RECEIVED
SEP 23 1987

HOMEWORK SET NO: 1

FOR EE 501

LINEAR SYSTEM THEORY

SEP. 11, 1987

Benmei Chen

P.O.Box 241, Gonzaga University

Spokane, WA 99258

Very good

10/10

HS

#1. (TEXT 1, KAILATH)

(a) P651, A.13, PART 1: SHOW THAT IF A IS $n\times n$, b is $n\times 1$, c is $1\times n$ AND d is a SCALAR, THEN

$$\det(sI-A)[c(sI-A)^{-1}b+d]=\det\left[\begin{array}{c|c} sI-A & b \\ \hline -c & d\end{array}\right]$$

SHOW: Because of $A: n\times n$, $b: n\times 1$, $c: 1\times n$ and d is a scalar, it is easy to see that

$$c(sI-A)^{-1}b+d : 1\times n \times n\times n \times n\times 1 + 1\times 1 = 1\times 1 \quad \text{is a scalar.}$$

So,

$$\det[c(sI-A)^{-1}b+d]=c(sI-A)^{-1}b+d \quad \cdots\cdots (1)$$

Now, observe that

$$\left[\begin{array}{c|c} sI-A & b \\ \hline -c & d\end{array}\right]=\left[\begin{array}{c|c} sI-A & 0 \\ \hline -c & d+c(sI-A)^{-1}b\end{array}\right]\cdot\left[\begin{array}{c|c} I & (sI-A)^{-1}b \\ \hline 0 & I\end{array}\right]$$

$$=\left[\begin{array}{c|c} sI-A & 0 \\ \hline 0 & 1\end{array}\right]\cdot\left[\begin{array}{c|c} I & 0 \\ \hline 0 & d+c(sI-A)^{-1}b\end{array}\right]\cdot\left[\begin{array}{c|c} I & 0 \\ \hline -[d+c(sI-A)^{-1}b]^{-1}c & I\end{array}\right]\cdot\left[\begin{array}{c|c} I & (sI-A)^{-1}b \\ \hline 0 & I\end{array}\right]$$

Recall that the determinant of the product of square matrices is equal to the product of the determinants of each matrix. And the determinant of a triangular matrix is equal to the product of the diagonal elements. So, that

$$\det\left[\begin{array}{c|c} sI-A & b \\ \hline -c & d\end{array}\right]=\det(sI-A)\cdot\det[d+c(sI-A)^{-1}b]$$

$$=\det(sI-A)\cdot[c(sI-A)^{-1}b+d] \qquad \text{[Eq.(1) above]}$$

END OF SHOW.

PART 2: IF $G(s) = c(sI - A)^{-1} b$, SHOW THAT WE CAN WRITE

$$G(s) = \frac{\det(sI - A + bc) - \det(sI - A)}{\det(sI - A)}$$

— — — — — — — —

SHOW:

$$\det(sI - A) \cdot [G(s) + 1]$$

$$= \det(sI - A) \cdot [c(sI - A)^{-1} b + 1]$$

$$= \det \left[\begin{array}{c|c} sI - A & b \\ \hline -c & 1 \end{array} \right] \quad \text{(Applied the result from Part 1)}$$

$$= \det \left(\left[\begin{array}{c|c} (sI - A) + bc & b \\ \hline 0 & 1 \end{array} \right] \cdot \left[\begin{array}{c|c} I & 0 \\ \hline -c & 1 \end{array} \right] \right)$$

$$= \det(sI - A + bc)$$

THAT IS,

$$\det(sI - A) \cdot [G(s) + 1] = \det(sI - A + bc)$$

Therefore,

$$G(s) = \frac{\det(sI - A + bc) - \det(sI - A)}{\det(sI - A)}$$

END OF SHOW!

(b) P.655, A.20: Let A be a nonsingular matrix. Let u and v be column matrices, and assume that $A + uv'$ is nonsingular, Verify that

$$(A + uv')^{-1} = A^{-1} - \frac{(A^{-1}u)(v'A^{-1})}{1 + v'A^{-1}u}$$

— — — — — — — —

SHOW ON NEXT PAGE.

SHOW: For this particular problem, we can assume

$$A^{-1}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix} \quad (\because A \text{ is nonsingular})$$

$$u=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n\end{bmatrix}, \quad v=\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n\end{bmatrix}$$

Then

$$A^{-1}uv'=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}\cdot\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n\end{bmatrix}\cdot(v_1 \; v_2 \; \cdots v_n)$$

$$=\begin{bmatrix} a_{11}u_1+a_{12}u_2+\cdots+a_{1n}u_n \\ a_{21}u_1+a_{22}u_2+\cdots+a_{2n}u_n \\ \vdots \\ a_{n1}u_1+a_{n2}u_2+\cdots+a_{nn}u_n\end{bmatrix}(v_1, v_2 \cdots v_n)$$

$$=\begin{bmatrix} v_1\cdot\sum_{i=1}^n a_{1i}u_i & v_2\cdot\sum_{i=1}^n a_{1i}u_i & \cdots & v_n\cdot\sum_{i=1}^n a_{1i}u_i \\ v_1\cdot\sum_{i=1}^n a_{2i}u_i & v_2\cdot\sum_{i=1}^n a_{2i}u_i & \cdots & v_n\cdot\sum_{i=1}^n a_{2i}u_i \\ \vdots & & & \vdots \\ v_1\cdot\sum_{i=1}^n a_{3i}u_i & v_2\cdot\sum_{i=1}^n a_{ni}u_i & \cdots & v_n\cdot\sum_{i=1}^n a_{ni}u_i\end{bmatrix}$$

LET $\sum_{i=1}^n a_{ki}u_i=\alpha_k$, $k=1,2,\cdots,n$, so that

$$A^{-1}uv'A^{-1}uv'=\begin{bmatrix} v_1\alpha_1 & v_2\alpha_1 & \cdots & v_n\alpha_r \\ v_1\alpha_2 & v_2\alpha_2 & \cdots & v_n\alpha_2 \\ \vdots & \vdots & & \vdots \\ v_1\alpha_n & v_2\alpha_n & \cdots & v_n\alpha_n\end{bmatrix}\cdot\begin{bmatrix} v_1\alpha_1 & v_2\alpha_1 & \cdots & v_n\alpha_1 \\ v_1\alpha_2 & v_2\alpha_2 & \cdots & v_n\alpha_2 \\ \vdots & \vdots & & \vdots \\ v_1\alpha_n & v_2\alpha_n & \cdots & v_n\alpha_n\end{bmatrix}$$

$$=\begin{bmatrix} v_1\alpha_1\sum_{r=1}^n v_r\alpha_r & v_2\alpha_1\sum_{r=1}^n v_r\alpha_r & \cdots & v_n\alpha_1\cdot\sum_{r=1}^n v_r\cdot\alpha_r \\ v_1\alpha_2\sum_{r=1}^n v_r\alpha_r & v_2\alpha_2\sum_{r=1}^n v_r\alpha_r & \cdots & v_n\alpha_2\cdot\sum_{r=1}^n v_r\cdot\alpha_r \\ \vdots & \vdots & & \vdots \\ v_1\alpha_n\sum_{r=1}^n v_r\alpha_r & v_2\alpha_n\sum_{r=1}^n v_r\alpha_r & \cdots & v_n\alpha_n\sum_{r=1}^n v_r\cdot\alpha_r\end{bmatrix}$$

$$=\begin{bmatrix} v_1\alpha_1 & v_2\alpha_1 & \cdots & v_n\alpha_1 \\ v_1\alpha_2 & v_2\alpha_2 & \cdots & v_n\alpha_2 \\ \vdots & \vdots & \cdots & \vdots \\ v_1\alpha_n & v_2\alpha_n & \cdots & v_n\alpha_n\end{bmatrix}\cdot\sum_{r=1}^n v_r\alpha_r$$

$$=(A^{-1}uv')\cdot(v_1 \; v_2 \; \cdots \; v_n)\begin{pmatrix}\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n\end{pmatrix}$$

Not needed

CONT. NEXT PAGE

#1 (b) (CONT.)

$$= (A^{-1}uv')\,v' \cdot \begin{pmatrix} a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n \\ a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n \\ \vdots \\ a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n \end{pmatrix}$$

$$= (A^{-1}uv')\,v' \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$= (A^{-1}uv')\,v'A^{-1}u$$

not needed

That is,

$$A^{-1}uv'A^{-1}uv' = A^{-1}uv'(v'A^{-1}u) \quad \cdots\cdots \quad (1)$$

Now, look at (since $v'A^{-1}u$: $1 \times n \times n \times n \times n \times 1 = 1 \times 1$ is a scalar),

$v'A^{-1}u$ is a number!!

$$\left(A^{-1} - \frac{(A^{-1}u)(v'A^{-1})}{1 + v'A^{-1}u}\right)(A + uv')$$

a number

$$= I + A^{-1}uv' - \frac{A^{-1}uv'}{1 + v'A^{-1}u} - \frac{A^{-1}u\underbrace{v'A^{-1}u}v'}{1 + v'A^{-1}u}$$

$$= I + A^{-1}uv' - A^{-1}uv'\,\frac{1 + v'A^{-1}u}{1 + v'A^{-1}u}$$

$$= I.$$

$$\cancel{= I + \frac{1}{1 + v'A^{-1}u} \cdot [A^{-1}uv' + A^{-1}uv'(v'A^{-1}u) - A^{-1}uv' - A^{-1}uv'A^{-1}uv']}$$

$$\cancel{= I + \frac{1}{1 + v'A^{-1}u}[A^{-1}uv'(v'A^{-1}u) - A^{-1}uv'A^{-1}uv']}$$

$$\cancel{= I + \frac{1}{1 + v'A^{-1}u}[0] = I} \quad \text{~~(Applied eq. (1))~~}$$

Since $A + uv'$ is nonsingular, it's ease to see,

$$(A + uv')^{-1} = A^{-1} - \frac{(A^{-1}u)(v'A^{-1})}{1 + v'A^{-1}u} \qquad \underline{\text{QED!}}$$

Dr. Hsu, please give me a simple method to show this problem.

I really don't know how to do it in a easy way. ✓

Yes

(c) P656, A.22, PART 2: IF A^{-1} exists, show that

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}+E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix} = M$$

Where $\Delta = B - CA^{-1}D$, $E = A^{-1}D$, and $F = CA^{-1}$. Show that if B^{-1} exists, the (1,1) block element of the inverse can also be written as $[A - DB^{-1}C]^{-1}$. Δ is known as the schur complement of A.

SHOW: From A.22, Part 1, if A^{-1}, B^{-1} exist, we have

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1} \end{bmatrix} \text{ and } \begin{bmatrix} A & D \\ 0 & B \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}DB^{-1} \\ 0 & B^{-1} \end{bmatrix} \quad \cdots (1)$$

Now we rewrite, (Because A^{-1} exists)

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & \underbrace{B - CA^{-1}D}_{\Delta} \end{bmatrix} \begin{bmatrix} I & \overbrace{A^{-1}D}^{E} \\ 0 & I \end{bmatrix}$$

alternatively, multiple $M \cdot \begin{bmatrix} A & D \\ C & B \end{bmatrix}$ show the product equals to I.

In this problem, we have to assum that $\begin{bmatrix} A & D \\ C & B \end{bmatrix}$ is nonsingular. yes,

So as $\Delta = B - CA^{-1}D$ must be nonsingular. (Otherwise the matrix $\begin{bmatrix} A & 0 \\ C & B - CA^{-1}D \end{bmatrix}$ must be singular and $\begin{bmatrix} I & A^{-1}D \\ 0 & I \end{bmatrix}$ is a triangular matrix with the diagonal elements of 1's, $\begin{bmatrix} A & D \\ C & B \end{bmatrix}$ have to be singular.)

So,

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} I & E \\ 0 & I \end{bmatrix}^{-1} \cdot \begin{bmatrix} A & 0 \\ C & \Delta \end{bmatrix}^{-1}$$

✓ (applied eq.(1))

$$= \begin{bmatrix} I & -E \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} A^{-1} & 0 \\ -\Delta^{-1}\underbrace{CA^{-1}}_{F} & \Delta^{-1} \end{bmatrix} = \begin{bmatrix} A^{-1}+E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix}$$ QED!

IF B^{-1} exists, we can $\Delta = A - DB^{-1}C$

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ B^{-1}C & I \end{bmatrix}^{-1} \cdot \begin{bmatrix} \Delta & D \\ 0 & B \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ -B^{-1}C & I \end{bmatrix} \cdot \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}DB^{-1} \\ 0 & B^{-1} \end{bmatrix} = \left[\begin{array}{c|c} \Delta^{-1} & -\Delta^{-1}DB^{-1} \\ \hline -B^{-1}C\Delta^{-1} & B^{-1}C\Delta^{-1}DB^{-1} + B^{-1} \end{array}\right]$$ QED

#2. (TEXT 1, P.174, 2.5~12)

SHOW that an underlying continus-time realization with a piecewise constant input,

$$\dot{x}(t) = A\,x(t) + b\,u(t), \quad y = c\,x(t), \quad u(t) = u_k, \quad k\Delta < t \le (k+1)\Delta$$

may be replaced by the discrete-time system

$$x_{k+1} = \Phi x_k + \Gamma u_k, \quad y_k = c\,x_k$$

where

$$y_k = y(k\Delta), \quad x_k = x(k\Delta)$$

$$\Phi = \exp A\Delta, \quad \Gamma = \int_0^\Delta (\exp A\tau)\,d\tau \cdot b$$

Δ is a fixed sampling interval.

SHOW: From the equation $\dot{x}(t) = Ax(t) + bu(t)$, we have the solution for $x(t)$,

$$x(t) = e^{A(t-s)}x(s) + \int_s^t e^{A(t-\tau)}\,b\,u(\tau)\,d\tau$$

Let $s = k\Delta$, and $t = (k+1)\Delta$, where Δ is a fixed sampling interval and must satisfy the sampling theorum. So,

$$x(k\Delta+\Delta) = e^{A\Delta}x(k\Delta) + \int_{k\Delta}^{k\Delta+\Delta} e^{A(k\Delta+\Delta-\tau)} \cdot b\,u(\tau)\,d\tau \quad \cdots (1)$$

Since the input $u(t)$ is piecewise constant

$$u(t) = u_k \qquad k\Delta \le t < (k+1)\Delta$$

The equation (1) becomes,

$$x(k\Delta+\Delta) = e^{A\Delta}x(k\Delta) + \int_{k\Delta}^{k\Delta+\Delta} e^{A(k\Delta+\Delta-\tau)}\,d\tau \cdot b \cdot u_k$$

$$= e^{A\Delta}x(k\Delta) + \int_0^\Delta e^{A\tau}\,d\tau \cdot b \cdot u_k$$

Thus,

$$x_{k+1} = \Phi x_k + \Gamma \cdot u_k$$

$$y_k = y(k\Delta+\Delta) = C\,x(k\Delta+\Delta) = C\,x_k$$

So, we have new discrete-time system,

$$x_{k+1} = \Phi x_k + \Gamma \cdot u_k, \quad y_k = C\,x_k$$

QED!

#3. (TEXT 2, P68, 2-46)

Consider the matrix equation

$$PEP + DP + PF + G = 0 \quad \text{------} \quad (1)$$

where all matrices are $n \times n$ constant matrices. It is called an algebraic Riccati equation. Define

$$M = \begin{bmatrix} -F & -E \\ G & D \end{bmatrix}$$

Let

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$$

consist of all generalized eigenvectors of M so that $Q^{-1}MQ = J$ is in a Jordan canonical form. We write

$$\begin{bmatrix} -F & -E \\ G & D \end{bmatrix} \cdot \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \cdot \begin{bmatrix} J_1 & O \\ O & J_2 \end{bmatrix}$$

Show that if Q_1 is nonsingular, then $P = Q_3 Q_1^{-1}$ is a solution of the Riccati equation.

SHOW: Because Q_1^{-1} is nonsingular, and from the results that we showed in #1 (c),

$$\begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}^{-1} = \begin{bmatrix} Q_1^{-1} + Q_1^{-1} Q_2 \Delta^{-1} \cdot Q_3 Q_1^{-1} & -Q_1^{-1} Q_2 \Delta^{-1} \\ -\Delta^{-1} Q_3 Q_1^{-1} & \Delta^{-1} \end{bmatrix}$$

where $\Delta = Q_4 - Q_3 Q_1^{-1} Q_2 \triangleq Q_4 - \tilde{P} Q_2$, $\tilde{P} \triangleq Q_3 Q_1^{-1}$

$$Q^{-1}MQ = \begin{bmatrix} Q_1^{-1} + Q_1^{-1} Q_2 \Delta^{-1} \tilde{P} & -Q_1^{-1} Q_2 \Delta^{-1} \\ -\Delta^{-1} \tilde{P} & \Delta^{-1} \end{bmatrix} \begin{bmatrix} -F & -E \\ G & D \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

$$= \begin{bmatrix} Q_1^{-1} + Q_1^{-1} Q_2 \Delta^{-1} \tilde{P} & -Q_1^{-1} Q_2 \Delta^{-1} \\ -\Delta^{-1} \tilde{P} & \Delta^{-1} \end{bmatrix} \cdot \begin{bmatrix} -FQ_1 - EQ_3 & -FQ_2 - EQ_4 \\ GQ_1 + DQ_3 & GQ_2 + DQ_4 \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

(CONT. NEXT PAGE.)

(CONT.)

$$= \left[\begin{array}{c|c} X_1 & X_2 \\ \hline \Delta^{-1}\tilde{P}(FQ_1+EQ_3)+\Delta^{-1}(GQ_1+DQ_3) & X_3 \end{array}\right] = \left[\begin{array}{c|c} J_1 & 0 \\ \hline 0 & J_2 \end{array}\right]$$

$$\therefore \quad \Delta^{-1}\tilde{P}(FQ_1+EQ_3)+\Delta^{-1}(GQ_1+DQ_3)=0$$

$$\tilde{P}FQ_1+\tilde{P}EQ_3+GQ_1+DQ_3=0$$

Again, since Q_1 is nonsingular,

$$0=(\tilde{P}FQ_1+\tilde{P}EQ_3+GQ_1+DQ_3)Q_1^{-1}$$

$$=\tilde{P}F+\tilde{P}EQ_3Q_1^{-1}+G+DQ_3Q_1^{-1}$$

$$(\tilde{P}\triangleq Q_3Q_1^{-1}) \quad 0 = (Q_3Q_1^{-1})E(Q_3Q_1^{-1})+D(Q_3Q_1^{-1})+(Q_3Q_1^{-1})F+G \quad \cdots \quad (2)$$

Compare the equation (2) to (1) on the previous page, we have now.

$P=Q_3Q_1^{-1}$ is a solution of the equation (1). QED

#4. GIVEN system

$$\dot{X}(t)=AX(t)+Bu(t) \qquad X(0)=X_0$$

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_1 & -a_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_3 & a_4 & 0 \end{bmatrix}, \qquad B=\begin{bmatrix} 0 \\ b_1 \\ 0 \\ -b_2 \end{bmatrix}$$

Determine the following

(a) $|A|$, A^{-1}, $\lambda_i(A)$, $i=1,2,3,4$

$$|A| = 1\cdot(-1)^{1+2}\cdot\begin{vmatrix} 0 & -a_2 & 0 \\ 0 & 0 & 1 \\ 0 & a_4 & 0 \end{vmatrix} = 0$$

$$A^{-1}=\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & -a_1 & -a_2 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & a_3 & a_4 & 0 \end{array}\right]^{-1}$$ does not exist, because $|A|=0$

$$|\lambda I-A|=\begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda+a_1 & a_2 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & -a_3 & -a_4 & \lambda \end{vmatrix}=\lambda\begin{vmatrix} \lambda+a_1 & a_2 & 0 \\ 0 & \lambda & -1 \\ -a_3 & -a_4 & \lambda \end{vmatrix}+\begin{vmatrix} 0 & a_2 & 0 \\ 0 & \lambda & -1 \\ 0 & -a_4 & \lambda \end{vmatrix}$$

TO BE CONT.

CONT. $\therefore\ |\lambda I - A| = \lambda[\lambda^2(\lambda+a_1)+a_2a_3-a_4(\lambda+a_1)]$

$$= \lambda(\lambda^3+a_1\lambda^2-a_4\lambda+a_2a_3-a_1a_4)$$

Let $a=\frac{1}{3}(-3a_4-a_1^2)$, $b=\frac{1}{27}(2a_1^3+9a_1a_4+27a_2a_3-27a_1a_4)$

AND

$$A=\sqrt[3]{-\frac{b}{2}+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}}\ , \quad B=\sqrt[3]{\frac{b}{2}+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}}$$

From formulars in CRC Handbook of Mathematical Sceience , we have

$$\lambda_1=0,\quad \lambda_2=A+B+\frac{a_1}{3},\quad \lambda_3=-\frac{A+B}{2}+\frac{A-B}{2}\sqrt{-3}+\frac{a_1}{3},\quad \lambda_4=-\frac{A+B}{2}-\frac{A-B}{2}\sqrt{-3}+\frac{a_1}{3}$$

(b)

$$(SI-A)^{-1}=\left[\begin{array}{cc|cc} S & -1 & 0 & 0\\ 0 & S+a_1 & a_2 & 0\\ \hline 0 & 0 & S & -1\\ 0 & -a_3 & -a_4 & S\end{array}\right]^{-1}$$

LEA $\tilde{A}\triangleq\begin{bmatrix} S & -1\\ 0 & S+a_1\end{bmatrix}$, $\tilde{B}\triangleq\begin{bmatrix} S & -1\\ -a_4 & S\end{bmatrix}$, $\tilde{C}\triangleq\begin{bmatrix} 0 & 0\\ 0 & -a_3\end{bmatrix}$, $\tilde{D}\triangleq\begin{bmatrix} 0 & 0\\ a_2 & 0\end{bmatrix}$

$$\tilde{A}^{-1}=\frac{1}{S(S+a_1)}\begin{bmatrix} S+a_1 & 1\\ 0 & S\end{bmatrix}$$

$$\Delta\triangleq\tilde{B}-\tilde{C}\tilde{A}^{-1}\tilde{D}=\begin{bmatrix} S & -1\\ -a_4 & S\end{bmatrix}-\begin{bmatrix} 0 & 0\\ 0 & -a_3\end{bmatrix}\begin{bmatrix} S+a_1 & 1\\ 0 & S\end{bmatrix}\begin{bmatrix} 0 & 0\\ a_2 & 0\end{bmatrix}/S(S+a_1)$$

$$=\begin{bmatrix} S & -1\\ \frac{-a_4S+a_2a_3-a_1a_4}{S+a_1} & S\end{bmatrix}$$

$$\Delta^{-1}=\frac{S+a_1}{S^3+a_1S^2-a_4S+a_2a_3-a_1a_4}\cdot\begin{bmatrix} S & 1\\ \frac{a_4S+a_1a_4-a_2a_3}{S+a_1} & S\end{bmatrix}$$

$$E\triangleq\tilde{A}^{-1}\tilde{D}=\begin{bmatrix} S+a_1 & 1\\ 0 & S\end{bmatrix}\cdot\begin{bmatrix} 0 & 0\\ a_2 & 0\end{bmatrix}/S(S+a_1)=\begin{bmatrix} 1 & 0\\ S & 0\end{bmatrix}\cdot\frac{a_2}{S(S+a_1)}$$

$$F\triangleq\tilde{C}\tilde{A}^{-1}=\begin{bmatrix} 0 & 0\\ 0 & -a_3\end{bmatrix}\begin{bmatrix} S+a_1 & 1\\ 0 & S\end{bmatrix}/S(S+a_1)=\begin{bmatrix} 0 & 0\\ 0 & -1\end{bmatrix}\cdot\frac{a_3}{S+a_1}$$

$$\therefore\quad \tilde{A}^{-1}+E\Delta^{-1}F=\begin{bmatrix} \frac{1}{S} & \frac{1}{S(S+a_1)}\\ 0 & \frac{1}{S+a_1}\end{bmatrix}+\frac{a_2a_3}{S(S+a_1)(S^3+a_1S^2-a_4S+a_2a_3-a_1a_4)}\begin{bmatrix} 1 & 0\\ S & 0\end{bmatrix}\cdot\begin{bmatrix} S & 1\\ \frac{a_4S+a_1a_4-a_2a_3}{S+a_1} & S\end{bmatrix}\cdot\begin{bmatrix} 0 & 0\\ 0 & -1\end{bmatrix}$$

$$=\left[\begin{array}{c|c} \frac{1}{S} & \frac{S^3+a_1S^2-a_4S-a_1a_4}{S(S+a_1)(S^3+a_1S^2-a_4S+a_2a_3-a_1a_4)}\\ \hline 0 & \frac{S^3+a_1S^2-a_4S-a_1a_4}{(S+a_1)(S^3+a_1S^2-a_4S+a_2a_3-a_1a_4)}\end{array}\right]$$

$$-E\Delta^{-1} = -\frac{a_2}{s(s+a_1)}\begin{bmatrix} 1 & 0 \\ s & 0 \end{bmatrix}\cdot\frac{s+a_1}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4}\begin{bmatrix} s & 1 \\ \frac{a_4s+a_1a_4-a_2a_3}{s+a_1} & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-a_2}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} & \frac{-a_2}{s(s^3+a_1s^2-a_4s+a_2a_3-a_1a_4)} \\ \frac{-a_2s}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} & \frac{-a_2}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} \end{bmatrix}$$

$$-\Delta^{-1}F = -\frac{s+a_1}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4}\begin{bmatrix} s & 1 \\ \frac{a_4s+a_1a_4-a_2a_3}{s+a_1} & s \end{bmatrix}\cdot\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\cdot\frac{a_3}{s+a_1}$$

$$= \begin{bmatrix} 0 & \frac{a_3}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} \\ 0 & \frac{a_3s}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} \end{bmatrix}$$

Now, apply the results we showed in #1, (c), we have,

$$(sI-A)^{-1} = \begin{bmatrix} \tilde{A} & \tilde{D} \\ \tilde{C} & \tilde{B} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{A}^{-1}+E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c|c} \frac{1}{s} & \frac{s^3+a_1s^2-a_4s-a_1a_4}{s(s+a_1)(s^3+a_1s^2-a_4s+a_2a_3-a_1a_4)} & \frac{-a_2}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} & \frac{-a_2}{s(s^3+a_1s^2-a_4s+a_2a_3-a_1a_4)} \\ \hline 0 & \frac{s^3+a_1s^2-a_4s-a_1a_4}{(s+a_1)(s^3+a_1s^2-a_4s+a_2a_3-a_1a_4)} & \frac{-a_2s}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} & \frac{-a_2}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} \\ \hline 0 & \frac{a_3}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} & \frac{s(s+a_1)}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} & \frac{s+a_1}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} \\ \hline 0 & \frac{a_3s}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} & \frac{a_4s+a_1a_4-a_2a_3}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} & \frac{s(s+a_1)}{s^3+a_1s^2-a_4s+a_2a_3-a_1a_4} \end{array}\right]$$

✓

(c)

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_1 & -a_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_3 & a_4 & 0 \end{bmatrix}\cdot\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_1 & -a_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_3 & a_4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a_1 & -a_2 & 0 \\ 0 & a_1^2 & a_1a_2 & -a_2 \\ 0 & a_3 & a_4 & 0 \\ 0 & -a_1a_3 & -a_2a_3 & a_4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & -a_1 & -a_2 & 0 \\ 0 & a_1^2 & a_1a_2 & -a_2 \\ 0 & a_3 & a_4 & 0 \\ 0 & -a_1a_3 & -a_2a_3 & a_4 \end{bmatrix}\cdot\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_1 & -a_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_3 & a_4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_1^2 & a_1a_2 & -a_2 \\ 0 & -a_1^3-a_2a_3 & -a_1^2a_2-a_2a_4 & a_1a_2 \\ 0 & -a_1a_3 & -a_2a_4 & a_4 \\ 0 & a_1^2a_3+a_3a_4 & a_1a_2a_3+a_4^2 & -a_2a_3 \end{bmatrix}$$

$$\therefore \quad e^{At} \cong I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6}$$

$$= \begin{bmatrix} 1 & t - \frac{1}{2}a_1t^2 + \frac{1}{6}a_1^2t^3 & \frac{1}{2}a_2t^2 + \frac{1}{6}a_1a_2t^3 & -\frac{1}{6}a_2t^3 \\ 0 & 1 - a_1t + \frac{1}{2}a_1^2t^2 - \frac{1}{6}(a_1^3 + a_2a_3)t^3 & -a_2t + \frac{1}{2}a_1a_2t^2 - \frac{a_2}{6}(a_1^2 + a_4)t^3 & -\frac{1}{2}a_2t^2 + \frac{1}{6}a_1a_2t^3 \\ 0 & \frac{1}{2}a_3t^2 - \frac{1}{6}a_1a_3t^3 & 1 + \frac{1}{2}a_4t^2 - \frac{1}{6}a_2a_4t^3 & t + \frac{1}{6}a_4t^3 \\ 0 & a_3t - \frac{1}{2}a_1a_3t^2 + \frac{a_3}{6}(a_1^2 + a_4)t^3 & a_4t - \frac{1}{2}a_2a_3t^2 + \frac{1}{6}(a_1a_2a_3 + a_4^2)t^3 & 1 + \frac{1}{2}a_4t^2 - \frac{1}{6}a_2a_3t^3 \end{bmatrix}$$

$$\int_0^T e^{At}dt = \begin{bmatrix} T & \frac{T^2}{2} - \frac{a_1T^3}{6} + \frac{a_1^2T^4}{24} & \frac{a_2T^3}{6} + \frac{a_1a_2T^4}{24} & -\frac{a_2T^4}{24} \\ 0 & T - \frac{a_1T^2}{2} + \frac{a_1^2T^3}{6} - \frac{(a_1^3 + a_2a_3)T^4}{24} & -\frac{a_2T^2}{2} + \frac{a_1a_2T^3}{6} - \frac{a_2(a_1^2 + a_4)T^4}{24} & -\frac{a_2T^3}{6} + \frac{a_1a_2T^4}{24} \\ 0 & \frac{a_3T^3}{6} - \frac{a_1a_3T^4}{24} & T + \frac{a_4T^3}{6} - \frac{a_2a_4T^4}{24} & \frac{T^2}{2} + \frac{a_4T^4}{24} \\ 0 & \frac{a_3T^2}{2} - \frac{a_1a_3T^3}{6} + \frac{a_3(a_1^2 + a_4)T^4}{24} & \frac{a_4T^2}{2} - \frac{a_2a_3T^3}{6} + \frac{(a_1a_2a_3 + a_4^2)T^4}{24} & T + \frac{a_4T^3}{6} - \frac{a_2a_3T^4}{24} \end{bmatrix}$$

✓

(d)

$$\mathscr{C} = [B, AB, A^2B, A^3B]$$

$$AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_1 & -a_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_3 & a_4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ b_1 \\ 0 \\ -b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ -a_1b_1 \\ -b_2 \\ a_3b_1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & -a_1 & -a_2 & 0 \\ 0 & a_1^2 & a_1a_2 & -a_2 \\ 0 & a_3 & a_4 & 0 \\ 0 & -a_1a_3 & -a_2a_3 & a_4 \end{bmatrix} \begin{bmatrix} 0 \\ b_1 \\ 0 \\ -b_2 \end{bmatrix} = \begin{bmatrix} -a_1b_1 \\ a_1^2b_1 + a_2b_2 \\ a_3b_1 \\ -a_1a_3b_1 - a_4b_2 \end{bmatrix}$$

$$A^3B = \begin{bmatrix} 0 & a_1^2 & a_1a_2 & -a_2 \\ 0 & -a_1^3 - a_2a_3 & -a_1^2a_2 - a_2a_4 & a_1a_2 \\ 0 & -a_1a_3 & -a_2a_4 & a_4 \\ 0 & a_1^2a_3 + a_3a_4 & a_1a_2a_3 + a_4^2 & -a_2a_3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ b_1 \\ 0 \\ -b_2 \end{bmatrix} = \begin{bmatrix} a_1^2b_1 + a_2b_2 \\ -a_1^3b_1 - a_2a_3b_1 - a_1a_2b_2 \\ -a_1a_3b_1 - a_4b_2 \\ a_1^2a_3b_1 + a_3a_4b_1 - a_2a_3b_2 \end{bmatrix}$$

$$\therefore \quad \mathscr{C} = \begin{bmatrix} 0 & b_1 & -a_1b_1 & a_1^2b_1 + a_2b_2 \\ b_1 & -a_1b_1 & a_1^2b_1 + a_2b_2 & -a_1^3b_1 - a_2a_3b_1 - a_1a_2b_2 \\ 0 & -b_2 & a_3b_1 & -a_1a_3b_1 - a_4b_2 \\ -b_2 & a_3b_1 & -a_1a_3b_1 - a_4b_2 & a_1^2a_3b_1 + a_3a_4b_1 - a_2a_3b_2 \end{bmatrix}$$

✓

RECEIVED
OCT 19 1987

HOMEWORK SET NO: 2

EE 501

LINEAR SYSTEM Theory

SEP. 26, 1987

Benmei Chen
P.O.Box 241, Gonzaga University
Spokane, Washington 99258

10/10

Excellent job

Hsu

I'm very much impressed by your excellent homework

Hsu

1. (TEXT 1, KAILATH, P.76 2.2-20) INVERSE OF A REALIZATION

a. If $\{A, b, c, d\}$, $d \neq 0$, is a realization with $H(s) = d + c(sI-A)^{-1}b$, show that $\{A-(bc/d), b/d, -c/d, 1/d\}$ is a realization for a system with transfer function $1/H(s)$

SHOW: FROM THE GIVENS, WE HAVE TO KNOW THE SYSTEMS MUST BE 'SISO'; OTHERWISE, b/d and $1/H(s)$ ARE MEANINGLESS. HOWEVER, WE CAN TREAT $H(s)$ AS A 1×1 MATRIX. SO, $1/H(s) = [H(s)]^{-1}$, APPLY THE RESULT A.21 ON PAGE 656 IN TEXT 1, WE HAVE

$$\frac{1}{H(s)} = [d + c(sI-A)^{-1}b]^{-1}$$

$$= \frac{1}{d}[1 + c(sI-A)^{-1}\cdot(\tfrac{b}{d})]^{-1}$$

$$= 1/d \cdot [1 - c(sI-A+(\tfrac{b}{d})\cdot c)^{-1}\cdot(\tfrac{b}{d})] \qquad (A.21)$$

$$= \frac{1}{d} + (-\frac{c}{d})[sI-(A-\frac{bc}{d})]^{-1}\cdot(\frac{b}{d})$$

FROM THE EQUATION ABOVE, IT IS EASY TO SEE THE SYSTEM HAS A REALIZATION, $\{A-(bc/d), b/d, -c/d, 1/d\}$. Q.E.D.

---

b. If we are given $\{A, b, c, d\}$, $d \neq 0$, show that the zeros of $c(sI-A)^{-1}b+d$ can be computed as the eigenvalues of the matrix $A - bd^{-1}c$.

SHOW: FROM THE FORM $c(sI-A)^{-1}b+d$, WE KNOW A IS A SQUARE MATRIX. SO IS THE $A-bd^{-1}c$. RECALLED THE RESULTS IN THE LECTURE ON SEP. 22, 1987, WE HAVE

$$(sI - A + bd^{-1}c)^{-1} = \sum_{i=1}^{n} \frac{R_i}{s-\lambda_i}$$

WHERE λ_i, $i = 1, 2, \cdots, n$ IS THE e.v. of MATRIX $A - bd^{-1}c$

TO BE CONT.

1. b. (CONT.)

FROM THE RESULTS IN PART a. λ_i, $i=1,2,\cdots,n$ ALSO IS THE POLE OF SYSTEM $1/H(s)$. THUS, λ_i, $i=1,2,\cdots,n$, MUST BE THE ZERO OF $H(s)$. Q.E.D.

$$\left[\quad \frac{1}{H(s)} = \frac{1}{d} + \left(-\frac{c}{d}\right)[sI - A + bd^{-1}c]^{-1}\left(\frac{b}{d}\right)\right.$$

$$= d^{-1} - cd^{-1} \cdot \sum_{i=1}^{n} \frac{R_i}{s-\lambda_i} \cdot bd^{-1}$$

$$\left. = \frac{d^{-1}\prod_{i=1}^{n}(s-\lambda_i) - cd^{-1}\left[\sum_{i=1}^{n} R_i \cdot \prod_{j\neq i}(s-\lambda_j)\right] \cdot bd^{-1}}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)} \right]$$

C. Show also that the zeros can be computed by solving the generalized e.v. problem

$$(\lambda E - F)P = 0$$

where

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} A & -b \\ c & -d \end{bmatrix}$$

SHOW:

$$\lambda E - F = \begin{bmatrix} \lambda I & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & -b \\ c & -d \end{bmatrix} = \begin{bmatrix} \lambda I - A & b \\ -c & d \end{bmatrix}$$

APPLY THE RESULT OF A.11 (2) ON P 650, TEXT 1, WE GET

IF. $d \neq 0$,
$$|\lambda E - F| = \det\begin{bmatrix} \lambda I - A & b \\ -c & d \end{bmatrix}$$

$$= d \cdot \det[\lambda I - A + bd^{-1}c] = 0$$

$$\det[\lambda I - (A - bd^{-1}c)] = 0$$

$\therefore$ λ IS THE e.v. OF MATRIX $A - bd^{-1}c$.

FROM PART b, WE KNOW λ IS ALSO THE ZERO OF SYSTEM.

IF $d=0$, IT HAS BEEN SHOWN FROM EQ. (21) TO (22) ON PAGE 448 IN TEXT 1.

Q.E.D.

---

2. (TEXT 1, KAILATH P.115 2.3-24) SECOND-ORDER VECTOR DIFFERENTIAL EQUATIONS

To analyze systems with small damping it is often convenient to use sets of coupled second-order equations (e.g. in vibration and circuit anal.):

$$\ddot{x} + D\dot{x} + Kx = Gu$$

Where x = an n-vector of generalized coordinates and u = an n-vector of control variables.

a. For a conservative system without gyroscopic coupling, $D=0$, and K is symmetric. Show that the e.v.s of such a system occur in pairs $\pm\sigma$ or $\pm j\omega$, where σ, ω are real constants, and the E.V.s are orthogonal to each other.

SHOW: $D=0$, SO,

$$\ddot{x} + Kx = Gu$$

LET $x_1 = \dot{x}$, $x_2 = x$

$$\therefore \begin{cases} \dot{x}_1 = \ddot{x} = -Kx + Gu = -Kx_2 + Gu \\ \dot{x}_2 = \dot{x} = x_1 \end{cases}$$

REWRITE THE EQUATIONS AS

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} G \\ 0 \end{pmatrix} \cdot u \quad , \quad \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} \text{ is } 2n \times 2n$$

NOW, WE ARE ABLE TO COMPUTE THE e.v.s OF THE SYSTEM

$$\det\left[\lambda I_{2n} - \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix}\right] = 0 \quad , \quad \lambda \text{ IS THE e.v. OF } \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix}$$

$$= \det\begin{bmatrix} \lambda I_n & K \\ -I_n & \lambda I_n \end{bmatrix} \qquad \text{(A.11 (2) ON P.650)}$$

$$= \det[\lambda I_n] \cdot \det[\lambda I_n + I_n(\lambda I_n)^{-1}K]$$

$$= \lambda^n \cdot \det[\lambda I_n + \lambda^{-1}K] \qquad (\det(cA) = c^n \det A)$$

(TO BE CONT.)

2. (a) (CONT.)

$$\therefore \quad \det\left[\lambda I_{2n} - \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix}\right]$$

$$= \lambda^n \det[\lambda I_n + \lambda^{-1} K]$$

$$= \det[\lambda^2 I_n + K]$$

SINCE K IS SYMMETRIC, FROM THEOREM E-4 IN TEXT 2 (CHEN), WE KNOW:

there EXISTS A NONSINGULAR MATRIX P, $P^{-1}KP = \Lambda$, WHERE Λ IS A DIAGONAL MATRIX, WITH REAL ELEMENTS IN THE DIAGONAL. SO,

$$\det\left[\lambda I_{2n} - \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix}\right]$$

$$= \det(P^{-1}) \cdot \det[\lambda^2 I_n + K] \cdot \det(P)$$

$$= \det[\lambda^2 P^{-1} I_n P + P^{-1}KP]$$

$$= \det[\lambda^2 I_n + \Lambda]$$

$$= \det \begin{bmatrix} \lambda^2+k_1 & & & 0 \\ & \lambda^2+k_2 & & \\ & & \ddots & \\ 0 & & & \lambda^2+k_n \end{bmatrix}$$

$$= (\lambda^2+k_1)(\lambda^2+k_2)\cdots(\lambda^2+k_n)$$

$\therefore$ WE HAVE

$$\underline{\lambda_i, \lambda_{n+i} = \pm\sqrt{k_i} \triangleq \pm\sigma}, \quad \text{IF} \quad k_i \geq 0$$

$$\underline{\lambda_i, \lambda_{n+i} = \pm\sqrt{k_i} \triangleq \pm j\omega}, \quad \text{IF} \quad k_i < 0$$

FOR $i = 1, 2, \ldots, n$.

LET $\lambda_i \neq \lambda_j$, AND P_i, P_j ARE THE E.V.s ASSOCIATED TO λ_i AND λ_j, SEPARATELY. ACCORDING THE DEFINITION,

$$\begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} P_i = \lambda_i P_i, \quad \lambda_i P_i' = P_i' \cdot \begin{pmatrix} 0 & I_n' \\ -K' & 0 \end{pmatrix} = P_i' \begin{pmatrix} 0 & I_n \\ -K & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} P_j = \lambda_j P_j \qquad \lambda_j P_j' = P_j' \begin{pmatrix} 0 & I_n \\ -K & 0 \end{pmatrix}$$

TO BE CONT.

DR. HSU:

IN HOMEWORK SET 2, PROBLEM 2. PART (a) (PAGE 4 – 5 IN MY HOMEWORK SOLUTION), IN ORDER TO SHOW THE EIGENVECTORS ARE ORTHOGONAL, I THINK I HAVE TO SHOW AS FOLLOWING:

ACCORDING THE DEFINITION:

$$\begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} P_i = \lambda_i P_i \quad , \quad \lambda_i P_i' = P_i' \cdot \begin{pmatrix} 0 & I_n' \\ -K' & 0 \end{pmatrix} = P_i' \begin{pmatrix} 0 & I_n \\ -K & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} P_j = \lambda_j P_j \quad ,$$

THEN

$$\lambda_i^2 P_i' = \lambda_i \cdot (\lambda_i P_i') = \lambda_i \cdot \left[P_i' \cdot \begin{pmatrix} 0 & I_n \\ -K & 0 \end{pmatrix} \right]$$

$$= \lambda_i P_i' \cdot \begin{pmatrix} 0 & I_n \\ -K & 0 \end{pmatrix}$$

$$= P_i' \cdot \begin{pmatrix} 0 & I_n \\ -K & 0 \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -K & 0 \end{pmatrix}$$

$$= P_i' \cdot \begin{pmatrix} -K & 0 \\ 0 & -K \end{pmatrix}$$

$$\lambda_j^2 P_j = \lambda_j (\lambda_j P_j) = \lambda_j \cdot \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} P_j = \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} \lambda_j P_j$$

$$= \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} P_j$$

$$= \begin{pmatrix} -K & 0 \\ 0 & -K \end{pmatrix} P_j$$

SO, $\lambda_i^2 P_i' P_j = P_i' \begin{pmatrix} -K & 0 \\ 0 & -K \end{pmatrix} P_j = P_i' \lambda_j^2 P_j = \lambda_j^2 \cdot P_i' P_j$

$\therefore \ (\lambda_i^2 - \lambda_j) \ P_i' P_j = 0$

IF $\lambda_i \neq \lambda_j$ THEN $P_i' P_j = 0$ THEY ARE ORTHOGONAL.

Yes

DR. HSU, MAY I HAVE MY HOMEWORK CORRECTED ? THANKS A LOT!.

BENMEI CHEN

Oct. 6, 1987. (TUESDAY).

2(a). (CONT.)

$$\therefore \quad \lambda_i^2 P_i' P_i = \lambda_i P_i' \cdot \lambda_i P_i = P_i' \begin{pmatrix} 0 & I_n \\ -K & 0 \end{pmatrix} \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} P_i$$

$$= P_i' \begin{pmatrix} I_n & 0 \\ 0 & K^2 \end{pmatrix} P_i$$

SINCE $P_i \neq 0$, WE HAVE

$$\lambda_i^2 P_i' = P_i' \begin{pmatrix} I_n & 0 \\ 0 & K^2 \end{pmatrix}, \qquad \lambda_i^2 P_i = \begin{pmatrix} I_n & 0 \\ 0 & K^2 \end{pmatrix} P_i$$

WITH SAME REASON,

$$\lambda_j^2 P_j' = P_j' \begin{pmatrix} I_n & 0 \\ 0 & K^2 \end{pmatrix}, \qquad \lambda_j^2 P_j = \begin{pmatrix} I_n & 0 \\ 0 & K^2 \end{pmatrix} P_j$$

$$\therefore \quad \lambda_i^2 P_i' P_j = P_i' \begin{pmatrix} I_n & 0 \\ 0 & K^2 \end{pmatrix} P_j = P_i' \cdot \lambda_j^2 \cdot P_j = \lambda_j^2 \cdot P_i' \cdot P_j$$

$$\therefore \quad (\lambda_i^2 - \lambda_j^2) P_i' \cdot P_j = 0$$

A. IF $\lambda_i \neq -\lambda_j$, THEN $\lambda_i^2 - \lambda_j^2 \neq 0$, THUS $\underline{P_i' \cdot P_j = 0}$

B. IF $\lambda_i = -\lambda_j \neq 0$ (OTHERWISE $\lambda_i = \lambda_j$), LET $\begin{pmatrix} \gamma_i \\ S_i \end{pmatrix}$ be E.V. associated with λ_i

$$\therefore \quad \lambda_i \begin{pmatrix} \gamma_i \\ S_i \end{pmatrix} = \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \gamma_i \\ S_i \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda_i \gamma_i \\ \lambda_i S_i \end{pmatrix} = \begin{pmatrix} -K S_i \\ \gamma_i \end{pmatrix} \Rightarrow \begin{cases} \gamma_i = \lambda_i S_i \\ \lambda_i \gamma_i = -K S_i \end{cases}$$

$$\text{CHECK:} \quad \begin{pmatrix} 0 & -K \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \gamma_i \\ -S_i \end{pmatrix} = \begin{pmatrix} K S_i \\ \gamma_i \end{pmatrix} = \begin{pmatrix} -\lambda_i \gamma_i \\ \lambda_i S_i \end{pmatrix} = -\lambda_i \begin{pmatrix} \gamma_i \\ -S_i \end{pmatrix} = \lambda_j \begin{pmatrix} \gamma_i \\ -S_i \end{pmatrix}$$ O.K.

$\therefore$ $\begin{pmatrix} \gamma_i \\ -S_i \end{pmatrix}$ IS E.V. ASSOCIATED WITH $\lambda_j = -\lambda_i$.

$$(\gamma_i' \;\; S_i') \begin{pmatrix} \gamma_i \\ -S_i \end{pmatrix} = \gamma_i' \cdot \gamma_i - S_i' S_i = \lambda_i^2 \cdot S_i' S_i - S_i' S_i = (\lambda_i^2 - 1) S_i' S_i$$

① IF $\lambda_i = \pm 1$ THEN THE E.V.s of λ_i, λ_j <u>ARE ORTHOGONAL.</u>

② IF $\lambda_i \neq \pm 1$, WE ASSUME A PARTICULAR $K = -4$, THEN

$$\det \begin{pmatrix} \lambda & -4 \\ -1 & \lambda \end{pmatrix} = \lambda^2 - 4 = 0 \quad \Rightarrow \quad \lambda_1 = 2, \; \lambda_2 = -2$$

AND $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ARE E.V.S ASSOCIATED WITH $\lambda_1 = 2$ and $\lambda_2 = -2$, SEPERATELY.

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}' \begin{pmatrix} 2 \\ -1 \end{pmatrix} = (2 \;\; 1) \begin{pmatrix} 2 \\ -1 \end{pmatrix} = +3 \neq 0$$ <u>NOT ORTHOGONAL!</u>

(b) *For a conservative system with gyroscopic coupling, D is antisymmetric (i.e. $D^T = -D$), and K is symmetric. Show that the eigenvalues of such a system are located symmetrically about both the real & imaginary axes.*

SHOW ON NEXT PAGE!

2(b) (CONT.)

<u>SHOW:</u> GIVENS $D^T = -D$, $K^T = K$

$$\ddot{x} + D\dot{x} + Kx = Gu$$

LET $x_1 = \dot{x}$, $x_2 = x$

$$\therefore \quad \dot{x}_1 = \ddot{x} = -D\dot{x} - Kx + Gu = -Dx_1 - Kx_2 + Gu$$

$$\dot{x}_2 = \dot{x} = x_1$$

$$\therefore \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -D & -K \\ I_n & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} G \\ 0 \end{pmatrix} \cdot u$$

IF λ is a e.v. OF THE SYSTEM , THEN

$$\det \begin{bmatrix} \lambda I_n + D & K \\ -I_n & \lambda I_n \end{bmatrix} = 0$$

$$= \lambda^n \cdot \det[\lambda I_n + D + K\lambda^{-1} I_n \cdot I_n] \qquad \text{(A.11 ON TEXT 1)}$$

$$= \det[\lambda^2 I_n + \lambda \cdot D + K]$$

$$= \det[\lambda^2 I_n + \lambda \cdot D + K]^T \qquad (\det A = \det A^T)$$

$$= \det[\lambda^2 I_n + \lambda \cdot D^T + K^T]$$

$$= \det[\lambda^2 I_n - \lambda \cdot D + K]$$

$$= \det[(-\lambda)^2 I_n + (-\lambda) \cdot D + K] = \det \begin{bmatrix} (-\lambda) I_n + D & K \\ -I_n & (-\lambda) I_n \end{bmatrix} = 0$$

$\therefore$ $-\lambda$ IS A e.v. OF system. ------ (1)

FROM DEFINITION, AND ASSUME P IS A E.V. ASSOCIATED WITH λ,

$$\lambda P = \begin{pmatrix} -D & -K \\ I_n & 0 \end{pmatrix} P$$

DEFINE M^* AS THE COMPLEX CONJUGATE TRANSPOSE OF M, WHERE M may be a complex number, vector or matrix. So,

$$P^* \lambda^* = (\lambda P)^* = \left[\begin{pmatrix} -D & -K \\ I_n & 0 \end{pmatrix} P \right]^* = P^* \cdot \begin{pmatrix} -D & -K \\ I_n & 0 \end{pmatrix}^* \qquad \text{(APPENDIX E IN TEXT II)}$$

$\begin{pmatrix} -D & -K \\ I_n & 0 \end{pmatrix}$ IS REAL VALUED. $\rightarrow$ $= P^* \begin{pmatrix} -D & -K \\ I_n & 0 \end{pmatrix}^T$

$$\therefore \quad (P^* \lambda^*)^T = \lambda^* \cdot (P^*)^T = \left[P^* \begin{pmatrix} -D & -K \\ I_n & 0 \end{pmatrix}^T \right]^T = \begin{pmatrix} -D & -K \\ I_n & 0 \end{pmatrix} (P^*)^T$$

SO, λ^* IS A e.v. OF SYSTEM TOO, WITH E.V. $(P^*)^T$. FROM (1) ABOVE, WE HAVE a e.v. OF $-\lambda^*$. <u>ALL $\lambda, -\lambda, \lambda^*, -\lambda^*$ ARE e.v.s OF system.</u> <u>Q.E.D.</u>

(IF $\lambda = \sigma + j\omega$, THEN $-\lambda = -\sigma - j\omega$, $\lambda^* = \sigma - j\omega$, $-\lambda^* = -\sigma + j\omega$)

2.(c). A frictionless spinning top is an example of a conservative system with gyroscopic coupling. The equations of motion may be normalized to

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \begin{bmatrix}0 & p\\ -p & 0\end{bmatrix}\cdot\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} - \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0$$

where x_1, x_2 are orthogonal lateral displacements from the vertical position and p is proportional to the spin rate. What is the minimum value of p for which the e.v.s are pure imaginary?

SOLUTION: LET

$$\begin{bmatrix}t_1\\ t_2\end{bmatrix} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}, \quad \begin{bmatrix}t_3\\ t_4\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}\dot{t}_1\\ \dot{t}_2\end{bmatrix} = \begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = -\begin{bmatrix}0 & p\\ -p & 0\end{bmatrix}\cdot\begin{bmatrix}t_1\\ t_2\end{bmatrix} + \begin{bmatrix}+1 & 0\\ 0 & +1\end{bmatrix}\cdot\begin{bmatrix}t_3\\ t_4\end{bmatrix}$$

$$\begin{bmatrix}\dot{t}_3\\ \dot{t}_4\end{bmatrix} = \begin{bmatrix}t_1\\ t_2\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}\dot{t}_1\\ \dot{t}_2\\ \dot{t}_3\\ \dot{t}_4\end{bmatrix} = \underbrace{\begin{bmatrix}0 & -p & +1 & 0\\ +p & 0 & 0 & +1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}}_{A}\cdot\begin{bmatrix}t_1\\ t_2\\ t_3\\ t_4\end{bmatrix}$$

e.v.s:

$$\det(\lambda I - A) = \begin{vmatrix}\lambda & +p & -1 & 0\\ -p & \lambda & 0 & -1\\ -1 & 0 & \lambda & 0\\ 0 & -1 & 0 & \lambda\end{vmatrix} = -\begin{vmatrix}\lambda & p & -1\\ -1 & 0 & \lambda\\ 0 & -1 & 0\end{vmatrix} + \lambda\cdot\begin{vmatrix}\lambda & p & -1\\ -p & \lambda & 0\\ -1 & 0 & \lambda\end{vmatrix}$$

$$= \lambda^4 + (p^2-2)\lambda^2 + 1 = 0$$

$$= \left(\lambda^2 + \frac{p^2-2}{2}\right)^2 - \frac{p^4-4p^2}{2^2}$$

$$= \left(\lambda^2 + \frac{p^2-2}{2} + \frac{\sqrt{p^2-4}}{2}\cdot p\right)\left(\lambda^2 + \frac{p^2-2}{2} - \frac{\sqrt{p^2-4}}{2}\cdot p\right) = 0$$

good

λ IS PURE IMAGINARY, IIF $\sqrt{p^2-4} \geq 0$, $\frac{p^2-2}{2} + \frac{\sqrt{p^2-4}}{2}\cdot p > 0$ AND $\frac{p^2-2}{2} - \frac{\sqrt{p^2-4}}{2}\cdot p > 0$

WE HAVE $p \geq 2$ OR $|p| \geq 2$, SO $\underline{p_{min} = 2}$. OR $\underline{|p|_{min} = 2}$

IF $p = 2$, THEN $\lambda_{1,2} = \pm j$, $\lambda_{3,4} = \pm j$ PURE imaginary.

3. Let $W(t) = \int_0^t e^{A\sigma} BB' e^{A'\sigma} d\sigma$

$$\hat{A} = \begin{pmatrix} -A & BB' \\ 0 & A' \end{pmatrix} \quad \text{and} \quad e^{\hat{A}t} = \begin{pmatrix} E_1(t) & E_2(t) \\ 0 & E_3(t) \end{pmatrix}$$

(a) Show that $W(t) = E_3'(t) E_2(t)$

SHOW:

$$\hat{A} = \begin{pmatrix} -A & BB' \\ 0 & A' \end{pmatrix}, \quad \hat{A}^2 = \begin{pmatrix} -A & BB' \\ 0 & A' \end{pmatrix}\begin{pmatrix} -A & BB' \\ 0 & A' \end{pmatrix} = \begin{pmatrix} A^2 & -ABB' + BB'A' \\ 0 & A'^2 \end{pmatrix}$$

$$\hat{A}^3 = \begin{pmatrix} -A & BB' \\ 0 & A' \end{pmatrix}\begin{pmatrix} A^2 & -ABB' + BB'A' \\ 0 & A'^2 \end{pmatrix} = \begin{pmatrix} -A^3 & A^2BB' - ABB'A' + BB'A'^2 \\ 0 & A'^3 \end{pmatrix}$$

$$e^{\hat{A}t} = I + \hat{A}t + \hat{A}^2 t^2/2 + \cdots$$

$$= \begin{pmatrix} I - At + \frac{A^2t^2}{2} - \cdots & 0 + BB't - (ABB' - BB'A')t^2/2! + \cdots \\ 0 & I + A't + A'^2t^2/2! + \cdots \end{pmatrix}$$

$$e^{\hat{A}t} = \begin{pmatrix} e^{-At} & \Sigma \cdot t \\ 0 & e^{A't} \end{pmatrix}$$

WHERE $\Sigma = BB' - (ABB' - BB'A')t/2 + \cdots$

$$E_2(t) = \Sigma \cdot t \qquad E_3(t) = e^{A't}, \quad E_3'(t) = e^{At}$$

$$\frac{d}{dt} e^{\hat{A}t} = \hat{A} e^{\hat{A}t} = \begin{pmatrix} -A & BB' \\ 0 & A' \end{pmatrix}\begin{pmatrix} e^{-At} & \Sigma t \\ 0 & e^{A't} \end{pmatrix} = \begin{pmatrix} -Ae^{-At} & -A\Sigma t + BB'e^{A't} \\ 0 & A'e^{A't} \end{pmatrix}$$

$$= \frac{d}{dt}\begin{pmatrix} E_1(t) & E_2(t) \\ 0 & E_3(t) \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}E_1(t) & \frac{d}{dt}E_2(t) \\ 0 & \frac{d}{dt}E_3(t) \end{pmatrix}$$

$$\frac{d}{dt} E_2(t) = -A\Sigma t + BB'e^{A't}, \quad \frac{d}{dt} E_3(t) = A'e^{A't}$$

$$\frac{d}{dt} E_3'(t) = \left[\frac{d}{dt} E_3(t)\right]' = e^{At} A$$

$$\frac{d}{dt}\left(E_3'(t) \cdot E_2(t)\right) = \frac{d}{dt} E_3'(t) \cdot E_2(t) + E_3'(t) \cdot \frac{d}{dt} E_2(t)$$

$$= e^{At} \cdot A \cdot \Sigma t + e^{At}(-A \cdot \Sigma t + BB'e^{A't})$$

$$= e^{At} BB' e^{A't}$$

$$e^{\hat{A}t} = \mathcal{L}^{-1}\left[(sI - \hat{A})^{-1}\right] = \mathcal{L}^{-1}\left[\begin{pmatrix} sI + A & -BB' \\ 0 & sI - A' \end{pmatrix}^{-1}\right] = \mathcal{L}^{-1}\begin{bmatrix} (sI+A)^{-1} & \cdots \\ 0 & (sI-A')^{-1} \end{bmatrix}$$

etc.

easier

HSW

TO BE CONT.

3. (a) (CONT.)

$$\therefore \quad W(t) = \int_0^t e^{A\sigma} BB' e^{A'\sigma} d\sigma$$

$$= \int_0^t \cdot \frac{d}{d\sigma} (E_3'(\sigma) E_2(\sigma)) d\sigma$$

$$= E_3'(\sigma) E_2(\sigma) \Big|_0^t$$

$$= E_3'(t) E_2(t) - E_3'(0) E_2'(0)$$

$$= E_3'(t) E_2(t) - E_3'(0) X \cdot 0$$

$$= E_3'(t) E_2(t)$$

Q.E.D.

(b) Refer to H.W. set NO:1, PROBLEM #4, Let

$a_1 = 3.878$, $a_2 = 0.1243$, $a_3 = 9.23$ and $a_4 = 23.62$

$b_1 = 2.771$, $b_2 = 6.595$

use (a), compute $W(T)$, $T = 0.1$, or choose other reasonable values.

SOLUTION: FROM PROBLEM #4 IN H.W. SET 1, AND THE GIVENS,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -3.878 & -0.1243 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 9.23 & 23.62 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2.771 \\ 0 \\ -6.595 \end{bmatrix}$$

$$\therefore \quad BB' = \begin{bmatrix} 0 \\ 2.771 \\ 0 \\ -6.595 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2.771 & 0 & -6.595 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 7.678 & 0 & -18.275 \\ 0 & 0 & 0 & 0 \\ 0 & -18.275 & 0 & 43.494 \end{bmatrix}$$

$$\therefore \quad \hat{A} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.878 & 0.1243 & 0 & 0 & 7.678 & 0 & -18.275 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -9.23 & -23.62 & 0 & 0 & -18.275 & 0 & 43.494 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3.878 & 0 & 9.23 \\ 0 & 0 & 0 & 0 & 0 & -0.1243 & 0 & 23.62 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

3(b) (CONT.) FROM MATLAB, I GET

$$e^{\hat{A}\times 0.1} = \begin{bmatrix} 1.0000 & -0.1222 & -0.0007 & 0.0000 & -0.0013 & -0.0389 & 0.0031 & 0.0944 \\ 0 & 1.4740 & 0.0157 & -0.0007 & 0.0389 & 0.7880 & -0.0944 & -1.9480 \\ 0 & 0.0537 & 1.1207 & -0.1040 & 0.0031 & 0.0944 & -0.0075 & -0.2289 \\ 0 & -1.1682 & -2.4629 & 1.1207 & -0.0944 & -1.9479 & 0.2289 & 4.8104 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0829 & 0.6784 & 0.0416 & 0.7985 \\ 0 & 0 & 0 & 0 & -0.0006 & -0.0108 & 1.1203 & 2.4508 \\ 0 & 0 & 0 & 0 & -0.0000 & -0.0006 & 0.1040 & 1.1203 \end{bmatrix}$$

$$W(0.1) = \begin{bmatrix} 0.0019 & 0.0264 & -0.0047 & -0.0671 \\ 0.0264 & 0.5346 & -0.0641 & -1.3217 \\ -0.0047 & -0.0641 & 0.0115 & 0.1628 \\ -0.0671 & -1.3217 & 0.1628 & 3.2725 \end{bmatrix}$$

✓

3(c). COMPUTE EIGENVALUES OF $A - \underline{b}\underline{b}' W^{-1}(T)$ FOR SEVERAL VALUES OF T. CHECK WHETHER ALL EIGENVALUES YOU OBTAINED ARE IN THE LHP. EXPLAIN YOUR RESULTS.

Would you re-try this problem by

COMPUTE THE e.v.s OF $A - \underline{b}\underline{b}' W^{-1}(T)$ FOR $T = 0.1, 0.5, 2, 10$

WE HAVE THE e.v.s FOR (FROM MATLAB)

$W(T) = \int_0^T e^{-A\sigma} BB' e^{-A'\sigma} d\sigma$

$T = 0.1$: $81.7809 \pm j \cdot 99.5776$, -19.3527, 28.1082

$T = 0.5$: -54.0643, $2.9276 \pm j\, 5.9116$, 6.2815

$T = 5$: 0.1898, -3.0991, -4.9300, 4.8465

$T = 10$: $\underline{0}$, $\underline{-3.7573}$, $\underline{-4.9672}$, $\underline{4.8465}$

then all e.v. of $A - \underline{b}\underline{b}' W^{-1}(T)$ will be in LHP.

THE e.v.s of A ARE:

$\underline{0}$, $\underline{-3.7573}$, $\underline{-4.9672}$, $\underline{4.8465}$

WE CAN SEE NOT ALL OF THE e.v.s ARE IN TH LHP.

Yes

THE REASON FOR THESE RESULTS COME OUT IS THAT MATRIX A IS NOT STABLE.

MATRIX A HAS A e.v. 4.8465, WHICH IS IN RHP. $W(T) = E_3'(T) E_2(T) = e^{AT} \cdot X \cdot T$

$|W(T)| \to \infty$ WHEN $T \to \infty$. SO, $W^{-1}(T) \to 0$, $T \to \infty$.

O.K.

$$A - \underline{b}\underline{b}' W^{-1}(T) \to A$$

$\therefore$ THE e.v.s OF $A - \underline{b}\underline{b}' W^{-1}(T)$ BECOME CLOSED TO THE e.v.s OF A. OF COURSE, CLOSE TO 4.8465 TOO.

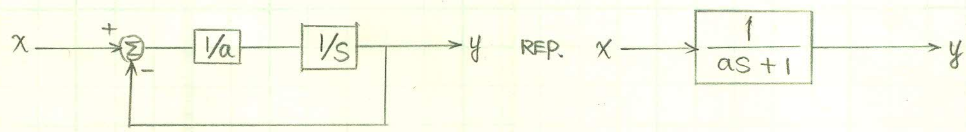
4. *Refer to the attached F-14 benchmark control problem, in particular, Fig A-1.*

(a) *Define appriciate state variables from Fig. A-1. Obtain state space data* $\{A, \underline{b}, \underline{c}, d\}$

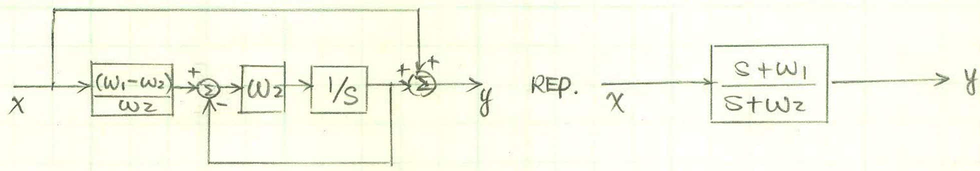
SOLUTION: FOR FIG A-1, WE DEFINE $\alpha_{command}$ AS SYSTEM INPUT, AND α AS SYSTEM OUTPUT (SO CALLED CLOSE-LOOP IN THAT PAPER)

WE USE THE FOLLOWING REPLACEMENTS.

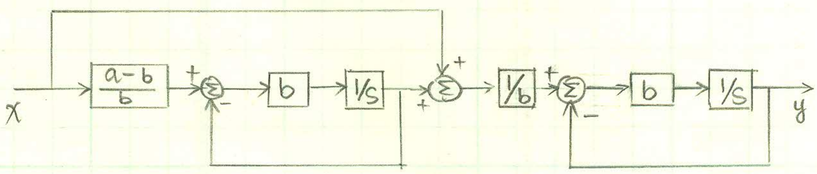
①



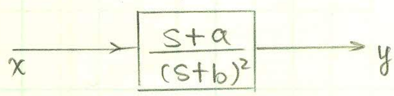
②



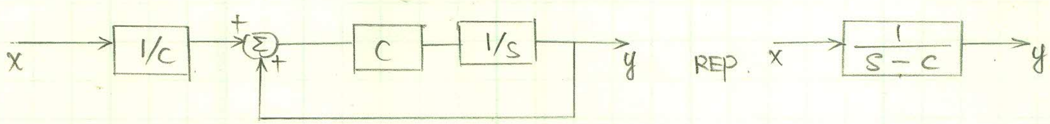
③



REPLACE:



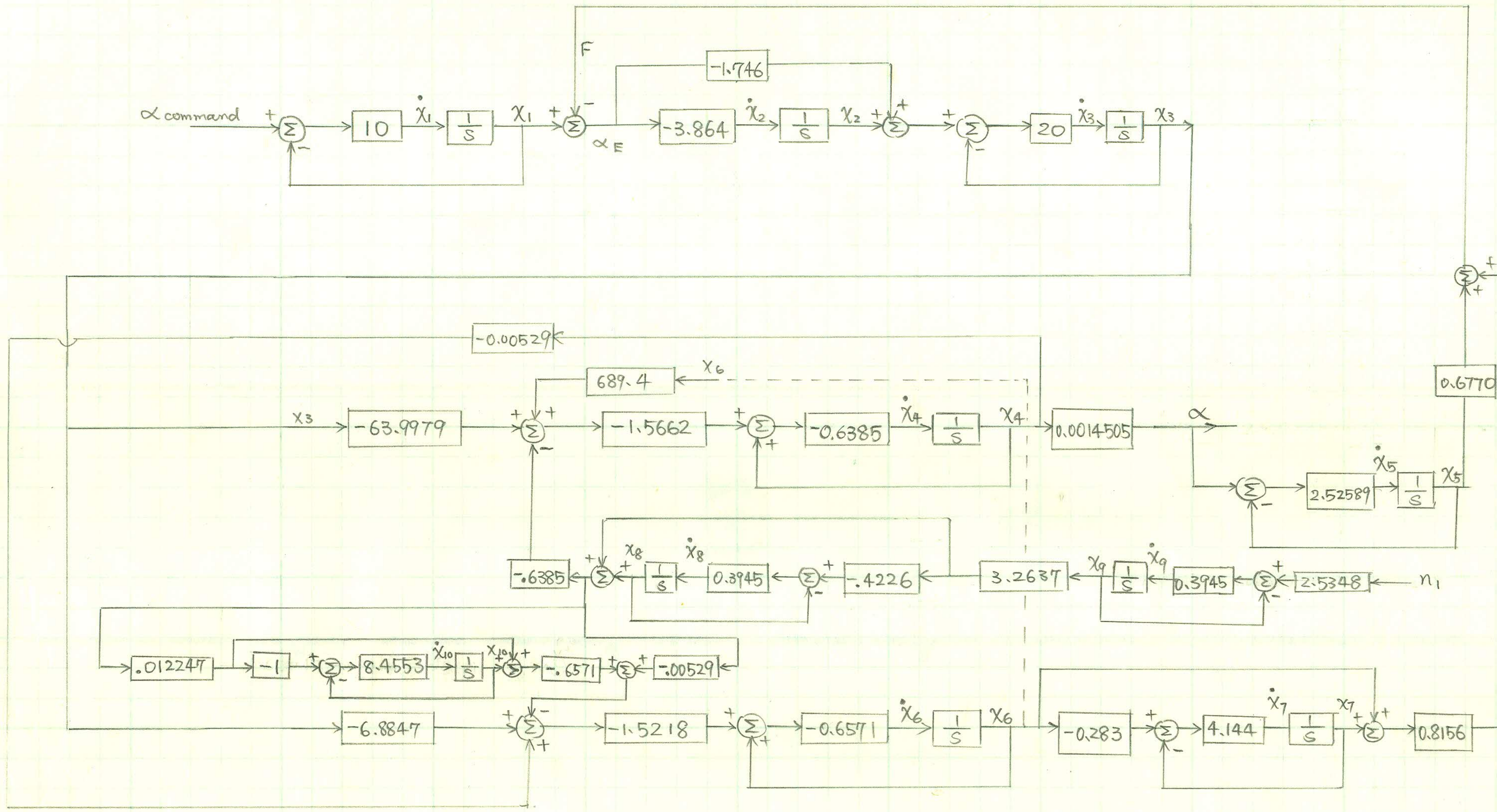
④



USE THIS REPLACEMENTS, WE REDRAW THE BLOCK DIAGRAM WITH THE GIVEN DATA AS BELOW.

TO BE CONT.

4 (a) (CONT.)



BLOCK DIAGRAM WITH DETAIL DATA AND STATE ASSIGNMENT

$$\alpha_{command} = 0.03491\left(e^{-0.01t} - e^{-426.4352t}\right)$$

$$\alpha_{command}(S) = 0.03491\left(\frac{1}{S+0.01} - \frac{1}{S+426.4352}\right)$$

4(a) (CONT.) FROM THE BLOCK DIAGRAM, I THINK 10 STATES ARE GOOD ENOUGH. FOR CLOSE-LOOP.
(NOT INCLUDED TWO FROM $\alpha_{command}$)

Yes

$$\dot{x}_1 = (\alpha - x_1)\times 10 = -10x_1 + 10\cdot\alpha$$

$$\dot{x}_2 = (x_1 - F)\times(-3.864) = -3.864x_1 + 3.864(0.677x_5 + 0.8156x_6 + 0.8156x_7)$$

$$= -3.864\cdot x_1 + 2.61593x_5 + 3.15148\cdot x_6 + 3.15148\cdot x_7$$

$$\dot{x}_3 = [x_2 - 1.746(x_1 - F) - x_3]\cdot 20 = -34.92x_1 + 20x_2 - 20x_3 + 34.92(0.677x_5 + .8156x_6 + .8156x_7)$$

$$= -34.92x_1 + 20x_2 - 20x_3 + 23.64084x_5 + 28.48075x_6 + 28.48075x_7$$

$$\dot{x}_4 = -0.6385[x_4 - 1.5662\cdot\{-63.998x_3 + 689.4x_6 + 0.6385(x_8 + 3.2637x_9)\}]$$

$$= -63.9992x_3 - 0.6385x_4 + 689.4x_6 + 0.6385\cdot x_8 + 2.08387\cdot x_9$$

$$\dot{x}_5 = 2.52589\cdot[0.0014505x_4 - x_5] = 0.003664x_4 - 2.52589x_5$$

$$\dot{x}_6 = -0.6571\cdot\{x_6 - 1.5218\cdot[-6.8847x_3 - 0.00529x_4 + 0.00529(x_8 + 3.2637x_9) +$$

$$0.6571\cdot(x_{10} + 0.012247\cdot\langle x_8 + 3.2637x_9\rangle)]\}$$

$$= -6.8847x_3 - 0.00529x_4 - 0.6571\cdot x_6 + 0.013338x_8 + 0.04353x_9 + 0.6571\cdot x_{10}$$

$$\dot{x}_7 = 4.144\cdot[-x_7 - 0.283x_6] = -1.17275x_6 - 4.144\cdot x_7$$

$$\dot{x}_8 = 0.3945\cdot[-x_8 - 0.4226\times 3.2637x_9] = -0.3945\cdot x_8 - 0.54411\cdot x_9$$

$$\dot{x}_9 = 0.3945\cdot[-x_9 + 2.5348\cdot n_1] = -0.3945\cdot x_9 + n_1$$

$$\dot{x}_{10} = 8.4553\cdot[-x_{10} - 0.12247\cdot(x_8 + 3.2637\cdot x_9)]$$

$$= -1.03552x_8 - 3.37963x_9 - 8.4553\cdot x_{10}$$

SO, ($\alpha_c = \alpha_{command}$) THE CLOSE-LOOP SYSTEM EQUATIONS.

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\\ \dot{x}_7\\ \dot{x}_8\\ \dot{x}_9\\ \dot{x}_{10}\end{pmatrix} = \underbrace{\begin{pmatrix} -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -3.864 & 0 & 0 & 0 & 2.6159 & 3.15148 & 3.15148 & 0 & 0 & 0\\ -34.92 & 20 & -20 & 0 & 23.64084 & 28.48075 & 28.48075 & 0 & 0 & 0\\ 0 & 0 & -63.998 & -.6385 & 0 & 689.4 & 0 & .6385 & 2.08387 & 0\\ 0 & 0 & 0 & .003664 & -2.5259 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -6.8847 & -.00529 & 0 & -.6571 & 0 & 0.01334 & 0.04353 & .6571\\ 0 & 0 & 0 & 0 & 0 & -1.17275 & -4.144 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.3945 & -.54411 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.3945 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.03552 & -3.3796 & -8.4553 \end{pmatrix}}_{A} \cdot \begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7\\ x_8\\ x_9\\ x_{10}\end{pmatrix} + \underbrace{\begin{pmatrix}10\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}}_{b}\alpha_c + \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix}\cdot n_1$$

noise input

$$\alpha = \underbrace{[\,0 \quad 0 \quad 0 \quad 0.0014505 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,]}_{c}\cdot X + \underbrace{0}_{d}\cdot\alpha_c$$

4(a) (CONT.) FOR THE OPEN-LOOP — LOOP BROKEN BEFORE PROPORTIONAL PLUS-INTEGRAL COMPENSATOR. THE SYSTEM INPUT IS THE α ERROR α_E, AND OUTPUT F.

COMPARING THE OPEN TO CLOSE-LOOP, WE SEE THERE ARE ONLY STATE x_2 AND x_3 DIFFERENT FROM THOSE IN CLOSE-LOOP.

$$\dot{x}_2 = -3.864\,\alpha_E$$

$$\dot{x}_3 = 20\,(-x_3 - 1.746\alpha_E + x_2) = -20x_3 - 34.92\cdot\alpha_E + 20x_2$$

THE OPEN-LOOP SYSTEM EQUATIONS:

$$\begin{bmatrix}\dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\\ \dot{x}_7\\ \dot{x}_8\\ \dot{x}_9\\ \dot{x}_{10}\end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 20 & -20 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -63.998 & -.6385 & 0 & 689.4 & 0 & .6385 & 2.08387 & 0\\ 0 & 0 & .003664 & -2.5259 & 0 & 0 & 0 & 0 & 0\\ 0 & -6.8847 & -.00529 & 0 & -.6571 & 0 & .01334 & .04353 & .6571\\ 0 & 0 & 0 & 0 & -1.17275 & -4.144 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -.3945 & -.54411 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.3945 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1.03552 & -3.3796 & -8.4553\end{bmatrix}}_{A_{open}} \cdot \begin{bmatrix}x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7\\ x_8\\ x_9\\ x_{10}\end{bmatrix} + \underbrace{\begin{bmatrix}-3.864\\ -34.92\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}}_{\underline{b}_{open}} \alpha_E + \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix} \cdot n_1$$

$$F = \alpha_{RETURN} = 0.6770x_5 + 0.8156\,(x_7 + x_6) = 0.6770x_5 + 0.8156x_6 + 0.8156x_7$$

$$F = \overbrace{[\,0 \quad 0 \quad 0 \quad 0.677 \quad 0.8156 \quad 0.8156 \quad 0 \quad 0 \quad 0\,]}^{C_{open}}\ \underline{X}_{open} + \overbrace{0}^{d_{open}} \cdot \alpha_E$$

(b) FROM THE MATLAB, WE HAVE OPEN-LOOP POLES: NEXT PAGES

OPEN-LOOP ZEROS: NEXT PAGES

(c) FROM THE MATLAB, WE HAVE CLOSE-LOOP POLES: NEXT PAGES

CLOSE-LOOP ZEROS: NEXT PAGES

It's in the Control System Toolbox, which will be sent to you soon. HS

4. (b) + (c) (CONT.)

BECAUSE THERE IS NO FUNCTION IN MATLAB TO COMPUTE THE GENERALIZED e.v.s. AND IN THIS CASE, $d=0$, WE CAN'T USE THE RESULTS FROM PROBLEM 1 (b). HOWEVER, WE CAN ASSUME $d=1$ FIRST, THEN USE THE RESULTS COMPUTING THE ZEROS OF SYSTEM $\{A, \underline{b}, \underline{c}, 1\}$, FOLLOWING SHOWS HOW TO FIND ZEROS FOR SYSTEM $\{A, \underline{b}, \underline{c}\}$ FROM ZEROS OF SYSTEM $\{A, \underline{b}, \underline{c}, 1\}$:

Assuming, the transfer function for system $\{A, \underline{b}, \underline{c}\}$ is

$$h_1(s) = \frac{a(s)}{b(s)} = c(sI-A)^{-1}b$$

For system $\{A, \underline{b}, \underline{c}, 1\}$ is

$$h_2(s) = \frac{c(s)}{d(s)} = c(sI-A)^{-1}b + 1$$

$$\therefore \quad \frac{a(s)}{b(s)} = \frac{c(s)}{d(s)} - 1 = \frac{c(s)-d(s)}{d(s)}$$

$$a(s) = (c(s) - d(s)) \cdot \frac{b(s)}{d(s)}$$

$$\because \quad b(s) = d(s) = \det(sI - A) \triangleq b(s)$$

$$\therefore \quad a(s) = c(s) - b(s) = -[b(s) - c(s)]$$

WE HAVE SHOWN IN #1 (b), $c(s)$ CAN BE FOUND BY COMPUTING THE e.v.s OF MATRIX $A - \underline{b}\underline{c}$, AND $b(s)$ CAN BE FOUND BY COMPUTING e.v.s OF MATRIX A. THEREFORE, WE CAN FIND $a(s)$, THEN THE ZEROS OF SYSTEM $\{A, \underline{b}, \underline{c}\}$.

(THESE WERE ALSO SHOWN IN #1, (b) IN HOMEWORK SET NO:1 BY USING THE OTHER METHOD)

4. (b) Compute open-loop poles and zeros.

As we showed the results above, we can compute the poles of system { Ao,bo,Co,0} by computing the eigenvalues of matrix Ao. And we can compute the zeros of the system by computing the roots of polynomial a(s)=b(s)-c(s), where a(s) , b(s) & c(s) are defined on the previous page. In MATLAB, we can find b(s) & c(s) as :

b(s) = poly(Ao)

c(s) = poly(Ao-Bo*Co)

Using the ROOTS function in MATLAB, we can find the zeros of system {Ao,bo,Co,0}. The following are poles and zeros I obtained and the results from the artical:

POLES FOR OPEN-LOOP		ZEROS FOR OPEN-LOOP	
COMPUTED	FROM ARTICAL	COMPUTED	FROM ARTICAL
-4.1440	-4.1440	-1.3221+/- j 1.4326	-1.3193+/- j 1.4308
-2.5259	-2.5259		
-0.6478+/- j 1.9097	-0.6478+/- j 2.0202	-3.4705	-3.4700
-20.000	-20.000	-2.2131	-2.2131

Canceled poles and zeros (three pairs) :

-8.4553, -0.3945, -0.3945

Conclusion: quite close.

4.(c) Compute close-loop poles and zeros.

Use the same method as part (b). Following are the poles and zeros I obtained and the results in the artical:

POLES FOR CLOSE-LOOP		ZEROS FOR CLOSE-LOOP	
COMPUTED	FROM ARTICle	COMPUTED	FROM ARTICle
-9.8651+/- j 9.5684	-9.8432+/- j 9.5718	-74.8205	-74.8207
		-4.1440	-4.1440
-1.6191+/- j 1.6046	-1.6717+/- j 1.5970	-2.5259	-2.5259
		-2.2131	-2.2131
-3.1400	-3.1231		
-1.8572	-1.8126		
-10.000	-10.000		

Canceled poles and zeros(three pairs):

-8.4553, -0.3945, -0.3945

Conclusion: O.K.

10/10 HS

HOMEWORK SET NO: 3

EE 501

LINEAR SYSTEM THEORY

OCT. 26, 1987

Benmei Chen

P.O.Box 241, Gonzaga University

Spokane, WA 99258

Phone : 327-6071

Benmei,

I'm very pleased with your excellent job. Keep on your good work. Someday you'll be an outstanding control engineer. If I can be of any assistance to advance your graduate studies or career planning, please ~~fell~~ feel free to let me know —

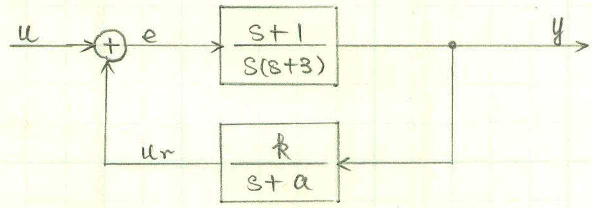
Hsu

PROBLEM 1: (a) PAGE 111 IN TEXT ONE 2.3-3

CONSIDER THE SYSTEM ILLUSTRATED IN THE FIGURE.

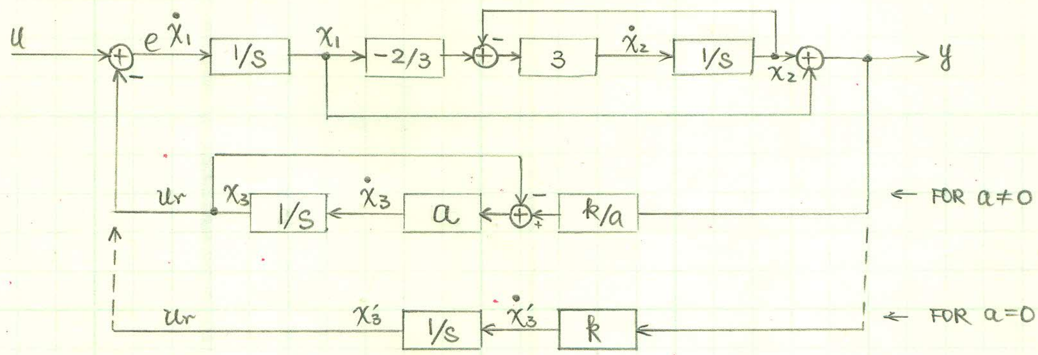
a. GIVE A STATE-VARIABLE REALIZATION OF THIS SYSTEM.

b. IS THERE ANY CHOICE OF PARAMETERS k AND/OR a FOR WHICH THIS REALIZATION LOSES CONTROLLABILITY OR OBSERVABILITY OR BOTH?



SOLUTION: a. *Use the replacements as those in problem #4 in homework set NO: 2,*

Redraw the system block diagram as below for both $a \neq 0$ & $a = 0$ separately.



From the diagram above, we have the equations, ($a \neq 0$ & $a = 0$)

$$
\begin{cases}
\dot{x}_1 = u - x_3 = -x_3 + u \\
\dot{x}_2 = 3(-\tfrac{2}{3}x_1 - x_2) = -2x_1 - 3x_2 \\
\dot{x}_3 = a(\tfrac{k}{a}x_1 + \tfrac{k}{a}x_2 - x_3) = kx_1 + kx_2 - ax_3
\end{cases}
$$

$$y = x_1 + x_2$$

$$
\therefore \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -2 & -3 & 0 \\ k & k & -a \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot u , \quad y = [1 \quad 1 \quad 0] \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
$$

OR
$$
A = \begin{bmatrix} 0 & 0 & -1 \\ -2 & -3 & 0 \\ k & k & -a \end{bmatrix} , \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \quad c = [1 \quad 1 \quad 0]
$$

✓

PROBLEM 1: (a) b. From the state variable realization in part a., we have the Controllability matrix and Observability matrix:

$$\mathcal{C} = [b, Ab, A^2b] = \begin{bmatrix} 1 & 0 & -k \\ 0 & -2 & 6 \\ 0 & k & -2k-ak \end{bmatrix}$$

$$\det \mathcal{C} = 4k + 2ak - 6k = 2k\cdot(a-1)$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -3 & -1 \\ 6-k & 9-k & 2+a \end{bmatrix}$$

$$\det \mathcal{O} = -6 - 3a - 6 + k + 4 + 2a + 9 - k = 1 - a$$

So, WHEN $a=1$, THE SYSTEM REALIZATION LOSES BOTH CONTROLLABILITY AND OBSERVABILITY.

WHEN $k=0$ & $a \neq 1$, THE SYSTEM ONLY LOSES CONTROLLABILITY.

FOR $k \neq 0$ AND $a \neq 1$, THE REALIZATION IS BOTH C.C. & C.O.

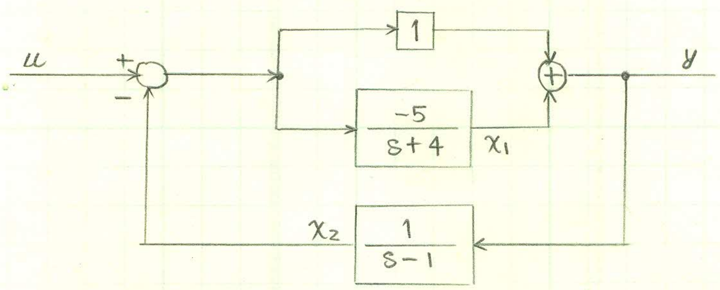
PROBLEM 1. (b) PAGE 111 IN TEXT ONE 2.3-4.

CHOOSE STATE VARIABLES AS SHOWN FOR THE SYSTEM SHOWN IN THE FIGURE.

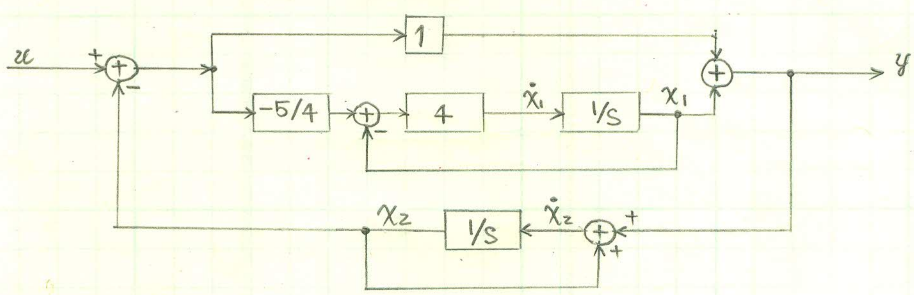
a. WRITE THE STATE EQUATIONS.

b. IS THIS SYSTEM REALIZATION CONTROLLABLE? OBSERVABLE?

c. WHAT IS THE TRANSFER FUNCTION FROM $u(\cdot)$ TO $y(\cdot)$?



SOLUTION: Use some replacements. It will ~~make us much~~ be easier to write the state equations. So, redraw the block diagram as:



a. THE STATE EQUATIONS

$$\begin{cases} \dot{x}_1 = 4(-x_1 + 5/4\, x_2 - 5/4\, u) = -4x_1 + 5x_2 - 5u \\ \dot{x}_2 = x_2 + x_1 + u - x_2 = x_1 + u \end{cases}$$

$$y = x_1 + u - x_2 = x_1 - x_2 + u$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -5 \\ 1 \end{pmatrix} u, \quad y = [1 \ \ -1] \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u \quad ✓$$

b. $C = [b, Ab] = \begin{pmatrix} -5 & 25 \\ 1 & -5 \end{pmatrix}$, $\det C = 0$; $\mathcal{O} = \begin{bmatrix} c \\ cA \end{bmatrix} = \begin{pmatrix} 1 & -1 \\ -5 & 5 \end{pmatrix}$, $\det \mathcal{O} = 0$

So, the system is neither complete controllabler nor observable.

c. $$Y(s)/U(s) = c(sI - A)^{-1} b + d = 1 + (1 \ \ -1) \begin{pmatrix} s+4 & -5 \\ -1 & s \end{pmatrix}^{-1} \begin{pmatrix} -5 \\ 1 \end{pmatrix}$$

$$= 1 - \frac{6(s-1)}{(s+5)(s-1)} = \frac{s-1}{s+5} \quad ✓$$

pole-zero cancellation

PROBLEM 1. (C) PAGE 113–114, TEXT ONE, 2.3–14

SHOW THAT THE PAIR

$$\left\{ \begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix} \right\}$$

IS CONTROLLABLE IF AND ONLY IF $\{A, b\}$ IS CONTROLLABLE AND

$$\begin{bmatrix} A & b \\ c & 0 \end{bmatrix}$$

HAS FULL RANK.

PROOF. (IF PART): $\{A,b\}$ is controllable, thus (WLOG: SISO, $A = n \times n$, SO, $\begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}$: $(n+1)\times(n+1)$)

$$\rho \mathcal{C}_{Ab} = \rho\,[b, Ab, \cdots, A^{n-1}b] = n \quad \text{and} \quad \rho \begin{bmatrix} A & b \\ c & 0 \end{bmatrix} = n+1$$

$$\mathcal{C}\left\{\begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}\right\} = \left[\begin{pmatrix} b \\ 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} b \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} A & 0 \\ c & 0 \end{pmatrix}^n \begin{pmatrix} b \\ 0 \end{pmatrix} \right],$$

$$= \begin{bmatrix} b & Ab & A^2b & \cdots & A^n b \\ 0 & cb & cAb & \cdots & cA^{n-1}b \end{bmatrix}$$

$$= \begin{bmatrix} b & A(b, Ab, \cdots A^{n-1}b) \\ 0 & c(b, Ab, \cdots A^{n-1}b) \end{bmatrix} = \begin{bmatrix} b & A\mathcal{C}_{Ab} \\ 0 & c\,\mathcal{C}_{Ab} \end{bmatrix} = \begin{bmatrix} A & b \\ c & 0 \end{bmatrix}\begin{bmatrix} 0 & \mathcal{C}_{Ab} \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} b & A \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \mathcal{C}_{Ab} \end{bmatrix} \quad \text{------} \quad (*)$$

~~Since matrix $\begin{bmatrix} b & A \\ 0 & c \end{bmatrix}$ has the same rank as matrix $\begin{bmatrix} A & b \\ c & 0 \end{bmatrix}$~~ and $\rho \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{C}_{Ab} \end{bmatrix} = n+1$

So, $\rho\, \mathcal{C}\left\{\begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}\right\} = n+1$. $\left\{\begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}\right\}$ is controllable.

(ONLY IF PART): From the equation (*) above,

$$\mathcal{C}\left\{\begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}\right\} = \begin{bmatrix} b & A \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{C}_{Ab} \end{bmatrix}$$

If $\left\{\begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}\right\}$ is controllable, then $\rho \cdot \mathcal{C}\left\{\begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}\right\} = n+1$

From matrix theory, we have $\rho \begin{bmatrix} b & A \\ 0 & c \end{bmatrix} = n+1$, $\rho \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{C}_{Ab} \end{bmatrix} = n+1$

Furthermore, we have matrix $\begin{bmatrix} A & b \\ c & 0 \end{bmatrix}$ to be full rank, because $\begin{bmatrix} A & b \\ c & 0 \end{bmatrix}$ is $(n+1)\times(n+1)$

& $\rho\, \mathcal{C}_{Ab} = n \Rightarrow \{A, b\}$ is controllable.

Q.E.D.

PROBLEM 1: (d) PAGE 155 IN TEXT ONE. 2.4-3 (De Bra)

AN INVERTED PENDULUM, OF MASS m, IS HINGED AT A. A GYRO WITH SPIN ANGULAR MOMENTUM, h, IS ATTACHED TO THE PENDULUM BUT IS FREE TO ROTATE ABOUT THE PENDULUM AXIS (ANGLE ϕ) AS SHOWN IN FIGURE. A CONTROL TORQUE, Q, CAN BE APPLIED TO THE GYRO FROM THE PENDULUM. THE EQUATIONS OF MOTION ARE

$$I\ddot{\theta} = mgl\,\theta - h\dot{\phi}$$

AND

$$J\ddot{\phi} = h\dot{\theta} + Q$$

WHERE I = THE MOMENT OF INERTIA OF THE PENDULUM PLUS GYRO ABOUT A.

J = THE MOMENT OF INERTIA OF THE GYRO ABOUT AXIS AC, AND

C = THE MASS CENTER OF THE PENDULUM PLUS GYRO.

a. COMPUTE THE TRANSFER FUNCTIONS FROM $u(\cdot)$ TO $\phi(\cdot)$ AND $u(\cdot)$ TO $\theta(\cdot)$

b. SHOW THAT THE SYSTEM IS CONTROLLABLE BY Q, OBSERVABLE WITH ϕ, AND UNOBSERVABLE WITH θ.

c. SHOW THAT THE SYSTEM IS ALWAYS UNSTABLE.

SOLUTION: From the problem statement, we don't know what is the definition for $u(\cdot)$. But I think it must be same as Q. Yes.

a.
$$I\ddot{\theta} = mgl\,\theta - h\dot{\phi}$$

$$J\ddot{\phi} = h\dot{\theta} + u \qquad \text{WLOG: LET } \theta(0)=0,\ \phi(0)=0 \quad ✓$$

$$\therefore \begin{cases} I\cdot s^2\,\theta(s) = mgl\,\theta(s) - hs\cdot\phi(s) & \cdots\cdots (1) \\ J\cdot s^2\,\phi(s) = h\cdot s\,\theta(s) + u(s) & \cdots\cdots (2) \end{cases}$$

From Eq. (1), $\phi(s) = \dfrac{mgl - Is^2}{h\cdot s}\cdot\theta(s) \quad \cdots\cdot (3)$

(2) & (3), we have $\dfrac{J\cdot s\,(mgl - Is^2)}{h}\,\theta(s) = h\cdot s\cdot\theta(s) + u(s)$

$$\therefore \frac{\theta(s)}{u(s)} = \frac{h}{-IJ\cdot s^3 + (Jmgl - h^2)\cdot s}$$

$$\frac{\phi(s)}{u(s)} = \frac{I\cdot s^2 - mgl}{IJ\cdot s^4 - (Jmgl - h^2)\cdot s^2} \quad ✓$$

PROBLEM 1: (d) b. ~~WE have to LINEARIZE THE MOTION EQUATIONS~~

LET $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = \phi$, $x_4 = \dot{\phi}$

$$\dot{x}_1 = \dot{\theta} = x_2$$

$$\dot{x}_2 = \ddot{\theta} = mgl/I \cdot \theta - h/I \cdot \dot{\phi} = mgl/I \cdot x_1 - h/I \cdot x_4$$

$$\dot{x}_3 = \dot{\phi} = x_4$$

$$\dot{x}_4 = \ddot{\phi} = h/J\ \dot{\theta} + Q = h/J \cdot x_2 + Q/J$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ mgl/I & 0 & 0 & -h/I \\ 0 & 0 & 0 & 1 \\ 0 & h/J & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/J \end{bmatrix} \cdot Q$$

$$\phi = x_3 = [\,0\ \ 0\ \ 1\ \ 0\,] \cdot X \triangleq C_\phi \cdot X$$

$$\theta = x_1 = [\,1\ \ 0\ \ 0\ \ 0\,] \cdot X \triangleq C_\theta \cdot X$$

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & -h/I\cdot J & 0 \\ 0 & -h/I\cdot J & 0 & -mglh/I^2J + h^3/I^2J^2 \\ 0 & 1/J & 0 & -h^2/IJ^2 \\ 1/J & 0 & -h^2/IJ^2 & 0 \end{bmatrix}$$

$$\det \mathcal{C} = h/IJ^3\ (h^3/I^2J^2 - mglh/I^2J) - h^4/I^3J^5 = -\frac{mglh^2}{I^3J^4} \neq 0$$

So, the system is controllable by the input OR controller Q.

If the output is ϕ ,

$$\mathcal{O}_\phi = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & h/J & 0 & 0 \\ mglh/IJ & 0 & 0 & -h^2/IJ \end{bmatrix}, \quad \det \mathcal{O}_\phi = -\frac{mglh^2}{IJ^2} \neq 0$$

So, the system is observable with ϕ .

If the output is θ ,

$$\mathcal{O}_\theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ mgl/I & 0 & 0 & -h/I \\ 0 & mgl/I - h^2/IJ & 0 & 0 \end{bmatrix}, \quad \det \mathcal{O}_\theta = 0$$

So, the system unobservable with θ .

Q.E.D.

PROBLEM 1 : (d) C. FROM the state equations, we have

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ mgl/I & 0 & 0 & -h/I \\ 0 & 0 & 0 & 1 \\ 0 & h/J & 0 & 0 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ -mgl/I & \lambda & 0 & h/I \\ 0 & 0 & \lambda & -1 \\ 0 & -h/J & 0 & \lambda \end{vmatrix}$$

$$= \lambda \cdot \begin{vmatrix} \lambda & 0 & h/I \\ 0 & \lambda & -1 \\ -h/J & 0 & \lambda \end{vmatrix} + \begin{vmatrix} -mgl/I & 0 & h/I \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{vmatrix}$$

$$= \lambda \cdot (\lambda^3 + h^2/I \cdot J \cdot \lambda) - mgl/I \cdot \lambda^2$$

$$= \lambda^2 \cdot (\lambda^2 + h^2/I \cdot J - mgl/I)$$

$$= \lambda^2 \cdot \left(\lambda^2 - \frac{mgl \cdot J - h^2}{I \cdot J}\right)$$

According the definition and theorem on page 177 IN text one :

A system realization will be stable if and only if

$$\mathrm{Re}[\lambda_i(A)] < 0$$

However, in this particular problem, at least $\exists$ three e.v. of A

$$\mathrm{Re}[\lambda_i(A)] \geq 0$$

So, the system is always unstable. Q.E.D.

$\lambda = 0, 0,$

PROBLEM 2: GIVEN AN UNCONTROLLABLE PAIR $\{A, B\}$, WHERE

$$A = \begin{bmatrix} -3 & -3 & 1 & 0 \\ 26 & 36 & -3 & -25 \\ 30 & 39 & -2 & -27 \\ 30 & 43 & -3 & -32 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 3 \\ -2 & -1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$$

COMPUTE P S.T. $PAP^{-1} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}$ AND $PB = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}$

(a) WITHOUT THE USE OF SVD.

(b) WITH THE AID OF SVD

ALSO COMPUTE $\hat{A}_{11}$, $\hat{A}_{12}$, $\hat{A}_{22}$ AND $\hat{B}_1$ FOR (a) & (b). IF THE ANSWERS ARE NOT THE SAME, EXPLAIN THEM.

SOLUTION: a. Use MATLAB, we have the controllability matrix $\mathcal{C}$.

$$\mathcal{C} = \begin{bmatrix} 3 & 3 & -3 & -3 & 3 & 3 & -3 & -3 \\ -2 & -1 & 6 & 8 & 2 & 6 & 14 & 22 \\ 0 & 3 & 12 & 18 & 12 & 24 & 36 & 60 \\ 0 & 1 & 4 & 6 & 4 & 8 & 12 & 20 \end{bmatrix}$$

rank $\mathcal{C} = 2$, so, we choose Q as below

$$Q = \begin{bmatrix} 3 & 3 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

And then $P_a = Q^{-1}$

$$P_a = \begin{bmatrix} -0.3333 & -1.0000 & 0.0000 & 0 \\ 0.6667 & 1.0000 & -0.0000 & 0 \\ -2.0000 & -3.0000 & 1.0000 & 0.0000 \\ -0.6667 & -1.0000 & 0 & 1.0000 \end{bmatrix}$$

$$\hat{A}_a = P_a A P_a^{-1} = \begin{bmatrix} -5.0000 & -7.0000 & 2.6667 & 25.0000 \\ 4.0000 & 6.0000 & -2.3333 & -25.0000 \\ -0.0000 & 0.0000 & 5.0000 & 48.0000 \\ -0.0000 & -0.0000 & -0.6667 & -7.0000 \end{bmatrix} \qquad \hat{B}_a = P_a B = \begin{bmatrix} 1.0000 & -0.0000 \\ -0.0000 & 1.0000 \\ 0.0000 & -0.0000 \\ 0.0000 & 0 \end{bmatrix}$$

$$\hat{A}_{11} = \begin{bmatrix} -5 & -7 \\ 4 & 6 \end{bmatrix} \qquad \hat{A}_{12} = \begin{bmatrix} 2.6667 & 25 \\ -2.333 & -25 \end{bmatrix}$$

$$\hat{A}_{22} = \begin{bmatrix} 5 & 48 \\ -0.6667 & -7 \end{bmatrix} \qquad \hat{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

✓

PROBLEM 2: b. Compute $@ = U \Sigma V'$ in MATLAB, we have

$$P_b = U' = \begin{bmatrix} 0.0405 & -0.3257 & -0.8961 & -0.2987 \\ 0.8430 & -0.4964 & 0.1966 & 0.0655 \\ 0.2528 & 0.3793 & -0.3927 & 0.7988 \\ -0.4731 & -0.7097 & 0.0639 & 0.5181 \end{bmatrix}$$

And then compute $\hat{A} = P_b A P_b^{-1}$, $\hat{B} = P_b B$

$$\hat{A}_b = P_b A P_b^{-1} = \begin{bmatrix} 1.7901 & -4.3748 & -1.8443 & 85.2917 \\ -0.1339 & -0.7901 & -2.3610 & 13.3221 \\ 0.0000 & -0.0000 & -1.0507 & -45.5507 \\ 0.0000 & -0.0000 & -0.0878 & -0.9493 \end{bmatrix}$$

$$\hat{B}_b = P_b B = \begin{bmatrix} 0.7727 & -2.5401 \\ 3.5218 & 3.6808 \\ 0.0000 & 0.0000 \\ -0.0000 & -0.0000 \end{bmatrix}$$

So,

$$\hat{A}_{11} = \begin{bmatrix} 1.7901 & -4.3748 \\ -0.1339 & -0.7901 \end{bmatrix} \qquad \hat{A}_{12} = \begin{bmatrix} -1.8443 & 85.2917 \\ -2.3610 & 13.3221 \end{bmatrix}$$

$$\hat{A}_{22} = \begin{bmatrix} -1.0507 & -45.5507 \\ -0.0878 & -0.9493 \end{bmatrix} \qquad \hat{B}_1 = \begin{bmatrix} 0.7727 & -2.5401 \\ 3.5218 & 3.6808 \end{bmatrix}$$

✓

EXPLAIN: It is easy to see that the answers in (a) & (b) are different. And they don't have to be the same. However, there exists some relationships between $\hat{A}_a$ and $\hat{A}_b$. We can find a nonsingular matrix T, s.t. $\hat{A}_a = T \hat{A}_b T^{-1}$. We know

$$\hat{A}_a = P_a A P_a^{-1} \Rightarrow A = P_a^{-1} \hat{A}_a P_a$$

$$\hat{A}_b = P_b A P_b^{-1} \Rightarrow A = P_b^{-1} \hat{A}_b P_b$$

$$\therefore \quad P_a^{-1} \hat{A}_a P_a = P_b^{-1} \hat{A}_b P_b \Rightarrow \hat{A}_a = (P_a P_b^{-1}) \cdot \hat{A}_b \cdot (P_a P_b^{-1})^{-1}$$

$\Rightarrow \quad T = P_a P_b^{-1}$, and $T^{-1} \hat{B}_a = P_b \cdot P_a^{-1} \hat{B}_a = P_b \cdot B = \hat{B}_b$. From MATLAB, we have

$$T = P_a \cdot P_b^{-1} = \begin{bmatrix} 0.3122 & 0.2154 & -0.4636 & 0.8674 \\ -0.2987 & 0.0655 & 0.5478 & -1.0251 \\ 0.0000 & 0.0000 & -2.0362 & 3.1391 \\ -0.0000 & 0.0000 & 0.2509 & 1.5432 \end{bmatrix}$$

Good ✓

Such a T is called similarity transformation.

PROBLEM 3: CONSIDER $\{A, b\}$. GIVEN $\underline{x}(t_0) = \underline{x}_1$ AT $t = t_0$. WE WOULD LIKE TO FIND A PIECE-WISE CONSTANT $u(t)$ SUCH THAT $\underline{x}(t_1) = \underline{x}_2$ AT $t = t_1$.

FOLLOWING THE LECTURE NOTE, WE HAVE

$$\int_{t_0}^{t_1} e^{-A\sigma} b\, u(\sigma) d\sigma = \underline{f} \triangleq e^{-At_1}\underline{x}_2 - e^{-At_0}\underline{x}_1 \qquad (*)$$

(a) VERIFY (*). LET'S PARTITION $[t_0, t_1]$ BY $\Delta = (t_1 - t_0)/n = t_{(i+1)\Delta} - t_{i\Delta}$ AND $u_i(t) = u_i$ ON $[t_{i\Delta}, t_{(i+1)\Delta})$, $i = 0, 1, \cdots, n-1$

(b) STARTING FROM (*) DERIVE A FORMULA WHICH YIELDS A COMPUTATIONAL PROCEDURE TO DETERMINE u_i

(c) USE THE FORMULA IN (b) TO TEST THE EXAMPLE (P. 213, TEXT 1) WITH

$$\underline{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad t_0 = 0, \quad t_1 = 4, \quad \Delta = 1$$

COMPUTE u_0, u_1, u_2, AND u_3

ALSO FINISH THE TABLE FOR STATES:

SOLUTION: (a) From system $\{A, b\}$ and assume $\underline{x}(0) = \underline{x}_0$, we have

$$\underline{x}(t) = e^{At}\underline{x}_0 + \int_0^t e^{A(t-\sigma)} b\, u(\sigma)\, d\sigma$$

$$\therefore \begin{cases} \underline{x}(t_0) = e^{At_0}\underline{x}_0 + \int_0^{t_0} e^{A(t_0-\sigma)} b\, u(\sigma) d\sigma \\ \underline{x}(t_1) = e^{At_1}\underline{x}_0 + \int_0^{t_1} e^{A(t_1-\sigma)} b\, u(\sigma) d\sigma \end{cases}$$

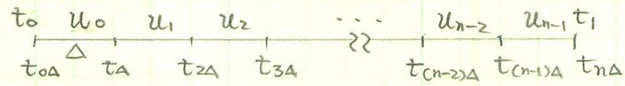
$$\Rightarrow \begin{cases} e^{-At_0}\underline{x}(t_0) = \underline{x}_0 + \int_0^{t_0} e^{-A\sigma} b\, u(\sigma) d\sigma & \cdots (1) \\ e^{-At_1}\underline{x}(t_1) = \underline{x}_0 + \int_0^{t_1} e^{-A\sigma} b\, u(\sigma) d\sigma & \cdots (2) \end{cases}$$

$$(2)-(1) \Rightarrow \int_{t_0}^{t_1} e^{-A\sigma} b\, u(\sigma) d\sigma = e^{-At_1}\underline{x}(t_1) - e^{-At_0}\underline{x}(t_0)$$

$$= e^{-At_1}\underline{x}_2 - e^{-At_0}\underline{x}_1$$

VERIFIED!

PROBLEM 3. (b)



$$\therefore \int_{t_0}^{t_1} e^{-A\sigma} b\, u(\sigma)\, d\sigma = \int_{t_{0\Delta}}^{t_\Delta} e^{-A\sigma} b\, u(\sigma) d\sigma + \int_{t_\Delta}^{t_{2\Delta}} e^{-A\sigma} b\, u(\sigma) d\sigma + \cdots + \int_{t_{(n-1)\Delta}}^{t_{n\Delta}} e^{-A\sigma} b\, u(\sigma) d\sigma$$

$$= \left[\int_{t_{0\Delta}}^{t_\Delta} e^{-A\sigma} d\sigma\right] \cdot b u_0 + \left[\int_{t_\Delta}^{t_{2\Delta}} e^{-A\sigma} d\sigma\right] \cdot b u_1 + \cdots + \left[\int_{t_{(n-1)\Delta}}^{t_{n\Delta}} e^{-A\sigma} d\sigma\right] \cdot b \cdot u_{n-1}$$

Let $\Gamma_i = \int_{t_{i\Delta}}^{t_{(i+1)\Delta}} e^{-A\sigma} d\sigma$, so,

$$\int_{t_0}^{t_1} e^{-A\sigma} b \cdot u(\sigma) d\sigma = \Gamma_0 \cdot b \cdot u_0 + \Gamma_1 \cdot b \cdot u_1 + \cdots + \Gamma_{n-1} \cdot b \cdot u_{n-1}$$

$$= [\Gamma_0 b \;\; \Gamma_1 \cdot b \;\; \cdots \;\; \Gamma_{n-1} b] \cdot \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}$$

$$= e^{-At_1} \underline{x_2} - e^{-At_0} \underline{x_1} \qquad \cdots\cdots \quad (**)$$

Now we are going to derive a computational formula for Γ_i, $i=0,1,\dots,n-1$

PRE-multiply A to both sides of equation $(**)$, we have

$$[A\Gamma_0 b \;\; A\Gamma_1 b \;\; \cdots \;\; A\Gamma_{n-1} b] \cdot \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix} = A\left(e^{-At_1} \underline{x_2} - e^{-At_0} \underline{x_1}\right) \cdots (***)$$

$$A\Gamma_i = A\int_{t_{i\Delta}}^{t_{(i+1)\Delta}} e^{-A\sigma} d\sigma = -\int_{t_{i\Delta}}^{t_{(i+1)\Delta}} -Ae^{-A\sigma} d\sigma$$

$$= -\int_{t_{i\Delta}}^{t_{(i+1)\Delta}} d e^{-A\sigma}$$

$$= -\left(e^{-At_{(i+1)\Delta}} - e^{-At_{i\Delta}}\right)$$

$$= -e^{-At_{i\Delta}} \cdot \left[e^{-A(t_{(i+1)\Delta} - t_{i\Delta})} - I\right]$$

$$= e^{-At_{i\Delta}} \cdot (I - e^{-A\Delta})$$

$$[Ae^{At} = e^{At}A] \rightarrow \quad = (I - e^{-A\Delta}) \cdot e^{-At_{i\Delta}} \qquad \text{-----} \quad (****)$$

Rewrite $(***)$ with $A\Gamma_i$, $i=0,1,\dots,n-1$, replaced by $(****)$, we have

$$(I - e^{-A\Delta}) \cdot \left[e^{-At_0} b,\; e^{-At_\Delta} b,\; \dots,\; e^{-At_{(n-1)\Delta}} b\right] \cdot \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}$$

$$= A \cdot \left(e^{-At_1} \underline{x_2} - e^{-At_0} \underline{x_1}\right)$$

PROBLEM 3 (b) (CONT.)

FORMULA 1: $$[e^{-At_0}b, e^{-At_\Delta}b, \dots, e^{-At_{(n-1)\Delta}}b]\cdot\begin{bmatrix}u_0\\u_1\\\vdots\\u_{n-1}\end{bmatrix} = (I-e^{-A\Delta})^{-1}\cdot A\cdot[e^{-At_1}x_2 - e^{-At_0}x_1]$$

IF $\rho\,[e^{-At_0}b, e^{-At_\Delta}b, \dots, e^{-At_{(n-1)\Delta}}b] = n$, and $A: n\times n$, then

FORMULA 2: $$\begin{bmatrix}u_0\\u_1\\\vdots\\u_{n-1}\end{bmatrix} = [e^{-At_0}b, e^{-At_\Delta}b, \dots, e^{-At_{(n-1)\Delta}}b]^{-1}\cdot(I-e^{-A\Delta})^{-1}\cdot A\cdot[e^{-At_1}x_2 - e^{-At_0}x_1]$$

Otherwise FORMULA 1 may have infinite pairs of solution.

PROBLEM 3 (c): The example on page 213 in text one has

$$A = \begin{bmatrix}0 & 1 & 0 & 0\\ 2.5 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -2.5 & 0 & -2 & 0\end{bmatrix}, \quad b = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$$

$$x_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad x_2 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \quad t_0=0, \; t_1=4, \; \Delta=1$$

From Formula 2 & MATLAB, WE HAD $u_0=-6.6775$, $u_1=5.7554$, $u_3=-10.2322$, $u_3=2.3629$

USE the results in H.W. NO:1 PROBLEM 2 & USE CONTROL TOOLBOX to compute Φ, Γ and then:

the table:

TIME	x_1	x_2	x_3	x_4
0.00	1.0000	1.0000	0.0000	0.0000
0.25	1.1250	0.0179	0.1244	0.9716
0.50	1.0143	-0.9004	0.4755	1.8194
0.75	0.6725	-1.8465	1.0269	2.5817
1.00	0.0803	-2.9217	1.7635	3.3118
1.25	-0.4177	-1.0923	2.2969	0.9626
1.50	-0.4751	0.6255	2.2491	-1.3353
1.75	-0.1038	2.3580	1.6350	-3.5650
2.00	0.7162	4.2385	0.4729	-5.7202
2.25	1.5375	2.3682	-0.7200	-3.8090
2.50	1.9113	0.6321	-1.4229	-1.7946
2.75	1.8525	-1.1169	-1.6069	0.3442
3.00	1.3401	-3.0202	-1.2398	2.6152
3.25	0.7132	-2.0369	-0.6818	1.8682
3.50	0.3048	-1.2573	-0.2986	1.2077
3.75	0.0744	-0.6000	-0.0740	0.5938
4.00	0.2342×10^{-11}	0.3156×10^{-11}	-0.1566×10^{-11}	-0.2019×10^{-11}

LINEAR SYSTEM THEORY

EE 501

HOMEWORK SET NO:4

NOV. 15,87

Benmei Chen
Box 241, Gonzaga University
Spokane, WA 99258
(509) 327-6071

Dr. Hsu:
I didn't have grade
for Homework #4.
Ben

10/10 HS

PROBLEM 1: (a) PROBLEM 2.2-17, P75, TEXT 1.

2.2-17. More Resolvent Identities

a. Show, when all inverses exist, that

$$(sI-A)^{-1}-(sI-B)^{-1}=(sI-A)^{-1}(A-B)(sI-B)^{-1} \quad \cdots\cdots (*)$$

and

$$(sI-A)^{-1}-(\nu I-A)^{-1}=(sI-A)^{-1}(\nu-s)(\nu I-A)^{-1} \quad \cdots\cdots (**)$$

SHOW: (i) $(sI-A)^{-1}(A-B)(sI-B)^{-1}=(sI-A)^{-1}[(sI-B)-(sI-A)]\cdot(sI-B)^{-1}$

$$=(sI-A)^{-1}-(sI-A)^{-1}$$ SHOWED (*)

(ii) $(sI-A)^{-1}(\nu-s)(\nu I-A)^{-1}=(sI-A)^{-1}(\nu-s)\cdot I\cdot(\nu I-A)^{-1}$

$$=(sI-A)^{-1}(\nu I-sI)(\nu I-A)^{-1}$$

$$=(sI-A)^{-1}\cdot[(\nu I-A)-(sI-A)]\cdot(\nu I-A)^{-1}$$

$$=(sI-A)^{-1}-(\nu I-A)^{-1}$$ SHOWN (**) Q.E.D

---

b. Use the above results to show that for a realization $\{A, b, c\}$ with

$$u(t)=e^{\nu t}\cdot 1(t)$$

the output can be written as

$$\mathcal{L}[y(t)]=c(sI-A)^{-1}[x_0-(\nu I-A)^{-1}b]+c(\nu I-A)^{-1}(s-\nu)^{-1}b$$

SHOW: FROM EQ. (H-2), P625, IN TEXT 2, we have

$$y(s)=c(sI-A)^{-1}x_0+c(sI-A)^{-1}b\,u(s)$$

$$u(t)=e^{\nu t}1(t), \quad \text{SO,} \quad u(s)=(s-\nu)^{-1}$$

$$\therefore\ y(s)=c(sI-A)^{-1}x_0+c(sI-A)^{-1}b\cdot(s-\nu)^{-1}=c(sI-A)^{-1}x_0+c(sI-A)^{-1}(s-\nu)^{-1}b$$

From (**) in (a), we see that, $(sI-A)^{-1}=(\nu I-A)^{-1}-(sI-A)^{-1}(s-\nu)(\nu I-A)^{-1}$

$$\therefore\ \mathcal{L}[y(t)]=y(s)=c(sI-A)^{-1}x_0+c\cdot[(\nu I-A)^{-1}-(sI-A)^{-1}(s-\nu)(\nu I-A)^{-1}]\cdot(s-\nu)^{-1}\cdot b$$

$$=c(sI-A)^{-1}x_0-c(sI-A)^{-1}\cdot(\nu I-A)^{-1}\cdot b+c(\nu I-A)^{-1}\cdot(s-\nu)^{-1}\cdot b$$

$$=c\cdot(sI-A)^{-1}\cdot[x_0-(\nu I-A)^{-1}\cdot b]+c(\nu I-A)^{-1}\cdot(s-\nu)^{-1}b$$ Q.E.D

ok ✓

PROBLEM 1 : a. (c)

C. Show that a necessary and sufficient condition for an input $u(t) = e^{zt} g$, $t \geq 0$, to yield $y(t) \equiv 0$, $t \geq 0$, is that exist $\{x_0, g\}$ such that

$$\begin{bmatrix} zI - A & -b \\ c & 0 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Show that in this case we also have

$$x(t) = e^{zt} x_0 \quad , \quad t \geq 0$$

SHOW: From (b), we have (in this problem $u(t) = e^{zt} \cdot g$, $\therefore u(s) = g \cdot (s-z)^{-1}$)

$$y(s) = c(sI - A)^{-1} x_0 + c(sI - A)^{-1} \cdot b \cdot u(s)$$

$$= c(sI - A)^{-1} x_0 + c(sI - A)^{-1} \cdot b \cdot g(s-z)^{-1}$$

same as (b) $\longrightarrow$ $= c \cdot (sI - A)^{-1} \cdot [x_0 - (zI - A)^{-1} \cdot b \cdot g] + c \cdot (zI - A)^{-1} \cdot b \cdot g \cdot (s-z)^{-1}$ ...... (Δ)

(i) "necessary condition" : if

$$\begin{bmatrix} zI - A & -b \\ c & 0 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad , \quad \text{then}$$

$$(zI - A) x_0 - b \cdot g = 0 \quad , \quad c x_0 = 0 \quad .$$

Thus, $\quad x_0 = (zI - A)^{-1} \cdot b \cdot g \quad , \quad c \cdot (zI - A)^{-1} \cdot b \cdot g = 0$

Look at eq. (Δ), we have

$$y(s) = c \cdot (sI - A)^{-1} \cdot [x_0 - x_0] + 0 \cdot (s-z)^{-1} = 0 \Rightarrow y(t) \equiv 0 \ , t \geq 0 .$$

(ii) "sufficient condition" : $y(t) \equiv 0$, $t \geq 0$, thus $y(s) \equiv 0$.

Let $x_0 = (zI - A)^{-1} \cdot b \cdot g \quad , \quad \therefore \quad (zI - A) x_0 - b \cdot g = 0$

From equation (Δ) above, we have

$$y(s) = c(sI - A)^{-1} \cdot [x_0 - (zI - A)^{-1} \cdot b \cdot g)] + c \cdot (zI - A)^{-1} \cdot b \cdot g \cdot (s-z)^{-1}$$

$$= c(sI - A)^{-1} \cdot [x_0 - x_0] + c \cdot x_0 \cdot (s-z)^{-1} = c \cdot x_0 \cdot (s-z)^{-1} = 0 .$$

$\therefore \quad c x_0 = 0 \quad , \Rightarrow \exists \ x_0 = (zI - A)^{-1} \cdot b \cdot g \quad$ s.t. $\begin{bmatrix} zI - A & -b \\ c & 0 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Q.E.D. the first part.

TO BE CONTINUED!

O.K.

PROBLEM 1. a. (c) (CONT.)

$$\dot{x} = Ax + bu$$

$$\therefore \quad sX(s) - x_0 = AX(s) + bu(s)$$

$$X(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}\cdot b\cdot u(s)$$

In this problem, $u(t) = e^{zt}\cdot g$, $x_0 = (zI-A)^{-1}b\cdot g$

$$\therefore \quad X(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}\cdot b\cdot g\cdot(s-z)^{-1}$$

From (a), we have

$$(sI-A)^{-1} = (zI-A)^{-1} - (sI-A)^{-1}\cdot(s-z)\cdot(zI-A)^{-1}$$

$$\therefore \quad X(s) = (sI-A)^{-1}x_0 + [(zI-A)^{-1} - (sI-A)^{-1}\cdot(s-z)\cdot(zI-A)^{-1}]\cdot b\cdot g(s-z)^{-1}$$

$$= (sI-A)^{-1}\cdot x_0 - (sI-A)^{-1}\cdot(zI-A)^{-1}\cdot b\cdot g + (zI-A)^{-1}\cdot b\cdot g\cdot(s-z)^{-1}$$

$$= (sI-A)^{-1}\cdot[x_0 - (zI-A)^{-1}\cdot b\cdot g] + (zI-A)^{-1}\cdot b\cdot g\cdot(s-z)^{-1}$$

$$= 0 + x_0\cdot(s-z)^{-1}$$

$$= (s-z)^{-1}x_0$$

$$x(t) = \mathcal{L}^{-1}[(s-z)^{-1}\cdot x_0] = e^{zt}\cdot x_0$$ Q.E.D

o.K.

PROBLEM 1. (b) PROBLEM 2.2-18 PP 75-76, TEXT 1.

2.2-18. *Dynamical Interpretation of Poles and Zeros.*

a. Let $H(s) = c(sI-A)^{-1}\cdot b$, with $a(s) = \det(sI-A)$, $b(s) = C\cdot \mathrm{Adj}(sI-A)\cdot b$ being coprime. Suppose ν is not an eigenvalue of A. Show that there exists an initial state x_0 such that the response to $u(t) = e^{\nu t}\cdot 1(t)$ is $y(t) = H(\nu)e^{\nu t}\cdot 1(t)$.

SHOW: From (a) b. (problem 2.2.-18), we have (ν is not e.v. of A $\therefore (\nu I - A)^{-1}$ exists)

$$y(s) = c(sI-A)^{-1}\cdot[x_0-(\nu I-A)^{-1}\cdot b] + C(\nu I-A)^{-1}\cdot b\cdot(s-\nu)^{-1}$$

Because $u(t) = e^{\nu t}\cdot 1(t)$, thus $u(s) = (s-\nu)^{-1}$

$$y(t) = H(\nu)\cdot e^{\nu t}\cdot 1(t), \quad y(s) = H(\nu)\cdot(s-\nu)^{-1} = c(\nu I - A)^{-1}\cdot b\cdot(s-\nu)^{-1}$$

So, we have that

$$C\cdot(\nu I-A)^{-1}\cdot b\cdot(s-\nu)^{-1} = c(sI-A)^{-1}[x_0-(\nu I-A)^{-1}b] + C(\nu I-A)^{-1}\cdot b\cdot(s-\nu)^{-1}$$

$$C(sI-A)^{-1}\cdot[x_0-(\nu I-A)^{-1}\cdot b] = 0$$

From the equation above, it is easy to see that

$$x_0 = (\nu I - A)^{-1}\cdot b$$

Satisfies the condition. But not necessary unique unless $\{A, B\}$ is C.O. Q.E.D

---

b. *What happens if ν is a zero of $H(s)$?*

SHOW: If ν is a zero of $H(s)$. thus $H(\nu) = 0$

$$y(t) = H(\nu)\cdot e^{\nu\cdot t}\cdot 1(t) = 0 \qquad t \geq 0$$

But there still exists a initial state. (Because $(\nu I - A)^{-1}$ still exists)

$$x_0 = (\nu I - A)^{-1} b$$

Such that the response to $u(t) = e^{\nu t}\cdot 1(t)$ is $y(t) \equiv 0$, $t \geq 0$.

So, the statement in a. is still true.

O.K.

PROBLEM 1(b) C.

C. Suppose v is an eigenvalue of A and therefore a pole of $H(s)$. Show that there exists an initial state x_0 that with no input $[u(t)=0]$ the response $y(t)$ has the form $\alpha \cdot e^{v \cdot t} \cdot 1(t)$, α = some constant. Assume A has distinct eigenvalues.

SHOW: $y(t)=\alpha \cdot e^{v \cdot t} \cdot 1(t)$, then

$u(t)=0$ so, we have

$$\underline{X}(t)=e^{At}\cdot x_0+\int_0^t e^{A(t-s)}\cdot b\cdot u(s)ds$$

$$=e^{At}\cdot x_0$$

$$y(t)=\alpha \cdot e^{v\cdot t}=cX(t)=ce^{At}\cdot x_0$$

Because A has distinct eigenvalues, it had been in the lecture 9-15-87, that $\exists$ matrix P, and $|P|\neq 0$ s.t.

$$P^{-1}AP=\begin{bmatrix} v & & & 0 \\ & v_2 & & \\ & & \ddots & \\ 0 & & & v_n \end{bmatrix} \triangleq \Sigma \quad \text{OR} \quad A=P\Sigma P^{-1} \qquad \therefore e^{At}=Pe^{\Sigma t}P^{-1}$$

$$\therefore \alpha \cdot e^{v\cdot t}=C\cdot P\cdot e^{\Sigma t}\cdot P^{-1}\cdot x_0$$

$$=C\cdot P\cdot\begin{bmatrix} e^{vt} & & & 0 \\ & e^{v_2t} & & \\ & & \ddots & \\ 0 & & & e^{v_nt} \end{bmatrix} P^{-1}\cdot x_0$$

$$\therefore \alpha = C\cdot P\cdot\begin{bmatrix} 1 & & & 0 \\ & e^{(v_2-v)t} & & \\ & & \ddots & \\ 0 & & & e^{(v_n-v)t} \end{bmatrix}\cdot P^{-1}\cdot x_0$$

LET $P=[P_1, P_2, \cdots, P_n]$, then we check eq.(1) below will satisfy the statement,

$$x_0=P\cdot\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\cdot\frac{\alpha}{CP_1} \quad \text{-----} \quad (1)$$

CHECK:

$$y(t)=C\cdot e^{At}\cdot x_0=C\cdot P\cdot e^{\Sigma t}\cdot P^{-1}\cdot P\cdot\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\cdot\frac{\alpha}{CP_1}$$

$$=C\cdot P\cdot\begin{bmatrix} e^{vt} & & & 0 \\ & e^{v_2t} & & \\ & & \ddots & \\ 0 & & & e^{v_nt} \end{bmatrix}\cdot\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\cdot\frac{\alpha}{CP_1}$$

$$=C\cdot[P_1, P_2, \cdots, P_n]\cdot\begin{bmatrix}e^{vt}\\0\\ \vdots\\0\end{bmatrix}\cdot\frac{\alpha}{CP_1}$$

$$=CP_1\cdot e^{vt}\cdot\alpha/CP_1=\alpha e^{vt} \qquad (CP_1 \text{ is scalar})$$

O.K.

Q.E.D.

PROBLEM 1. (c) PROBLEM 2.4-6, P156, TEXT 1.

2.4.-6. *Controllability and Observability of Interconnected Subsystems.*

Let $\{A_i, b_i, C_i, i=1,2\}$ be realizations of order n_i of the transfer functions $H_i(s)=g_i(s)/a_i(s)$, also of order n_i

a. Show that if the realizations are controllable, then the series conbination of system 1 followed by 2 is controllable if and only if $g_1(s)$ and $a_2(s)$ are coprime.

SHOW:

$$S_1 : \begin{cases} \dot{x}_1 = A_1 x_1 + b_1 u \\ y_1 = c_1 x_1 \end{cases}$$

$$S_2 : \begin{cases} \dot{x}_2 = A_2 x_2 + b_2 y_1 \\ y = C_2 x_2 \end{cases}$$

So, we have the series conbination system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ b_2 C_1 & A_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \cdot u$$

$$y = [0 \;\; C_2] \cdot [x_1' \;\; x_2']'$$

$$\therefore \left(sI - \begin{bmatrix} A_1 & 0 \\ b_2 C_1 & A_2 \end{bmatrix}\right)^{-1} \cdot \begin{bmatrix} b_1 \\ 0 \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_1 & 0 \\ -b_2 C_1 & sI_{n_2} - A_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (sI_{n_1}-A_1)^{-1} & 0 \\ (sI_{n_2}-A_2)^{-1} b_2 C_1 (sI_{n_1}-A_1)^{-1} & (sI_{n_2}-A_2)^{-1} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (sI_{n_1}-A_1)^{-1} b_1 \\ (sI_{n_2}-A_2)^{-1} b_2 C_1 (sI_{n_1}-A_1)^{-1} b_1 \end{bmatrix} \quad \cdots\cdots \quad (*)$$

According to Example 2.4-3, and we know both S_1 and S_2 are controllable.

Thus, $$(sI_{n_1} - A_1)^{-1} b_1 = \frac{[P_{11}(s), P_{12}(s), \cdots, P_{1n_1}(s)]'}{a_1(s)}$$

$$(sI_{n_2} - A_2)^{-1} b_2 = \frac{[P_{21}(s), P_{22}(s), \cdots, P_{2n_2}(s)]'}{a_2(s)}$$

and $\{P_{11}(s), P_{12}(s), \cdots, P_{1n_1}(s), a_1(s)\}$ has no nontrivial common factors. $\cdots$ (a)

$\{P_{21}(s), P_{22}(s), \cdots, P_{2n_2}(s), a_2(s)\}$ has no nontrivial common factors. $\cdots$ (b)

also $C_1(sI_{n_1} - A_1)^{-1} b_1 = g_1(s)/a_1(s)$

(TO BE CONTNUED.)

PROBLEM 1. (c) a.(CONT.)

Thus, the equation $(*) = \begin{bmatrix} \dfrac{[P_{11}(s), P_{12}(s), \cdots, P_{1n_1}(s)]'}{a_1(s)} \\ \dfrac{[P_{21}(s), P_{22}(s), \cdots, P_{2n_1}(s)]'}{a_1(s)a_2(s)} \cdot g_1(s) \end{bmatrix}$

$$\det\left(sI - \begin{bmatrix} A_1 & 0 \\ b_2C_1 & A_2 \end{bmatrix}\right) = a_1(s)a_2(s) \rightarrow \quad = \frac{[a_2(s)P_{11}(s), a_2(s)P_{12}(s), \cdots, a_2(s)P_{1n_1}(s), g_1(s)P_{21}(s), g_1(s)P_{22}(s), \cdots, g_1(s)P_{2n_2}(s)]'}{a_1(s)a_2(s)} \cdots (**)$$

☆ From equations (a), (b) and (**) above, and if $a_2(s)$ and $g_1(s)$ ar coprime, then $\{a_2(s)P_{11}(s), a_2(s)P_{12}(s), \cdots a_2(s)P_{1n_1}(s), g_1(s)P_{21}(s), \cdots, g_1(s)P_{2n_2}(s), a_1(s)a_2(s)\}$ has no nontrivial common factors, thus the combined system is controllable.

If the combined system is controllable, then

$$\{a_2(s)P_{11}(s), a_2(s)P_{12}(s), \cdots, a_2(s)P_{1n_1}(s), g_1(s)P_{21}(s), \cdots, g_1(s)P_{2n_2}(s), a_1(s)a_2(s)\}$$

has no nontrivial common factors.

Following we show $a_2(s)$ and $g_1(s)$ are coprime.

Assume $a_2(s)$ and $g_1(s)$ are not coprime, then $\exists\ \lambda$, s.t

$$a_2(\lambda) = 0 \quad \text{and} \quad g_1(\lambda) = 0.$$

The $a_1(\lambda)a_2(\lambda) = 0$ and $[a_2(\lambda)P_{11}(\lambda), \cdots a_2(\lambda)P_{1n_1}(\lambda), g_1(\lambda)P_{21}(\lambda), \cdots, g_1(\lambda)P_{2n_2}(\lambda)] = 0$

Contradiction! So, $a_2(s)$ and $g_1(s)$ are coprime

Q.E.D.

☆: If $g_1(s)$ and $a_2(s)$ are coprime, and for any λ

(i) $a_1(\lambda) = 0$, $a_2(\lambda) = 0$, So, $a_1(\lambda)a_2(\lambda) = 0$ and $g_1(\lambda) \neq 0$

From eq (b), we have $[g_1(\lambda)P_{21}(\lambda), g_1(\lambda)P_{22}(\lambda), \cdots, g_1(\lambda)P_{2n_2}(\lambda)] \neq 0$, Thus

$$[a_2(\lambda)P_{11}(\lambda), \cdots, a_2(\lambda)P_{1n_1}(\lambda), g_1(\lambda)P_{21}(\lambda), \cdots, g_1(\lambda)P_{2n_2}(\lambda)] \neq 0.$$

(ii) $a_1(\lambda) = 0$, $a_2(\lambda) \neq 0$, So, $a_1(\lambda)a_2(\lambda) = 0$.

From eq (a), we have $[a_2(\lambda)P_{11}(\lambda), a_2(\lambda)P_{12}(\lambda), \cdots, a_2(\lambda)P_{1n_1}(\lambda)] \neq 0$, thus

$$[a_2(\lambda)P_{11}(\lambda), \cdots, a_2(\lambda)P_{1n_1}(\lambda), g_1(\lambda)P_{21}(\lambda), \cdots, g_1(\lambda)P_{2n_2}(\lambda)] \neq 0.$$

(iii) $a_1(\lambda) \neq 0$, $a_2(\lambda) = 0$. So, $a_1(\lambda)a_2(\lambda) = 0$ and $g_1(\lambda) \neq 0$.

Same as (i). So, $\{a_2(s)P_{11}(s), \cdots, a_2(s)P_{1n_1}(s), g_1(s)P_{21}(s), \cdots, g_1(s)P_{2n_2}(s), a_1(s)a_2(s)\}$ has no nontrivial comman factors. $\Rightarrow$ combined system controllable.

PROBLEM 1 (c). b.

b. Show that if the realizations are observable (controllable), then the parallel combination is observable (controllable) if and only if $a_1(s)$ and $a_2(s)$ are coprime.

SHOW: $S_1: \begin{cases} \dot{x}_1 = A_1 x_1 + b_1 u \\ y_1 = c_1 x_1 \end{cases}$ $S_2: \begin{cases} \dot{x}_2 = A_2 x_2 + b_2 u \\ y_2 = c_2 x_2 \end{cases}$

The combined system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cdot u$$

$$y = [c_1 \ c_2] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We are going to show the observable case only. The controllable case will be similary.

According to Problem 2.4-9. a. (can be proven in a similary way as that in Example 2.4-3):

$\{c, A\}$ is observable if and only if the $n+1$ $\{g_i(s), a(s)\}$ defined by $c(sI-A)^{-1} = [g_1(s), \cdots, g_n(s)]/a(s)$ have no nontrivial common factors.

Since S_1 and S_2 are observable, we have

$$c_1(sI-A_1)^{-1} = [g_{11}(s), \cdots, g_{1n_1}(s)]/a_1(s) \quad \text{and}$$

(1) $\cdots$ $\{g_{11}(s), g_{12}(s), \cdots, g_{1n_1}(s), a_1(s)\}$ has no nontrivial common factors.

$$c_2(sI-A_2)^{-1} = [g_{21}(s), \cdots, g_{2n_2}(s)]/a_2(s) \quad \text{and}$$

(2) $\cdots$ $\{g_{21}(s), \cdots, g_{2n_2}(s), a_2(s)\}$ has no nontrivial common factors.

For combined system,

$$c(sI-A)^{-1} = [c_1 \ c_2] \cdot \begin{bmatrix} sI_{n_1}-A_1 & 0 \\ 0 & sI_{n_2}-A_2 \end{bmatrix}^{-1}$$

$$= [c_1 \ c_2] \cdot \begin{bmatrix} (sI_{n_1}-A_1)^{-1} & 0 \\ 0 & (sI_{n_2}-A_2)^{-1} \end{bmatrix} = [c_1(sI_{n_1}-A_1)^{-1} \ \ c_2(sI_{n_2}-A_2)^{-1}]$$

$$= [[g_{11}(s), \cdots, g_{1n_1}(s)]/a_1(s), \ [g_{21}(s), \cdots, g_{2n_2}(s)]/a_2(s)]$$

$\det(sI-A) = a_1(s)a_2(s) \longrightarrow$ $= [a_2 g_{11}(s), \cdots, a_2(s) g_{1n_1}(s), a_1(s) g_{21}(s), \cdots, a_1(s) g_{2n_2}(s)]/a_1(s)a_2(s)$

With the same procedures as we did in a., we show the statement. Q.E.D.

PROBLEM 1 (c). c. & d.

c. Show that if the realizations are observable (controllable), then the feedback configuration with system 1 in the forward path and 2 in the feedback path is observable (controllable) if and only if $g_1(s)$ and $a_2(s)$ are coprime.

d. Extend the results to the case of systems with direct feedthrough from input to output.

We are going to show the observable case in c. and controllable case for d. (Omit controllable case in c. and observable in d., because they can be showed with almost the same procedures.)

c. realizations are observable and feedback connection. So

$$S_1: \begin{cases} \dot{x}_1 = A_1 x_1 + b_1(u - y_2) \\ y_1 = c_1 x_1 \end{cases} \qquad S_2: \begin{cases} \dot{x}_2 = A_2 x_2 + b_2 y_1 \\ y_2 = c_2 x_2 \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = A_1 x_1 + b_1 c_2 x_2 + b_1 u \\ y_1 = c_1 x_1 \end{cases} \qquad \Rightarrow \begin{cases} \dot{x}_2 = A_2 x_2 + b_2 c_1 x_1 \\ y_2 = c_2 x_2 \end{cases}$$

The combined system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & b_1 c_2 \\ b_2 c_1 & A_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \cdot u \quad \text{and} \quad y = [c_1 \ \ 0] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$C(sI - A)^{-1} = [c_1 \ \ 0] \cdot \begin{bmatrix} sI_{n_1} - A_1 & -b_1 c_2 \\ -b_2 c_1 & sI_{n_2} - A_2 \end{bmatrix}^{-1}$$

$$= [c_1 \ \ 0] \cdot \begin{bmatrix} \tilde{A}^{-1} + E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix} = [c_1(\tilde{A}^{-1} + E\Delta^{-1}F), \ -c_1 E\Delta^{-1}] \cdots (-)$$

wher $\tilde{A}^{-1} = (sI_{n_1} - A_1)^{-1}$, $\Delta^{-1} = [(sI_{n_2} - A_2) - b_2 c_1 (sI_{n_1} - A_1)^{-1} b_1 c_2]^{-1}$

$$= [(sI_{n_2} - A_2) + b_2 \cdot g_1(s)/a_1(s) \cdot c_2]^{-1}$$

(A-21 on P656) $$= (sI_n - A_2)^{-1} + (sI_{n_2} - A_2)^{-1} b_2 \cdot \left(\frac{a_1(s)}{g_1(s)} - c_2(sI_{n_2} - A_2)^{-1} b_2\right)^{-1} \cdot c_2 (sI - A_2)^{-1}$$

$$= (sI_{n_2} - A_2)^{-1} + (sI_{n_2} - A_2)^{-1} b_2 c_2 (sI_{n_2} - A_2)^{-1} \cdot \frac{g_1(s) a_2(s)}{a_1(s) a_2(s) - g_1(s) g_2(s)}$$

$$E = -(sI_{n_1} - A_1)^{-1} \cdot b_1 c_2$$

$$F = -b_2 c_1 (sI_{n_1} - A_1)^{-1}$$

PROBLEM 1 (c). C (CONT.)

$$\therefore \quad C_1(\tilde{A}^{-1}+E\Delta^{-1}F) = C_1\cdot\Big[(sI_{n_1}-A_1)^{-1}+(sI_{n_1}-A_1)^{-1}\cdot b_1\cdot C_2\cdot \Big((sI_{n_2}-A_2)^{-1}+(sI_{n_2}-A_2)^{-1}b_2C_2(sI_{n_2}-A_2)^{-1}\cdot\frac{g_1(s)a_2(s)}{a_1(s)a_2(s)-g_1(s)g_2(s)}\Big)\cdot b_2C_1(sI_{n_1}-A_1)^{-1}\Big]$$

$$= C_1\cdot\Big[(sI_{n_1}-A_1)^{-1}+(sI_{n_1}-A_1)^{-1}b_1C_2(sI_n-A_2)^{-1}b_2C_1(sI_{n_1}-A_1)^{-1} + (sI_{n_1}-A_1)^{-1}b_1C_2\cdot(sI_{n_2}-A_2)^{-1}b_2C_2(sI_{n_2}-A_2)^{-1}b_2C_1(sI_{n_1}-A_1)^{-1}\cdot\frac{g_1(s)a_2(s)}{a_1(s)a_2(s)-g_1g_2}\Big]$$

$$= C_1\cdot(sI_{n_1}-A_1)^{-1}+\frac{g_1(s)}{a_1(s)}\cdot\frac{g_2(s)}{a_2(s)}\cdot C_1(sI_{n_1}-A_1)^{-1} + \frac{g_1(s)}{a_1(s)}\cdot\frac{g_2(s)}{a_2(s)}\cdot\frac{g_2(s)}{a_2(s)}\cdot\frac{g_1(s)a_2(s)}{a_1(s)a_2(s)-g_1(s)g_2(s)}\cdot C_1(sI_{n_1}-A_1)^{-1}$$

$$= \frac{a_1(s)a_2(s)}{a_1(s)a_2(s)-g_1(s)g_2(s)}\cdot\frac{[g_{11}(s),g_{12}(s),\cdots,g_{1n_1}(s)]}{a_1(s)} \quad \cdots\cdots \quad (=)$$

and

$$-C_1E\Delta^{-1} = C_1(sI_{n_1}-A_1)^{-1}b_1\cdot C_2\Big[(sI_{n_2}-A_2)^{-1}-(sI_{n_2}-A_2)^{-1}b_2C_2(sI_{n_2}-A_2)^{-1}\cdot\frac{g_1a_2}{a_1a_2-g_1g_2}\Big]$$

$$= \frac{g_1(s)}{a_1(s)}\cdot C_2(sI_{n_2}-A_2)^{-1}+\frac{g_1(s)}{a_1(s)}\cdot\frac{g_2(s)}{a_2(s)}\cdot\frac{g_1(s)a_2(s)}{a_1(s)a_2(s)-g_1(s)g_2(s)}\cdot C_2(sI_{n_2}-A_2)^{-1}$$

$$= \frac{a_2(s)g_1(s)}{a_1(s)a_2(s)-g_1(s)g_2(s)}\cdot C_2(sI_{n_2}-A_2)^{-1}$$

$$= \frac{a_2(s)g_1(s)}{a_1(s)a_2(s)-g_1(s)g_2(s)}\cdot\frac{[g_{21}(s),\cdots,g_{2n_2}(s)]}{a_2(s)} \quad \cdots \quad (\equiv)$$

Subtitute equation (=) and (≡) in equation (−), we have

$$C(sI-A)^{-1} = \frac{[a_2(s)g_{11}(s),\cdots,a_2g_{1n_1}(s),\ g_1(s)g_{21}(s),\cdots,g_1(s)g_{2n_2}(s)]}{a_1(s)a_2(s)-g_1(s)g_2(s)} \quad \cdots\cdots \quad (\equiv)$$

Since $C(sI-A)^{-1}b = \dfrac{a_2(s)[g_{11}(s),\cdots,g_{1n_1}(s)]\cdot b_1}{a_1(s)a_2(s)-g_1(s)g_2(s)}$, $\Rightarrow \det(sI-A) = a_1(s)a_2(s)-g_1(s)g_2(s)$

Recall the equation (1) and (2) in b. on page 8 and the equation (≡) above, with almost the same procedure as we did in part a. and b. we showed the statement.

Q.E.D.

PROBLEM 1 (c) d.

We only show the systems are connected as those in case a.

Now,

$$S1 : \begin{cases} \dot{x}_1 = A_1 x_1 + b_1 u \\ y_1 = c_1 x_1 + u \end{cases}$$

$$S2 : \begin{cases} \dot{x}_2 = A_2 x_2 + b_2 y_1 \\ y_2 = c_2 x_2 + y_1 \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = A_2 x_2 + b_2 c_1 x_1 + b_2 u \\ y_2 = c_1 x_1 + c_2 x_2 + u \end{cases}$$

The new system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ b_2 c_1 & A_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cdot u$$

$$y_2 = [c_1 \ c_2] \cdot [x_1 \ x_2]' + u$$

Similar as the results in part a. we have

$$\left(sI - \begin{bmatrix} A_1 & 0 \\ b_2 c_1 & A_2 \end{bmatrix}\right)^{-1} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} (sI_{n_1} - A_1)^{-1} & 0 \\ (sI_{n_2} - A_2)^{-1} b_2 c_1 (sI_{n_1} - A_1)^{-1} & (sI_{n_2} - A_2)^{-1} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \begin{bmatrix} (sI_{n_1} - A_1)^{-1} b_1 \\ (sI_{n_2} - A_2) b_2 c_1 (sI_{n_1} - A_1)^{-1} b_1 + (sI_{n_2} - A_2)^{-1} b_2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{[p_{11}(s), p_{12}(s), \cdots, p_{1n_1}(s)]'}{a_1(s)} \\ \dfrac{[p_{21}(s), p_{22}(s), \cdots, p_{2n_2}(s)]'}{a_1(s)\, a_2(s)} \cdot g_1(s) + \dfrac{[p_{21}(s), \cdots, p_{2n_2}(s)]'}{a_2(s)} \end{bmatrix}$$

$$= \frac{[p_{11}(s) a_2(s), \cdots, p_{1n_1}(s) a_2(s), (a_1(s) + g_1(s)) p_{21}(s), \cdots, [a_1(s) + g_1(s)] p_{2n_2}(s)]'}{a_1(s)\, a_2(s)}$$

So, we have the result: Two subsystem with direct feedthrough from input to output, and if both systems are controllable, then series combination of system 1 followed by 2 is controllable if and only if $a_2(s)$ and $a_1(s) + g_1(s)$ are coprime.

PROBLEM 1 (d): H-7 ON page 635, in TEXT one.

Show that

$$\text{rank}\begin{bmatrix} A-\lambda I & B \\ -C & E \end{bmatrix} = \text{rank}\begin{bmatrix} T(A+BK-\lambda I)T^{-1} & TB \\ -(C+EK)T^{-1} & E \end{bmatrix} \quad \cdots \quad (*)$$

for any nonsingular T and K. Can you conclude that the transmission zeros are invariant under any state feedback and any equivalence transformation? Can you arrive at the same conclusion from the facts that a state feed-back does not affect the numerator matrix of a transfer-function matrix and that an equivalence transformation does not affect a tranfer-function matrix?

NOTE: It is easy to show the first part of problem statement is wrong. Yes

Assume $T=1$, $A=1$, $B=2$, $C=3$, $E=4$ and feedback $K=-\frac{3}{4}$.

$$\det\begin{bmatrix} A-\lambda I & B \\ -C & E \end{bmatrix} = \det\begin{bmatrix} 1-\lambda & 2 \\ -3 & 4 \end{bmatrix} = 10-4\lambda = 10\cdot(2.5-\lambda) \quad \cdots \quad (1)$$

$$\det\begin{bmatrix} T(A+BK-\lambda I)T^{-1} & TB \\ -(C+EK)T^{-1} & E \end{bmatrix} = \det\begin{bmatrix} 1-\frac{3}{2}-\lambda & 2 \\ 0 & 4 \end{bmatrix} = 4(-0.5-\lambda) \quad \cdots \quad (2)$$

So, when $\lambda=2.5$ and/or $\lambda=-0.5$, the ranks of two matrices are not equal. Q.E.D.

I think the statement has to be changed as Yes

Show that

$$\text{rank}\begin{bmatrix} \lambda I-A & B \\ -C & E \end{bmatrix} = \text{rank}\begin{bmatrix} T(\lambda I-A-BK)T^{-1} & TB \\ -(C+EK)T^{-1} & E \end{bmatrix} \quad \cdots\cdots \quad (**)$$

for any nonsingular T and K. ......

SHOW: Now it is more easy to show this statement are correct.

$$\because \begin{bmatrix} T(\lambda I-A-BK)T^{-1} & TB \\ -(C+EK)T^{-1} & E \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}\cdot\begin{bmatrix} \lambda I-A & B \\ -C & E \end{bmatrix}\cdot\begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}\cdot\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}$$ ✓

and the first, third and fourth matrices on the right-hand side of equation above are full rank matrices if T is nonsingular. So, we show the statement. Q.E.D.

PROBLEM 1 (d) cont.

Yes. Now we can conclude that the transmission zeros are invariant under any state feedback and any equivalence transformation. Assume we have system

$$\dot{x} = Ax + Bu \qquad \cdots\cdots \text{(old)}$$
$$y = Cx + Eu.$$

If have feedback $u = K\cdot x + v$, where v is new input, then

$$\begin{cases} \dot{x} = Ax + B\cdot(K\cdot x + v) = (A+BK)x + B\cdot v \\ y = Cx + E(K\cdot x + v) = (C+EK)x + E\cdot v \end{cases} \qquad \cdots \text{(mid)}$$

And then equivalent transform system states $x \to T^{-1}\hat{x}$, thus

$$x = T^{-1}\hat{x}, \quad \dot{x} = T^{-1}\dot{\hat{x}}$$

Replace these into system (mid), we have

$$\begin{cases} T^{-1}\dot{\hat{x}} = (A+BK)T^{-1}\hat{x} + B\cdot v \\ y = (C+EK)T^{-1}\hat{x} + E\cdot v \end{cases}$$

That is new system.

$$\begin{cases} \dot{\hat{x}} = T(A+BK)T^{-1}\cdot\hat{x} + TB\cdot v \\ y = (C+EK)T^{-1}\hat{x} + E\cdot v \end{cases} \qquad \cdots \text{(new)}$$

According to the definition of transmission zeros, if s_0 is tzero of new system then

$$\text{rank}\begin{bmatrix} s_0 I - T(A+BK)T^{-1} & TB \\ -(C+EK)T^{-1} & E \end{bmatrix} = \text{rank}\begin{bmatrix} T(s_0 I - A - BK)T^{-1} & TB \\ -(C+EK)T^{-1} & E \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} s_0 I - A & B \\ -C & E \end{bmatrix}$$

So, we have s_0 is the tzero of new system iff s_0 is the tzero of old system.

Yes, we can arrive at the facts that a state feedback does not affect the numerator matrix of a transfer-function matrix and that an equivalence transformation does not affect a transfer-function matrix.

TO BE CONT.

PROBLEM 1 (d) CONT.

Show the equivalence transformation case first. We have eq. trans. system

$$\begin{cases} \dot{\hat{x}} = TAT^{-1}\hat{x} + TB \\ y = CT^{-1}\cdot\hat{x} + E \end{cases}$$

The transfer matrix:

$$\begin{aligned} G(s) &= CT^{-1}\cdot(sI - TAT^{-1})^{-1}\cdot TB + E \\ &= C\cdot T^{-1}[T(sI-A)\cdot T^{-1}]^{-1}\cdot TB + E \\ &= C\cdot T^{-1}\cdot T\cdot(sI-A)^{-1}\cdot T^{-1}\cdot TB + E \\ &= C\cdot(sI-A)^{-1}\cdot B + E \end{aligned}$$

Q.E.D.

It has been showed in section 7-2.1, eqation (1) ~ (5) that state feedback does not alter the numerator of the transfer matrix.

Q.E.D

PROBLEM 2: (Refer to the "Observer design algorithm" in the lecture)

Do problem 4.1-8, P268, TEXT 1.

Plot "ture" and "estimated" state trajectories, make sure that the "algorithm" works. Note: MATLAB needed for this problem.

4.1-8 Station-Keeping Satellite.

For the station-keeping satellite of Example 3.3-2, design an observer using measurements y(·) of azimuthal position perturbation. Place the observe poles at $s=-2\omega$, $s=-3\omega$, $s=-3\omega \pm j3\omega$, which means that the estimate errors will decay in about 2.5 days ($\doteq \frac{1}{2}\omega$, $\omega = 2\pi/29.3$ rad/day)

SOLUTION: From Example 3.3-2 on page 212, we have original system eq.

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{y}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 9\omega^2 & 0 & 0 & 2\omega\\ 0 & 0 & 0 & 1\\ 0 & -2\omega & -4\omega^2 & 0\end{bmatrix}\cdot\begin{bmatrix}x\\ \dot{x}\\ y\\ \dot{y}\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\cdot u$$

$$y = [0\ 0\ 1\ 0]\cdot[x\ \dot{x}\ y\ \dot{y}]'$$

$\omega = 2\pi/29.3 = 0.21444$, we have

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{y}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0.41387 & 0 & 0 & 0.42889\\ 0 & 0 & 0 & 1\\ 0 & -0.42889 & -.18394 & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ y\\ \dot{y}\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\cdot u$$

$$y = [0\ 0\ 1 0]\cdot[x\ \dot{x}\ y\ \dot{y}]'$$

As the specifications in 4.1-8, we have to choose the observer poles at $s=-2\omega$, $s=-3\omega$, $s=-3\omega \pm j3\omega$, so, we choose F as

$$F = \begin{bmatrix}-2\omega & 0 & 0 & 0\\ 0 & -3\omega & 0 & 0\\ 0 & 0 & -3\omega & -3\omega\\ 0 & 0 & 3\omega & -3\omega\end{bmatrix}$$

Observer Design:

STEP 1: Pick F as

$$F = \begin{bmatrix} -0.42889 & 0 & 0 & 0 \\ 0 & -0.64333 & 0 & 0 \\ 0 & 0 & -0.64333 & -0.64333 \\ 0 & 0 & 0.64333 & -0.64333 \end{bmatrix}$$

STEP 2: Choose

$$G = \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \end{bmatrix}, \quad \text{CHECK } \rho[G, FG, F^2G, F^3G] = 4, \ \& \ \text{cond}(@) \approx 98$$

STEP 3: Solve $F \cdot T - T \cdot A = -GC$, we have

$$T = \begin{bmatrix} -3.4977 & 3.6246 & 0.3886 & -4.5306 \\ 2.3315 & -3.6242 & 1.5544 & -0.0000 \\ -0.2210 & 0.7231 & 0.3179 & 0.6236 \\ -0.2442 & 0.0362 & 1.4149 & -1.5998 \end{bmatrix}$$

STEP 4:

$$H = TB = \begin{bmatrix} -4.5306 \\ -0.0000 \\ 0.6236 \\ -1.5998 \end{bmatrix}$$

STATE-PLOTS: *Because the original system is not stable, we can assume* (WOLG)

$u(t) = 0$, $x(0) = y(0) = 0$, $\dot{x}(0) = \dot{y}(0) = 3$. $\hat{\underline{x}}(0) = [5, 5, 5, 5]'$

$$\therefore \quad \underline{x}(t) = e^{At} \cdot \underline{x}_0$$

$$= \text{expm}\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ .41387 & 0 & 0 & .42889 \\ 0 & 0 & 0 & 1 \\ 0 & -.42889 & -.18394 & 0 \end{bmatrix} \cdot t \right) \cdot \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix}$$

$$y(t) = C \cdot \underline{x}(t) = [0 \ \ 0 \ \ 1 \ \ 0] \cdot \underline{x}(t).$$

$$\dot{\hat{x}}(t) = F \hat{x}(t) + G \cdot y(t)$$

Using discrete system, $\hat{x}_{k+1} = \Phi \hat{x}_k + \Gamma \cdot y_k$. *And then the plots.*

(By Hand-puter)

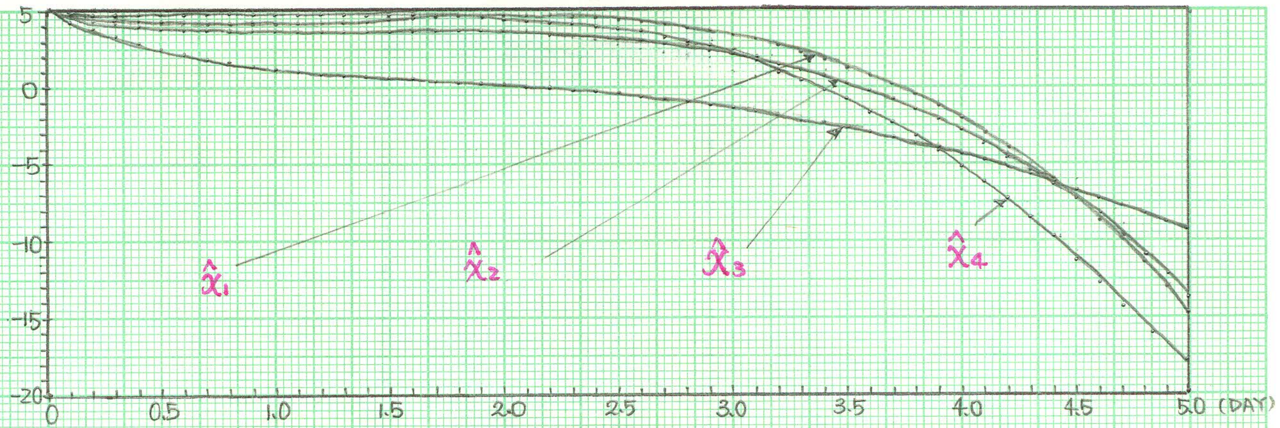


FIG. 1. ESTIMATED STATE PLOTS $\hat{x}$

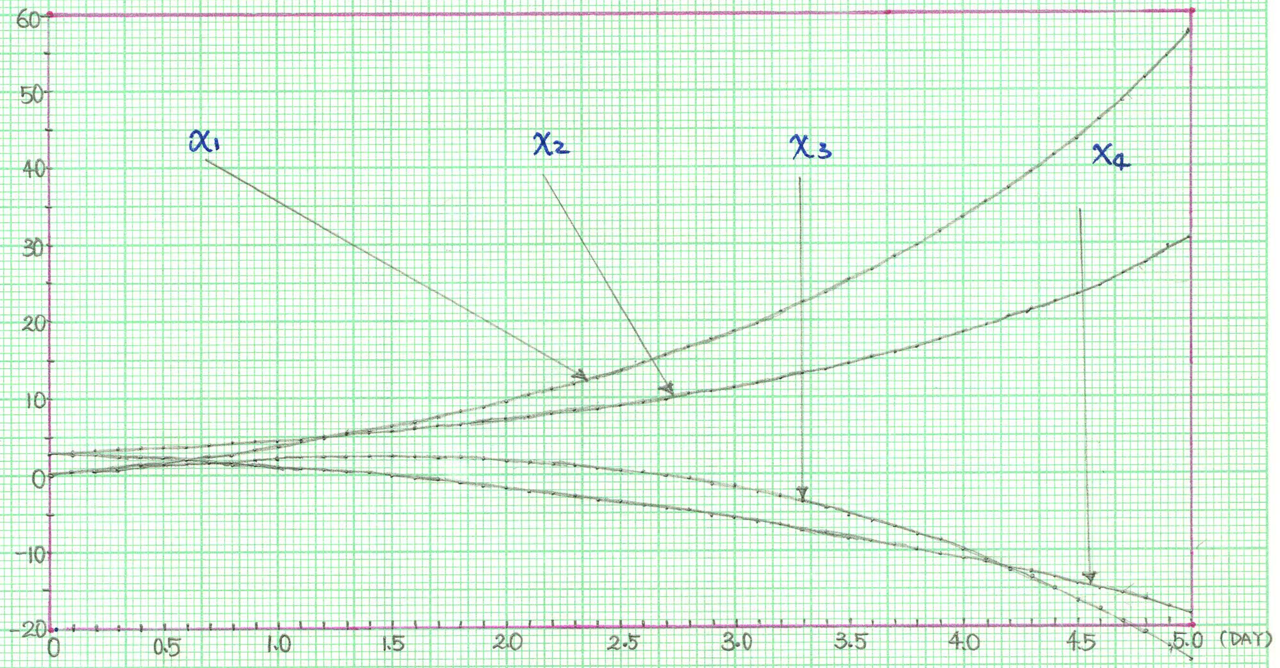


FIG 2 : REAL STATE TRAJECTORIES x

FIG 3 : OBSERVER ERROR $e = \hat{x} - T \cdot x$

PROBLEM 3 (a) Compute "transmission zeros" of a chemical reator system. Also compute $G(s)$, $\lambda_i(A)$, and find idz, odz, iodz, if any. Use MATLAB.

SYSTEM MTRICES FOR A REACTOR

$$A = \begin{bmatrix} 1.380 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.290 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

SOLUTION: From MATLAB, we obtain the transmission zeros:

$S_1 = 2.2708 \times 10^{17}$, $S_2 = 5.8491 \times 10^{15}$, $S_3 = -5.0394$, $S_4 = -1.1916$

And then, we check that all the transmission zeros are satisfied the condition

$$\rho \begin{bmatrix} S_0 I - A & B \\ -C & 0 \end{bmatrix} < 6$$

So, they are O.K.

Compute the eigenvalues of A, we have

$\lambda_1 = -8.6659$, $\lambda_2 = 1.9910$, $\lambda_3 = -5.0566$, $\lambda_4 = 0.0635$

And the transfer function matrix,

$$G(s) = \frac{\begin{bmatrix} 0 & -0.0000 & 0.0008 & 29.2264 & 263.4419 \\ 0 & 5.6790 & 42.6665 & -68.8304 & -106.8024 \end{bmatrix}}{S^4 + 11.6680 S^3 + 15.7538 S^3 - 88.2911 S + 5.5406}$$

Dr. Hsu, I don't have the control tool box manual. So, I don't know how to work out the num. polynomial of G(s) from the data I got.

Check the condition $\rho[\lambda_i I - A, B] < 4$, we have the idz: NONE.

Check $\rho[C', \lambda_i I - A'] < 4$. WE Have odz: NONE

And then the iodz: NONE

PROBLEM 3 (b) Plot step responses and Bode plots of the following transfer functions:

(i) $K = 67.5$; POLES: $-2\pm j, -3, -3, -6$; zeros: $-1, -4$

(ii) Gain $K = 68.3$; POLES: $-2.22 \pm 1.37j$, $-1.17 \pm 0.382j$, -6.17

zeros: $-0.917 \pm 0.3j$

(i)

$$H(S) = \frac{67.5(S+1)(S+4)}{(S+3)^2(S+6)(S+2-j)(S+2+j)} = \frac{67.5S^2+337.5S+270}{(S^2+6S+9)(S^2+4S+5)(S+6)}$$

$$= \frac{67.5S^2+337.5S+270}{S^5+16S^4+98S^3+294S^2+441S+270}$$

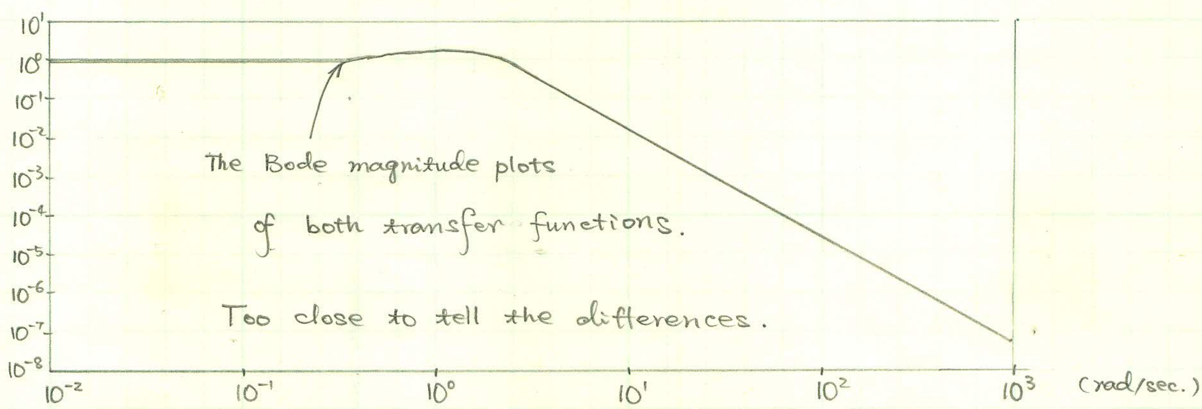
(ii)

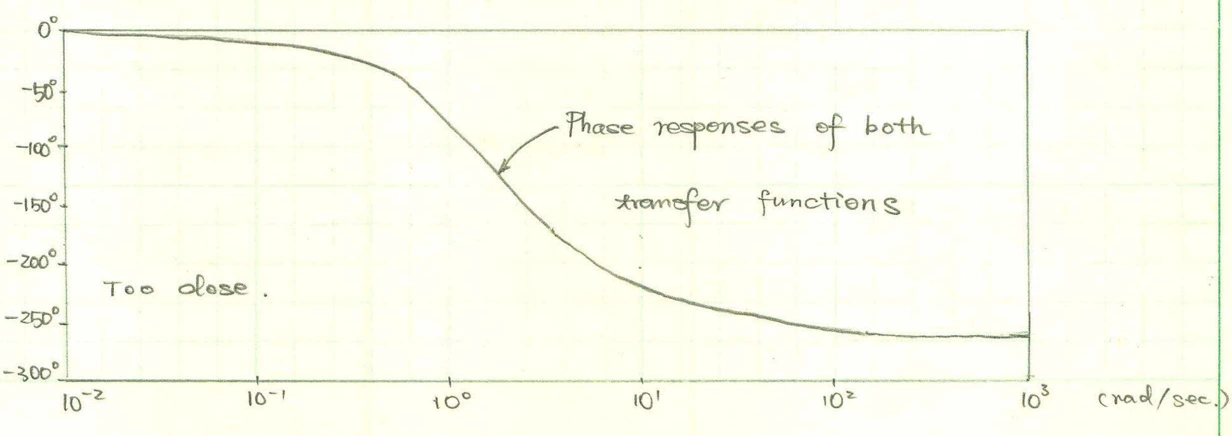
$$H(S) = \frac{68.3(S+0.917+0.3j)(S+0.917-0.3j)}{(S+2.22+1.37j)(S+2.22-1.37j)(S+1.17+0.382j)(S+1.17-0.382j)(S+6.17)}$$

$$= \frac{68.3S^2 + 125.2622S + 63.5797}{(S^2+4.44S+6.8053)(S^2+2.34S+1.5148)(S+6.17)}$$

$$= \frac{68.3 \cdot S^2 + 125.2622 \cdot S + 63.5797}{S^5 + 12.95S^4 + 60.5423S^3 + 138.0889S^2 + 150.0598S + 63.6047}$$

I have to use my hand-puter to do the Bode-plots:





LINEAR SYSTEM THEORY

EE501

BONUS PROBLEM

NOV. 3, 87

Benmei Chen

P.O.Box 241, Gonzaga University

Spokane, WA 99258

(509) 327-6071

6/6

Excellent job!

I'm pleased to have you in my class. Keep on going, you will be an outstanding control engineer.

H Su

PROBLEM 1 (2 POINT):

$$\ddot{x} + x = u, \quad x(0) = x_0, \quad \dot{x}(0) = v_0, \quad x(t_f) = 0, \quad \dot{x}(t_f) = 0$$

a. DETERMINE A MINIMUM ENERGY CONTROLLER (MEC), HINT: $u(t) = \underline{b}' e^{-A't} \hat{W}_c^{-1} \underline{f}$

b. LET $t_f = \pi/4$, $x_0 = 1$, $v_0 = 5$, PLOT $x_1(t)$, $x_2(t)$ & $u(t)$, $0 \le t \le \pi/4$

SOLUTION: a. LET $x_1 = x$, $x_2 = \dot{x}$, SO,

$$\begin{cases} \dot{x}_1 = \dot{x} = x_2 \\ \dot{x}_2 = \ddot{x} = -x + u = -x_1 + u \end{cases}$$

The system equation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underline{X}_0 = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}, \quad \underline{X}(t_f) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

According the lecture note of 10-1-87, we have

$$\underline{f} = e^{-At_f} \cdot [\underline{X}(t_f) - e^{At_f} \underline{X}_0]$$

$$= e^{-At_f} \cdot [-e^{At_f} \underline{X}_0] = -\underline{X}_0 = \begin{bmatrix} -x_0 \\ -v_0 \end{bmatrix}$$

And the minimum energy controller,

$$u(t) = b' \cdot e^{-A't} \cdot \left[\int_0^{t_f} e^{-A\sigma} b b' e^{-A'\sigma} d\sigma\right]^{-1} \underline{f} \quad \cdots\cdots \quad (*)$$

Now, we have to find out $e^{-A't}$ and e^{-At}, for this particular problem, ~~we can be~~ it is easy to show $-A' = A$.

$$\therefore \mathcal{L}(e^{-A't}) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$$

$$\therefore e^{-A't} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

$$e^{-At} = (e^{-A't})'$$

$$\cancel{\mathcal{L}(e^{-At}) = (sI + A)^{-1} = \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{-1}{s^2+1} \\ \frac{1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}}$$

$$\therefore e^{-At} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

PROBLEM 1 a. (CONT.) THE MEC

$$u(t) = [0\ \ 1]\cdot\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\cdot\left[\int_0^{t_f}\begin{bmatrix}\cos\sigma & -\sin\sigma\\ \sin\sigma & \cos\sigma\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix}\cdot[0\ \ 1]\cdot\begin{bmatrix}\cos\sigma & \sin\sigma\\ -\sin\sigma & \cos\sigma\end{bmatrix}d\sigma\right]^{-1}\cdot\begin{bmatrix}-x_0\\-v_0\end{bmatrix}$$

$$= [-\sin t\ \ \cos t]\cdot\left[\int_0^{t_f}\begin{bmatrix}\sin^2\sigma & -\sin\sigma\cos\sigma\\ -\sin\sigma\cos\sigma & \cos^2\sigma\end{bmatrix}d\sigma\right]^{-1}\cdot\begin{bmatrix}-x_0\\-v_0\end{bmatrix}$$

$$= [-\sin t\ \ \cos t]\cdot\begin{bmatrix}t_f/2-\sin 2t_f/4 & -\sin^2 t_f/2\\ -\sin^2 t_f/2 & t_f/2+\sin 2t_f/4\end{bmatrix}^{-1}\cdot\begin{bmatrix}-x_0\\-v_0\end{bmatrix}$$

$$= \frac{16\cdot[-\sin t\ \ \cos t]\cdot\begin{bmatrix}t_f/2+\sin 2t_f/4 & +\sin^2 t_f/2\\ +\sin^2 t_f/2 & t_f/2-\sin 2t_f/4\end{bmatrix}\cdot\begin{pmatrix}-x_0\\-v_0\end{pmatrix}}{4t_f^2+4\sin^2 t_f-\sin^2 2t_f}$$

MEC. $$u(t) = \frac{(8t_f x_0+4x_0\sin 2t_f+8v_0\sin^2 t_f)\sin t+(4v_0\sin 2t_f-8x_0\sin^2 t_f-8v_0 t_f)\cos t}{4t_f^2-4\sin^4 t_f-\sin^2 2t_f}$$

b. LET $t_f=\pi/4$, $x_0=1$, $v_0=5$, THEN $\sin 2t_f=1$, $\sin^2 t_f=\frac{1}{2}$

$$u(t) = \frac{(2\pi+4+20)\sin t+(20-4-10\pi)\cos t}{\pi^2/4-1-1}$$

$$= \underline{64.7906\cdot\sin t - 32.9822\cdot\cos t}$$

$$\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = e^{At}x_0+\int_0^t e^{A(t-\sigma)}b\,u(\sigma)\,d\sigma$$

$$= \begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\cdot\begin{bmatrix}1\\5\end{bmatrix}+\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\cdot\int_0^t\begin{bmatrix}\cos\sigma & -\sin\sigma\\ \sin\sigma & \cos\sigma\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix}\cdot u(\sigma)\,d\sigma$$

$$= \begin{bmatrix}\cos t+5\sin t\\ -\sin t+5\cos t\end{bmatrix}+\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\cdot\int_0^t\begin{bmatrix}-64.7906\sin^2\sigma+32.9822\sin\sigma\cos\sigma\\ 64.7906\sin\sigma\cos\sigma-32.9822\cos^2\sigma\end{bmatrix}d\sigma$$

$$= \begin{bmatrix}\cos t+5\sin t\\ -\sin t+5\cos t\end{bmatrix}+\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\cdot\begin{bmatrix}32.3953t+16.19764\sin 2t+16.4911\cdot\sin^2 t\\ 32.3953\sin^2 t-16.4911t-8.2455\sin 2t\end{bmatrix}$$

$\therefore$ $\underline{x_1(t) = \cos t+5\sin t-32.3953t\cdot\cos t+16.19764\sin 2t\cos t+16.4911\sin^2 t\cos t}$

$\underline{+32.3953\sin^3 t-16.4911\cdot t\cdot\sin t-8.2456\sin 2t\cdot\sin t}$

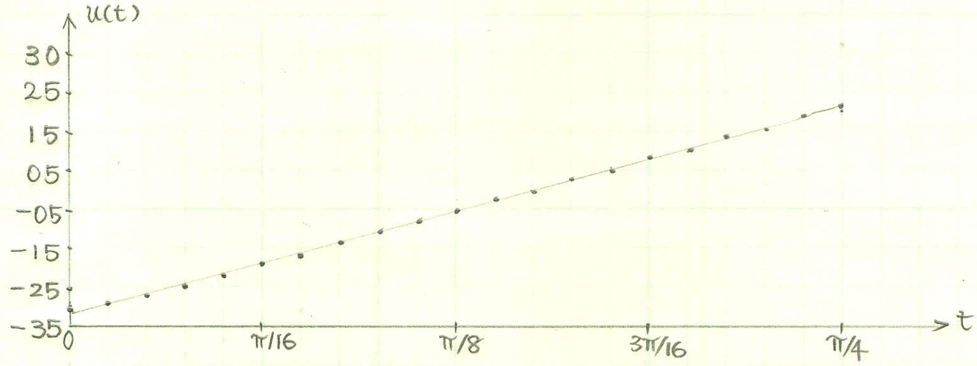
$\underline{x_2(t) = -\sin t+5\cos t+32.3953t\sin t-16.19764\sin 2t\sin t-16.4911\sin^3 t}$

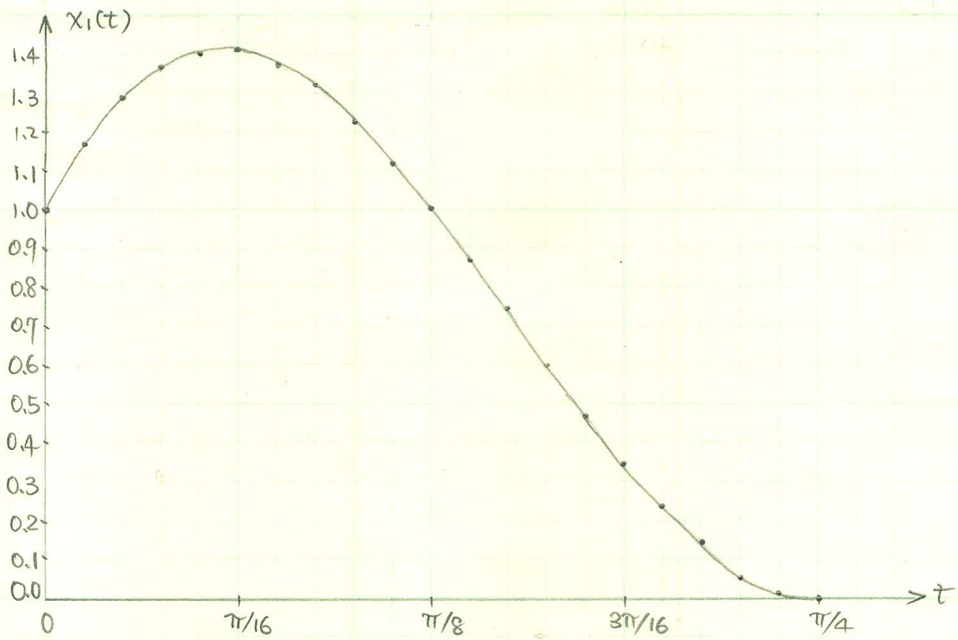
$\underline{+32.3953\sin^2 t\cos t-16.4911t\cos t-8.2455\sin 2t\cdot\cos t.}$

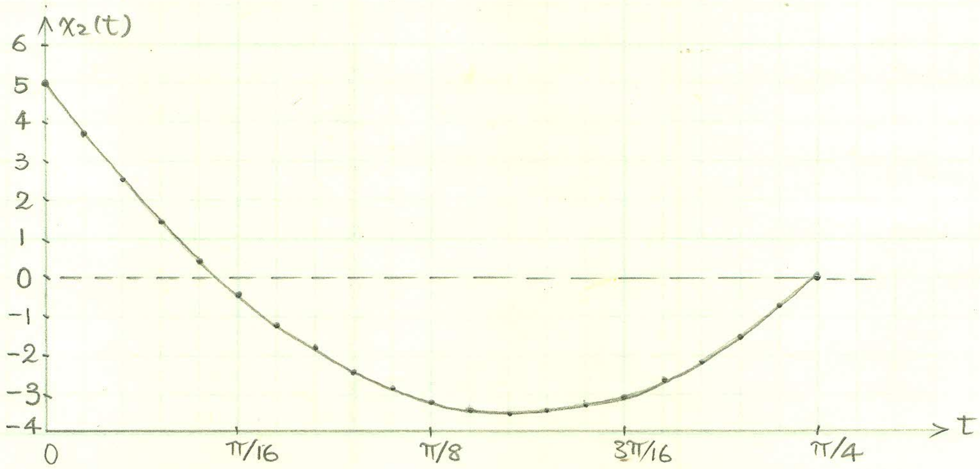
PROBLEM 1 b. (CONT.)

WE can use MATLAB to have nice plots. But, unfortunately, I won't be able to get a copy. So, the following plots are based on data I obtained from my HP-41CX programmable calculator.

why?







PROBLEM 2. A.60 Bounds on Eigenvalues.

If you use the "Rayleigh quotient", the proof is not lengthy

LET σ_{min} and σ_{max} be the smallest and largest singular values of a matrix A.

Hsu

Let $\{\lambda_i\}$ denote the eigenvalues of A. Show that $\sigma_{min} \le \min_i |\lambda_i| \le \max_i |\lambda_i| \le \sigma_{max}$.

$$\lambda_{max} = \max \frac{(x, Ax)}{(x, x)}$$

etc.

PROOF: IN ORDER TO PROVE THE STATEMENT ABOVE, I AM GOING TO USE Theorem 8 on page 440, In A.C. Zaanen's Linear Analysis, 1960

(A copy of the theorem is attached to the last page.)

Lemma 1: A is $n \times n$ nonsingular matrix, Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A and $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n| > 0$, Then there exist n singular values of A, $\sigma_1 \ge \sigma_2 \ge \cdots \sigma_n > 0$, $\forall s > 0$, we have

$$\sum_{i=1}^{n} |\lambda_i|^s \le \sum_{i=1}^{n} \sigma_i^s$$

Reference: Introduction to Matrix Analysis R. Bellman

PROOF: Use singular value decomposition, we have

$$A = U \Sigma V'$$

$$AA' = U \Sigma V' \cdot V \cdot \Sigma' U' = U \Sigma^2 U'$$

$$\det(AA') = [\det A]^2 = \det(\Sigma^2) = \sigma_1^2 \cdot \sigma_2^2 \cdots \sigma_n^2 \ne 0$$

So, we have n singular values $\sigma_1 \ge \sigma_2 \ge \cdots \sigma_n > 0$ (always can be arranged)

Then, according the Theorem 8 attached to the last page, we immediately have the results. Q.E.D.

Lemma 2: Let σ_1 be the largest singular value of A (nonsingular), and $\xi_1, \xi_2, \cdots, \xi_n$ be the eigenvalues of $\frac{1}{\sigma_1} A$. Then

$$|\xi_i| \le 1, \quad \text{for } i = 1, 2, \cdots n$$

PROOF: LET $A = U \Sigma V' = U \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix} V'$, $\lambda_i, i = 1, \cdots, n$ are singular values of A

$$\frac{1}{\sigma_1} A = U \begin{pmatrix} 1 & & & \\ & \sigma_2/\sigma_1 & & 0 \\ & & \ddots & \\ & 0 & & \sigma_n/\sigma_1 \end{pmatrix} V' \triangleq U \begin{pmatrix} 1 & & & \\ & x_2 & & 0 \\ & & \ddots & \\ & 0 & & x_n \end{pmatrix} V'$$

So, $1, x_2, \cdots, x_n$ are singular values of $\frac{1}{\sigma_1} A$ and $0 < x_i \le 1$, $i = 2, \cdots, n$.

TO BE CONT.

Lemma 2 (CONT.) According Lemma 1, we have

$$\sum_{i=1}^{n} |\xi_i|^s \leqslant 1 + \sum_{i=2}^{n} x_i^s \qquad \forall\ s > 0 \quad \cdots\cdots (1)$$

LET $s \to \infty$, $\sum_{i=2}^{n} x_i^s \leqslant n-1$, so,

$\lim_{s\to\infty} \sum_{i=1}^{n} |\xi_i|^s \leqslant n$, Thus $|\xi_i| \leqslant 1$, $i = 1, 2, \cdots, n$.

Otherwise $\lim_{s\to\infty} \sum_{i=1}^{n} |\xi_i|^s \to \infty$. Q.E.D.

Lemma 3: For a nonsingular matrix A, $\sigma_1 \triangleq \max(\sigma_i) \geqslant |\lambda_1| \triangleq \max(|\lambda_i|)$.

PROOF: Let ξ be any eigenvalue of $\frac{1}{\sigma_1} A$, with eigenvector P.

$$\frac{1}{\sigma_1} A P = \xi P.$$

Then $$AP = (\sigma_1 \xi) P = \lambda P$$

We see that $\lambda = \sigma_1 \xi$ is the eigenvalue of A.

From Lemma 2, we have $|\xi| \leqslant 1$. so,

$$|\lambda| = |\sigma_1 \xi| = \sigma_1 |\xi| \leqslant \sigma_1$$ Q.E.D

Lemma 4: For a nonsingular matrix A, $\sigma_{min} = \min(\sigma_i) \leqslant \min(|\lambda_i|)$

PROOF: For $|A| \neq 0$, Let $A = U \Sigma V'$, $\Sigma = \begin{pmatrix} \sigma_{max} & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \ \sigma_{min} \end{pmatrix}$

$$A^{-1} = V \Sigma^{-1} U'$$

$$\Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_{max}} & & 0 \\ & \frac{1}{\sigma_2} & \\ 0 & & \ddots \ \frac{1}{\sigma_{min}} \end{pmatrix}$$

$$\therefore \quad \sigma_{max}(A^{-1}) = \frac{1}{\sigma_{min}}$$

ALSO, IF A has e.v.s $\lambda_1, \lambda_2, \cdots \lambda_n$, A^{-1} has e.v.s $\xi_1 = \lambda_1^{-1}, \cdots, \xi_n = \lambda_n^{-1}$

$$\therefore \quad \max(|\xi_i|) = 1 / \min(|\lambda_i|)$$

Apply Lemma 3 to A^{-1}, we have

$$\sigma_{max}(A^{-1}) \geqslant \max(|\xi_i|)$$

$$\frac{1}{\sigma_{min}} \geqslant \frac{1}{\min(|\lambda_i|)} \quad \Rightarrow \quad \sigma_{min} \leqslant \min(|\lambda_i|)$$

Q.E.D

RESULT 1: FOR ANY NONSINGULAR MATRIX A, LET σ_{min} and σ_{max} BE THE SMALLEST AND LARGEST SINGULAR VALUES OF A, LET $\{\lambda_i\}$ DENOTE THE EIGENVALUES OF A. WE HAVE $\sigma_{min} \leq \min_i |\lambda_i| \leq \max_i |\lambda_i| \leq \sigma_{max}$

PROOF: IT'S ALWAYS TRUE THAT $\min_i |\lambda_i| \leq \max_i |\lambda_i|$, THEN FROM Lemma 3 and Lemma 4, WE PROVE THE RESULT. Q.E.D

Lemma 5: For any singular matrix A: $n\times n$ has a rank k, we can alway find a $k\times k$ nonsingular matrix B, such that

(i) The eigenvalues of B are also all nonzero eigenvalues of A.

(ii) The singular values of B are all nonzero singular values of A.

PROOF: (i) Use the singular value decomposition again, Let

$$A = U\Sigma V' = U\cdot \Sigma V'\cdot U\cdot U'$$

$$= U\cdot \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \cdot U'$$

$$= U\cdot \begin{bmatrix} \Sigma_k X_1 & \Sigma_k X_2 \\ 0 & 0 \end{bmatrix} \cdot U'$$

LET $B \triangleq \Sigma_k X_1$, $C = \Sigma_k X_2$.

$$A = U\cdot \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix} U' \quad \text{and} \quad \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix} \text{ has the same e.v.s as } A.$$

$$\det\left(\lambda I - \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix}\right) = \det \begin{bmatrix} \lambda I_k - B & -C \\ 0 & \lambda I_{n-k} \end{bmatrix} = \lambda^{n-k}\cdot \det(\lambda I_k - B) \qquad (A.11)$$

So, The eigenvalues of B are also all nonzero eigenvalues of A and $|B| \neq 0$

(ii) From (i), we have

$$A' = U\cdot \begin{bmatrix} B' & 0 \\ C' & 0 \end{bmatrix} \cdot U'$$

$$A'A = U\cdot \begin{bmatrix} B' & 0 \\ C' & 0 \end{bmatrix} U'\cdot U \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix} U' = U\cdot \begin{bmatrix} B'B & B'C \\ C'B & C'C \end{bmatrix} \cdot U'$$

$A'A$ and $\begin{bmatrix} B'B & B'C \\ C'B & C'C \end{bmatrix}$ have same eigenvalues.

TO BE CONT.

Lemma 5 (CONT.)

$$\det\left(\lambda I - \begin{bmatrix} B'B & B'C \\ C'B & C'C \end{bmatrix}\right) = \det\begin{bmatrix} \lambda I_k - B'B & -B'C \\ -C'B & \lambda I_{n-k} - C'C \end{bmatrix}$$

$$= \det[\lambda I_k - B'B] \cdot \det[\lambda I_{n-k} - C'C - C'B(\lambda I_k - B'B)^{-1}B'C] = 0$$

$$\therefore \quad \det[\lambda I_k - B'B] = 0 \qquad \text{and} \quad |B| \neq 0 \Rightarrow |B'B| \neq 0$$

Thus the eigenvalues of $B'B$ are nonzero eigenvalues of $A'A$.

Because $\rho(A'A) = \rho(A) = k$. thus $A'A$ can only have k nonzero eigenvalues.

So, the eigenvalues of $B'B$ are all nonzero eigenvalues of $A'A$

The following shows that the eigenvalues of $A'A \geq 0$

$$A = U\Sigma V' \quad , \quad A' = V\Sigma' U' = V\Sigma U'$$

$$A'A = V\Sigma U' U\Sigma V' = V\Sigma^2 V'$$

Then, according the definition of singular values of A in A.57 on page 667 in text one, we have

The singular values of B are all nonzero singular values of A.

Q.E.D

FINAL RESULT: FOR ANY MATRIX $A = n\times n$, WE HAVE.

$$\sigma_{min} \leq \min_i|\lambda_i| \leq \max_i|\lambda_i| \leq \sigma_{max}.$$

PROOF: IF A is nonsingular, it is showed in Result 1.

IF A is singular, from Lemma 5, we can find a matrix B. s.t. $\sigma_{max}(A) = \sigma_{max}(B) \geq \max_B|\lambda_i| = \max_A|\lambda_i|$

and $\min_A|\lambda_i| = 0$. $\sigma_{min} = 0$. (otherwise $\sigma_{min} > \min_A|\lambda_i|$?)

Then, we have $\sigma_{min} \leq \min_i|\lambda_i| \leq \max_i|\lambda_i| \leq \sigma_{max}$

Q.E.D

END OF PROBLEM 2.

PROBLEM 3. A.41 on Page 662 IN TEXT one.

IF A is an $m\times m$ matrix with e.v.s $\{\lambda_1, \cdots, \lambda_m\}$ and B is an $n\times n$ matrix with e.v.s $\{\mu_1, \mu_2, \cdots, \mu_n\}$, show that

1. The e.v.s of $A\otimes B$, the Kronecker product of A and B are $\{\lambda_i\mu_j\}$.
2. Det $A\otimes B = (\det A)^n \cdot (\det B)^m$
3. The equation $AX+XB=C$ can be written as $(A\otimes I_n + I_m\otimes B)\mathcal{X} = \mathcal{Y}$, where $\{\mathcal{X}, \mathcal{Y}\}$ are column vectors of mn elements formed from the rows of X, C (respectively) taken in order; i.e.

$$\mathcal{X} = [x_{11} \;\; x_{12} \cdots x_{1n} \cdots x_{mn}]'$$

E. Use 1 and 3 to show that the matrix equation has a unique solution if and only if A and $-B$ have no e.v.s in common.

4. Show that the matrix equation $P - FPF' = Q$, where F and P are given $n\times n$ matries, can be written as $(I - F\otimes F)\mathcal{X} = \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are obtained from P and Q by "stacking" the rows as in 3.

SHOW: SHOWS ARE ON NEXT SEVEN PAGES.

PROBLEM 3.1.

SHOW: we only have to show that: $\forall \lambda, \mu$ where λ is e.v. of A and μ is e.v. of B, then $\lambda\mu$ is e.v. of $A \otimes B$.

LET

$$P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_m \end{bmatrix}$$ be a E.V. of A associated with λ

and Q is a E.V. of B associated with μ.

Then

$$AP = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \cdot \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_m \end{bmatrix} = \begin{bmatrix} a_{11}P_1 + a_{12}P_2 + \cdots + a_{1m}P_m \\ a_{21}P_1 + a_{22}P_2 + \cdots + a_{2m}P_m \\ \vdots \\ a_{m1}P_1 + a_{m2}P_2 + \cdots + a_{mm}P_m \end{bmatrix} = \begin{bmatrix} \lambda P_1 \\ \lambda P_2 \\ \vdots \\ \lambda P_m \end{bmatrix} \cdots (1)$$

Now, we are going to show that $\lambda\mu$ and $P \otimes Q$ are e.v. and E.V. of $A \otimes B$.

$$(A \otimes B)(P \otimes Q) = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & & \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix} \cdot \begin{bmatrix} P_1Q \\ P_2Q \\ \vdots \\ P_mQ \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}P_1BQ + a_{12}P_2BQ + \cdots + a_{1m}P_mBQ \\ a_{21}P_1BQ + a_{22}P_2BQ + \cdots + a_{2m}P_mBQ \\ \vdots \\ a_{m1}P_1BQ + a_{m2}P_2BQ + \cdots + a_{mm}P_mBQ \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}P_1 + a_{12}P_2 + \cdots + a_{1m}P_m)\mu Q \\ (a_{21}P_1 + a_{22}P_2 + \cdots + a_{2m}P_m)\mu Q \\ \vdots \\ (a_{m1}P_1 + a_{m2}P_2 + \cdots + a_{mm}P_m)\mu Q \end{bmatrix}$$

FROM EQ.(1)

$$= \begin{bmatrix} \lambda P_1 \mu Q \\ \lambda P_2 \mu Q \\ \vdots \\ \lambda P_m \mu Q \end{bmatrix} = \lambda\mu \cdot \begin{bmatrix} P_1Q \\ P_2Q \\ \vdots \\ P_mQ \end{bmatrix} = \lambda \cdot \mu \cdot (P \otimes Q)$$

So, $\lambda_i \mu_j$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ are the e.v.s of $A \otimes B$ and $A \otimes B$ is $mn \times mn$ matrix which can only has mn e.v.s. ✓

Q.E.D.

PROBLEM 3.2.

SHOW: From 1., we know $\{\lambda_i \mu_j,\ i=1,2,\cdots m;\ j=1,2,\cdots n\}$ are e.v.s of $A\otimes B$. and $\{\lambda_1,\lambda_2,\cdots,\lambda_m\}$ are e.v.s of A, $\{\mu_1,\cdots \mu_n\}$ are e.v.s of B.

And from CRC math handbook, we have

$$\det A = \lambda_1\cdot\lambda_2\cdots\lambda_m \quad \cdots\ (2)$$

$$\det B = \mu_1\cdot\mu_2\cdots\mu_n \quad \cdots\ (3)$$

$$\det A\otimes B = \prod_{i=1}^{m}\cdot\prod_{j=1}^{n}\lambda_i\cdot\mu_j$$

$$= (\lambda_1\cdot\lambda_2\cdots\lambda_m)^n\cdot(\mu_1\cdot\mu_2\cdots\mu_n)^m$$

$$= (\det A)^n\cdot(\det B)^m \qquad \text{Q.E.D}$$

PROBLEM 3.3: SHOW:

LET $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m}\\ a_{21} & a_{22} & \cdots & a_{2m}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mm}\end{bmatrix}$, $B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2n}\\ \vdots & \vdots & & \\ b_{n1} & b_{n2} & \cdots & b_{nn}\end{bmatrix}$

$X=\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & & \\ x_{m1} & x_{m2} & \cdots & x_{mn}\end{bmatrix}$, $C=\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n}\\ c_{21} & c_{22} & \cdots & c_{2n}\\ \vdots & \vdots & & \\ c_{m1} & c_{m2} & \cdots & c_{mn}\end{bmatrix}$, $x=\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_m\end{bmatrix}$, $y=\begin{bmatrix} c_1\\ c_2\\ \vdots\\ c_m\end{bmatrix}$

$$\therefore\ AX+XB=\begin{bmatrix} \sum_{i=1}^{m} a_{1i}x_{i1} & \sum_{i=1}^{m} a_{1i}x_{i2} & \cdots & \sum_{i=1}^{m} a_{1i}x_{in}\\ \sum_{i=1}^{m} a_{2i}x_{i1} & \sum_{i=1}^{m} a_{2i}x_{i2} & \cdots & \sum_{i=1}^{m} a_{2i}x_{in}\\ \vdots & \vdots & & \vdots\\ \sum_{i=1}^{m} a_{mi}x_{i1} & \sum_{i=1}^{m} a_{mi}x_{i1} & \cdots & \sum_{i=1}^{m} a_{mi}x_{in}\end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n} x_{1i}b_{i1} & \cdots & \sum_{i=1}^{n} x_{1i}b_{in}\\ \sum_{i=1}^{n} x_{2i}b_{i1} & \cdots & \sum_{i=1}^{n} x_{2i}b_{in}\\ \vdots & & \vdots\\ \sum_{i=1}^{n} x_{mi}b_{i1} & \cdots & \sum_{i=1}^{n} x_{mi}b_{in}\end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{m} a_{1i}x_{i1}+\sum_{i=1}^{n} x_{1i}b_{i1} & \cdots & \sum_{i=1}^{m} a_{1i}x_{in}+\sum_{i=1}^{n} x_{1i}b_{in})\\ \vdots & & \vdots\\ \sum_{i=1}^{m} a_{mi}x_{i1}+\sum_{i=1}^{n} x_{mi}b_{i1}) & \cdots & \sum_{i=1}^{m} a_{mi}x_{in}+\sum_{i=1}^{n} x_{mi}b_{in}\end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n}\\ \vdots & & \vdots\\ c_{m1} & \cdots & c_{mn}\end{bmatrix} = C$$

So, we have the linear equation:

$$\begin{cases} \sum_{i=1}^{m} a_{1i}x_{i1}+\sum_{i=1}^{n} x_{1i}b_{i1}) = c_{11}\\ \vdots\\ \sum_{i=1}^{m} a_{1i}x_{in}+\sum_{i=1}^{n} x_{1i}b_{in}) = c_{1n}\\ \vdots\\ \sum_{i=1}^{m} a_{mi}x_{in}+\sum_{i=1}^{n} x_{mi}b_{in} = c_{mn}\end{cases} \quad \cdots\cdots\ (1)$$

PROBLEM 3.3 (CONT.) Now, we start from equation below

$$(A\otimes I_n + I_m \otimes B')\mathcal{X} = \left(\begin{bmatrix} a_{11}I_n & \cdots & a_{1m}I_n \\ \vdots & & \vdots \\ a_{m1}I_n & \cdots & a_{mm}I_n \end{bmatrix} + \begin{bmatrix} B' & & O \\ & B' & \\ O & & B' \end{bmatrix} \right)\mathcal{X}$$

$$= \begin{bmatrix} a_{11}I_n+B' & a_{12}I_n & \cdots & a_{1m}I_n \\ a_{21}I_n & a_{22}I_n+B' & \cdots & a_{2m}I_n \\ \vdots & & & \\ a_{m1}I_n & a_{m2}I_n & \cdots & a_{mm}I_n+B' \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \quad \cdots\cdots (2)$$

where $x_i = [x_{i1}, \cdots x_{in}]'$ $i=1,2,\cdots,m$. So, the eq.(2) above

$$(2) = \begin{bmatrix} (a_{11}I_n+B')x_1 + a_{12}I_n x_2 + \cdots + a_{1m}I_n x_m \\ \vdots \\ a_{m1}I_n x_1 + a_{m2}I_n x_2 + \cdots + (a_{mm}I_n + B')x_m \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m + B'x_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm}x_m + B'x_m \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_{11} + a_{12}x_{21} + \cdots + a_{1m}x_{m1} + \sum_{i=1}^{n} b_{i1}x_{1i} \\ \vdots \\ a_{11}x_{1n} + a_{12}x_{2n} + \cdots + a_{1m}x_{mn} + \sum_{i=1}^{n} b_{in}x_{1i} \\ \vdots \\ a_{m1}x_{1n} + a_{m2}x_{2n} + \cdots + a_{mm}x_{mn} + \sum_{i=1}^{n} b_{in}x_{mi} \end{bmatrix}$$

You are right, Thanks. HS ✓

$$= \begin{bmatrix} \sum_{i=1}^{m} a_{1i}x_{i1} + \sum_{i=1}^{n} x_{1i}b_{i1} \\ \vdots \\ \sum_{i=1}^{m} a_{1i}x_{in} + \sum_{i=1}^{n} x_{1i}b_{in} \\ \vdots \\ \sum_{i=1}^{m} a_{mi}x_{in} + \sum_{i=1}^{n} x_{mi}b_{in} \end{bmatrix} = \mathcal{Y} = \begin{bmatrix} c_{11} \\ \vdots \\ c_{1n} \\ \vdots \\ c_{mn} \end{bmatrix} \quad \cdots\cdots (3)$$

Compare eq. (1) and (3), we show that

$AX + XB = C$ can be written as $(A\otimes I_n + I_m\otimes B')\mathcal{X} = \mathcal{Y}$ #

DR. HSU; IN TEXT, THERE IS: $AX+XB=C \Rightarrow (A\otimes I_n + I_m \otimes B)\mathcal{X} = \mathcal{Y}$. BUT I THINK IT IS NOT CORRECT. FOR EXAMPLE: $A=a$, $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}$, $X=[x_1\ x_2]$, $C=[c_1, c_2]$

THEN $AX+BX = [ax_1 + x_1b_{11} + x_2b_{21},\ ax_2 + x_1b_{12} + x_2b_{22}] = C = [c_1\ c_2]$

AND $(A\otimes I_2 + I_1\otimes B)\mathcal{X} = \begin{pmatrix} a+b_{11} & b_{12} \\ b_{21} & a+b_{22}\end{pmatrix}\begin{pmatrix} x_1 \\ x_2\end{pmatrix} = \begin{pmatrix} ax_1 + b_{11}x_1 + b_{12}x_2 \\ b_{21}x_1 + ax_2 + b_{22}x_2\end{pmatrix} \neq \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathcal{Y}$, because $b_{21} \neq b_{12}$ *

E. SHOW: $AX + XB = C$ HAS A UNIQUE SOLUTION IF AND ONLY IF A and $-B$ HAVE NO e.v.s IN COMMON.

please refere to Text. 2. Appendix F) p. 572

1. (ONLY IF PART): $AX + XB = C$ HAS UNIQUE SOLUTION $\Rightarrow$ A and $-B$ HAVE no common e.v.s.

We prove it by contradiction: assuming A and $-B$ have a common eigenvalue λ and P is a E.V. of A, Q is a E.V. of $-B'$ (we know $-B$ and $-B'$ have the same e.v.s). s.t.

$$AP = \lambda P \quad \text{and} \quad -B'Q = \lambda Q \quad \text{THUS} \quad B'Q = -\lambda Q$$

From 1, we see that (I_m, I_n have e.v.s of 1 with E.V. $[0, \cdots, 1, \cdots 0]'$)

λ is e.v. of matrix $A \otimes I_n$ and $I_m \otimes B'$

$$(A \otimes I_n + I_m \otimes B') \cdot (P \otimes Q)$$

$$= \begin{bmatrix} a_{11} I_n + B' & a_{12} I_n & \cdots & a_{1m} I_n \\ a_{21} I_n & a_{22} I_n + B' & \cdots & a_{2m} I_n \\ \vdots & \vdots & & \vdots \\ a_{m1} I_n & a_{m2} I_n & \cdots & a_{mm} I_n + B' \end{bmatrix} \cdot \begin{bmatrix} P_1 Q \\ P_2 Q \\ \vdots \\ P_m Q \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} P_1 Q + a_{12} P_2 Q + \cdots + a_{1m} P_m Q + P_1 B'Q \\ a_{21} P_1 Q + a_{22} P_2 Q + \cdots + a_{2m} P_m Q + P_2 B'Q \\ \vdots \\ a_{m1} P_1 Q + a_{m2} P_2 Q + \cdots + a_{mm} P_m Q + P_m B'Q \end{bmatrix}$$

FROM 1, equation (1)

$$= \begin{bmatrix} (a_{11} P_1 + a_{12} P_2 + \cdots + a_{1m} P_m) Q - \lambda P_1 Q \\ (a_{21} P_1 + a_{22} P_2 + \cdots + a_{2m} P_m) Q - \lambda P_2 Q \\ \vdots \\ (a_{m1} P_1 + a_{m2} P_2 + \cdots + a_{mm} P_m) Q - \lambda P_m Q \end{bmatrix}$$

$$= \begin{bmatrix} \lambda P_1 Q - \lambda P_1 Q \\ \lambda P_2 Q - \lambda P_2 Q \\ \vdots \\ \lambda P_m Q - \lambda P_m Q \end{bmatrix} = 0$$

Because P, Q are E.V.s, $P \otimes Q \neq 0$, Thus

$$\det (A \otimes I_n + I_m \otimes B') = 0,$$

That is the matrix equation has no unique solution. CONTRADICTION! So, A and $-B$ have no e.v.s in common. SHOWED ONLY IF!

PROBLEM 3. E. (XTRA).

The "if" part: If A and $-B$ have no e.v.s in common.

We are going to show $\{\lambda_i - \mu_j\}$ are the eigenvalue of $A\otimes I_n + I_m \otimes B'$:

$\forall$ λ, the e.v. of A, and μ, the e.v. of $-B$ (is e.v. of $-B'$ too).

And P is E.V. of A with λ, Q is E.V of $-B'$ with μ.

$$(A\otimes I_n + I_m\otimes B')(P\otimes Q)$$

$$= \begin{bmatrix} a_{11}I_n + B' & a_{12}I_n & \cdots & a_{1m}I_n \\ a_{21}I_n & a_{22}I_n + B' & \cdots & a_{2m}I_n \\ \vdots & \vdots & & \vdots \\ a_{m1}I_n & a_{m2}I_n & \cdots & a_{mm}I_n + B' \end{bmatrix} \begin{bmatrix} P_1Q \\ P_2Q \\ \vdots \\ P_mQ \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}P_1 + a_{12}P_2 + \cdots + a_{1m}P_m)Q + P_1B'Q \\ (a_{21}P_1 + a_{22}P_2 + \cdots + a_{2m}P_m)Q + P_2B'Q \\ \vdots \\ (a_{m1}P_1 + a_{m2}P_2 + \cdots + a_{mm}P_m)Q + P_mB'Q \end{bmatrix}$$

FROM 1, eq. (1) $\rightarrow$ $$= \begin{bmatrix} \lambda P_1Q - \mu P_1Q \\ \lambda P_2Q - \mu P_2Q \\ \vdots \\ \lambda P_mQ - \mu P_mQ \end{bmatrix} = (\lambda - \mu)\cdot \begin{bmatrix} P_1Q \\ P_2Q \\ \vdots \\ P_mQ \end{bmatrix} = (\lambda-\mu)(P\otimes Q)$$

$\therefore$ $\lambda_i - \mu_j$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ are the e.v.s of matrix $(A\otimes I_n + I_m\otimes B')$. Because A and $-B$ have no common e.v.s.

Thus $\lambda_i - \mu_j \neq 0$. $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$

And $A\otimes I_n + I_m\otimes B'$ is $mn\times mn$ matrix which can only have mn e.v.s.

So, $\lambda_i - \mu_j$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ are all the e.v.s of $A\otimes I_n + I_m\otimes B'$

$$\therefore \quad \det(A\otimes I_n + I_m\otimes B') = \prod_{i=1}^{m}\prod_{j=1}^{n}(\lambda_i - \mu_j) \neq 0$$

<u>THE MATRIX EQUATION HAS A UNIQUE SOLUTION!</u> <u>END OF IF PART.</u> Q.E.D.

PROBLEM 3.4.

SHOW: LET

$$P=\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \\ p_{n1} & p_{n2} & \cdots & p_{nn}\end{bmatrix} \triangleq \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}, \quad F=\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn}\end{bmatrix}, \quad Q=\begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{n1} & \cdots & q_{nn}\end{bmatrix} \triangleq \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$$

$$\therefore \quad \mathcal{X}=[x_1' \; x_2' \; \cdots \; x_n']', \quad y=[y_1' \; y_2' \; \cdots \; y_n']'$$

$$P-FPF' = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn}\end{bmatrix} - \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn}\end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn}\end{bmatrix} \cdot \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n1} \\ f_{12} & f_{22} & \cdots & f_{n2} \\ \vdots & \vdots & & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{nn}\end{bmatrix}$$

$$= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn}\end{bmatrix} - \begin{bmatrix} \sum_{i=1}^{n} f_{1i}p_{i1} & \cdots & \sum_{i=1}^{n} f_{1i}p_{in} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} f_{ni}p_{i1} & \cdots & \sum_{i=1}^{n} f_{ni}p_{in}\end{bmatrix} \cdot \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n1} \\ f_{12} & f_{22} & \cdots & f_{n2} \\ \vdots & \vdots & & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{nn}\end{bmatrix}$$

$$= \begin{bmatrix} p_{11}-\sum_{j=1}^{n}\sum_{i=1}^{n}(f_{1i}p_{ij}f_{1j}) & \cdots & p_{1n}-\sum_{j=1}^{n}\sum_{i=1}^{n}(f_{1i}p_{ij}f_{nj}) \\ \vdots & \ddots & \vdots \\ p_{n1}-\sum_{j=1}^{n}\sum_{i=1}^{n}(f_{ni}p_{ij}f_{1j}) & \cdots & p_{nn}-\sum_{j=1}^{n}(f_{ni}p_{ij}f_{nj}) \end{bmatrix} = Q = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn}\end{bmatrix}$$

$$\therefore \left\{ \begin{array}{l} p_{11}-\sum_{j=1}^{n}\sum_{i=1}^{n} f_{1i}f_{1j}p_{ij} = q_{11} \\ \vdots \\ p_{1n}-\sum_{j=1}^{n}\sum_{i=1}^{n} f_{1i}f_{nj}p_{ij} = q_{1n} \\ \vdots \\ p_{nn}-\sum_{j=1}^{n}\sum_{i=1}^{n} f_{ni}f_{nj}p_{ij} = q_{nn} \end{array} \right. \quad \cdots\cdots \quad (a)$$

NOW, we start from equation $(I-F\otimes F)\mathcal{X}=y$

$$(I-F\otimes F)\mathcal{X} = \begin{bmatrix} I_n-f_{11}F & -f_{12}F & \cdots & -f_{1n}F \\ -f_{21}F & I_n-f_{22}F & \cdots & -f_{2n}F \\ \vdots & \vdots & & \\ -f_{n1}F & -f_{n2}F & \cdots & I_n-f_{nn}F \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1-(f_{11}Fx_1+f_{12}Fx_2+\cdots+f_{1n}Fx_n) \\ x_2-(f_{21}Fx_1+f_{22}Fx_2+\cdots+f_{2n}Fx_n) \\ \vdots \\ x_n-(f_{n1}Fx_1+f_{n2}Fx_2+\cdots+f_{nn}Fx_n) \end{bmatrix}$$

$$= \begin{bmatrix} x_1-\sum_{i=1}^{n} f_{1i}\cdot Fx_i \\ \vdots \\ x_n-\sum_{i=1}^{n} f_{ni}\cdot Fx_i \end{bmatrix} \quad \cdots\cdots \quad (b)$$

(TO BE CONT.)

PROBLEM 3.4 (CONT.)

$$FX_i = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \cdot \begin{bmatrix} P_{i1} \\ P_{i2} \\ \vdots \\ P_{in} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{n} f_{1j} P_{ij} \\ \vdots \\ \sum_{j=1}^{n} f_{nj} P_{ij} \end{bmatrix}$$

So, equation (b) above.

$$(b) = \begin{bmatrix} P_{11} - \sum_{i=1}^{n} f_{1i} \cdot \sum_{j=1}^{n} f_{1j} P_{ij} \\ \vdots \\ P_{1n} - \sum_{i=1}^{n} f_{1i} \cdot \sum_{j=1}^{n} f_{nj} P_{ij} \\ \vdots \\ P_{nn} - \sum_{i=1}^{n} f_{ni} \sum_{j=1}^{n} f_{nj} P_{ij} \end{bmatrix}$$

$$= \begin{bmatrix} P_{11} - \sum_{j=1}^{n} \sum_{i=1}^{n} f_{1i} f_{1j} P_{ij} \\ \vdots \\ P_{1n} - \sum_{j=1}^{n} \sum_{i=1}^{n} f_{1i} f_{nj} P_{ij} \\ \vdots \\ P_{nn} - \sum_{j=1}^{n} \sum_{i=1}^{n} f_{ni} f_{nj} P_{ij} \end{bmatrix} = y = \begin{bmatrix} q_{11} \\ \vdots \\ q_{1n} \\ \vdots \\ q_{nn} \end{bmatrix} \quad \cdots\cdots \ (c)$$

Compare equation (c) to eq. (a) on the previous page, we have the conclusion:

MATRIX EQUATION $P - FPF' = Q$ CAN BE WRITTEN AS $(I - F \otimes F)\mathcal{X} = y$

END OF PROBLEM 3 (BONUS).

please keep in mind "bonus problems" are optional. Don't kill too much of your time. HS

Conclusion : ?/2 bonus points = 7 pages = 2 days.

DR. HSU, THANK YOU FOR ENCOURAGING ME IN EVERY HOMEWORK ASSIGNMENT. I DO REALLY LIKE YOUR LECTURES AND HOMEWORK ASSIGNMENTS, AS WELL AS BONUS PROBLEMS.

Ben.

Since generally $\Phi(x+h) = \Phi(x) + h\Phi'(x) + \frac{1}{2}h^2\Phi''(x+\theta h)$ with $0 < \theta < 1$, we find $\Phi(x+h) - \Phi(x) \geqslant h\Phi'(x)$, hence

$$\{\Phi(x+h) - \Phi(x)\}/h \geqslant \Phi'(x) \text{ for positive } h,$$

$$\{\Phi(x) - \Phi(x+h)\}/|\, h \,| \leqslant \Phi'(x) \text{ for negative } h.$$

This implies

$$\frac{\Phi(x_2) - \Phi(x_3)}{x_2 - x_3} \geqslant \Phi'(x_3) \geqslant \frac{\Phi(x_3) - \Phi(x_1)}{x_3 - x_1},$$

hence $(x_3 - x_1)\{\Phi(x_2) - \Phi(x_3)\} \geqslant (x_2 - x_3)\{\Phi(x_3) - \Phi(x_1)\}$ or $(x_2 - x_1)\Phi(x_3) \leqslant (x_2 - x_3)\Phi(x_1) + (x_3 - x_1)\Phi(x_2)$. This gives finally

$$\Phi(p_1x_1 + p_2x_2) = \Phi(x_3) \leqslant p_1\Phi(x_1) + p_2\Phi(x_2).$$

Theorem 8. *Let the bounded linear transformation T have the sequence $\xi_1, \xi_2, \ldots$ of characteristic values $\neq 0$, each characteristic value repeated according to its algebraic multiplicity, and arranged such that $|\,\xi_1\,| \geqslant |\,\xi_2\,| \geqslant \ldots$. Let furthermore $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$ be the sequence of singular values of T and T^*. Then, if $\omega(x)$, defined for $x \geqslant 0$, is continuous and non-decreasing and such that $\Phi(x) = \omega(e^x)$ is a convex function of x, we have*

$$\Sigma_1^n\, \omega(|\,\xi_i\,|) \leqslant \Sigma_1^n\, \omega(\lambda_i).$$

In particular, if $s > 0$ is arbitrary,

$$\Sigma_1^n\, |\,\xi_i\,|^s \leqslant \Sigma_1^n\, \lambda_i^s,$$

so that, if the number of characteristic values ξ_i is infinite, the series $\Sigma_1^\infty |\,\xi_i\,|^s$ converges if $\Sigma_1^\infty \lambda_i^s$ converges [12] [15].

Proof. Let $\omega(x)$ be defined for $x \geqslant 0$ as a continuous and non-decreasing function such that $\omega(e^x)$ is convex in x. Then $\Phi(x) = \omega(e^x)$ is defined for all real x as a continuous convex function which is non-decreasing. By Theorem 7 we have $\Pi_1^n |\,\xi_i\,| \leqslant \Pi_1^n \lambda_i$; hence; writing $a_i' = \log |\,\xi_i\,|$, $a_i = \log \lambda_i$,

$$\Sigma_1^n\, a_i' \leqslant \Sigma_1^n\, a_i.$$

Then, on account of Lemma χ,

$$\Sigma_1^n\, \Phi(a_i') \leqslant \Sigma_1^n\, \Phi(a_i),$$

$= \omega(e^x)$,

$$\Sigma_1^n\, \omega(|\,\xi_i\,|) \leqslant \Sigma_1^n\, \omega(\lambda_i).$$

$= x^s$, where $s > 0$, $x \geqslant 0$, the corresponding function

T is bounded
$\Rightarrow$ T^* exists.

For a matrix
$A = n \times n$
A' or A^* exist.
, of course.

NOTE: THE characteristic value definition in this book is the same as the definition of eigenvalue in other books.

RECEIVED
SEP 10 1987

BONUS PROBLEM FOR EE 501

LINEAR SYSTEM THEORY

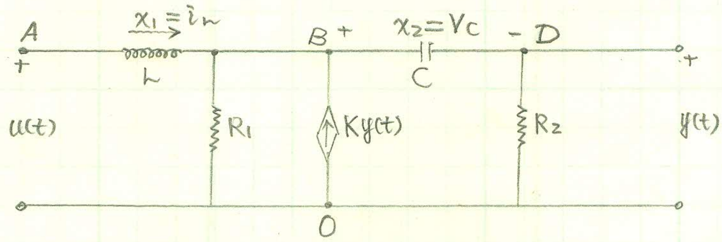
SEP. 8, 1987

Benmei Chen

Box 241, Gonzaga University

Spokane, Washington 99258

1. GIVE A COMPLETE STUDY OF THE CIRCUIT BELOW:



Good

2/2

IN the circuit above, we assume two state variables, $x_1 = i_L$, the current through the inductor, and $x_2 = V_c$, the voltage across the capacitor. Looking at the loop ABDO, we have

$$L \cdot \dot{x}_1 + x_2 + C \cdot R_2 \dot{x}_2 = u(t) \quad \text{------} \quad (1)$$

At the node B,

$$x_1 + k \cdot y(t) = C\dot{x}_2 + (x_2 + y)/R_1 \quad \text{------} \quad (2)$$

And it is obvious that,

$$y(t) = R_2 \cdot C \cdot \dot{x}_2 \quad \text{------} \quad (3)$$

Rewrite the equation (1), (2) and (3) as,

$$L\dot{x}_1 + CR_2 \dot{x}_2 = -x_2 + u(t) \quad \text{---} \quad (1')$$

$$C \cdot (kR_1R_2 - R_1 - R_2)\dot{x}_2 = -R_1 x_1 + x_2 \quad \text{---} \quad (2')$$

$$y(t) = R_2/(kR_1R_2 - R_1 - R_2) \cdot (-R_1 x_1 + x_2) \quad \text{---} \quad (3')$$

Now we can write down the state variable system form:

$$S: \begin{bmatrix} L & CR_2 \\ 0 & C\cdot[kR_1R_2 - R_1 - R_2] \end{bmatrix} \cdot \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ -R_1 & 1 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot u(t)$$

$$y(t) = \left[-R_1R_2/(kR_1R_2 - R_1 - R_2) \quad R_2/(kR_1R_2 - R_1 - R_2) \right] \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Or $\{E, A, \underline{b}, \underline{c}, d\}$ with

$$E = \begin{bmatrix} L & CR_2 \\ 0 & C\cdot[kR_1R_2 - R_1 - R_2] \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 \\ -R_1 & 1 \end{bmatrix}, \quad \underline{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underline{c} = \frac{R_2}{kR_1R_2 - R_1 - R_2}[-R_1, \ 1] \ \& \ d = 0$$

Assume that, $kR_1R_2 - R_1 - R_2 \neq 0$, that is

$$K \neq (R_1 + R_2)/R_1R_2$$

The system can be simplified as

$$\dot{\underline{x}} = \begin{bmatrix} L & CR_2 \\ 0 & C\cdot[kR_1R_2 - R_1 - R_2] \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & -1 \\ -R_1 & 1 \end{bmatrix} \cdot \underline{x} + \begin{bmatrix} L & CR_2 \\ 0 & C(kR_1R_2 - R_1 - R_2) \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$

$$= \begin{bmatrix} \frac{1}{L} & -\frac{R_2}{L(kR_1R_2 - R_1 - R_2)} \\ 0 & \frac{1}{C(kR_1R_2 - R_1 - R_2)} \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ -R_1 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} \frac{1}{L} & -\frac{R_2}{L(kR_1R_2 - R_1 - R_2)} \\ 0 & \frac{1}{C(kR_1R_2 - R_1 - R_2)} \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$

$$S = \begin{cases} \dot{\underline{x}} = \begin{bmatrix} \frac{R_1R_2}{L(kR_1R_2 - R_1 - R_2)} & -\frac{kR_1R_2 - R_1}{L(kR_1R_2 - R_1 - R_2)} \\ -\frac{R_1}{C(kR_1R_2 - R_1 - R_2)} & \frac{1}{C(kR_1R_2 - R_1 - R_2)} \end{bmatrix} \underline{x} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \cdot u(t) & \cdots (a) \\ y(t) = \begin{bmatrix} \frac{-R_1R_2}{kR_1R_2 - R_1 - R_2} & \frac{R_2}{kR_1R_2 - R_1 - R_2} \end{bmatrix} \cdot \underline{x} & \cdots (b) \end{cases}$$

✓

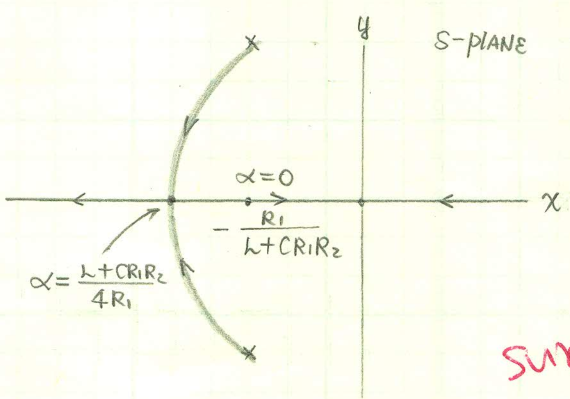
LET $\alpha = kR_1R_2 - R_1 - R_2 \neq 0$, we have the transform function

$$g(s) = \left[-\frac{R_1R_2}{\alpha}, \frac{R_2}{\alpha}\right] \cdot \begin{bmatrix} s - \frac{R_1R_2}{L\cdot\alpha} & \frac{kR_1R_2 - R_1}{L\cdot\alpha} \\ \frac{R_1}{C\cdot\alpha} & s - \frac{1}{C\cdot\alpha} \end{bmatrix}^{-1} \cdot \begin{pmatrix} 1/L \\ 0 \end{pmatrix}$$

$$= \frac{-CR_1R_2 \cdot s}{C\cdot L\cdot\alpha\cdot s^2 - (L + CR_1R_2)s - R_1} \triangleq \frac{as}{\alpha\cdot s^2 + bs + d}$$

Where $a = -R_1R_2/L$, $b = -(L + CR_1R_2)/C\cdot L$, $d = -R_1/C\cdot L$ and $\alpha = kR_1R_2 - R_1 - R_2$

Then the root-locus:



✓

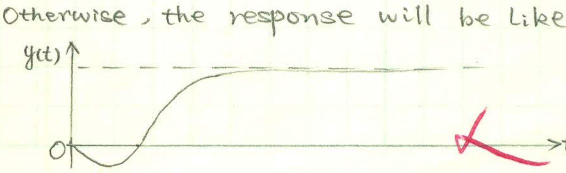
The unit step response,

$$y(s) = \frac{a}{\alpha\cdot s^2 + bs + d}$$

IF $b^2 - 4\alpha\cdot d \geq 0$, the unit step-response will look like,

sure ?

Otherwise, the response will be like,



#

$\frac{1}{2}$

2. DISSCUSS HOW TO COMPUTE THE INTEGRAL

$$\int_0^t e^{A\sigma}d\sigma \quad \text{IF} \quad |A| = 0$$

---

definition

DISSCUSSION: According to the defination,

$$e^{A\sigma} \triangleq I + A\sigma + \frac{A^2}{2!}\sigma^2 + \frac{A^3}{3!}\sigma^3 + \cdots + \frac{A^n}{n!}\sigma^n + \cdots, \quad \forall A$$

So,

$$\int_0^t e^{A\sigma}d\sigma = \int_0^t \left(I + A\sigma + \frac{A^2}{2!}\sigma^2 + \frac{A^3}{3!}\sigma^3 + \cdots\right)d\sigma$$

$$= I\cdot\int_0^t d\sigma + A\cdot\int_0^t \sigma\, d\sigma + \frac{A^2}{2!}\cdot\int_0^t \sigma^2 d\sigma + \cdots$$

$$= I\cdot t + \frac{A}{2!}t^2 + \frac{A^2}{3!}\cdot t^3 + \frac{A^3}{4!}\cdot t^4 + \cdots$$

$$\boxed{\int_0^t e^{A\sigma}d\sigma = t\cdot\left[I + \frac{At}{2}\left[I + \frac{At}{3}\left[I + \frac{At}{4}[\cdots]\right]\right]\right]}$$

$$\frac{At}{n} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty$$

So, the equation above can be easilly computed by using a recursive program. In practice, a finite n, say 10 or 100, will be good enough to product a close result to $\int_0^t e^{A\sigma}d\sigma$.

produce

Would you try one example 4x4 to verify your statement

HOMEWORK SET NO: 5

EE 501

LINEAR SYSTEM THEORY

DEC. 8, 1987

Benmei Chen

Box 241, Gonzaga U.

Spokane, WA 99258

10/10

PROBLEM 1: Repeat problem 5 of Test # 2

$$H_1(S)=\frac{1}{S+1}\ ,\qquad H_2(S)=\frac{S+1}{(S+2)(S+3)}=\frac{S+1}{S^2+5S+6}$$

We can realize both systems by CR-3 . and it's easy to check that both realizations are c.o.

$$S1:\ \begin{cases}\dot{x}_1=-x_1+u_1\\ y_1=x_1\end{cases}\qquad S2:\ \left\{\begin{pmatrix}\dot{x}_2\\ \dot{x}_3\end{pmatrix}=\begin{pmatrix}-5 & -6\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}x_2\\ x_3\end{pmatrix}+\begin{pmatrix}1\\ 0\end{pmatrix}u_2\ ,\quad y_2=[1,1]\begin{pmatrix}x_2\\ x_3\end{pmatrix}\right.$$

a. $H_1(S)\cdot H_2(S)$, Use the results in problem 1 (c) of Homework #4 , part a. , we have the connected system

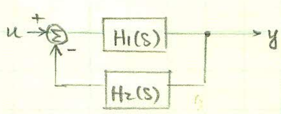
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 1 & -5 & -6\\ 0 & 1 & 0\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\cdot u\quad \&\quad y=[0\ 1\ 1]\cdot\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$\lambda(A_{H_1H_2})=\{-1,-2,-3\}$ CHECK $\rho[\lambda_i I-A\ \ B]$ and $\rho\begin{bmatrix}C\\ \lambda_i I-A\end{bmatrix}$, we have odz $=-1$ ✓

$H_2(S)\cdot H_1(S)$, then the connected system is

$$\begin{bmatrix}\dot{x}_2\\ \dot{x}_3\\ \dot{x}_1\end{bmatrix}=\begin{bmatrix}-5 & -6 & 0\\ 1 & 0 & 0\\ 1 & 1 & -1\end{bmatrix}\cdot\begin{bmatrix}x_2\\ x_3\\ x_1\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\quad \&\quad y=[0\ 0\ 1]\cdot\begin{bmatrix}x_2\\ x_3\\ x_1\end{bmatrix}$$

Again, use MATLAB to check the conditions . we have idz $=-1$ ✓

b. 

The transfer function for this connected system is

$$H(S)=\frac{H_1(S)}{1+H_1(S)H_2(S)}=\frac{(S+2)(S+3)}{(S+1)(S^2+5S+7)}$$

✓

H(S) is irreducible and deg $a(s)=3$, from Thm 2.4-4 in text 1 , we have no idz and odz .

And then vice versa: Use the results in part c , problem 1 (c) of H.W. #4 , we have

$$\begin{bmatrix}\dot{x}_2\\ \dot{x}_3\\ \dot{x}_1\end{bmatrix}=\begin{bmatrix}-5 & -6 & 1\\ 1 & 0 & 0\\ 1 & 1 & -1\end{bmatrix}\cdot\begin{bmatrix}x_2\\ x_3\\ x_1\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\cdot u\quad \&\quad y=[1\ 1\ 0]\cdot\begin{bmatrix}x_2\\ x_3\\ x_1\end{bmatrix}$$

CHECKED , we have ~~idz = -1 and odz = -1~~ iodz $=-1$ ✓

c. $H_1(S)=\frac{1}{S-1}$ and $H_2(S)=\frac{S-1}{S+1}$

EXPLAIN : From the results of part a, problem 1 (c) of H.W. #4 , we know

$H_1(S)\cdot H_2(S)$ is C.C. and not C.O. and $H_2(S)\cdot H_1(S)$ is not C.C. ✓

<u>PROBLEM 2:</u> (a) $A = -(B + \alpha I)$, Show that A is stable if $\alpha > \|B\|_2$

<u>SHOW:</u> Let λ be the e.v. of A, then

$$|\lambda I - A| = |\lambda I + B + \alpha I| = |(\lambda + \alpha)I - (-B)| = 0$$

So, $\lambda + \alpha$ is an e.v of $-B$.

According A.60 on page 668 in text 1,

$$|\lambda + \alpha| \le \bar{\sigma}(-B) = \bar{\sigma}(B) = \|B\|_2 \text{ ------ } (1)$$

If $Re(\lambda) \geq 0$, and from statement we have that $\alpha > \|B\|_2$, However from (1)

$$\|B\|_2 \geq |\lambda + \alpha| = |(\alpha + Re(\lambda)) + i \cdot Im(\lambda)| = \sqrt{(\alpha + Re(\lambda))^2 + Im^2(\lambda)} \geq \alpha + Re(\lambda) \geq \alpha \Rightarrow \alpha \leq \|B\|_2$$

CONTRADICTION! Thus $Re(\lambda)$ must be negative and we showed that A is stable. <u>Q.E.D</u>

(b) $|A| \neq 0$, $|A+E| \neq 0$, Show that $\|(A+E)^{-1} - A^{-1}\| \leq \|E\| \cdot \|A^{-1}\| \cdot \|(A+E)^{-1}\|$

<u>SHOW:</u> Rewrite ~~$A+E = A + E \cdot I \cdot I$ and then use A.21 on page 656 in text 1,~~

~~$(A+E)^{-1} = (A+E\cdot I \cdot I)^{-1} = A^{-1} - A^{-1} \cdot E(A^{-1} \cdot E + I)^{-1} A^{-1} = A^{-1} - A^{-1} \cdot E \cdot (E + A)^{-1}$~~

$$\Rightarrow \quad (A+E)^{-1} - A^{-1} = -A^{-1} \cdot E \cdot (A+E)^{-1}$$

$$\|(A+E)^{-1} - A^{-1}\| = \|-A^{-1} \cdot E \circ (A+E)^{-1}\| = |-1| \cdot \|A^{-1} \cdot E \cdot (A+E)^{-1}\|$$

$$\leq \|A^{-1}\| \cdot \|E\| \cdot \|(A+E)^{-1}\| \qquad \text{(2-92 on P58 in text 2)}$$

$$= \|E\| \cdot \|A^{-1}\| \cdot \|(A+E)^{-1}\| \qquad \text{(norm is scalar)}$$

<u>Q.E.D</u>

$$(A+E)\left[(A+E)^{-1} - A^{-1}\right] = I - (A+E)A^{-1} = -EA^{-1}$$

$$\therefore \quad (A+E)^{-1} - A^{-1} = -(A+E)^{-1} E A^{-1}$$

PROBLEM 3 Consider the matrix equation , $P > 0$, $Q > 0$

$$-\alpha A^*P - \alpha PA + A^*PA + (\alpha^2 - \gamma^2)P = -Q$$ $*$ denotes the transpose conjugate.

Show that eigenvalues of A are within a circle with radius γ .

SHOW: Assume λ is an e.v. of A and $\underline{e}$ is an E.V. associated with it. So

$$A\underline{e} = \lambda\underline{e} \Rightarrow \underline{e}^*A^* = \lambda^*\underline{e}^*$$

Then, pre- and post-multiply $\underline{e}^*$ and $\underline{e}$, respectly, to the equation in statement.

$$\underline{e}^* \cdot [-\alpha A^*P - \alpha PA + A^*PA + (\alpha^2-\gamma^2)P] = -\underline{e}^*Q\underline{e}$$

$$-\alpha\underline{e}^*A^*P\underline{e} - \alpha\underline{e}^*PA\underline{e} + \underline{e}^*A^*PA\underline{e} + (\alpha^2-\gamma^2)\cdot\underline{e}^*P\underline{e} = -\underline{e}^*Q\underline{e}$$

$$-\alpha\lambda^*\cdot\underline{e}^*P\underline{e} - \alpha\cdot\lambda\cdot\underline{e}^*P\underline{e} + \lambda^*\cdot\lambda\cdot\underline{e}^*P\underline{e} + (\alpha^2-\gamma^2)\cdot\underline{e}^*P\underline{e} = -\underline{e}^*Q\underline{e}$$

Thus, $[-\alpha\lambda^* - \alpha\lambda + \lambda^*\lambda + \alpha^2 - \gamma^2]\cdot\underbrace{\underline{e}^*P\underline{e}}_{>0} = \underbrace{-\underline{e}^*Q\underline{e}}_{<0}$ $(\because P>0,\ Q>0)$

$\therefore$ $-\alpha\lambda^* - \alpha\lambda + \lambda^*\lambda + \alpha^2 - \gamma^2 < 0$

Let $\lambda = a + i\cdot b$, Then we have

$$-2a\alpha + a^2 + b^2 + \alpha^2 < \gamma^2.$$

OR

$$(a-\alpha)^2 + b^2 < \gamma^2$$

So, we have shown that the eigenvalues of matrix A are within a circle with radius γ and center point $(\alpha, 0)$ or $\alpha + j0$. Q.E.D

PROBLEM 4: Refer to: Computer control of a double inverted pendulum.

(i) Use the model (6a)*, by "try and ERROR on T" determine an appropriate feedback gain matrix (OR vector) $\underline{K} \triangleq B'W_T^{-1}$

(ii) "Compare" your results with those given in (10) and (15), by computing e.v.s of closed-loop system, root-locus, Nyquist plot etc as your wish.

— — — — — — — — — — —

SOLUTION: From the paper, we have

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -5.46 & 0.188 & -3.43 & 0.0120 & -0.000426 \\ 0 & 27.9 & -12.7 & 3.67 & -0.0862 & 0.0288 \\ 0 & -31.1 & 53.8 & -4.10 & 0.180 & -0.122 \end{bmatrix} \cdot \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 26.7 \\ -28.6 \\ 32.0 \end{bmatrix} \cdot u$$

(i) Using results of problem 3 in H.W. # 2 to compute $W_T = \int_0^T e^{-A\sigma}BB'e^{-A'\sigma}d\sigma$ & MATLAB.

a) PICK $T = 0.5$, we have $K_{0.5} = [983.6 \quad 2476.6 \quad 1497.9 \quad 273.2 \quad 543.7 \quad 260.2]$

b) PICK $T = 1.5$, we have $K_{1.5} = [2.3554 \quad 19.0920 \quad 33.5352 \quad 1.8729 \quad 7.0616 \quad 5.6999]$

c) Pick $T = 2.0$, we have $K_{2.0} = [0.0739 \quad 1.1449 \quad 10.8526 \quad -0.0127 \quad 1.1202 \quad 1.6705]$

(ii) Because of limitation of hardware at G.U. (By the way, they only allow me using MATLAB in one particular P.C.), I can only do the comparison by computing the eigenvalues:

O.K.

e.v.s in (10)	e.v.s in (15)	e.v.s FOR $K_{0.5}$	e.v.s FOR $K_{1.5}$	e.v.s FOR $K_{2.0}$
$-0.134 + 0.909j$	-1.0040	$-8.3548 + 15.9913j$	$-3.2390 + 3.7170j$	-8.2828
$-0.134 - 0.909j$	$-4.483 + 2.854j$	$-8.3548 - 15.9913j$	$-3.2390 - 3.7170j$	-7.5670
$-1.27 + 4.49j$	$-4.483 - 2.854j$	$-13.5938 + 9.2453j$	$-8.1282 + 0.0689j$	-4.7885
$-1.27 - 4.49j$	$-23.062 + 0.8825j$	$-13.5938 - 9.2453j$	$-8.1282 - 0.0689j$	$-0.7765 + 1.2946j$
$-22.2 + 45.2j$	$-23.062 - 0.8825j$	$-15.8704 + 3.0301j$	$-5.6720 + 1.4389j$	$-0.7765 - 1.2946j$
$-22.2 - 45.2j$	-81.1	$-15.8704 - 3.0301j$	$-5.6720 - 1.4389j$	-2.5241