^2 = 0 \quad \text{FOR} \quad n_1 - n_2 \neq 0$$

$$= R_X(n_1,n_2) - p^2 = p - p^2 = p(1-p) \quad \text{FOR} \quad n_1 - n_2 = 0 \qquad \text{QED}$$

4.2.2 Binomial Counting Process

$Y_n \triangleq \sum_{i=1}^{n} X_i$ $\quad X_i$ is Bernoulli Process in 4.2.1

Equation (4.15): $m_Y(n) = E[Y_n] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} 1\cdot p$

$$= [1\cdot p + 1\cdot p + \cdots + 1\cdot p] = n\cdot p \qquad \text{QED.}$$

Equation (4.16): $R_Y(n_1,n_2) = E[Y_{n_1}, Y_{n_2}]$

$$= E\left[\sum_{i=1}^{n_1} X_i \cdot \sum_{j=1}^{n_2} X_j\right]$$

$$= E\left[\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} X_i\cdot X_j\right]$$

$$= \sum_{\substack{i=1\\ i\neq j}}^{n_1}\sum_{j=1}^{n_2} (1\cdot1\cdot p\cdot p) \cancel{= n_1 n_2 p^2} + \sum_{i=1}^{n_1} \cdot p$$

$$= (n_1 n_2 - n_1)\cdot p^2 + n_1 p$$

$$= n_1\cdot n_2 p^2 - n_1 p^2 + n_1 p = p(1-p)\min\{n_1,n_2\} + n_1 n_2 p^2$$

$$K_Y(n_1,n_2) = R_Y(n_1,n_2) - n_1 n_2 p^2 = p(1-p)\min\{n_1,n_2\}$$

4.2.3. Random-Walk Process

$$Z_n \triangleq \begin{cases} +1 & \text{FOR SUCCESS in } n\text{th trial} \\ -1 & \text{FOR FAILURE IN } n\text{th trial} \end{cases}$$

$$W_n \triangleq \sum_{i=1}^{n} Z_i$$

Equation (4.22): $m_W(n) = E[W_n] = E[\sum_{i=1}^{n} Z_i]$

$$= \sum_{i=1}^{n} (p + (-1)\cdot(1-p)) = \sum_{i=1}^{n} (2p-1) = n(2p-1) \quad \text{QED.}$$

$$=0 \quad \text{FOR } p = \tfrac{1}{2}$$

Equation (4.23): $K_W(n_1,n_2) = E[W_{n_1} W_{n_2}] - m_W(n_1)\, m(n_2)$

$$= E\left[\sum_{i=1}^{n_1} \cdot \sum_{j=1}^{n_2} Z_i \cdot Z_j\right] - n_1 n_2 (2p-1)^2$$

$$= [n_1 \cdot n_2 - \min\{n_1,n_2\}] \cdot [1*1*p^2 - 1*1*p(1-p) - 1*1*p(1-p) + 1*(1-p)^2]$$

$$\min\{n_1,n_2\} \cdot (0p^2 - 0p + 1) \quad \cdots \quad - n_1 n_2 (2p-1)^2$$

$$= [n_1 n_2 - \min\{n_1,n_2\}] \cdot [p^2 - 2p + 2p^2 + 1 - 2p + p^2] - n_1 n_2 (2p-1)^2$$

$$= 4p(1-p) \min\{n_1, n_2\} \quad \text{QED.}$$

$$R_W(n_1,n_2) = \min\{n_1,n_2\} \quad \text{FOR } p = \tfrac{1}{2}$$

2. Verify Equations (4.38) to (4.41). Compare this result with (4.25) and (4.26).

Hint: To obtain (4.39), use the identity $\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$

4.28. Amplitude-Modulated Sine Wave.

$Z(t) = Y(t) \sin(\omega_0 t + \Phi)$; assume $Y(t)$ and Φ are independent.

Equation (4.38): $m_Z(t) = E[Z(t)] = E[Y(t)\sin(\omega_0 t + \Phi)]$ independent.

$$= E[Y(t)] \cdot E[\sin(\omega_0 t + \Phi)] = m_Y(t)\, E\{\sin(\omega_0 t + \Phi)\} \quad \text{QED.}$$

Equation (4.39): $R_Z(t_1,t_2) = R_Y(t_1,t_2) \cdot [\frac{1}{2}\cos[\omega_0(t_1-t_2)] - \frac{1}{2}E\{\cos[\omega_0(t_1+t_2)+2\Phi]\}]$

$$R_Z(t_1,t_2) = E[Z(t_1)\cdot Z(t_2)] \cancel{- m_Z(t_1)\, m_Z(t_2)}$$

$$= E[Y(t_1)\sin(\omega_0 t_1 + \Phi)\, Y(t_2)\sin(\omega_0 t_2 + \Phi)] \cancel{- m_Y(t_1) m_Y(t_2) E\{\sin(\omega_0 t_1 + \Phi)\} E\{\sin(\omega_0 t_2 + \Phi)\}}$$

$$= E[Y(t_1)Y(t_2)] \cdot E[\tfrac{1}{2}\cos[\omega_0(t_1-t_2) - \tfrac{1}{2}\cos[\omega_0(t_1+t_2)+2\Phi]]$$

$$= R_Y(t_1,t_2) \cdot [\tfrac{1}{2}\cos[\omega_0(t_1-t_2)] - \tfrac{1}{2}E\{\cos[\omega_0(t_1+t_2)+2\Phi]\}] \quad \text{QED.}$$

Equation (4.40): $m_Z(t) = m_Y(t) \cdot E\{\sin(\omega_0 t + \Phi)\} = m_Y(t) \cdot \int_{-\pi}^{\pi} \sin(\omega_0 t + \Phi) \cdot \frac{1}{2\pi}\, d\Phi = 0.$

Equation (4.41): $R_Z(t_1,t_2) = \frac{1}{2} R_Y(t_1,t_2) \cos[\omega_0(t_1-t_2)]$

$$R_Z(t_1,t_2) = R_Y(t_1,t_2) \cdot [\tfrac{1}{2}\cos(\omega_0(t_1-t_2)) - \tfrac{1}{2}\int_{-\pi}^{\pi} \cos[\omega_0(t_1+t_2)+2\Phi] \cdot \tfrac{1}{2\pi}\, d\Phi]$$

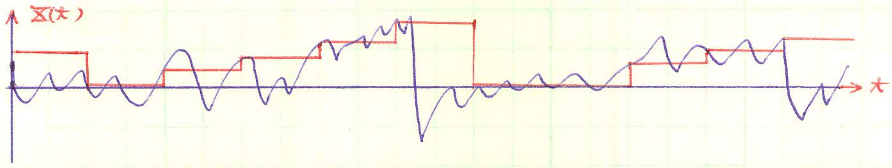
$$= R_Y(t_1,t_2) \cdot \tfrac{1}{2}\cos(\omega_0(t_1-t_2)). \quad \text{QED.}$$

3. Verify Equations (4.31) and (4.35). Then draw graphs [as surfaces above the (t_1, t_2) plane] of the functions $K_X(t_1,t_2)$ in (4.31), and $K_Y(t_1-t_2)$ in (4.35). Describle the paths, in the (t_1,t_2) plane, along which K_X is periodic and K_Y is constant. Hint: To obtain (4.35), use the indetity

$$\sum_{n=-\infty}^{\infty}\int_{t-nT-T/2}^{t-nT+T/2} f(\phi)\,d\phi = \int_{-\infty}^{\infty} f(\phi)\,d\phi,$$ which holds for any integrable $f(\cdot)$.

4.2.6. Sampled-and-Held Noise Process

$$X(t) = \sum_{n=-\infty}^{\infty} V(nT)\,h(t-nT)$$



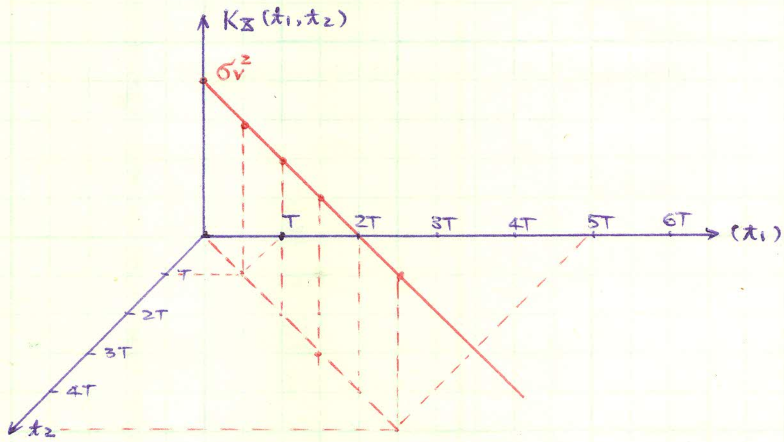
Equation (4.31): $K_X(t_1,t_2) = E[X(t_1)X(t_2)] - m_X(t_1)m_X(t_2)$

$$= E\Big[\sum_{n=-\infty}^{\infty} V(nT)h(t_1-nT)\cdot\sum_{j=-\infty}^{\infty} V(jT)h(t_2-jT)\Big]$$

$$= \sum_{n=-\infty}^{\infty}\cdot\sum_{j=-\infty}^{\infty} E[V(nT)h(t_1-nT)V(jT)h(t_2-jT)]$$

$$= \sum_{n=-\infty}^{\infty}\cdot\sum_{j=-\infty}^{\infty} E[V(nT)V(jT)]\cdot h(t_1-nT)\,h(t_2-jT)$$

$$= \sum_{n=-\infty}^{\infty} \sigma_V^2\, h(t_1-nT)\,h(t_2-nT)$$



Equation (4.35): $Y(t) = \sum_{n=-\infty}^{\infty} V(nT+\Theta)h(t-nT-\Theta)$; $f_\Theta(\theta) = \begin{cases}\frac{1}{T}, & \frac{-T}{2}\le\theta<\frac{T}{2}\\ 0 & \text{otherwise}\end{cases}$

Phase-Randomized Sampled-and-Held Process

On page 77 in Gardener. is too difficult.

4. As a generalization of the result in exercise 2, verify that the autocorrelation function for the product of two independent processes, $Y(t) = X(t) \cdot Z(t)$, is the product of their autocorrelations, $R_Y = R_X \cdot R_Z$.

$$\cancel{R_Y = E[Y(t)] = E[X(t) \cdot Z(t)] =}$$

$$R_Y(t_1, t_2) = E[Y(t_1)Y(t_2)] = E[X(t_1)Z(t_1) \cdot X(t_2) \cdot Z(t_2)]$$

$$= E[X(t_1)X(t_2)] \cdot E[Z(t_1)Z(t_2)]$$

$$= R_X(t_1, t_2) \cdot R_Z(t_1, t_2) \quad \text{QED.}$$

5. Consider the quadrature-amplitude-modulated signal

$$Y(t) = X(t)\cos(w_0 t) - Z(t)\sin(w_0 t)$$

where $X(t)$ and $Z(t)$ are zero-mean independent processes with identical autocorrelation functions, $R_X = R_Z$. Determine $R_Y(t_1, t_2)$, and show that if $R_X(t_1, t_2) = R_X(t_1 - t_2)$, then $R_Y(t_1, t_2) = R_Y(t_1 - t_2)$.

SOL:

$$R_Y(t_1, t_2) = E[Y(t_1)Y(t_2)] = E\left[\{X(t_1)\cos(w_0 t_1) - Z(t_1)\sin(w_0 t_1)\} \cdot \{X(t_2)\cos(w_0 t_2) - Z(t_2)\sin(w_0 t_2)\}\right]$$

$$= E\left[X(t_1)X(t_2)\cos w_0 t_1 \cos w_0 t_2 - X(t_1)Z(t_2)\cos w_0 t_1 \sin w_0 t_2 - Z(t_1)X(t_2)\sin w_0 t_1 \cos w_0 t_2 + Z(t_1)Z(t_2)\sin w_0 t_1 \sin w_0 t_2\right]$$

$$= R_X(t_1, t_2)\cos w_0 t_1 \cos w_0 t_2 + R_Z(t_1, t_2)\sin w_0 t_1 \sin w_0 t_2$$

$$= R_X(t_1, t_2)\left[\cos w_0 t_1 \cos w_0 t_2 + \sin w_0 t_1 \sin w_0 t_2\right]$$

$$= R_X(t_1, t_2)\cos[w_0(t_1 - t_2)]$$

IF $R_X(t_1, t_2) = R_X(t_1 - t_2)$, THEN

$$R_Y(t_1, t_2) = R_X(t_1 - t_2)\cos(w_0(t_1 - t_2)) = R_Y(t_1 - t_2) \quad \text{QED.}$$

6. Let $X(t)$ and $Z(t)$ be independent processes. Determine the autocorrelation functions (in terms of m_X, m_Z, R_X, R_Z) of

$$Y_1(t) = X(t) + Z(t) \quad \text{and} \quad Y_2(t) = X(t) - Z(t).$$

Then determine the cross-correlation function of $Y_1(t)$ and $Y_2(t)$. Finally, consider the special case for which $R_X = R_Z$ and $m_X = m_Z$, and simplify the preceding results. Hint: Use the result of exercise 4.

SOL:

$$R_{Y_1}(t_1,t_2) = E[Y_1(t_1)\cdot Y_1(t_2)] = E[(X(t_1)+Z(t_1))\cdot(X(t_2)+Z(t_2))]$$

$$= E[X(t_1)X(t_2) + X(t_1)Z(t_2) + Z(t_1)X(t_2) + Z(t_1)Z(t_2)]$$

$$= R_X(t_1,t_2) + R_Z(t_1,t_2) + E[X(t_1)]\cdot E[Z(t_2)] + E[Z(t_1)]\cdot E[X(t_2)]$$

$$= R_X(t_1,t_2) + R_Z(t_1,t_2) + m_X(t_1)\cdot m_Z(t_2) + m_Z(t_1)\cdot m_X(t_2)$$

$$R_{Y_2}(t_1,t_2) = R_X(t_1,t_2) + R_Z(t_1,t_2) - m_X(t_1)m_Z(t_2) - m_Z(t_1)\cdot m_X(t_2)$$

$$R_{Y_1Y_2}(t_1,t_2) = E[Y_1(t_1)\cdot Y_2(t_2)] = E[\{X(t_1)+Z(t_1)\}\cdot\{X(t_2)-Z(t_2)\}]$$

$$= E[X(t_1)X(t_2) - X(t_1)\cdot Z(t_2) + Z(t_1)\cdot X(t_2) - Z(t_1)\cdot Z(t_2)]$$

$$= R_X(t_1,t_2) - R_Z(t_1,t_2) - m_X(t_1)\cdot m_Z(t_2) + m_Z(t_1)m_X(t_2)$$

FOR THE SPECIAL CASE, $R_X = R_Z$, and $m_X = m_Z$, then

$$R_{Y_1}(t_1,t_2) = 2R_X(t_1,t_2) + 2m_X(t_1)m_X(t_2)$$

$$R_{Y_2}(t_1,t_2) = 2R_X(t_1,t_2) - 2m_X(t_1)m_X(t_2)$$

$$R_{Y_1Y_2}(t_1,t_2) = 0$$

7. Determine the autocorrelation function for the process

$$Y(t) = X(t) - X(t-T),$$

for the case in which

$$R_X(t_1,t_2) = R_X(t_1-t_2).$$

$$R_Y(t_1,t_2) = E[Y(t_1)Y(t_2)] = E[\{X(t_1)-X(t_1-T)\}\cdot\{X(t_2)-X(t_2-T)\}]$$

$$= E[X(t_1)X(t_2) - X(t_1)X(t_2-T) - X(t_1-T)X(t_2) + X(t_1-T)X(t_2-T)]$$

$$= R_X(t_1-t_2) - R_X(t_1-t_2+T) - R_X(t_1-t_2-T) + R_X(t_1-t_2)$$

$$R_Y(\tau) = 2R_X(\tau) - R_X(\tau+T) - R_X(\tau-T).$$

Exercises Chapter 5 Random Processes Benmei Chen

1. (a) Apply the result (5.58) to show that for a zero-mean Gaussian process $X(t)$, the fourth joint moment is given by

$$E\{X(t_1)X(t_2)X(t_3)X(t_4)\} = K_X(t_1,t_2)K_X(t_3,t_4) + K_X(t_1,t_3)K_X(t_2,t_4) + K_X(t_2,t_3)K_X(t_1,t_4)$$

<u>SOLUTION:</u> For the even-order joint moments of a set of jointly Gaussian zero-mean variables, Formula (5.58) gives

$$E\{X_1 X_2, \cdots X_r\} = \Sigma\, E\{X_{j_1} X_{j_2}\} \cdots E\{X_{j_{r-1}} X_{j_r}\}$$

so,

$$E\{X(t_1)X(t_2)X(t_3)X(t_4)\} = E\{X(t_1)\cdot X(t_2)\} + E\{X(t_1)\cdot X(t_3)\} + E\{X(t_1)\cdot X(t_4)\}$$
$$+ E\{X(t_2)X(t_3)\} + E\{X(t_2)X(t_4)\} + E\{X(t_3)X(t_4)\}$$
$$= E\{X(t_1)\cdot X(t_2)\}\cdot E\{X(t_3)\cdot X(t_4)\} + E\{X(t_1)X(t_3)\}\cdot E\{X(t_2)X(t_4)\}$$
$$+ E\{X(t_2)X(t_3)\}\cdot E\{X(t_1)X(t_4)\}$$
$$= K_X(t_1,t_2)K_X(t_3,t_4) + K_X(t_1,t_3)K_X(t_2,t_4) + K_X(t_2,t_3)K_X(t_1,t_4)$$

(b) Derive the result (5.59) by starting with the identity (5.57)

FROM THE EQUATION (5.57), AND LET $K_1=K_2=K_3=K_4=1$, $r=4$

$$i^r m(k_1,k_2,k_3,k_4) = E\{X_1X_2X_3X_4\}$$
$$= \left.\frac{\partial^4 \Phi(\omega_1,\omega_2,\omega_3,\omega_4)}{\partial\omega_1\,\partial\omega_2,\partial\omega_3\partial\omega_4}\right|_{\omega_1=\omega_2=\omega_3=\omega_4=0}$$
$$= \left.\frac{\partial E\{e^{i(\omega_1X_1+\omega_2X_2+\omega_3X_3+\omega_4X_4)}\}}{\partial\omega_1\,\partial\omega_2\,\partial\omega_3\,\partial\omega_4}\right|_{\omega_1=\omega_2=\omega_3=\omega_4=0}$$

$$e^{i(\omega_1X_1+\omega_2X_2+\omega_3X_3+\omega_4X_4)} = 1 + i(\omega_1X_1+\omega_2X_2+\omega_3X_3+\omega_4X_4)$$
$$+ \frac{i^2}{2}(\omega_1X_1+\omega_2X_2+\omega_3X_3+\omega_4X_4)^2 + \cdots$$

$$\left.\frac{\partial e^{i(\omega_1X_1+\omega_2X_2+\omega_3X_3+\omega_4X_4)}}{\partial\omega_1\,\partial\omega_2\,\partial\omega_3\,\partial\omega_4}\right|_{\omega_1=\omega_2=\omega_3=\omega_4=0}$$

$$= \cdots$$

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2. To illustrate that the result of exercise 4 in chapter 4 is invalid if the processes $X(t)$ and $Z(t)$ are dependent, let $Z(t) = X(t)$ be a Gaussian process, and apply (5.59) to determine R_Y, for $Y(t) = X(t)Z(t) = X^2(t)$.

$$Y(t) = X^2(t)$$

$$R_Y(t_1, t_2) = E\{Y(t_1)Y(t_2)\} = E\{X^2(t_1)X^2(t_2)\}$$

$$= E\{X(t_1)X(t_1)X(t_2)X(t_2)\}$$

$$= R_X(t_1, t_1)R_X(t_2, t_2) + R_X(t_1, t_2)\cdot R_X(t_1, t_2) + R_X(t_1, t_2)\cdot R_X(t_1, t_2)$$

$$\neq R_X(t_1, t_2)R_X(t_1, t_2)$$

3. (a) If the processes in Section 4.2.4 and 4.2.6. are Gaussian, then their phase-randomized versions in Sections 4.2.5 and 4.2.7 cannot be Gaussian. Verfify this.

$$Y(t) = A\sin(\omega_0 t + \Theta)$$

$$= A\,(\sin\omega_0 t \cdot \cos\theta + \sin\theta\cos\omega_0 t)$$

$$= A\sin\omega_0 t \cdot \cos\theta + \vdash A\,\cancel{\cos}\sin(\omega_0 t - \tfrac{\pi}{2})\,\cancel{\cos}\sin\theta$$

$$= X(t)\cdot\cos\theta - X\!\left(t - \frac{\pi}{2\omega_0}\right)\sin\theta$$

(b)

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