EE 570 MID-TERM EXAM PART 1

Give short answers, either verbal or mathematical, to the following questions:

1. How does a random process differ from a random variable?

2. What is the correlation coefficient?

3. How is the power spectral density related to the autocorrelation function of a stationary random process?

4. What is the definition of a stationary random process?

5. What is the common perception of an ergodic random process?

6. Under what conditions does zero correlation imply statistical independence of two random variables?

7. Statistically independent random variables are always uncorrelated (true or false).

8. Statistically orthogonal random variables are always uncorrelated (true or false).

9. Which of the following functions are valid mathematical models for the autocorrelation of a random process:

   a) $e^{-\tau^2}$

   b) $|\tau| e^{-|\tau|}$

   c) $10 e^{-(\tau+2)}$

   d) $\left[\frac{\sin \pi\tau}{\pi\tau}\right]^2$

   e) $\frac{\tau^2+4}{\tau^2+8}$

$\omega = 2\pi f$

$T = \quad f = \frac{\omega}{}$

10. Consider a white noise process with a power spectral density of 10 volts-squared/Hz.

   a) Justify the name "power spectral density" for a quantity with units "volts-squared/Hz".

   b) Give the associated autocorrelation function with the correct units.

EE570 MID-TERM EXAM PART 2

425

1. The resistance values of resistors nominally rated at 1000 ohms are uniformly distributed between 900 and 1100 ohms. If two resistors are selected randomly and independently, what is the probability that the resistance value of the series connection of the two resistors is between 1900 and 2100 ohms.

2. The power spectral density of a stationary random process is given below. What is the associated autocorrelation function.

$$S_x(j\omega) = \frac{768}{\omega^4 + 20\omega^2 + 64}$$

3. Three samples of a stationary, Gaussian process taken at one unit intervals are to be considered. The autocorrelation function of the process is given below. What is the joint probability density function relating the three random variables.

$$R_x(\tau) = 9 + 10e^{-|\tau|}$$

4. Consider the bandpass bandlimited white noise process with power spectral density

$$S_x(j\omega) = .005 \qquad 200\pi \le |\omega| \le 250\pi$$
$$= 0 \qquad \text{Elsewhere}$$

a) Sketch the spectral density.

b) What is the bandwidth in Hz?

c) What is the mean-square value of the process?

Benmei Chen

-9

Part one

1. X is random variable $X(t)$ is random process $E[X]=m$ (number) ; $E[X(t)]=m_X(t)$

2. For two random variables , the correlation coef. $\triangleq \sigma_{XY}^2/\sigma_X\sigma_Y$

$$\sigma_{XY}^2 = E\{(X-m_X)(Y-m_Y)\} \quad \sigma_X=\sqrt{E[X^2]} \ , \ \sigma_Y=\sqrt{E[Y^2]}$$

For a random process : $\rho(\tau)=R_X(\tau)/\sigma_X^2$

3. For a random (stationary) process $X(t)$

$S_X(j\omega)=\mathfrak{F}_1\{R_X(\tau)\}$ WHERE $\mathfrak{F}_1$ is Fourier tranform

4. A random process $X(t)$ is said to be stationary if $E[X(t)]=m_X$ (number)

and $E[X(t_1)X(t_2)]=R_X(t_2-t_1)=R_X(\tau)$

5. For a ergodic random process , the time average = ensemble Average , Thus $E[X(t)]=\lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)\,dt$

and $R_X(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)x(t+\tau)\,dt$

6. $m_X=0$, $m_Y=0$

7. TRUE

8. FALSE

9. a) . d) .

10. a) $S(\omega)=\mathfrak{F}\{R(\tau)\}=\int_{-\infty}^{\infty}R(\tau)e^{-j\omega\tau}\,d\tau$

$$R(\tau)=E[X(t)X(t+\tau)] = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)x(t+\tau)\,dt = V^2$$

$$\therefore S(\omega) = \int_{-\infty}^{\infty}R(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty}\frac{R(\tau)}{e^{j\omega\tau}}d\tau = V^2/Hz$$

OR $p=V^2=\int_{-\infty}^{\infty}S(\omega)\,d\omega$ $S(\omega)=V^2/Hz$ QED.

b) $R(\tau) = 10\,\delta(\omega) = 10\,\delta(\omega)\ V^2$

I: -9

II: -10

85%

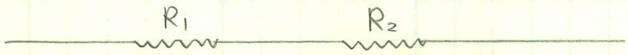
-19

~~78%~~

Part II:

10

1.



The PDF for the values of R_1, R_2

$$f_{R_1}(r_1)=f_{R_2}(r_2)=\frac{1}{1100-900}=0.005 \quad \text{FOR} \quad 900\le r_1, r_2 \le 1100$$

$$=0 \qquad \text{eleswhere.}$$

FOR series. $R=R_1+R_2$

$$f_R(r)=\int_{-\infty}^{\infty} f_{R_1R_2}(r-r_2, r_2)\,dr_2$$

$$=\int_{-\infty}^{\infty} f_{R_1}(r-r_2)\cdot f_{R_2}(r_2)\,dr_2 \quad , \quad R_1,R_2 \text{ independent.}$$

$$=\begin{cases} 2.5\times10^{-5}r-0.045 & \text{FOR } r\le 2000 \\ -2.5\times10^{-5}r+0.055 & \text{FOR } r\ge 2000 \end{cases}$$

$$P_R[1900\le r\le 2100]$$

$$=\int_{1900}^{2000}[2.5\times10^{-5}r-0.045]\,dr$$

$$+\int_{2000}^{2100}[-2.5\times10^{-5}r+0.055]\,dr$$

$$=0.75$$

0
0
10
0
-10

2. $R_X(\tau) = \mathcal{F}^{-1}\{S_X(j\omega)\}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega)e^{j\omega\tau}\,d\omega$$

$$S_X(j\omega) = \frac{768}{\omega^4+20\omega^2+64} = \frac{768}{(\omega^2+4)(\omega^2+16)}$$

$$= \frac{64}{\omega^2+4} - \frac{64}{\omega^2+16}$$

$$R_X(\tau) = \mathcal{F}^{-1}\left\{\frac{64}{\omega^2+4}\right\} - \mathcal{F}^{-1}\left\{\frac{64}{\omega^2+16}\right\}$$

$$= \mathcal{F}^{-1}\left\{\frac{16\times 4}{\omega^2+2^2}\right\} - \mathcal{F}^{-1}\left\{\frac{8\times 8}{\omega^2+4^2}\right\}$$

$$= \underline{16\cdot e^{-2|\tau|} - 8e^{-4|\tau|}}$$

-10 2

3. $X(t)$, three samples : $X(t_1)$, $X(t_2)$, $X(t_3)$

LET $\mathbf{X} = \begin{pmatrix} X(t_1) \\ X(t_2) \\ X(t_3) \end{pmatrix}$ and $t_2 - t_1 = t_3 - t_2 = 1$

-10 3

$$P = E[\mathbf{X}\mathbf{X}^T] = E\begin{bmatrix} X(t_1)X(t_1) & X(t_1)X(t_2) & X(t_1)X(t_3) \\ X(t_2)X(t_1) & X(t_2)X(t_2) & X(t_2)X(t_3) \\ X(t_3)X(t_1) & X(t_3)X(t_2) & X(t_3)X(t_3) \end{bmatrix}$$

$$= \begin{bmatrix} R_X(0) & R_X(1) & R_X(2) \\ R_X(-1) & R_X(0) & R_X(1) \\ R_X(-2) & R_X(-1) & R_X(0) \end{bmatrix} = \begin{bmatrix} 19 & 12.68 & 10.35 \\ 12.68 & 19 & 12.68 \\ 10.35 & 12.68 & 19 \end{bmatrix}$$

$C = P - m^2$

Joint PDF : $P_X(x) = \dfrac{1}{2\pi\sqrt{2\pi}\,|\cancel{P}\,C|^{1/2}} e^{-\frac{1}{2}(\mathbf{X}^*)^T \cancel{P}^{-1}\mathbf{X}^*}$

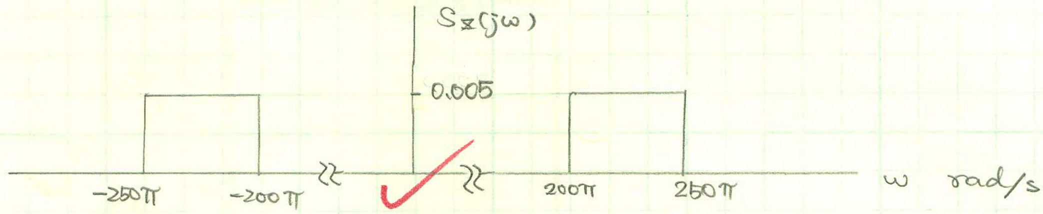
WHERE $\mathbf{X}^* = \begin{pmatrix} X(t_1) - \bar{X}(t_1) \\ X(t_2) - \bar{X}(t_2) \\ X(t_3) - \bar{X}(t_3) \end{pmatrix} = \begin{pmatrix} X(t_1) - 3 \\ X(t_2) - 3 \\ X(t_3) - 3 \end{pmatrix}$

Because $m_X = \sqrt{R(\tau)} = 3$ for stationary process.

$$\therefore P_X(x) = \frac{1}{2\pi\sqrt{2\pi}\times 45.19}\, e^{-\frac{1}{2}(\mathbf{X}^*)^T \begin{bmatrix} 19 & 12.68 & 10.35 \\ 12.68 & 19 & 12.68 \\ 10.35 & 12.68 & 19 \end{bmatrix}^{-1}\cdot \mathbf{X}^*}$$

remove $m^2 = 9$

4. a)



b) Bandwidth $= (250 - 200)\pi$ rad/s $= 50\pi$ rad/sec. $= 25$ Hz

c) $E[X^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega)\,d\omega$

$= \frac{1}{\pi}\int_{0}^{\infty} S_X(j\omega)\,d\omega$ IN THIS CASE

$= \frac{1}{\pi}\int_{200\pi}^{250\pi} 0.005\,d\omega$

$= 0.005/\pi \cdot 50\pi = 0.25$

-0

4

Benmei Chen

EE 507 (WSU) / EE 570 (UI)

INTRODUCTION TO RANDOM PROCESSES

FINAL EXAMINATION

SPRING 1987

FOUR PROBLEMS , ~~TWO~~ THREE HOURS , CLOSED BOOK*, OPEN NOTES

* BOOKS MAY BE USED FOR INTEGRATION FORMULAE , ETC.

1. Continuous observations z of an unknown constant x are corrupted by multiplicative noise v i.e.,

$$z = (1+v)x$$

where

$$E[x] = E[v] = E[vx] = 0$$

$$E[x^2] = s_x^2 \ , \ E[v^2] = s_v^2$$

(a) Find the optimal linear estimate of x based on a measurement z. That is, for $\hat{x} = k z$, find k and use it to estimate x.

(b) Find the rms error in this estimate.

(c) Find the bias in this estimate.

2. Consider the system

$$x(t)' = -2\, x(t) + w(t) \qquad x(0) = 1$$

with discrete measurements

$$z_k = x_k + v_k$$

made at one second intervals. w(t) is continuous white noise with mean 0 and power spectral density 2. v_k is zero mean and has covariance 2.

Design a filter to give minimum variance estimates of x at one second intervals where the filter is mistakenly, but unknowingly, is based on the system equation

$$x(t)' = -1\, x(t) + w(t)$$

and the initial value of x is known to be 1.

(a) Compare the actual mean value of x with the estimated mean value of x when t is two seconds.

(b) Compare the actual variance in x with the estimated variance in x at two seconds.

(c) Comment on the results of (a) and (b).

3. A system with transfer function

$$T(s) = 1 / (s + 3)$$

is driven by a white noise input with power spectral density 2.

(a) Find the power spectral density of the output.

(b) Find the autocorrelation of the output.

(c) Find the rms value of the output.

(d) Find the crosscorrelation between the input and the output.

4. Assume the system of problem 3 is drive by correlated noise with power spectral density

$$S(jw) = 18 / (w^2 + 9)$$

(a) Find the power spectral density of the output.

(b) Find the autocorrelation of the output.

(c) Find the rms value of the output.

(d) Find the crosscorrelation between the input and the output.

#1 5
2 7
3 1
4 3

-16

84%

$$\frac{A}{\omega^2+9} + \frac{B}{\omega^2+3}$$

$-3t_2 + 3t_1 - 6t_1$

$A\omega^2 + 3A + B\omega^2 + 9B$

$3A + 3B = 0$

$3A + 9B = 1$

$6B = 1$

$B = \frac{1}{6}$

$A = -\frac{1}{6}$

1.

$$\dot{x} = 0$$

$$z = x + vx$$

$$F=0 \qquad G=0 \qquad H=1$$

Assume x and v are indepdemt, then

$$E[vx] = 0$$

$$E[(vx)^2] = E[v^2 \cdot x^2] = E[v^2] \cdot E[x^2] = S_v^2 \cdot S_x^2$$

LET $v' = vx$, $v' \sim N(0, S_v^2 S_x^2)$

THE NEW SYSTEM becomes

$$\dot{x} = 0$$

$$z = x + v'$$

(a) Using the solution of the Ricati equation

$$\begin{bmatrix} \dot{y} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 0 & (S_v^2 S_x^2)^{-1} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ \lambda \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & (S_v^2 S_x^2)^{-1} \\ 0 & 0 \end{bmatrix} \cdot t.$$

$$= \begin{bmatrix} 1 & (S_v^2 S_x^2)^{-1} \cdot t \\ 0 & 1 \end{bmatrix}$$

-5

$$P(t) = [0 + S_x^2] \cdot \left[1 + \frac{t}{S_v^2 S_x^2} \cdot S_x^2\right]^{-1}, \qquad \cancel{P(0) = S_x^2}$$

$$S_x^2 = E[x(0)^2] \neq E[\hat{x}(0)^2] = P(0).$$

$$= \frac{S_v^2 S_x^2}{t + S_v^2}$$

Otherwise, very good.

$$K(t) = P(t) \cdot H^T(t) \cdot R^{-1}(t) = \frac{S_v^2 S_x^2}{t + S_v^2} \cdot \frac{1}{S_v^2 \cdot S_x^2} = \frac{1}{t + S_v^2}$$

(b) rms error $= \sqrt{P(t)} = S_v \cdot S_x \cdot \sqrt{\dfrac{1}{t + S_v^2}} \approx 0 \quad t \to \infty$

(c) $E[\hat{x}] = E[Kz] = E[K(x + Vx)]$

$= E[Kx + KVx]$

$= E[Kx] + E[KVx]$

$= KE[x] + KE[Vx] = 0$ ✓

-5

2. FOR THE 'MISTAKEN' SYSTEM. THE KALMAN FILTER BASE ON THIS SYSTEM IS

$F = -1$ ✓ $G = 1.$ ✓

$\Phi(t) = \mathcal{L}^{-1}\{sI - F\} = \mathcal{L}^{-1}\{\frac{1}{s+1}\} = e^{-t}$ ✓

$\Phi = \Phi(t)|_{t=1} = 0.3679$ ✓

$\Gamma Q_k \Gamma^T = \int_0^1 e^{-(1-\tau)} \cdot 2 \cdot e^{-(1-\tau)} d\tau$

-1

$e^{-2}(e^2 - 1) = 1 - e^{-2} = .8647$

$= 2e^{-2} \cdot \int_0^1 e^{2\tau} d\tau = e^{-2} \cdot e^{2\tau}|_0^1 = \cancel{2.3505}$

but $P_0^- = 0$ ~~is~~ as a given so this form of P_0^+ is indeterminant

-3

$P_0^{-1}(+) = P_0^{-1}(-) + H_0^T R_0^{-1} H_0 = 0.5$

$P_0^+ = \cancel{2}$

$P_1^- = 0.3679 \times 2 \times 0.3679 + 2.3505 = 2.6212$

$P_1^{-1}(+) = 0.3815 + 0.5 = 0.8815$ $P_1(+) = 1.1344$

$P_2(-) = 0.3679 \times 1.1344 \times 0.3679 + 2.3505 = 2.5040.$

$P_2'(+) = 1.1119$

ok

MISTAKEN KALMAN GAIN
$$\begin{cases} K_0 = P_0^+ \times 1 \times 0.5 = 1 \\ K_1 = 0.5672 \\ K_2 = 0.5560 \end{cases}$$

ok MISTAKEN FILTER $\hat{X}_k(+) = \hat{X}_k(-) + K_k[Z_k - H_k \hat{X}_k(-)]$

FOR THE SYSTEM WE CONSIDER.

$\Phi_1(t) = e^{-2t}$, $\Phi_1 = 0.1353.$ $\Gamma Q_k \Gamma = 7.2537$

$X_{k+1} = 0.1353\, X_k$

$X_0 = 1$

$X_1 = 0.1353$

✓ $X_2 = 0.0183$

(a) Actual mean value of $x_2 = 0.0183$ at $t = 2$ sec.

No estimates are made of the "actual" system's state. It only generates measurements $z_1, z_2, \ldots$

estimated value

$$E[\hat{X}_0^+] = 1 + 1\cdot[1-1] = 1$$

$$E[\hat{X}_1^-] = 0.3679 \times 1 = 0.3679$$

$$E[\hat{X}_1^+] = 0.3679 + 0.5672\cdot[0.1353 - 0.3679] = 0.2360$$

$$E[\hat{X}_2^-] = 0.3679 \times 0.2360 = 0.0868.$$

$$E[\hat{X}_2^+] = 0.0868 + 0.5560\,[0.0183 - 0.0868] = 0.0487$$

$$E[\hat{X}_2^{+2}] = 0.0024 \quad \text{at } t = 2 \text{ sec.}$$

-1

(b) Actual variance.

$$P_2 = 1.5691$$ where did this come from?

Estimated $P_2' = 1.1119$ (computed from mistaken filter)

P_2 is computed from the filter base on the actual system we considered

$P_0(+) = 0.5$; $P_1(-) = 0.1353 \times 2 \times 0.1353 + 7.2537 = 7.2903$

$P_1(+) = 1.5694$

$P_2(-) = 0.1353 \times 1.5694 \times 0.1353 + 7.2537 = 7.2824$

$P_2(+) = 1.5691$

-1

(c) If k is large the mean value may go closed but the variance can never be the same.

-1 Incomplete

-7

3.

$$T(S) = 1/(S+3)$$

(a) Assume the system is in steady-state. $S_w(j\omega) = 2$

$$|T(j\omega)|^2 = 1/(\omega^2+9)$$

$$S_y(j\omega) = S_w(j\omega)\cdot|T(j\omega)|^2 = \frac{2}{\omega^2+9}$$ ✓

(b) The autocorrelation,

$$R_y(\tau) = \mathcal{F}^{-1}\{S_y(j\omega)\} = \mathcal{F}^{-1}\left\{\frac{2}{\omega^2+9}\right\}$$

$$= \frac{1}{3}e^{-3|\tau|}$$ ✓

(c) $E[y^2] = R_y(0) = \frac{1}{3}$ ✓

(d) From the equation in notes, $h(t) = e^{-3t}$

$$R_{wy}(\tau) = \int_0^\infty Q\,\delta(\tau-\alpha)h(\alpha)\,d\alpha$$ -1

$$= Q\cdot h(\tau) = 2\cdot e^{-3\tau} \quad \tau \geq 0$$

$$= 0 \quad \tau < 0$$

$\boxed{-1}$

4.

$$S_w(j\omega) = 18/(\omega^2+9)$$ ✓

(a)

$$S_y(j\omega) = S_w(j\omega)\cdot|T(j\omega)|^2 = \frac{18}{\omega^2+9}\cdot\frac{1}{\omega^2+9} = \frac{18}{\omega^4+18\omega^2+81}$$ ✓

(b)

$$R_y(\tau) = \mathcal{F}^{-1}\{S_y(j\omega)\}$$

$$= \frac{1}{2}\mathcal{F}^{-1}\left\{\frac{2\cdot3}{\omega^2+9}\cdot\frac{2\cdot3}{\omega^2+9}\right\}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-3|\tau|}\cdot e^{-3|t-\tau|}\,d\tau$$

$$= \begin{cases} \frac{1}{2}e^{-3\tau}\cdot\tau + \frac{1}{12}e^{-3\tau} - \frac{1}{12}e^{3\tau} & \tau > 0 \\ \frac{1}{2}e^{3\tau}(-\tau) + \frac{1}{12}e^{3\tau} - \frac{1}{12}e^{-3\tau} & \tau < 0 \end{cases}$$ -2

$$= \frac{1}{2}e^{-3|\tau|}\cdot|\tau| + \frac{1}{12}e^{-3|\tau|} \cancel{-\frac{1}{12}e^{3|\tau|}} \quad \text{FOR ALL } \tau$$

(c) $E[y^2] = R_y(0) = \frac{1}{2}e^{-3}\cdot 0 + \frac{1}{12}e^{-0} - \frac{1}{12}e^{0} \cancel{\neq 0} = \frac{1}{6}$ ✓

(d) $R_{uy}(\tau) = \int_0^\infty R_u(\tau - \alpha)\, h(\alpha)\, d\alpha$ ✓

$$= \int_0^\infty 3e^{-3|\tau-\alpha|} \cdot e^{-3\alpha}\, d\alpha$$

$$= \int_0^\tau 3e^{-3\tau+3\alpha} \cdot e^{-3\alpha}\, d\alpha + \int_\tau^\infty 3\cdot e^{-3\alpha+3\tau} \cdot e^{-3\alpha}\, d\alpha$$

$$= 3e^{-3\tau}\cdot\tau + 3e^{+3\tau} \cdot \int_\tau^\infty e^{-6\alpha}\, d\alpha$$

$$= 3e^{-3\tau}\cdot\tau + \frac{1}{2}e^{-3\tau} \cancel{- \frac{1}{2}e^{3\tau}} \qquad \tau \geq 0$$

$$= \frac{1}{2}e^{3\tau}, \quad \tau \leq 0$$

-1

$\boxed{-3}$