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EE 501 | Test #1 | Fall 1987

Notes:

1. 75-minute exam: 9:10am to 10:25am, Thursday, October 8, 1987

2. Closed-book exam, 5-page crib pages are allowed, along with "main results of Chapter 1 and Chapter 2" handout.

3. CRC Table and a calculator are permitted.

Name Benmei Chen

ID# G. U.

Grade

26.5 / 30

Class rank

2 / 33

#1. (20 points, 2.5 points each)

a) Show that $|e^{At}| = e^{Tr(A)\cdot t}$

<u>Hint</u>: $P^{-1}AP = \Lambda$

SHOW : Assuming Matrix A has n distinct e.v. so, $\exists\ P = [P_1, P_2, \cdots, P_n]$

AND $Q = P^{-1} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$

$$e^{At} = e^{P\Lambda P^{-1}t} = Pe^{\Lambda t}P^{-1}$$

$$= P \cdot e^{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}t} P^{-1} = P\begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}P^{-1}$$

$$|e^{At}| = |P|\cdot|P^{-1}|\, e^{(\lambda_1 t \cdots \lambda_n)t} = e^{Tr(A)\cdot t}.$$

b) Let $\underline{V} \in R^n$ be non zero. An nxn matrix H of the form

$$H = I - 2\frac{\underline{V}\ \underline{V}'}{\underline{V}'\ \underline{V}}$$

is known as a Householder transformation. Show that H is an orthogonal matrix.

$$H' = I' - 2\left(\frac{\underline{V}\ \underline{V}'}{\underline{V}'\ V}\right)'$$

a number

$$= I - 2\cdot\frac{\underline{V}\ \underline{V}'}{\underline{V}'\ V}$$

$$= H.$$

$\therefore$ H IS O.G. MATRIX.

−2

(Show : $H'H = I$

c) $A = A'$, $A\underline{x}_i = \lambda_i B\underline{x}_i$, $A\underline{x}_j = \lambda_j B\underline{x}_j$, $\lambda_i \neq \lambda_j$ show that $\underline{x}_i' B \underline{x}_j = 0$

SHOW:
$$\lambda_j x_i' B x_j = x_i'(\lambda_j B x_j) = x_i'(A x_j)$$

$A' = A$
$$= x_i' A \cdot x_j$$

$\Rightarrow B = B'$
$$= (A x_i)' \cdot x_j \qquad (A = A')$$
$$= (\lambda_i B x_i)' x_j = \lambda_i x_i' B' x_j$$
$$= \lambda_i x_i' B x_j$$
$$\lambda_i \neq \lambda_j \Rightarrow \quad x_i' B x_j = 0.$$

d) $e^{At} = \begin{bmatrix} -e^{-t} + \alpha e^{-2t} & -e^{-t} + \beta e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$

Determine α, β, A and A^{100}

$\mathcal{L}^{-1}$

$$(sI - A)^{-1} = \begin{bmatrix} -\frac{1}{s+1} + \alpha \frac{1}{s+2} & \frac{-1}{s+1} + \frac{\beta}{s+2} \\ \frac{2}{s+1} - \frac{2}{s+2} & \frac{2}{s+1} - \frac{1}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(\alpha-1)s + (\alpha-2)}{(s+1)(s+2)} & \frac{(\beta-1)s + (\beta-2)}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix}$$

$$\therefore \quad sI - A = \begin{bmatrix} s+3 & (1-\beta)s + (2-\beta) \\ -2 & (\alpha-1)s + \alpha - 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & -(1-\beta)s - (2-\beta) \\ 2 & (1-\alpha)s + 2 - \alpha \end{bmatrix}$$

$\therefore \quad \beta = 1 \qquad \alpha = 2$, $\quad A = \begin{bmatrix} -3 & \cancel{0}\ -1 \\ 2 & 0 \end{bmatrix}$

$$A^{100} = \begin{bmatrix} (-3)^{100} & 0 \\ 2\cdot(-3)^{99} & 0 \end{bmatrix}$$

$q'A = \lambda q'$ $q'B=0$

e) Show that if {A, B} is not C. C., then {A+αI, B } is not C.C.

{A, B} is not C. C. THEN

$$\mathcal{C} = [B, AB, A^2B, \cdots A^{n-1}B]$$

-0.5

FOR $\{A+\alpha I, B\}$, $\mathcal{C}' = [B, (A+\alpha I)B, \cdots, (A+\alpha I)^{n-1}B]$.

$\forall\ i\ ;\ i=0, \cdots, n-1$.

$$(A+\alpha I)^i = A^i + a_1 A^{i-1} + \cdots + a_n I.$$

$$\therefore (A+\alpha I)^i B = A^i B + a_1 A^{i-1} B + \cdots + a_n B$$

∴ Any column of C' can be represented by the columns of $\mathcal{C}$.

⇒ If {A, B} is not C.C, the {A+αI, B} is not C.C.

f) Let $A_d = e^{AT} = e^{\frac{AT}{2}} \left[e^{\frac{-AT}{2}}\right]^{-1}$

Ignore H. O. T., derive a formula for A in terms of A_d and T.

$$Ad = e^{\frac{AT}{2}} \cdot \left[e^{-\frac{AT}{2}}\right]^{-1}$$

$$= \left(I + \frac{AT}{2} + \frac{(\frac{A}{2})^2 T^2}{2!} + \cdots\right)\left(I - \frac{AT}{2} + \frac{(\frac{A}{2})^2 T^2}{2!} - \cdots\right)^{-1}$$

$$\simeq \left(I + \frac{AT}{2}\right)\left(I - \frac{AT}{2}\right)^{-1}$$

$$Ad\left(I - \frac{AT}{2}\right) = \left(I + \frac{AT}{2}\right)$$

$$Ad - Ad \cdot \frac{AT}{2} = I + \frac{AT}{2}$$

$$Ad - I = A \cdot \frac{T}{2}(I + Ad)$$

$$A = \frac{2}{T}(Ad - I)(Ad + I)^{-1}$$

g) $S = R(A)$, show that $S^{\perp} \subset N(A')$

Assuming $A = m \times n$.

$\therefore$ LET $z_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $z_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ $\cdots$, $z_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

According definition:

$$Az_1,\ Az_2,\ \cdots\ Az_n \in R(A)$$

$\therefore \forall\ x \in S^{\perp} = R^{\perp}(A)$

$$x'Az_1 = x'Az_2 = \cdots = x'Az_n = 0.$$

$$z_1'A'x = z_2'A'x = \cdots = z_n'A'x = 0.$$

$$\begin{pmatrix} z_1'A'x \\ \vdots \\ z_n'Ax \end{pmatrix} = \begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{pmatrix} A'x = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} A'x = IA'x = A'x = 0$$

$\therefore x \in N(A')$, THEN $S^{\perp} \subset N(A')$

h) $A:\ n \times n$, $|A| \neq 0$ show that

$\underline{\sigma}(A)\,\bar{\sigma}(A^{-1}) = 1$

$|A| \neq 0.$

$\exists\ T$, $T^{-1}AT = \begin{pmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{pmatrix}$

NO, use SVD

$$(T^{-1}AT)^{-1} = T^{-1}A^{-1}T = \begin{pmatrix} \frac{1}{\sigma_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_n} \end{pmatrix}$$

LET $\sigma_i = \min(\sigma_1, \cdots, \sigma_n) = \underline{\sigma}(A)$

THEN $\frac{1}{\sigma_i} = \max\left(\frac{1}{\sigma_1}, \cdots \frac{1}{\sigma_n}\right) = \bar{\sigma}(A^{-1})$

$$\therefore\ \underline{\sigma}(A) \cdot \bar{\sigma}(A^{-1}) = \sigma_i \cdot \frac{1}{\sigma_i} = 1$$

#2. (5 points)

Verify the following substitutions

$$A \longrightarrow (A_d - I)(A_d + I)^{-1}$$

$$Q \longrightarrow 2(A_d' + I)^{-1} Q_d (A_d + I)^{-1}$$

$$P \longrightarrow P_d$$

will convert the continuous-time Lyapunov equation $A'P + PA = -Q$ into its discrete-time counterpart

$$P_d - A_d' P_d A_d = Q_d$$

SHOW : $A' = [(A_d - I)(A_d + I)^{-1}]'$

$= [(A_d + I)^{-1}]' \cdot (A_d - I)'$

$= [(A_d + I)']^{-1} \cdot (A_d - I)' = (A_d' + I)^{-1} \cdot (A_d - I)'$

$A'P + PA = (A_d' + I)^{-1} \cdot (A_d - I)' \cdot P_d + P_d \cdot (A_d - I)(A_d + I)^{-1}$

$= -Q = -2(A_d' + I)^{-1} Q_d (A_d + I)^{-1}$

$(A_d - I)' P_d (A_d + I) + (A_d' + I) P_d (A_d - I) = -2Q_d$

$(A_d' - I) P_d \cdot (A_d + I) + (A_d' + I) P_d (A_d - I) = -2Q_d$

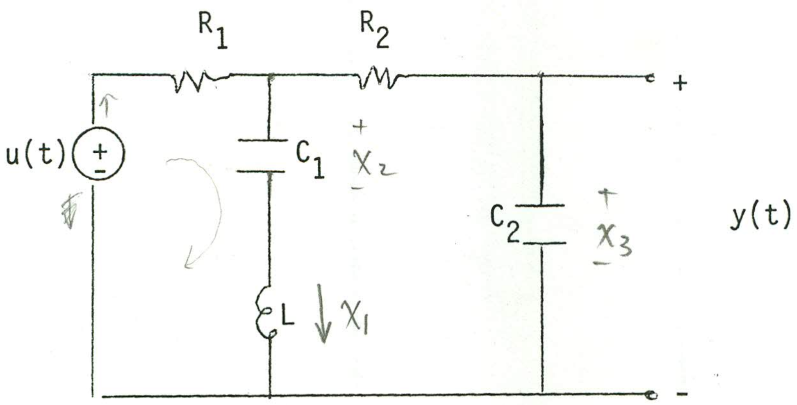
$A_d' P_d A_d + \cancel{A_d' P_d} - \cancel{P_d A_d} - P_d$

$+ A_d' P_d A_d - \cancel{A_d' P_d} + \cancel{P_d A_d} - P_d = -2Q_d$

$\therefore 2P_d - 2A_d' P_d A_d = 2Q_d$

$\Rightarrow P_d - A_d' P_d A_d = Q_d$ Q.E.D.

#3. (5 points)



WLOG, Let $R_1 = R_2 = 1$, $C_1 = C_2 = 1$, $L=1$,

a) Derive a model $\{A, \underline{b}, \underline{c}\}$ for the given circuit.

b) Is the circuit asymptotically stable? completely controllable? completely observable? Give explanation or justification of your results.

$$(C_2\dot{x}_3 + x_1)R_1 + x_2 + L\dot{x}_1 = u$$

$\dot{x}_1 = -x_1 - 2x_2 + x_3 + u$

$\dot{x}_2 = x_1$

$\dot{x}_3 = x_2 - x_3$

$\dot{x}_1 + \dot{x}_3 = -x_1 - x_2 + u$

$\dot{x}_1 - \dot{x}_3 = -x_2 + x_3$

$\dot{x}_2 = x_1$

$y = x_3$

$$\Rightarrow \cancel{S \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u}$$

a) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u$$

$$y = [0 \quad 0 \quad 1] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

?

b) we have $\lambda_1 = 0$, $\lambda_{2,3} = -1 \pm j1.4142$

① so the circuit is asymtotically stable.

② $[B \quad AB \quad A^2B] = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ $\det [B \quad AB \quad A^2B] = 1 \Rightarrow$ C.C.

③ $\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, $\det \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = -1 \Rightarrow$ C.O

EE 501 Test #2 Fall 1987

NOTES:

1. 75-minute exam: 9:10 a.m. to 10:25 a.m.
   Thursday November 19, 1987

2. Closed-book exam, 5-page crib pages are allowed, along with "main results of Chapters 1,2, & 3" handout.

3. CRC Table and a calculator are permitted.

Name Benmei Chen

ID# G. U.

Grade

28.5 / 30

Class rank

1 / 31

Excellent

$\{(\lambda^2+\alpha)I - A^2 - \alpha I\ B\}$

$= \{\lambda^2 I - A^2, B\}$

$= (\lambda I - A, B)\begin{pmatrix} \lambda I - A & 0 \\ 0 & I \end{pmatrix}$

1. (5points)

Use the PBH test to prove or disprove that, for $\alpha \varepsilon R^1$, $\{A,B\}$ C.C. implies $\{A^2+\alpha I,B\}$ C.C.

Counter example $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(A, B) C.C.

But (A^2, B)

$= (I, B)$ not C.C.

$\{A, B\}$ is C.C. $\Rightarrow \exists$ no $q' \neq 0$. $q'A = \lambda q'$, $q'B = 0$.

~~$\rho[\lambda I - A, B] = n$.~~

~~$\rho[\lambda I - \alpha I - A^2, B]$~~

~~$= \rho[(\lambda - \alpha)I - A^2, B]$~~

$\Rightarrow \exists q' \neq 0$, $q'(A^2 + \alpha I) = q'A^2 + \alpha q'$

$= \lambda \cdot q'A + \alpha q'$

$= \lambda^2 q' + \alpha q'$

$= (\lambda^2 + \alpha) q'$

$qB = 0$.

$\{A^2 + \alpha I, B\}$ is C.C. when $\alpha = -\lambda_i^2$, $\forall i$

otherwise. $\{A^2 + \alpha I, B\}$ is not C.C.

-1

2

2. (6 points)
Consider an SISO model $\{A, \underline{b}, \underline{c}\}$. Let $Q = \underline{b}\,\underline{c}$.
(a) show that Q has an eigenvalue $\lambda = \underline{c}\,\underline{b}$

$$Q\underline{b} = \underline{b}\,\underline{c}\,\underline{b} = (\underline{c}\underline{b})\,\underline{b}$$

$\underline{c}\underline{b}$ is scaal

$\therefore \underline{c}\,\underline{b}$ is a e.v. of Q.

(b) Show that if AQ=QA, then $\{A, \underline{b}\}$ is not C.C.

Assume:

$A = 1 \quad b = 1 \quad c = 1$

$Q = \underline{b}\underline{c} = 1$

Then

$AQ = QA = 1.$

$\mathcal{C} = [1]$

$\rho(\mathcal{C}) = 1 = n$

$\Rightarrow \{A, b\}$ is c.c.

why?

$n=1$

$\mathcal{C} = [\underline{b}]$

~~see solution~~

$$AQ = QA$$

$$AQ\underline{b} = QA\underline{b}$$

$$\rightarrow A \cdot (\underline{c}\underline{b})\,b = QA\underline{b}$$

$$\Rightarrow \underline{c}\underline{b}\ A\underline{b} = Q\underline{A}\underline{b}$$

$$\{(\underline{c}\underline{b})I - Q\}\,A\underline{b} = 0$$

$\Rightarrow A\underline{b}$ is E.V. of Q associated with $\underline{c}\underline{b} \triangleq \lambda$.

b is E.V of with λ.

$\Rightarrow \{A, b\}$ is not c.c.

$\{b, Ab, \cdots\}$

3. (4 points)
A system is described by its transfer function

$$g(s)=\frac{1}{s+1}\cdot e^{\alpha sT},\ T>0,\ \alpha<0$$

Is the system BIBO-stable? Verify your answer.

1+

Yes.

$g(s) \longleftrightarrow h(t)$

$$\int_0^\infty |h(t)|^2 dt = \int_0^\infty |g(s)|^2 ds$$

$$= \int_0^{j\infty} \left|\frac{1}{s+1}\right|^2 \cdot \left|e^{2\alpha sT}\right| ds$$

Because $T>0$ $\alpha<0$

$$d\frac{1}{(s+1)} = \frac{1}{(s+1)^2}$$

$\therefore$ $e^{2\alpha sT} \leq k$, $0<k<-\infty$

$$< K\int_0^{j\infty} \frac{1}{(s+1)^2}\cdot ds$$

-1

$$= K\int_0^{j\infty} -d\frac{1}{s+1}$$

$$\cancel{\frac{-s+1}{(s+1)^2}}$$

$$= \left(-\frac{1}{s+1}\Big|_0^{j\infty}\right)K = K$$

All solution

$$\therefore \left(\int_0^\infty |h(t)|dt\right)^2 < \int_0^\infty |h(t)|^2 dt < K$$

$\Rightarrow$ BIBO stable.

−0.5

4. (5 points)

A: mxn, B: pxn

$\{\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_r\}$ and $\{\underline{w}_1, \underline{w}_2 \ldots, \underline{w}_q\}$ are O.N. bases of N(A) and N(BZ), respectively, where $Z=[\underline{z}_1, \underline{z}_2 \ldots, \underline{z}_r]$. Let $W=[\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_q]$ Show that the columns of ZW form a basis for N(A)∩N(B).

Let $w \in N(BZ)$.

$\Rightarrow BZw = 0$.

$Zw \in N(B)$.

but $Zw \in R(Z) \rightarrow Zw \in N(A)$

$\Rightarrow Zw \in N(A) \cap N(B)$. Q.E.D

②

5. (10 points)

a. Consider the cascade connections of minimal realizations of $H_1(s)$ and $H_2(s)$ as $H_1(s)H_2(s)$ and $H_2(s)H_1(s)$, where $H_1(s)=1/(s+1)$ and $H_2(s)=(s+1)/(s+2)(s+3)$. For each connection, determine the uncontrollable and unobservable modes, if any.

b. Repeat for the realizations connected in feedback form, first with $H_1(s)$ in the feedforward path and $H_2(s)$ in the feedback path, and then vice versa.

c. We noted that the behavior of the cascade connection of systems with $H_1(s)=1/(s-1)$ and $H_2(s)=(s-1)/(s+1)$ depended on the order in which they were connected. Can you give a simple explanation of the differences?

a. $H_1 = \dfrac{1}{s+1}$ $\qquad H_2 = \dfrac{s+1}{(s+2)(s+3)}$

$H_1 \cdot H_2 \Rightarrow$ odz $= -1$

$H_2 \cdot H_1 \Rightarrow$ idz $= -1$, ~~odz $= -1$~~

b.

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} = \frac{\frac{1}{s+1}}{1 + \frac{1}{(s+2)(s+3)}}$$

$$= \frac{\frac{1}{s+1}}{\frac{s^2+5s+7}{(s+2)(s+3)}}$$

$$= \frac{(s+2)(s+3)}{(s+1)(s^2+5s+7)}$$

Answer. none.

c. $H_1(s) = \dfrac{g_1(s)}{a_1(s)}$ $\qquad H_2(s) = \dfrac{g_2(s)}{a_2(s)}$

As we did in H.W. 4. the controllable of connected systhem only depend on $g_1(s)$ and $a_2(s)$

$\boxed{H_1(s)} - \boxed{H_2(s)} \Rightarrow$ 6 c.c. $\qquad \boxed{H(s)} - \boxed{H(s)} \Rightarrow$ not c.c.

EE 501 Final Exam Fall 1987

Notes:

1. 2-hour exam: 10:10 a.m. - 12:10 a.m.
   Wednesday, December 16, 1987

2. Open-book exam: Open to everything except cheating. Violation of academic integrity may lead to "electrocution."

3. Budget your time, if you wish, as:

	(minutes)
#1	20
#2	15
#3	15
#4	20

Grade:

42 / 50

Rank:

/ 31

Name: Benmei Chen

ID#: G. U.

#1. (20 points)

Determine whether the given statements are true. Give a brief, yet precise, justification for each of your answers. Answers without justification will not be graded.

(a) A: nxn, A = -A', then e^{At} is an orthogonal matrix for every t.

+

Yes.

$$\because \ (e^{At})' = e^{A't} = e^{-At}$$

$$(e^{At})\cdot(e^{At})' = e^{At}\cdot e^{-At} = e^{(A-A)t} = I.$$

(b) $\dot{\underline{x}} = A\underline{x}$, $\underline{x}(t_0) = \underline{x}_0$

$\dot{\underline{z}} = -A'\underline{z}$, $\underline{z}(t_1) = \underline{z}_1$, $t_1 > t_0 \geq 0$

then

$\underline{z}'(t)\underline{x}(t)$ equals to a constant for all t.

$$\frac{d(z'x)}{dt} = \dot{z}'x + z'\dot{x} = -z'\cdot A\cdot x + z'\cdot A\cdot x = 0$$

$$\Rightarrow z'(t)x(t) = \text{constant}$$

If we define $\underline{x}(t) = \underline{0}$ for $t < t_0$ and $z(t) = \underline{0}$ for $t < t_1$,

Then the statement is not true.

$\cancel{x(t) = e^{At}\cdot x_0 + 0 = e^{At}\cdot \underline{x}_0}$ for $t \geq t_0$ $x(t) = e^{A(t-t_0)}\underline{x}_0$

$\cancel{z(t) = e^{-A't}\cdot z_1 + 0 = e^{-A't} z_1}$ for $t \geq t_1$ $\underline{z}(t) = e^{-A'(t-t_1)}\underline{z}_0$

$z'(t) = z_1'\cdot e^{-At}$.

$\Rightarrow z'(t)\,x(t) = \text{constant}.$

$$\underline{z}'(t)\cdot\underline{x}(t) = \begin{cases} 0 & \text{for } t < t_1 \\ z_1'\cdot e^{-At}\cdot e^{At}\cdot \underline{x}_0 = z_1'\cdot \underline{x}_0 = a & \text{for } t \geq t_1 \end{cases}$$

-3.5

(c) {A, B} C.C., $\rho(B) = 1$, $B = \underline{\ell}\,\underline{s}'$, where $\underline{\ell}$: nx1, $\underline{s}'$: 1xm.

Then $\{A, \underline{\ell}\}$ is C.C.

Yes.

−0.5

$\{A, B\}$ is C.C. Then $\forall\ \lambda_i$, $i = 1, 2, \cdots, n$.

$$\rho[\lambda_i I - A, B] = n. \quad \forall\ i$$

?

$n \times n$

$$[\lambda_i I - A, \underline{\ell}] \cdot \begin{bmatrix} I & 0 \\ 0 & \underline{s}' \end{bmatrix} = [\lambda_i I - A, B] = [\lambda_i I - A, \underline{\ell}\,\underline{s}'] = \underbrace{[\lambda_i I - A, \underline{\ell}]}_{(n+1) \times n} \cdot \begin{bmatrix} I \\ \underbrace{\underline{s}'}_{1 \times m} \end{bmatrix}$$

Because $\cancel{\rho \begin{bmatrix} I \\ \underline{s}' \end{bmatrix} = n}$ $\therefore$

$$\rho[\lambda_i I - A, \underline{\ell}] = \rho \cdot [\lambda_i I - A, B] = n \Rightarrow \{A, \underline{\ell}\} \text{ C.C}$$

(d) A system can be C.C., BIBO and not A.S.

Yes.

Because a BIBO stable system is A.S. if and only if the system is C.C. and C.O. Thus A system can be C.C., BIBO, but not C.O. and not A.S.

#2. (10 points)

(a) Consider two systems $\{A,B\}$ and $\{\hat{A},\hat{B}\}$ with statevector $\underline{x}(t)$ and $\hat{\underline{x}}(t)$, respectively. Let $\hat{\underline{x}}(t) = T\underline{x}(t)$, show that

$$\hat{W}_c = TW_cT'$$

+

$\hat{\underline{x}}(t) = T\,\underline{x}(t)$. Then $\hat{A} = TAT^{-1}$, $\hat{B} = TB$.

$$\hat{W}_c = \int_0^\infty e^{\hat{A}t}\hat{B}\hat{B}'e^{\hat{A}'t}\,dt = \int_0^\infty e^{TAT^{-1}t}\cdot TB\cdot B'T'\cdot e^{(T^{-1})'A'T't}\,dt$$

$$= \int_0^\infty Te^{At}\cdot T^{-1}\cdot T\cdot B\cdot B'\cdot T'\cdot(T')^{-1}\cdot e^{A't}T'\,dt.$$

$$= T\cdot\int_0^\infty e^{At}\cdot BB'e^{A't}\cdot dt\cdot T' = TW_cT'$$

(b) $\{A,B\}$ C.C., $M>0$, show that $\{A-BM^{-1}C,\ BM^{-1}B'\}$ C.C.

Note: if you can't prove the above statement just give some explanation why you believe the statement is correct.
Hint: Feedback.

Note: A: nxn, B: nxm, M: mxm, C: mxn

$\{A,B\}$, C.C $\Rightarrow$ no $\exists$ $e^* \neq 0$ s.t.

$e^*A = \sigma^* e^*$ and $e^*B = 0 \Rightarrow B'B \neq 0$.

If $\{A-BM^{-1}C,\ BM^{-1}B'\}$ is not C.C. $\Rightarrow \exists\ g^* \neq 0$

s.t. $g^*(A-BM^{-1}C) = \lambda^* g^*$ and $g^*BM^{-1}B' = 0$

Because $M>0 \Rightarrow M^{-1}>0$. and $g^*BM^{-1}B' = 0$.

$$\Rightarrow g^*\cdot\begin{bmatrix} \underset{>0}{b_1'M^{-1}b_1} & \cdots & b_1'M^{-1}b_n \\ \vdots & \ddots & \\ b_n'M^{-1}b_1 & & \underset{>0}{b_n'M^{-1}b_n}\end{bmatrix} \triangleq g^*\cdot H$$

$\because$ $B'B \neq 0 \Rightarrow \lambda_e(H) \neq 0 \Rightarrow$ $g^*\cdot H = g^*BM^{-1}B \stackrel{?}{=} 0$ is impossible

#3. (10 points)

{A,C} not C.O., explain (or prove, if you wish)

(a) The unobservable subspace $S_{\bar{o}}$ can be expressed as

$$S_{\bar{o}} = \bigcap_{i=1}^{n} N(CA^{i-1}) \neq \{0\}$$

where

$N(\cdot)$ denotes the null space

$$\bigcap_{i=1}^{n} N(CA^{i-1}) = N(C) \cap N(CA) \cap \cdots \cap N(CA^{n-1})$$

$$\longrightarrow \quad = N \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad \longrightarrow \quad = N(\mathcal{O}') = R^{\perp}(\mathcal{O}) = S_{\bar{o}}$$

$\because$ {A, C} not C.O. $\therefore$ $S_{\bar{o}} = R^{\perp}(\mathcal{O}) \neq \{0\}$

(b) $S_{\bar{o}} \subset N(C)$

Because $S_{\bar{o}} = \bigcap_{i=1}^{n} N(CA^{i-1}) = N(C) \cap \cdots \cap N(CA^{n-1})$

$S_{\bar{o}} \subset N(CA^{i-1}) \quad i = 1, 2, \cdots n$

$\Rightarrow S_{\bar{o}} \subset N(C)$

(c) $S_{\bar{o}}$ is A-invariant

According to Cayley-Hamilton theorem:

$a(A) = A^n + a_1 A^{n-1} + \cdots + a_n I = 0$.

$\Rightarrow A^n = -a_1 A^{n-1} - \cdots - a_n I$ ...

$\forall\, x \in S_{\bar{o}} \Rightarrow x \in \bigcap_{i=1}^{n} N(CA^{i-1}) \Rightarrow x \in N(CA^{i-1}) \quad i = 1, \cdots, n$

$Ax\ N(CA^{i-1}) \quad i = 2, \cdots, n \qquad Ax \in N(CA^n) \Rightarrow Ax \in N(CA^{i-1}) \quad i = 1, 2 \ldots n$

$\Rightarrow S_{\bar{o}}$ is A-invariant.

#4. (10 points)

A_1 is anti-stable, i.e., $R_e[\lambda_i(A_1)] \geq 0$ for all i

A_2 is stable, i.e., $R_e[\lambda_j(A_2)] < 0$ for all j

$\dot{\underline{x}}_1 = A_1\underline{x}_1, \qquad \underline{x}_1(0) = \underline{x}_{10}$

$\dot{\underline{x}}_2 = A_2\underline{x}_2 + B_2\underline{x}_1, \; \underline{x}_2(0) = \underline{x}_{20}$

Define $\underline{z}(t) = F_1\underline{x}_1(t) - F_2\underline{x}_2(t)$

Where B_2 and F_2 are arbitrary show that there exists F_1 such that $\underline{z}(t) \to 0$ as $t \to \infty$ for all $\underline{x}_{10}$ and $\underline{x}_{20}$.

$z_1 = Qx_1 - x_2$

Hint: Let $Q(t) = \displaystyle\int_0^t e^{A_2 S} B_2 \, e^{-A_1 S} \, dS$

then $A_2Q(\infty) - Q(\infty)A_1 + B_2 = 0 \quad \Rightarrow \quad F_2A_2Q(\infty) - F_1A_1 + F_2B_2 = 0.$

Let $F_1 = F_2Q(\infty) \quad \Rightarrow \quad F_1A_1 - F_2B_2 = F_2A_2Q(\infty)$

Note: If you read the problem and the hint "carefully", you should be able to complete this problem within n minutes (for Dr. Hsu, n = 2, for Dr. Hsu's students, n<2).

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ B_2 & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

$$z = [F_1, -F_2]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$$z(t) = [F_1, -F_2] \cdot e^{\begin{pmatrix} A_1 & 0 \\ B_2 & A_2 \end{pmatrix}t} \cdot \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}.$$

$\downarrow \mathcal{L}$ Let $F_1 = F_2Q(\infty)$

$$z(s) = [F_2Q(\infty), -F_2] \cdot \left(sI - \begin{pmatrix} A_1 & 0 \\ B_2 & A_2 \end{pmatrix}\right)^{-1} \cdot \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

$$= [F_2Q(\infty), -F_2] \cdot \begin{bmatrix} (sI - A_1)^{-1} & 0 \\ -(sI-A_2)^{-1}B_2(sI-A_1)^{-1} & (sI-A_2)^{-1} \end{bmatrix}\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

$\downarrow \mathcal{L}^{-1}$

$$z(t) = F_2[Q(\infty), -I] \cdot \begin{bmatrix} e^{A_1 t} & \\ \int_0^t e^{A_2 s} & e^{A_2 t} \end{bmatrix}$$

Show: $\dot{z}(t) = F_1 \dot{x}_1(t) - F_2 \dot{\underline{x}}_2(t)$

$$= F_1 A_1 x_1 - F_2 A_2 \underline{x}_2 - F_2 B_2 \underline{x}_1$$

$$= (F_1 A_1 - F_2 B_2)\underline{x}_1 - F_2 A_2 \underline{x}_2.$$

$$= F_2 A_2 Q(\infty)\underline{x}_1 - F_2 A_2 \underline{x}_2$$

$$= F_2 A_2 [Q(\infty)\underline{x}_1 - \underline{x}_2]$$

$$= [F_2 Q(\infty) x_1 - F_2 x_2] A_2 \quad \times$$

$$= z(t) \cdot A_2$$

$$\Rightarrow \quad z(t) = 0 \qquad t \to \infty.$$

-4