1.1 Suppose a radar search antenna at San Francisco airport rotates at 6 rev/min and data points corresponding to the position of flight 1081 are plotted on the controller's screen once per antenna revolution. Flight 1081 is traveling directly toward the airport at 540 mi/hr. A feedback control system is established when the controller's gives course correction to the pilot. He wishes to do so each 9 mi of travel of the aircraft, and his instructions consist of course headings in integral degree values.

a) What is the sampling rate, in seconds, of the range signal plotted on the radar screen?

b) What is the sampling rate, of the controller's instructions, in seconds?

c) Identify the following signals as continuous, discrete, or digital:
   i) the aircraft's range from the airport,
   ii) the range data as plotted on the radar screen,
   iii) the controller's instructons to the pilot,
   iv) the pilot's actions on the aircraft control surface.

d) Is this a continuous, sampled-data, digital control system?

e) Show that it is possible for the pilot of flight 1081 to fly a zigzag course which would show up as a straight line on the controller's screen. What is the (lowest) frequency of a sinusoidal zigzag course which will be hidden from the controller's radar?

<u>SOLUTION:</u>

a) Sampling rate on the radar screen = 60/6 = 10 seconds.

b) Sampling rate of the controller's = 3600*9/540 = 60 sec.

c) i) is continuous.

   ii) is discrete.

   iii) is digital.

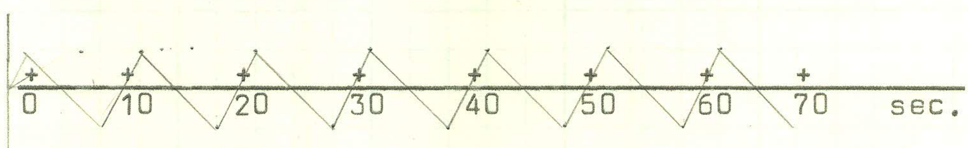
   iv) ~~is neither~~ continuous ~~or discrete or digital~~.

wow!

-1

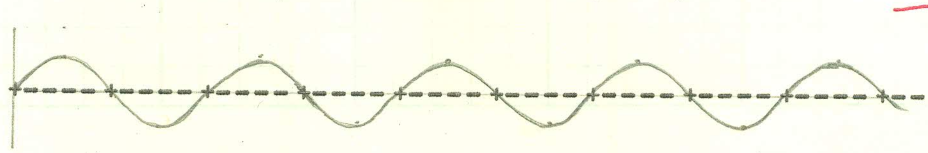
d) Sampled-data control system.

e) Because the sampling rate of the range signal plotted on the radar screen is 10 seconds, it is possible for the pilot to fly a zigzag course which would show up as a straight line on the controller's screen.(see the figuer below)



The lowest freq. of sin. zigzag course = 1/10 sec. = 0.1 Hz

For the case below, the lowest freq.= 1/20 sec. = <u>0.05</u> Hz

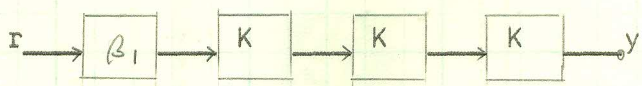


1.2 From Truxal(1965), page 122. An electronic designer has three component amplifiers with gain given by $K = +K_o + \delta K$, where the magnitude of δK is less than 10% of K_o and K_o is 1000. He wishes to design an overall amplifier with a very precise gain of 100. Three topologies are suggested, as sketched in Fig. 1.3. (Topology I is certainly not practical as it stands but is usefulto allow comprison of open-loop to closed-loop sensitivity) We define the sensitivity of an overall gain G as the ratio of the relative change in G to the relative change in the parameter. The sensitivity of G to K is, for infinitesimal changes, given by -3

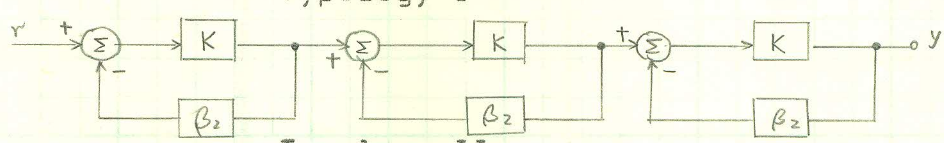
$$\mathcal{S}_G^K = \frac{\delta G/G}{\delta K/K} = \frac{K}{G}\cdot\frac{\partial G}{\partial K}$$

a) Compute the sensitivity of the three systems above to K if the β_i are selected to make $G_o = 100$ in each case.
b) Which has the lowest sensitivity ?
c) If each K changes by 10%, how close is this gain to 100?
d) Suppose each amplifier has an input noise w. Which system has the best disturbance rejection, i.e., the smallest output signal due to the disturbance alone, when r=0 ?
e) Considering part (d) above, if one of the amplifiers has significantly lower input noise than the other two, which position should it occupy, i.e., should it be at the input or the output end of the three-amplifier chain ?
f) Compute the sensitivity of the given systems with respect to changes in β, that is, $\mathcal{S}_G^\beta = \frac{\beta}{G}\cdot\frac{\partial G}{\partial \beta}$

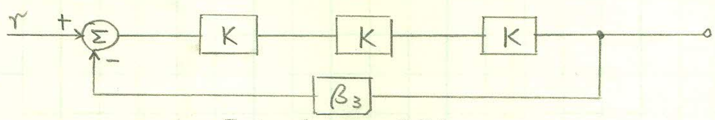
and compare the three topologies with respect to this fig. of merit.



Typology I



Typology II



Typology III

Figure 1.3

SOLUTION:

a) FOR THE TYPOLOGY I :

$$G = \beta_1 \cdot K \cdot K \cdot K = \beta_1 \cdot K^3$$

$$G_o = \beta_1 * K_o * K_o * K_o = \beta_1 \times 1000 \times 1000 \times 1000 = 100$$

$$\beta_1 = 10^{-7}$$

$$\mathcal{S}_G^K = \frac{K}{G}\cdot\frac{\partial G}{\partial K} = \frac{K}{G}\beta_1 3K^2 = \frac{3K^3}{\beta_1 \cdot K^3} = \frac{3}{\beta_1} = 3\times\cancel{10^7} = 3$$

(TO BE CONTINUED NEXT PAGE)

Benmei Chen

FOR THE TYPOLOGY II :

$$G = \frac{K}{1+K\cdot\beta_2}\cdot\frac{K}{1+K\cdot\beta_2}\cdot\frac{K}{1+K\cdot\beta_2} = \frac{K^3}{(1+K\cdot\beta_2)^3}$$

$$G_0 = \frac{1000^3}{(1+1000\,\beta_2)^3} = 100$$

$$\beta_2 = 0.21444$$

$$\mathcal{S}_G^K = \frac{K}{G}\cdot\frac{\partial G}{\partial K} = \frac{K}{G}\cdot\frac{3K^2}{(1+K\beta_2)^4} = \frac{(1+K\beta_2)^3}{K^2}\cdot\frac{3K^2}{(1+K\beta_2)^4}$$

$$= \frac{3}{1+K\beta_2} \doteq \frac{3}{1+1000\times 0.2144} = 0.01392 \quad \text{FOR } K=1000$$

$$= \frac{3}{1+900\times 0.2144} = 0.01546 \quad \text{FOR } K=K_0+\delta K,\ \delta K=-100$$

$$= \frac{3}{1+1100\times 0.2144} = 0.01266 \quad \text{FOR } K=K_0+\delta K,\ \delta K=100$$

FOR THE TYPOLOGY III :

$$G = \frac{K^3}{1+\beta_3 K^3}$$

$$G_0 = \frac{K_0^3}{1+\beta_3 K_0^3} = \frac{1000^3}{1+\beta_3\times 1000^3} = 100$$

$$\beta_3 = 0.01000$$

$$\mathcal{S}_G^K = \frac{K}{G}\cdot\frac{\partial G}{\partial K} = \frac{K}{G}\cdot\frac{3K^2}{(1+\beta_3K^3)^2} = \frac{(1+\beta_3K^3)\cdot 3K^2}{K^2\cdot(1+\beta_3K^3)^2}$$

$$= \frac{3}{1+\beta_3K^3} = \begin{cases} 3.0\times 10^{-7} & \delta K=0 \\ 4.12\times 10^{-7} \quad \text{FOR } K=K_0+\delta K & \delta K=-100 \\ 2.25\times 10^{-7} & \delta K=100 \end{cases}$$

b) TYPOLOGY III HAS THE LOWEST SENSITIVITY.

c) FOR TYP. I : $G = 10^{-7}\cdot K^3$

G changes by 30 % $\quad G' = \begin{cases} 10^{-7}\times 900^3 = 72.9 & \text{FOR } \delta K = -10\%\cdot K_0 \\ 10^{-7}\times 1100^3 = 133.1 & \text{FOR } \delta K = 10\%\cdot K_0 \end{cases}$

TYP. II : $G = K^3/(1+K * 0.21444)^3$

G changes by .139% $\quad G' = \begin{cases} 900^3/(1+900\times 0.21444)^3 = 99.85 & \text{FOR } \delta K=-10\% K_0 \\ 1100^3/(1+1100\times 0.21444)^3 = 100.13 & \text{FOR } \delta K = 10\% K_0 \end{cases}$

TYP. III : $G = K^3/(1+0.01\cdot K^3)$

G changes by .0003% $\quad G' = \begin{cases} 900^3/(1+0.01\times 900^3) = 99.999996 & \text{FOR } \delta K = -10\%\cdot K_0 \\ 1100^3/(1+0.01\times 1100^3) = 100.000003 & \text{FOR } \delta K = 10\% K_0 \end{cases}$

d) FOR TYP. I : $y^* = (K^3 + K^2 + K)\,\omega = 100100100\,\omega$ WHEN $K = K_0$

TYP. II : $y' = \dfrac{K}{1+K\cdot\beta_2}\cdot\omega = 4.6417\,\omega$ $\dfrac{Y}{\omega} = K^3$

$$y'' = \frac{K}{1+K\cdot\beta_2}\cdot(\omega + y') = 4.6417\,\omega + 21.5450\,\omega = 26.1867\,\omega$$

$$y^* = \frac{K}{1+K\cdot\beta_2}\cdot(\omega + y'') = 4.6417\,\omega + 21.5450\,\omega + 100.0048\,\omega = 126.1915\,\omega$$

$= 100$

TYP. III : $y^* = (K^3+K^2+K)\cdot\dfrac{1}{1+\beta_3 K^3}\cdot\omega$

$= 100.1001\cdot\omega$

SO, WE KNOW

THE OUTPUT OF TYP. III WOULD BE THE SMALLEST.

e) i) IN THE CASE OF TYP. I , THE LOWER-INPUT-NOISE AMPLIFIER MUST BE AT THE INPUT END OF THE THREE-AMPLIFIER CHAIN.   output end

ii) IN THE CASE OF TYP. II , THE ANSWER IS THE SAME AS i).

iii) FOR TYP. III , THE SAME.

f) TYP. I : $G = \beta_1\cdot K^3$

$$S_G^{\beta_1} = \frac{\beta_1}{\beta_1\cdot K^3}\cdot K^3 = 1$$

TYP II : $G = \dfrac{K^3}{(1+K\cdot\beta_2)^3}$

$$S_G^{\beta_2} = \frac{\beta_2(1+K\cdot\beta_2)^3}{K^3}\cdot\frac{-3K^4}{(1+K\beta_2)^4} = -\frac{3K\beta_2}{1+K\beta_2} = -2.98608$$

≈ -3

TYP. III : $G = \dfrac{K^3}{1+\beta_3 K^3}$

$$S_G^{\beta_3} = \frac{\beta_3(1+\beta_3K^3)}{K^3}\cdot\frac{-K^6}{(1+\beta_3\cdot K^3)^2} = \frac{-K^3\cdot\beta_3}{1+\beta_3\cdot K^3} \doteq -1$$

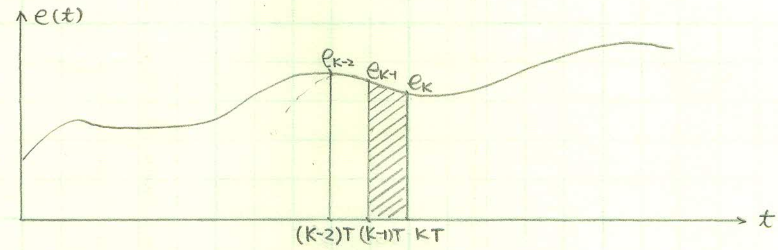
CONCLUTION : THE TYPOLOGY III IS THE BEST WITH RESPECT TO THE SENSITIVITY OF G TO K , and SENSITIVITY OF G TO β.

THE TYPOLOGY I IS THE WORST.

THE TYPOLOGY II IS BETWEEN.

20/20

2.1 a) Derive the difference equation corresponding to the approximation of integration found by fitting a parabola to the points e_{K-2}, e_{K-1}, e_K and taking the area under this parabola between $t=kT-T$ and $t=kT$ as the approximation to the integral of $e(t)$ over this range.



LET THE PARABOLA BE $y=at^2+bt+c$, SO

$$\begin{cases} e_{K-2} = a\cdot(K-2)^2T^2 + b\cdot(K-2)T + c \\ e_{K-1} = a\cdot(K-1)^2T^2 + b\cdot(K-1)T + c \\ e_K = a\cdot(K)^2T^2 + b\cdot K\cdot T + c \end{cases}$$

$$a = \frac{\begin{vmatrix} e_{K-2} & (K-2)T & 1 \\ e_{K-1} & (K-1)T & 1 \\ e_K & KT & 1 \end{vmatrix}}{\begin{vmatrix} (K-2)^2T^2 & (K-2)T & 1 \\ (K-1)^2T^2 & (K-1)T & 1 \\ K^2T^2 & KT & 1 \end{vmatrix}} = \frac{1}{2T^2}e_{K-2} - \frac{1}{T^2}e_{K-1} + \frac{1}{2T^2}e_K$$

$$b = \frac{\begin{vmatrix} (K-2)^2T^2 & e_{K-2} & 1 \\ (K-1)^2T^2 & e_{K-1} & 1 \\ K^2T^2 & e_K & 1 \end{vmatrix}}{\begin{vmatrix} (K-2)^2T^2 & (K-2)T & 1 \\ (K-1)^2T^2 & (K-1)T & 1 \\ K^2T^2 & KT & 1 \end{vmatrix}} = -\frac{(2K-1)}{2T}e_{K-2} + \frac{2K-2}{T}e_{K-1} - \frac{2K-3}{2T}e_K$$

$$c = \frac{\begin{vmatrix} (K-2)^2T^2 & (K-2)T & e_{K-2} \\ (K-1)^2T^2 & (K-1)T & e_{K-1} \\ K^2T^2 & KT & e_K \end{vmatrix}}{\begin{vmatrix} (K-2)^2T^2 & (K-2)T & 1 \\ (K-1)^2T^2 & (K-1)T & 1 \\ (KT)^2 & KT & 1 \end{vmatrix}} = \frac{K(K-1)}{2}e_{K-2} - K(K-2)e_{K-1} + \frac{(K-1)(K-2)}{2}e_K$$

SO, THE PARABOLA $y = \left(\frac{1}{2T^2}e_{K-2} - \frac{1}{T^2}e_{K-1} + \frac{1}{2T^2}e_K\right)t^2 + \left[-\frac{2K-1}{2T}e_{K-2} + \frac{2K-2}{T}e_{K-1} - \frac{2K-3}{2T}e_K\right]\cdot t$

$+ \left[\frac{K(K-1)}{2}e_{K-2} - K(K-2)e_{K-1} + \frac{1}{2}(K-1)(K-2)e_K\right]$

2.1 (a) CONT.

$$\Delta u_k = \int_{kT-T}^{kT} (at^2+bt+c)\,dt = \left[\tfrac{1}{3}at^3+\tfrac{1}{2}bt^2+ct\right]\Big|_{kT-T}^{kT}$$

$$= \tfrac{1}{3}a\cdot(kT)^3 - \tfrac{1}{3}a(k-1)^3T^3 + \tfrac{1}{2}b(kT)^2 - \tfrac{1}{2}b(k-1)^2T^2 + c\cdot kT - c\cdot(k-1)T$$

$$= \tfrac{1}{3}\cdot(3k^2-3k+1)T^3\cdot a + \tfrac{1}{2}\cdot(2k-1)\cdot T^2\cdot b + c\cdot T$$

$$= \tfrac{1}{3}\cdot(3k^2-3k+1)\cdot T^3\left[\frac{1}{2T^2}e_{k-2} - \frac{1}{T^2}e_{k-1} + \frac{1}{2T^2}e_k\right]$$

$$+\tfrac{1}{2}(2k-1)\cdot T^2\cdot\left[-\frac{2k-1}{2T}e_{k-2} + \frac{2k-2}{T}e_{k-1} - \frac{2k-3}{2T}e_k\right]$$

$$+\ T\cdot\left[\tfrac{1}{2}k(k-1)e_{k-2} - k(k-2)e_{k-1} + \tfrac{1}{2}(k-1)(k-2)e_k\right]$$

$$= -\frac{T}{12}e_{k-2} + \frac{2}{3}T\cdot e_{k-1} + \frac{5}{12}T\cdot e_k$$

SO, $u_k = u_{k-1} + \Delta u_k$

$$= u_{k-1} - \frac{1}{12}\cdot T\cdot e_{k-2} + \frac{2}{3}\cdot T\cdot e_{k-1} + \frac{5}{12}\cdot T\cdot e_k$$

(b) *Find the tranfer function of the resulting discrete system and plot the poles and zeros in z-plane.*

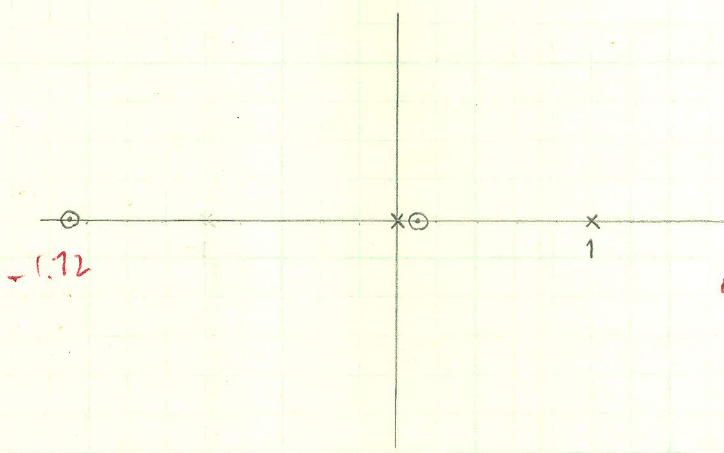
FROM $u_k = u_{k-1} - \frac{T}{12}e_{k-2} + \frac{2}{3}T\cdot e_{k-1} + \frac{5}{12}T\cdot e_k$

z-Xform: $U(z) = z^{-1}U(z) - \frac{T}{12}\cdot z^{-2}E(z) + \frac{2}{3}T\cdot z^{-1}E(z) + \frac{5}{12}T\cdot E(z)$

$$(1-z^{-1})\cdot U(z) = \frac{T}{12}(5+8z^{-1}-z^{-2})\cdot E(z)$$

$$H(z) = \frac{U(z)}{E(z)} = \frac{T}{12}\cdot\frac{5+8z^{-1}-z^{-2}}{1-z^{-1}}$$

$$= \frac{T}{12}\cdot\frac{5z^2+8z-1}{z^2-z} = \frac{T}{12}\cdot\frac{(z-0.11652)(z+1.71652)}{z(z-1)}$$



2.3 a) Compute and plot the unit pulse response of the system derived in 2.1.

From the result we had in 2.1

$$u_k = u_{k-1} - \frac{T}{12} e_{k-2} + \frac{2}{3} T e_{k-1} + \frac{5}{12} T \cdot e_k$$

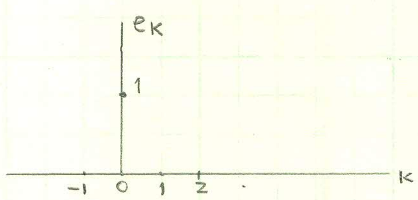
Unit pulse response : $e_k = 1$, $k=0$, $e_k = 0$, $k \neq 0$

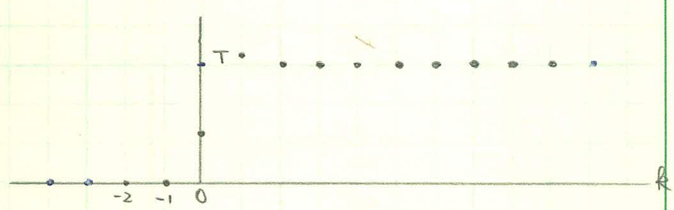
So, $$E(z) = \sum_{k=-\infty}^{\infty} e_k \cdot z^{-k} = 1$$

$$U(z) = H(z) \cdot E(z) = H(z) = \frac{T}{12} \cdot \frac{5+8z^{-1}-z^{-2}}{1-z^{-1}}$$

$$= \frac{T}{12}(5+8z^{-1}-z^{-2}) \cdot \sum_{k=0}^{\infty} z^{-k}$$

$$= \frac{5}{12}T + \frac{13}{12}T \cdot z^{-1} + T \cdot z^{-2} + T \cdot z^{-3} + \cdots = \sum_{i=0}^{\infty} h_i z^{-i}$$





b) Is this system BIBO stable?

This system is not BIBO stable. Because

good

$$H(z) = \frac{5}{12}T + \frac{13}{12}T \cdot z^{-1} + T \cdot z^{-2} + T \cdot z^{-3} + \cdots$$

$$\sum_{\ell=-\infty}^{\infty} |h_\ell| = T \cdot \left(\frac{5}{12} + \frac{13}{12} + 1 + 1 + \cdots\right) \longrightarrow \infty$$

17/20

2.4 The first-order system $(z-\alpha)/(1-\alpha)z$ has a zero at $z=\alpha$.

a) Plot the step response for this system for $\alpha = 0.8,\ 0.9,\ 1.1,\ 1.2,\ 2$.

FROM THE EQUATION (2.31) ON PAGE 27, WE HAVE THE Z-TRANSFORM OF STEP INPUT

$$E(z) = z/(z-1) \qquad |z|>1$$

FROM THE GIVEN, $H(z) = (z-\alpha)/(1-\alpha)z$.

SO, THE STEP RESPONSE $U(z)$,

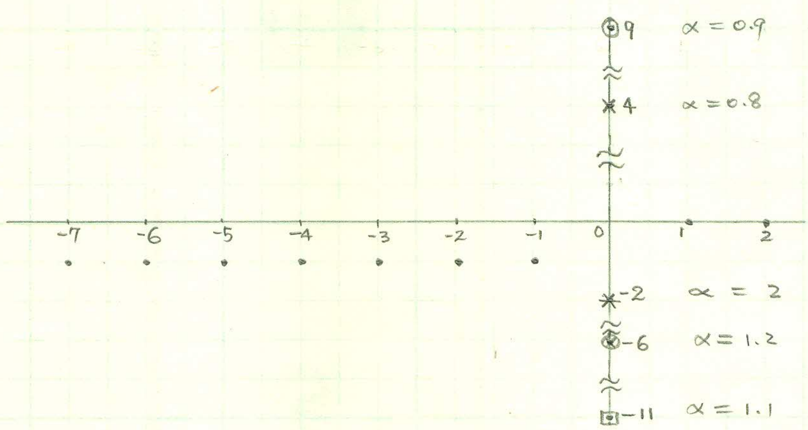
$$U(z) = H(z)E(z) = \frac{z-\alpha}{(1-\alpha)z}\cdot\frac{z}{z-1} = \frac{1}{(1-\alpha)}\cdot\frac{z-\alpha}{z-1}$$

$$= \frac{1}{1-\alpha}\cdot\frac{z-1+1-\alpha}{z-1}$$

$$= \frac{1}{1-\alpha} - \frac{1}{1-z}\cdot\frac{z^{-1}}{z^{-1}} =$$

$$= \frac{1}{1-\alpha} - (1+z+z^2+\cdots$$

$$= \frac{\alpha}{1-\alpha}\cdot z^{-0} - (z^{-1})^{-1} - (z^{-1})^{-2} - (z^{-1})^{-3} - \cdots$$



NOTE: X FOR $\alpha = 0.8$; ⊙ FOR $\alpha = 0.9$; ⊡ FOR $\alpha = 1.1$

⊗ FOR $\alpha = 1.2$; ✱ FOR $\alpha = 2$; AND • FOR ALL

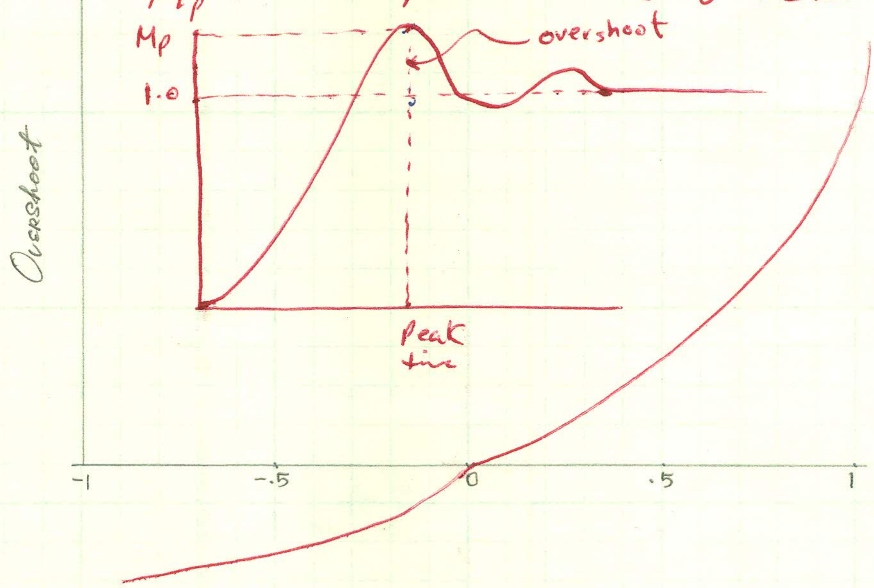
(b) Plot the overshoot of this system on the same coordinates as those appearing in Fig. 2.21 for $-1 < \alpha < 1$

(Can you give me the defination of overshoot ? I can't find it in text.)

The percent overshoot is defined as

$$P.O = \frac{M_p - 1}{1} \times 100\%$$ for a unit step input.

M_p : is the peak value of the time response.



,3

(c) In what way is the step response of this system unusual for $\alpha > 1$?

FOR $\alpha > 1$, THE STEP RESPONSE OF THIS SYSTEM

u_0 IS NEGATIVE, AND

$u_0 \to -1$ WHEN $\alpha \to \infty$

overshoot is negative

2.5 The one-sided z-transform is defined as

$$F(z) = \sum_{0}^{\infty} f(k) z^{-k} .$$

a) Show that the one-sided transform of $f(k+1)$ is

$$\mathcal{Z}\{f(k+1)\} = zF(z) - zf(0)$$

SHOW:

$$\begin{aligned}
\mathcal{Z}\{f(k+1)\} &= \sum_{0}^{\infty} f(k+1)\, z^{-k} \\
&= f(1) + f(2)\cdot z^{-1} + f(3)\cdot z^{-2} + \cdots \\
&= [f(0) + f(1) z^{-1} + f(2) z^{-2} + f(3) z^{-3} + \cdots]\cdot z - f(0)\cdot z \\
&= z \cdot \sum_{0}^{\infty} f(k) z^{-k} - z f(0) \\
&= z F(z) - z f(0)
\end{aligned}$$

b) Use the one-sided transform to solve the transforms of the Fibonacci numbers by writing (2.4) as $u_{k+2} = u_{k+1} + u_k$. Let $u_0 = u_1 = 1$.

$$\mathcal{Z}\{u_{k+2}\} = \mathcal{Z}\{u_{k+1}\} + \mathcal{Z}\{u_k\} \quad \cdots\cdots (1)$$

FROM THE RESULT IN PART (a), WE HAVE

$$\mathcal{Z}\{u_{k+1}\} = z\,U(z) - z\,u_0 = z\,U(z) - z \quad \cdots\cdots (2)$$

$$\begin{aligned}
\mathcal{Z}\{u_{k+2}\} &= z\,\mathcal{Z}\{u_{k+1}\} - z\cdot u_1 = z\cdot[z\,U(z) - z] - z \\
&= z^2\cdot U(z) - z^2 - z \quad \cdots\cdots (3)
\end{aligned}$$

FROM THE EQUATIONS (1), (2) and (3) ABOVE, WE HAVE

$$z^2 U(z) - z^2 - z = z\,U(z) - z + U(z)$$

$$U(z) = \frac{z^2}{z^2 - z - 1}$$

c) Compute the location of the poles of the transform of the Fibonacci num.

POLES: $z_1 = 1.61803$, $z_2 = -0.61803$

(d) Compute the inverse transform of the numbers.

FROM THE RESULT WE GOT IN PART (b),

$$U(z) = \frac{z^2}{z^2 - z - 1}$$

$$= \frac{z^2 - z - 1 + z + 1}{z^2 - z - 1} = 1 + \frac{z+1}{(z-1.61803)(z+0.61803)}$$

$$= 1 + \frac{1.17082}{z - 1.61803} - \frac{0.17082}{z + 0.61803}$$

$$= 1 + 1.17082 \frac{z^{-1}}{1 - \frac{1.61803}{z}} - 0.17082 \cdot \frac{z^{-1}}{1 - (-\frac{0.61803}{z})}$$

$$= 1 + 1.17082 \cdot z^{-1} [1 + 1.61803 z^{-1} + 1.61803^2 z^{-2} + 1.61803^3 z^{-3} + \cdots]$$

$$- 0.17082 \cdot z^{-1} [1 - 0.61803 z^{-1} + 0.61803^2 z^{-2} - 0.61803^3 z^{-3} + \cdots]$$

$$= 1 + 1 + 2 \cdot z^{-2} + 3 \cdot z^{-3} + \cdots + [1.17082 \times (1.61803)^{k-1} - 0.17082 \times (-0.61803)^{k-1}] \cdot z^{-(k-1)}$$

$$+ [1.17082 \times (1.61803)^{k} - 0.17082 \times (-0.61803)^{k}] \cdot z^{-k} + \cdots$$

SO, THE INVERSE transform of $U(z)$

$$u_{k+1} = [1.17082 \times (1.61803)^k - 0.17082 \times (-0.61803)^k]$$

(e) Show that if u_k is the kth Fibonacci number, then the ratio u_{k+1}/u_k will go to $(1+\sqrt{5})/2$, the golden ratio of the Greeks.

$$\frac{u_{k+1}}{u_k} = \frac{1.17082 \times (1.61803)^{k+1} - 0.17082 \times (-0.61803)^{k+1}}{1.17082 \times (1.61803)^k - 0.17082 \times (-0.61803)^k}$$

$$= \frac{1 - 0.14590 \times (-0.37920)^{k+1}}{0.61803 - 0.07702 \times (-0.37920)^k}$$

$$\longrightarrow \frac{1}{0.61803} = 1.61803 = \frac{1+\sqrt{5}}{2}$$

(f). Show that if we add a forcing term, $e(k)$, to (2.4) we can generate the Fibonacci numbers by a system which can be analyzed by the two-sided Xansform; i.e., Let $u_k = u_{k-1} + u_{k-2} + e_k$ and let $e(k) = \delta_0(k)$. Take the two-sided Xform and show that the same $U(z)$ results as in (b).

$$u_k = u_{k-1} + u_{k-2} + e_k$$

$$\mathcal{Z}\{u_k\} = \mathcal{Z}\{u_{k-1}\} + \mathcal{Z}\{u_{k-2}\} + \mathcal{Z}\{\delta_0(k)\}$$

$$U(z) = z^{-1}U(z) + z^{-2}U(z) + 1$$

$$U(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$

$$= \frac{z^2}{z^2 - z - 1}$$

QED

18/20

2.9 Consider a signal with the transform (which converges for $|z| > 2$)

$$U(z) = \frac{z}{(z-1)(z-2)}$$

a) What value is given by the formula (final-value therom) of (2.52) applied to this $U(z)$?

$$\lim_{k\to\infty} f(k) = \lim_{z\to 1} (z-1)U(z)$$

$$= \lim_{z\to 1} (z-1)\cdot\frac{z}{(z-1)(z-2)}$$

$$= -1$$

(b) Find the final value of $u(k)$ by taking the inverse transform of $U(z)$, using partial-fraction expansion and the tables

$$U(z) = \frac{z}{(z-1)(z-2)}$$

$$= \frac{z^{-1}}{(1-z^{-1})(1-2z^{-1})} = \frac{-1}{1-z^{-1}} + \frac{1}{1-2z^{-1}}$$

$$= -(1+z^{-1}+z^{-2}+\cdots) + (1+2z^{-1}+4z^{-2}+8z^{-3}+\cdots)$$

$$= z^{-1}+3z^{-2}+\cdots+(2^k-1)z^{-k}+\cdots\cdots$$

$$u(k) = (2^k-1) \longrightarrow \infty \quad , \quad k\to\infty$$

(C) Explain why the two results of (a) and (b) differ.

Because $U(z)$ does not satisfy the condition for (2.52). $(1-z)U(z)$ has a pole at $z=2$ which is not inside THE unit circle.

2.11 Compute the inverse transform, $f(k)$, for each of the following transforms:

(a) $F(z) = \dfrac{1}{1+z^{-2}}$ $\quad |z|>1 \quad , \quad |z^{-2}|<1$

$$= \frac{1}{1-(-z^{-2})} = 1+(-z^{-2})+(-z^{-2})^2+(-z^{-2})^3+(-z^{-2})^4+\cdots$$

$$= 1-z^{-2}+z^{-4}-z^{-6}+z^{-8}-\cdots$$

$$f(k) = \begin{cases} (-1)^{\frac{k}{2}} & k\geq 0 \text{ and } k=2n, \quad n=1,2,3,\cdots \\ 0 & \text{OTHERWISE} \end{cases}$$

2.11 (b) $F(z) = \dfrac{z(z-1)}{z^2 - 1.25z + 0.25}$, $|z| > 1$

$$= \frac{z^2 - 1.25z + 0.25 + 0.25z - 0.25}{z^2 - 1.25z + 0.25}$$

$$= \frac{0.25z - 0.25}{z^2 - 1.25z + 0.25} + 1 = 0.25 \cdot \frac{z-1}{(z-1)(z-0.25)} + 1$$

$$= 1 + \frac{0.25}{z - 0.25} = 1 + 0.25z^{-1}\frac{1}{1 - 0.25z^{-1}} \qquad |0.25z^{-1}| < 1$$

$$= 1 + 0.25z^{-1}(1 + 0.25z^{-1} + 0.25^2 \cdot z^{-2} + 0.25^3 z^{-3} + \cdots)$$

$$= 1 + 0.25z^{-1} + 0.25^2 z^{-2} + 0.25^3 z^{-3} + \cdots$$

$f(k) = 0.25^k \qquad k \geq 0$

$= 0 \qquad k < 0$

(c) $F(z) = \dfrac{z}{z^2 - 2z + 1}$ $\qquad |z| > 1$

$$
\begin{array}{r l}
 & z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \cdots \\
z^2 - 2z + 1 \,\big) & z \\
 & z - 2 + z^{-1} \\ \hline
 & 2 - z^{-1} \\
 & 2 - 4z^{-1} + 2z^{-2} \\ \hline
 & 3z^{-1} - 2z^{-2} \\
 & 3z^{-1} - 6z^{-2} + 3z^{-3} \\ \hline
 & 4z^{-2} - 3z^{-3} \\
 & 4z^{-2} - 8z^{-3} + 4z^{-4} \\ \hline
 & \cdots
\end{array}
$$

$f(k) = k \qquad k \geq 0$

$= 0 \qquad k < 0$

(d) $F(z) = \dfrac{z}{(z - \frac{1}{2})(z-2)}$, $\frac{1}{2} < |z| < 2 \qquad \frac{1}{2} < |z^{-1}| < 2$

$$= \frac{-\frac{1}{3}}{z - \frac{1}{2}} + \frac{\frac{4}{3}}{z-2}$$

$$= -\frac{1}{3}z^{-1} \cdot \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{4}{3} \cdot \frac{1}{2} \cdot \frac{-1}{1 - \frac{z}{2}}$$

$$= -\frac{1}{3}z^{-1}\left[1 + \frac{1}{2}z^{-1} + \frac{1}{2^2}z^{-2} + \cdots + \frac{1}{2^k}z^{-k} + \cdots\right]$$

$$- \frac{2}{3} \cdot \left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots + \frac{1}{2^k} \cdot z^k + \cdots\right]$$

$$= \cdots - \frac{1}{3} \cdot \frac{1}{2^{k-1}} z^k - \cdots - \frac{1}{3}z - \frac{2}{3} - \frac{1}{3}z^{-1} - \frac{1}{3} \cdot \frac{1}{2}z^{-2} - \cdots - \frac{1}{3} \cdot \frac{1}{2^k} z^{-k-1} \cdots$$

$f(k) = \frac{-1}{3} \cdot 2^{-|k|+1}$

-1

$f(k) = -\frac{2}{3}\frac{1}{2^k} \qquad k \geq 0$

$= -\frac{2}{3} 2^k \qquad k \leq -1$

2.12 Use the z-transform to solve the difference equation

$$y(k)-3y(k-1)+2y(k-2)=2u(k-1)-2u(k-2)$$

$$u(k)=k \quad k\geq 0$$
$$=0 \quad k<0$$

$$y(k)=0 \quad k<0$$

$$\mathcal{Z}\{y(k)-3y(k-1)+2y(k-2)\}=\mathcal{Z}\{2u(k-1)-2u(k-2)\}$$

$$Y(z)-3z^{-1}Y(z)+2z^{-2}Y(z)=2z^{-1}U(z)-2z^{-2}U(z)$$

$$(1-3z^{-1}+2z^{-2})Y(z)=2(z^{-1}-z^{-2})U(z)$$

$$Y(z)=\frac{2z^{-1}-2z^{-2}}{1-3z^{-1}+2z^{-2}}\cdot U(z)$$

$$U(z)=\sum_{k=0}^{\infty} kz^{-k}=\frac{z}{z^2-2z+1} \qquad \text{(FOR 2.11 (c))}$$

$$\text{SO,}\quad Y(z)=\frac{2z^{-1}-2z^{-2}}{1-3z^{-1}+2z^{-2}}\cdot\frac{z}{z^2-2z+1}$$

$$=\frac{2z^2-2}{(z-1)^3(z-2)}$$

$$=\frac{2(z+1)}{(z-1)^2(z-2)}=\frac{-4z}{(z-1)^2}-\frac{2}{z-1}+\frac{6}{z-2}$$

$$=-4\cdot\frac{z}{z^2-2z+1}-2\cdot z^{-1}\cdot\frac{1}{1-z^{-1}}+6z^{-1}\cdot\frac{1}{1-2z^{-1}}$$

$$=\sum_{k=0}^{\infty}(-4k)z^{-k}-\sum_{k=0}^{\infty}2\cdot z^{-k-1}+\sum_{k=0}^{\infty}6\cdot 2^k\cdot z^{-k-1}$$

$$=\sum_{k=0}^{\infty}(3\times 2^k-4k-2)z^{-k}-1$$

-1

THUS,

$$y(k)=\begin{cases}3\times 2^k-4k-2 & ; \ k\geq 1\\ 0 & \text{OTHERWISE}\end{cases}$$

$$=2^{k+1}-2k-2 \qquad k\geq 0$$

$\frac{18}{20}$

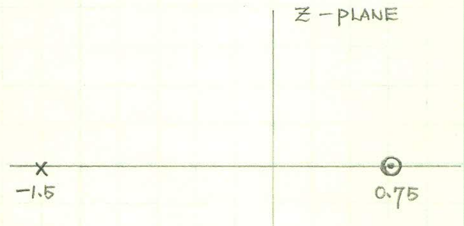
3.3 a) The following transfer function is a lead network designed to add about 60° lead at $\omega_1 = 3$ rad: $H(S) = (S+1)/(0.1S+1)$

For each of the following design methods compute the plot in the z-plane the pole and zero locations and compute the amount of phase lead given by the network at $z_1 = e^{j\omega_1 T}$. Let $T = 0.25$ sec.

1. Forward rectangular rule.

$$H(Z) = H(S)\Big|_{S=\frac{Z-1}{T}} = \frac{[1/T \cdot (Z-1)+1]}{0.1 \cdot \frac{1}{T}(Z-1)+1}$$

$$= \frac{4Z-3}{0.4Z+0.6} = \frac{10(Z-0.75)}{Z+1.5}$$



(i) ZERO: $Z = 0.75$

POLE: $Z = -1.5$

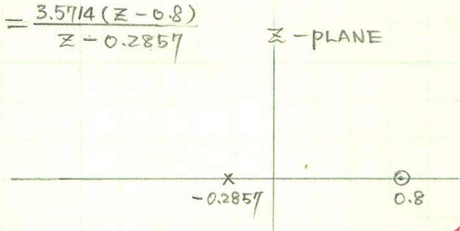
(ii) $H(Z)\big|_{Z=Z_1} = H(Z)\big|_{Z=e^{j\cdot 3\times 0.25}}$

$$= \frac{4\cdot e^{j0.75}-3}{0.4e^{j0.75}+0.6} = 2.92218 \angle{+74.55°}\text{; PHASE LEAD } 74.55°$$

2. Backward rectangular rule

$$H(Z) = H(S)\Big|_{S=\frac{Z-1}{TZ}} = \frac{\frac{Z-1}{TZ}+1}{0.1\cdot\frac{Z-1}{TZ}+1}$$

$$= \frac{Z-1+0.25Z}{0.1Z-0.1+0.25Z} = \frac{1.25Z-1}{0.35Z-0.1} = \frac{3.5714(Z-0.8)}{Z-0.2857}$$



i) ZERO: $Z = 0.8$

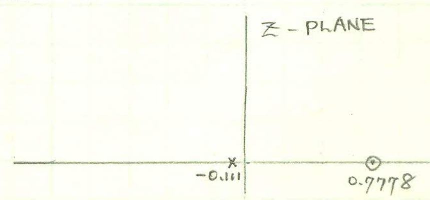
POLE: $Z = -0.2857$

ii) $H(Z)\big|_{Z=e^{j0.75}} = \frac{1.25e^{j0.75}-1}{0.35e^{j0.75}-0.1} = 3.00357\angle 38.918°$; PHASE LEAD 38.92°

3. Trapezoid rule

$$H(Z) = H(S)\Big|_{S=\frac{2}{T}\cdot\frac{Z-1}{Z+1}} = \frac{8\cdot\frac{Z-1}{Z+1}+1}{0.8\cdot\frac{Z-1}{Z+1}+1}$$

$$= \frac{9Z-7}{1.8Z+0.2} = \frac{5(Z-0.7778)}{Z+0.1111}$$



(i) ZERO: $Z = 0.7778$.

POLE: $Z = -0.1111$

ii) $H(z)\big|_{z=e^{j0.75}} = \dfrac{5\,(e^{j0.75}-0.7778)}{e^{j0.75}+0.1111} = 3.15143\,\angle 54.90^\circ$ PHASE LEAD 54.90°

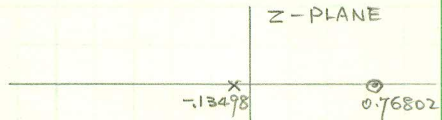
4. *Bilinear with prewarping* (use ω_1 as the warping frequency)

$$H(z)=H(s)\Big|_{s=\omega_1/\tan\omega_1 T/2\cdot\frac{z-1}{z+1}} = H(s)\Big|_{s=7.62144\,\frac{z-1}{z+1}}$$

$$= \frac{7.62144\cdot\frac{z-1}{z+1}+1}{0.762144\cdot\frac{z-1}{z+1}+1} = \frac{8.62144z-6.62144}{1.762144z+0.23786} = \frac{4.89259(z-0.76802)}{z+0.13498}$$

(i) ZERO : $z=0.76802$

POLE : $z=-0.13498$



(ii) $H(z)\big|_{z=e^{j0.75}} = \dfrac{4.89259(e^{j0.75}-0.76802)}{e^{j0.75}+0.13498} = 3.02688\,\angle 54.03^\circ$; PHASE LEAD 54.03°

5. *Pole-zero mapping.*

According to the rule, we have

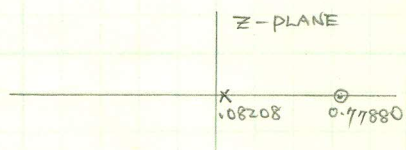
$$H(z)=\frac{k(z-e^{-0.25})}{z-e^{-2.5}} = \frac{k(z-0.77880)}{z-0.08208}$$

$H(z)\big|_{z=1} = 0.24098\,k = H(s)\big|_{s=0} = 1$, $k=4.14971$

$$\therefore\ H(z)=\frac{4.14971(z-0.77880)}{z-0.08208}$$

(i) ZERO : $z=0.77880$

POLE : $z=0.08208$



(ii) $H(z)\big|_{z=e^{j0.75}} = \dfrac{4.14971(e^{j0.75}-0.7788)}{e^{j0.75}-0.08208} = 2.99733\,\angle 54.62^\circ$; PHASE LEAD 54.62°

47.5

6. *Zero-order-hold equivalent*

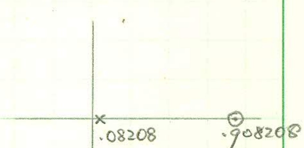
$$H(z) = (1-z^{-1})\,\mathcal{Z}\left\{\frac{H(s)}{s}\right\} = (1-z^{-1})\,\mathcal{Z}\left\{\frac{s+1}{s(0.1s+1)}\right\}$$

$$= (1-z^{-1})\cdot\mathcal{Z}\left\{\frac{1}{s}+\frac{9}{s+10}\right\}$$

$$= \frac{z-1}{z}\cdot\left[\frac{z}{z-1}+\frac{9z}{z-e^{-10\times0.25}}\right] = \frac{10(z-0.908208)}{z-0.08208}$$

(i) ZERO : $z=0.908208$

POLE : $z=0.08208$



(ii) $H(z)\big|_{z=e^{j0.75}} = \dfrac{10(e^{j0.75}-0.908208)}{e^{j0.75}-0.08208}$

$$= 7.4779\,\angle 58.14^\circ$$

PHASE LEAD 58.14°

b) Plot on log-log paper over the frequency range $\omega = 0.1 \to \omega = 100$ the amplitude Bode plots on each of the above equivalents.

A. Forward Rectangular Rule:

$$H(e^{j\omega T}) = \frac{10(e^{j\omega T} - 0.75)}{e^{j\omega T} + 1.5} = \frac{(10\cos .25\omega - 7.5) + j\sin .25\omega \times 10}{(\cos .25\omega + 1.5) + j\sin .25\omega}$$

$$|H(e^{j\omega T})|^2 = \frac{(10\cos .25\omega - 7.5)^2 + \sin^2 .25\omega \times 100}{(\cos .25\omega + 1.5)^2 + \sin^2 .25\omega}$$

$$H_{dB} = 10 \log |H(e^{j\omega T})|^2$$

B. Backward Rectangular Rule:

$$H(e^{j\omega T}) = \frac{3.5714\,(e^{j0.25\omega} - 0.8)}{e^{j0.25\omega} + 0.2857} = \frac{(3.5714\cos .25\omega - 2.85712) + j\sin 0.25\omega \times 3.5714}{(\cos 0.25\omega - 0.2857) + j\sin 0.25\omega}$$

$$|H(e^{j\omega T})|^2 = \frac{(3.5714\cos .25\omega - 2.85712)^2 + \sin^2 .25\omega \times 12.7549}{(\cos .25\omega - 0.2857)^2 + \sin^2 .25\omega}$$

C. Trapezoid Rule:

$$H(e^{j\omega T}) = \frac{5\,(e^{j0.25\omega} - 0.7778)}{e^{j0.25\omega} + 0.1111} = \frac{(5\cos 0.25\omega - 3.88889) + j\sin 0.25\omega \times 5}{(\cos .25\omega + 0.11111) + j\sin .25\omega}$$

$$|H(e^{j\omega T})|^2 = \frac{(5\cos .25\omega - 3.88889)^2 + \sin^2 .25\omega \times 25}{(\cos .25\omega + 0.11111)^2 + \sin^2 .25\omega}$$

D. Bilinear with prewarping.

$$H(e^{j\omega T}) = \frac{4.89259\,(e^{j0.25\omega} - 0.76802)}{e^{j0.25\omega} + 0.13498} = \frac{(4.89259\cos .25\omega - 3.75761) + j\sin .25\omega \times 4.89259}{\cos .25\omega + 0.13498 + j\sin .25\omega}$$

$$|H(e^{j\omega T})|^2 = \frac{(4.89259\cos .25\omega - 3.75761)^2 + \sin^2 .25\omega \times 23.93744}{(\cos .25\omega + 0.13498)^2 + \sin^2 .25\omega}$$

E. Pole-zero mapping

$$H(e^{j\omega T}) = \frac{4.14971\,(e^{j0.25\omega} - 0.77880)}{e^{j0.25\omega} - 0.08208} = \frac{(4.14971\cos .25\omega - 3.23179) + j\sin .25\omega \times 4.14971}{\cos .25\omega - 0.08208 + j\sin .25\omega}$$

$$|H(e^{j\omega T})|^2 = \frac{(4.14971\cos .25\omega - 3.23179)^2 + \sin^2 .25\omega \times 17.22009}{(\cos .25\omega - 0.08208)^2 + \sin^2 .25\omega}$$

F. Zero-order-hold equivalent

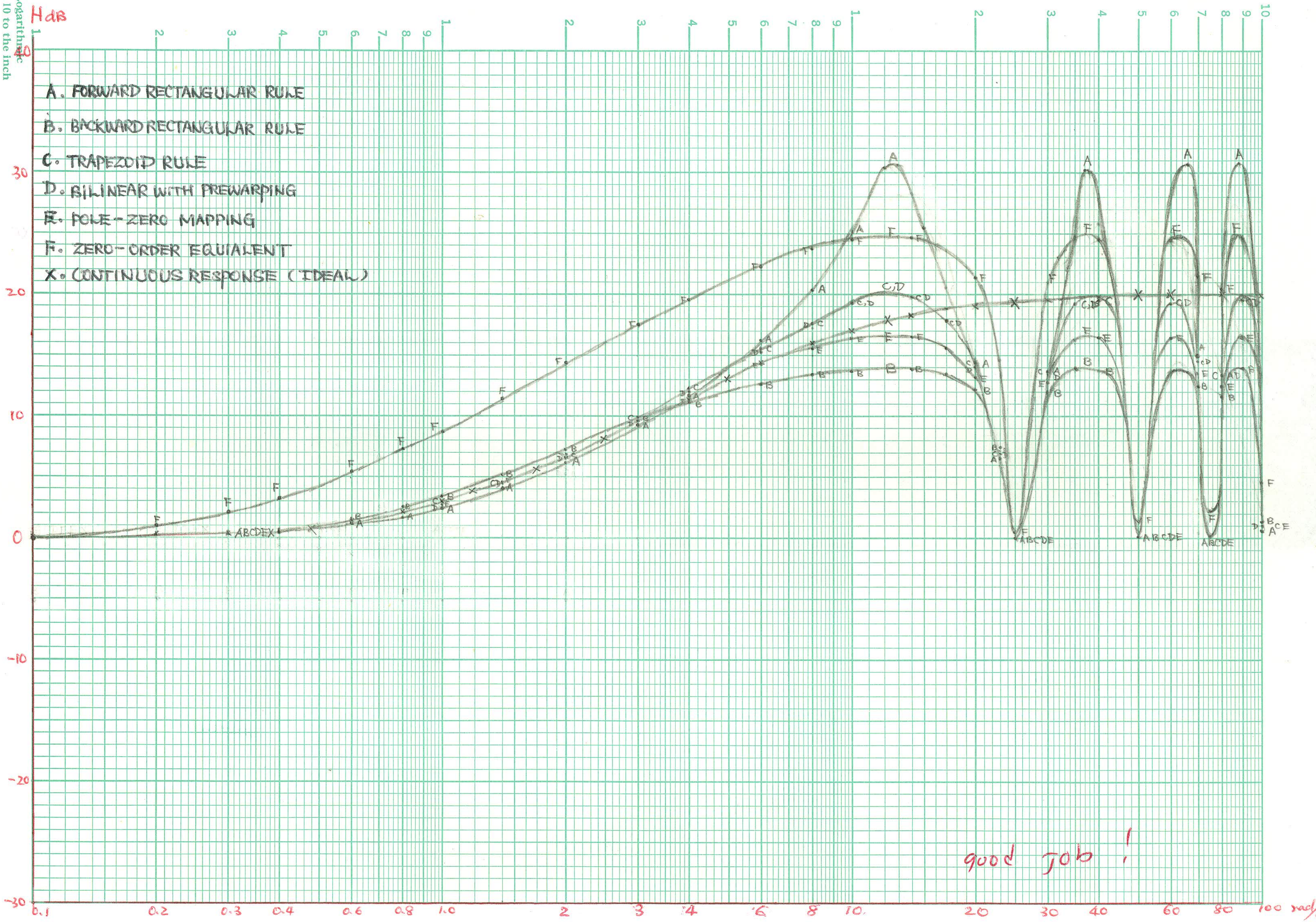
$$H(e^{j\omega T}) = \frac{10\,(e^{j0.25\omega} - 0.908208)}{e^{j0.25\omega} - 0.08208} = \frac{(10\cos .25\omega - 9.08208) + j\sin .25\omega \times 10}{\cos .25\omega - 0.08208 + j\sin .25\omega}$$

$$|H(e^{j\omega T})|^2 = \frac{(10\cos .25\omega - 9.08208)^2 + 100\sin^2 .25\omega}{(\cos .25 - 0.08208)^2 + \sin^2 .25\omega}$$

X. Continuous response (IDEAL)

$$H(j\omega) = \frac{j\omega + 1}{j0.1\omega + 1}$$

$$|H(j\omega)|^2 = \frac{\omega^2 + 1}{0.01\omega^2 + 1}$$


HdB
A. FORWARD RECTANGULAR RULE
B. BACKWARD RECTANGULAR RULE
C. TRAPEZOID RULE
D. BILINEAR WITH PREWARPING
E. POLE-ZERO MAPPING
F. ZERO-ORDER EQUIVALENT
X. CONTINUOUS RESPONSE (IDEAL)
40
30
20
10
0
-10
-20
-30
0.1
0.2
0.3
0.4
0.6
0.8
1.0
2
3
4
6
8
10
20
30
40
60
80
100 rad/s
good Job !

SP 1 SKETCH MAG & PHASE FOR $u_k = -0.5u_{k-1} + 5e_{k-1}$

$$u_k = -0.5u_{k-1} + 5e_{k-1}$$

$$U(z) = -0.5z^{-1} \cdot U(z) + 5z^{-1} \cdot E(z)$$

$$H(z) \triangleq U(z)/E(z) = 5z^{-1}/(1+0.5z^{-1})$$

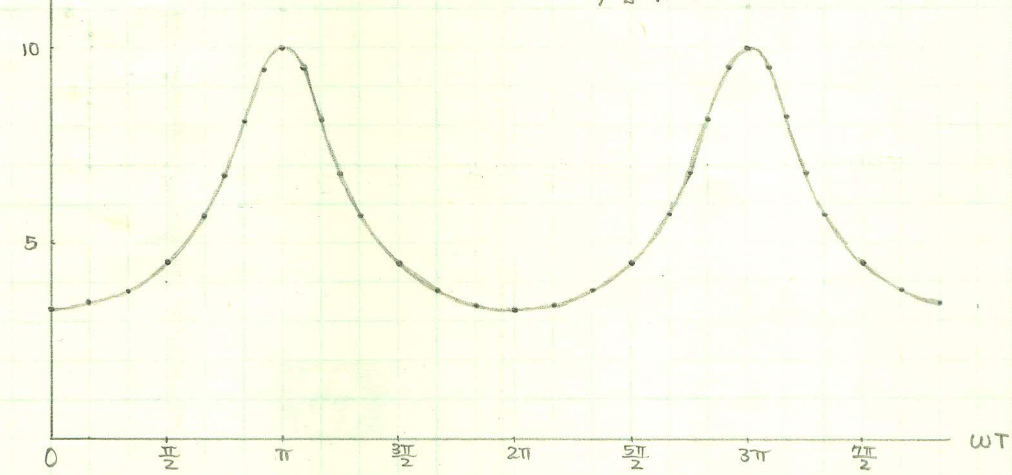
$$= 5/(z+0.5)$$

$$|H(e^{j\omega T})| = 5/|(e^{j\omega T}+0.5)|$$

$$= 5/|(0.5+\cos\omega T + j\sin\omega T)| = 5/\sqrt{(0.5+\cos\omega T)^2 + \sin^2\omega T}$$

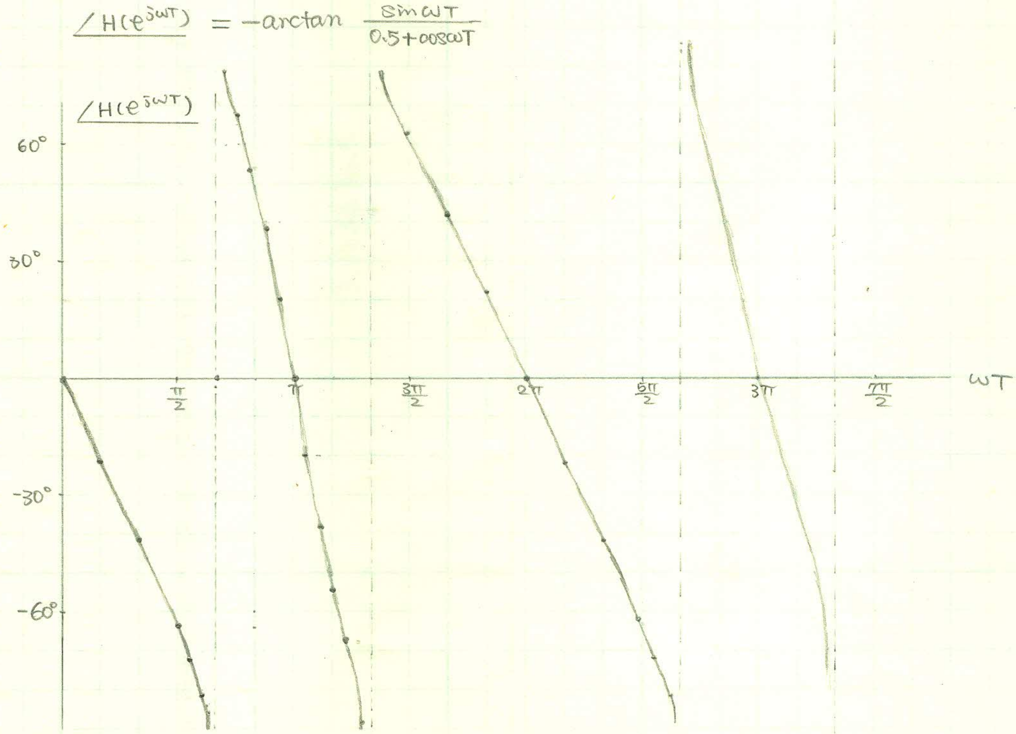
$|H(e^{j\omega T})|$

$$= 5/\sqrt{1.25 + \cos\omega T}$$



$$H(e^{j\omega T}) = \frac{5}{(0.5+\cos\omega T)^2 + \sin^2\omega T} \cdot (0.5+\cos\omega T - j\sin\omega T)$$

$$\underline{\angle H(e^{j\omega T})} = -\arctan \frac{\sin\omega T}{0.5+\cos\omega T}$$



SP 2. Sketch MAG & PHASE for $H(z) = 2z^{-1}/[(1-\sqrt{2}z^{-1}+z^{-2})(1+\sqrt{2}z^{-1}+z^{-2})]$

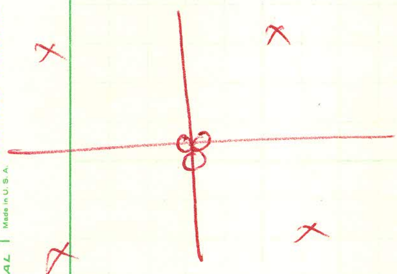
$$H(z) = 2z^3/[(z^2-\sqrt{2}z+1)(z^2+\sqrt{2}z+1)]$$

$$H(e^{j\omega T}) = 2e^{j\omega T}/[(e^{j2\omega T}-\sqrt{2}e^{j\omega T}+1)(e^{j2\omega T}+\sqrt{2}e^{j\omega T}+1)]$$

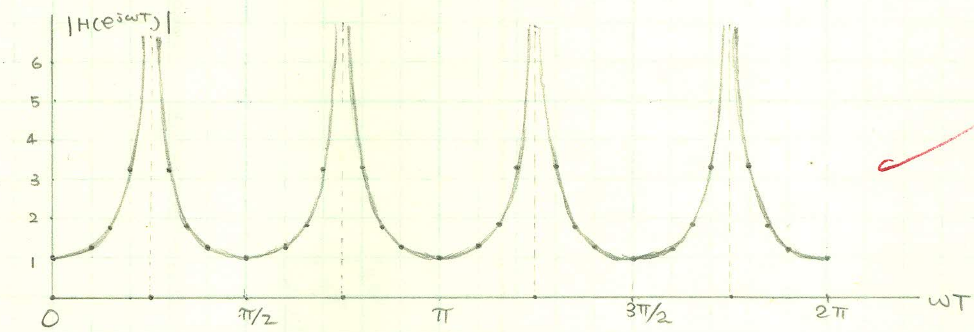
$$= 2e^{j\omega T}/[e^{j4\omega T}+\cancel{\sqrt{2}e^{j3\omega T}}+\cancel{e^{j2\omega T}}-\cancel{\sqrt{2}e^{j3\omega T}}-2e^{j2\omega T}-\cancel{\sqrt{2}e^{j\omega T}}+\cancel{e^{j2\omega T}}+\cancel{\sqrt{2}e^{j\omega T}}+1]$$

$$= 2e^{j\omega T}/(e^{j4\omega T}+1)$$

$$= \frac{e^{j\omega T}/e^{j2\omega T}}{\frac{e^{j2\omega T}+e^{-j2\omega T}}{2}} = \frac{e^{-j\omega T}}{\cos 2\omega T}$$

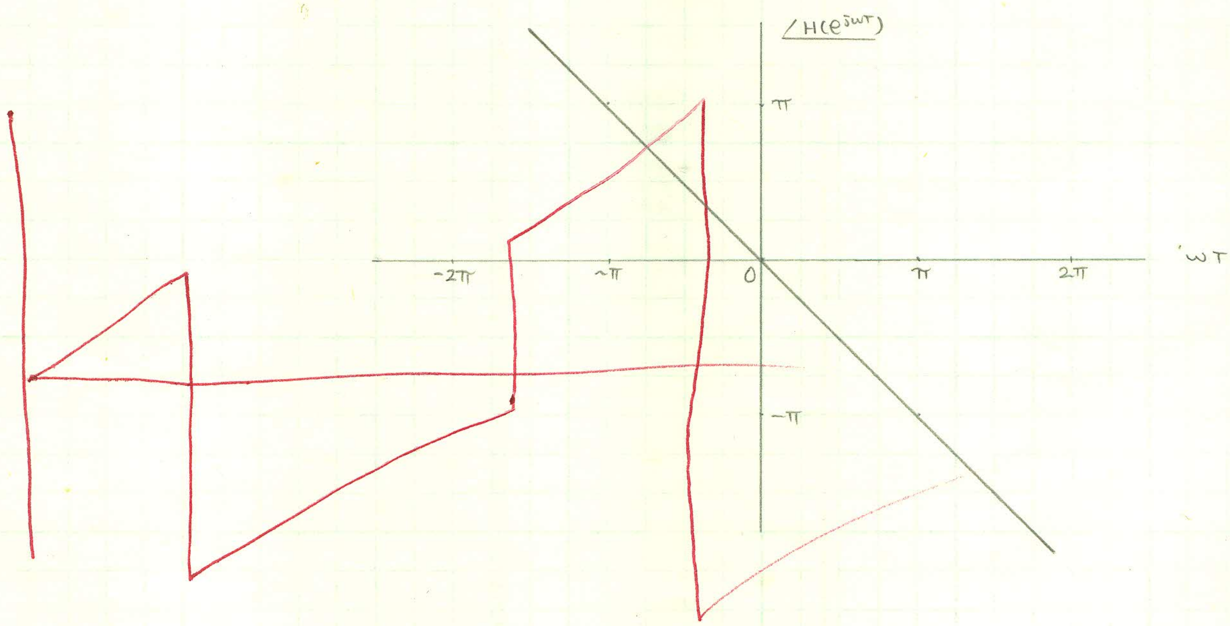


$$|H(e^{j\omega T})| = \frac{1}{|\cos 2\omega T|}$$



$$H(e^{j\omega T}) = \frac{1}{\cos 2\omega T}\cdot(\cos\omega T - j\sin\omega T)$$

$$\underline{/H(e^{j\omega T})} = -\tan^{-1}\cdot\frac{\sin\omega T}{\cos\omega T} = -\tan^{-1}\tan\omega T = -\omega T$$



SP 3 GIVE: $H(z) = \dfrac{1-a^{-1}z^{-1}}{1-a z^{-1}}$.

FIND: a) DIFF. EQ.

$$H(z) = \frac{U(z)}{E(z)} = \frac{1-a^{-1}z^{-1}}{1-a z^{-1}}$$

$$U(z)(1-az^{-1}) = E(z)(1-a^{-1}z^{-1})$$

$$u_k - a u_{k-1} = e_k - a^{-1}e_{k-1}$$

THE DIFF. EQ.: $u_k = a u_{k-1} + e_k - a^{-1}e_{k-1}$

b) RANGE OF a FOR STABLE SYSTEM.

$$H(z) = \frac{1-a^{-1}z^{-1}}{1-az^{-1}} = \frac{z-a^{-1}}{z-a}$$

$$a^* \triangleq \lim_{z\to 1}(z-1)\cdot H(z) = \lim_{z\to 1}(z-1)\frac{(z-a^{-1})}{z-a} \quad \cdots\cdots \quad (1)$$

$$H(z) = \frac{z-a+a-a^{-1}}{z-a} = 1 + (a-a^{-1})\cdot\frac{1}{z-a}$$

$$= 1 + (a-a^{-1})\cdot z^{-1}\cdot\frac{1}{1-az^{-1}}$$

$$= 1 + (a-a^{-1})z^{-1}(1+az^{-1}+a^2z^{-2}+\cdots)$$

$$= 1 + (a-a^{-1})z^{-1} + a(a-a^{-1})z^{-2} + \cdots + a^k\cdot(a-a^{-1})z^{-k} + \cdots$$

$$h_k = a^k(a-a^{-1}) \qquad k \geq 1$$
$$= 1 \qquad k = 0$$
$$= 0 \qquad k < 0$$

$$b^* \triangleq \lim_{k\to\infty} h_k = \lim_{k\to\infty} a^{k-1}(a^2-1) \quad \cdots\cdots \quad (2)$$

IN ORDER TO LET THE POLES OF $(z-1)H(z)$ BE INSIDE UNIT CYCLE, FROM EQ.(1),

$$|a| \leq 1$$

FOR $|a| \leq 1$, EQ (1). $a^* =$ EQ.(2) $b^* = 0$. SO,

THE SYSTEM IS STABLE WITHIN $-1 \leq a \leq 1$.

C) MAG VS FREQ.

$$H(z) = \frac{z-a^{-1}}{z-a}$$

$$H(e^{j\omega T}) = \frac{e^{j\omega T}-a^{-1}}{e^{j\omega T}-a} = \frac{(\cos\omega T - a^{-1}) + j\sin\omega T}{(\cos\omega T - a) + j\sin\omega T}$$

$$|H(e^{j\omega T})| = \frac{\sqrt{(\cos\omega T - a^{-1})^2 + \sin^2\omega T}}{\sqrt{(\cos\omega T - a)^2 + \sin^2\omega T}} = \sqrt{\frac{1+a^{-2}-2a^{-1}\cos\omega T}{1+a^2-2a\cos\omega T}}$$

$$= \frac{1}{a}$$

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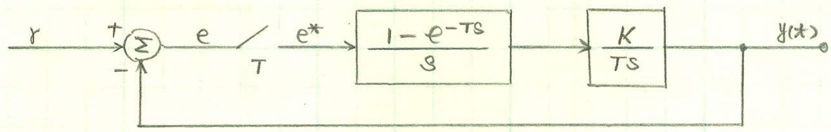
4.4 *Sketch the step response $y(t)$ of the system shown in Fig. 4.12 for $K = \frac{1}{2}, 1, 2$*



SOLUTION:

$$E(S) = R(S) - Y(S)$$

$$E^*(S) = \sum_{-\infty}^{\infty} e(kT)\, e^{-skT} = R^*(S) - Y^*(S)$$

$$Y(S) = E^*(S) \cdot \frac{1 - e^{-TS}}{S} \cdot \frac{K}{TS}$$

$$Y^*(S) = E^*(S) \cdot (1 - e^{-TS}) \cdot \frac{K}{T} \left(\frac{1}{S^2}\right)^*$$

$$= \frac{K}{T} \cdot E^*(S) \cdot (1 - e^{-TS}) \cdot \frac{T e^{ST}}{(e^{ST} - 1)^2}$$

$$= K \cdot E^*(S) \cdot (1 - e^{-TS}) \cdot \frac{e^{-TS}}{(1 - e^{-TS})^2}$$

$$= K \cdot E^*(S) \cdot \frac{e^{-TS}}{1 - e^{-TS}}$$

$$= K \cdot (R^*(S) - Y^*(S)) \cdot \frac{e^{-TS}}{1 - e^{-TS}}$$

$$\left(1 + K \cdot \frac{e^{-TS}}{1 - e^{-TS}}\right) \cdot Y^*(S) = K \cdot \frac{e^{-TS}}{1 - e^{-TS}} \cdot R^*(S)$$

$$\therefore \quad Y^*(S) = \frac{K \cdot e^{-TS}}{1 + (K-1) e^{-TS}} \cdot R^*(S)$$

$$Y(z) = Y^*(S)\Big|_{e^{ST} = z} = \frac{K \cdot z^{-1}}{1 + (K-1) z^{-1}} \cdot R(z) \qquad \text{(UNIT STEP)}$$

$$= \frac{K z^{-1}}{1 + (K-1) z^{-1}} \cdot \frac{z}{z - 1} \qquad |z| > 1$$

$$= \frac{K z^{-1}}{(1 + (K-1) z^{-1}) \cdot (1 - z^{-1})}$$

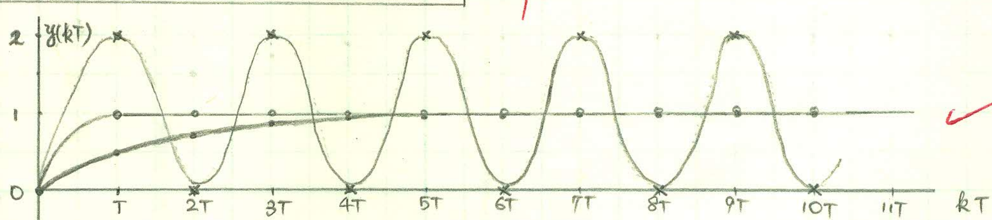
$$= \frac{1}{1 - z^{-1}} - \frac{1}{1 - (1-K) \cdot z^{-1}} \qquad |z| > 1$$

$$= \sum_{k=0}^{\infty} z^{-k} - \sum_{k=0}^{\infty} (1-K)^k \cdot z^{-k} \quad ; \ |z| > 1 \ \& \ |(1-K) z^{-1}| < 1 \ \text{FOR } K = \tfrac{1}{2}, 1, 2$$

$$= \sum_{k=0}^{\infty} (1 - (1-K)^k) \cdot z^{-k}$$

$$\boxed{\therefore \quad Y(kT) = 1 - (1-K)^k}$$

good



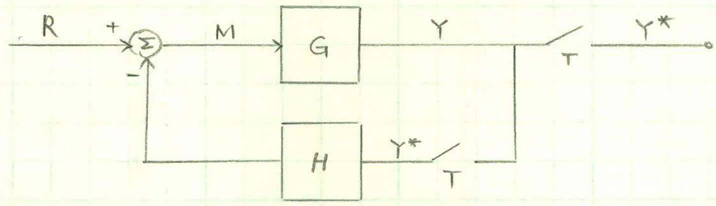
NOTE : • $y(t)$ with $K = \frac{1}{2}$

∘ $y(t)$ with $K = 1$

× $y(t)$ with $K = 2$

4.6. Find the transform of the output, $Y(s)$, and its samples $Y^*(s)$ for the block diagrams shown in Fig 4.14. Indicate if a transfer function exists in each case.

(a)



SOLUTION:

$M = R - HY^*$

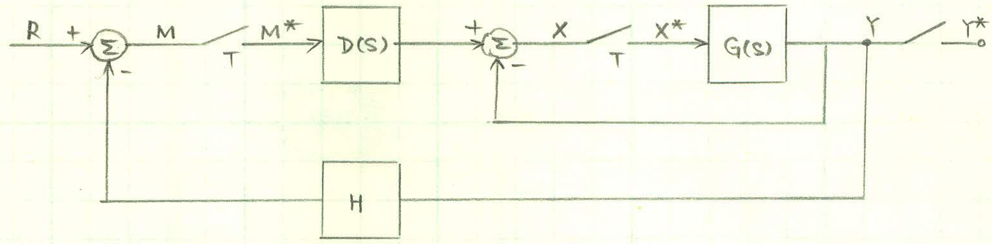
$Y = G \cdot M = G \cdot (R - HY^*)$

$Y(s) = G(s) \cdot [R(s) - H(s) \cdot Y^*(s)]$; WE CAN'T HAVE A TRANSFER FUNCTION IN THIS CASE BECAUSE OF $Y^*(s)$

$Y^* = (GR - GHY^*)^* = (GR)^* - (GH)^* Y^*$

$\therefore Y^*(s) = \dfrac{(GR)^*}{1 + (GH)^*}$; WE CAN'T HAVE A TRANSFER FUNCTION IN THIS CASE TOO BECAUSE WE CAN NOT PULL OUT R^* FROM $(GR)^*$.

(b)



SOLUTION:

$M = R - HY$; $M^* = R^* - (HY)^*$

$X = M^* \cdot D - Y$; $X^* = M^* \cdot D^* - Y^* = [R^* - (HY)^*] \cdot D^* - Y^*$

$Y = X^* \cdot G = R^* D^* G - (HY)^* \cdot D^* \cdot G - Y^* G$

$Y(s) = R^*(s) D^*(s) G(s) - [H(s) Y(s)]^* \cdot D^*(s) \cdot G(s) - Y^*(s) \cdot G(s)$

WE CAN'T FIND OUT THE TRANSFER FUNCTION IN THIS CASE. BECAUSE THERE IS NO $R(s)$ IN THE EQATION ABOVE. (CAN'T WRITE AS THE FORM OF $Y(s)/R(s)$).

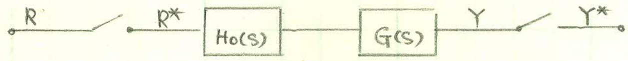
-1

$Y^* = X^* \cdot G^* = R^* D^* G^* - (HY)^* \cdot D^* G^* - Y^* \cdot G^*$

$Y^* = \dfrac{R^* D^* G^* - (HY)^* \cdot D^* \cdot G^*}{1 + G^*} = \dfrac{R^* D^* G^*}{1 + (GH)^* D^* + G^*}$

BECAUSE WE HAVE $(HY)^*$ IN EQUATION ABOVE ; Y^* CAN'T BE PULLED OUT FROM IT, WE CAN'T EVEN DETERMINE THE $Y^*(s)$ FROM THE SYSTEM. THEREFORE NO TRANSFER FUNCTION EXISTS.

4.7 In Appendix A are sketched several process transfer functions. Assume they are preceded by a zero-order hold and compute the resulting discrete transfer function for



$$Y = R^* \cdot H_0 \cdot G \quad ; \quad Y^* = R^* \cdot (H_0 \cdot G)^* \quad ; \quad H_0(s) = (1-e^{-sT})/s$$

$$\frac{Y^*}{R^*} = (H_0 G)^* \quad \& \quad G(z) = \left.\frac{Y^*}{R^*}\right|_{e^{sT}=z} = (H_0 G)^* \Big|_{e^{sT}=z}$$

(a) $G_1(s) = 1/s^2$

$$(H_0 G_1)^* = \left(\frac{1-e^{-sT}}{s^3}\right)^* = (1-e^{-sT})\cdot\left(\frac{1}{s^3}\right)^* = (1-e^{-sT})\cdot\frac{T^2 e^{sT}(e^{sT}+1)}{2(e^{sT}-1)^3}$$

$$\therefore G_1(z) = (1-z^{-1})\cdot\frac{T^2}{2}\cdot\frac{z(z+1)}{(z-1)^3} = \frac{T^2}{2}\cdot\frac{z+1}{(z-1)^2} \quad \underline{\underline{\text{QED.}}}$$

(b) $G_2(s) = e^{-1.5s}/(s+1)$

$$(H_0 G_2)^* = \left[\frac{e^{-1.5s}(1-e^{-s})}{s(s+1)}\right]^*$$

$$G_2(z) = \mathcal{Z}\left[\frac{e^{-2s}(1-e^{-s})\cdot e^{0.5s}}{s(s+1)}\right] = z^{-2}\cdot(1-z^{-1})\cdot\mathcal{Z}\left[\frac{e^{0.5s}}{s} - \frac{e^{0.5s}}{s+1}\right]$$

$$= \frac{z-1}{z^2}\left[\frac{z}{z-1} - \frac{e^{-0.5}\cdot z}{z-e^{-1}}\right] \qquad \text{FROM EQ. A.16 ON PAGE 298}$$

$$= \frac{1}{z^2} - \frac{0.6065\,z}{z-0.3679}\cdot\frac{z-1}{z^2} = \frac{0.3935\,z + 0.2386}{z^2(z-0.3679)} = \frac{0.3935(z+0.6065)}{z^2(z-0.3679)} \quad \underline{\underline{\text{QED.}}}$$

(c) $G_3(s) = 1/s(s+1)$

$$(H_0 G_3)^* = \left[\frac{1-e^{-sT}}{s^2(s+1)}\right]^* = (1-e^{-sT})\cdot\left[\frac{1}{s^2(s+1)}\right]^*$$

$$\therefore G_3(z) = (1-z^{-1})\cdot\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right] = (1-z^{-1})\cdot\frac{z[(T-1+e^{-T})z + (1-e^{-T}-Te^{-T})]}{(z-1)^2(z-e^{-T})}$$

$$= \frac{(T-1+e^{-T})z + (1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})} \quad \underline{\underline{\text{QED.}}}$$

(d) $G_7(s) = e^{-1.5s}/s(s+1)$; Assume $T = 1.0$ in this case.

$$G_7(z) = \mathcal{Z}[H_0(s)\cdot G_7(s)] = \mathcal{Z}\left[\frac{(1-e^{-s})e^{-2s}\cdot e^{0.5s}}{s^2(s+1)}\right] = \frac{z-1}{z^3}\cdot\mathcal{Z}\left[\frac{e^{0.5s}}{s^2(s+1)}\right]$$

$$= \frac{z-1}{z^3}\,\mathcal{Z}\left[\frac{e^{0.5s}}{s^2} - \frac{e^{0.5s}}{s} + \frac{e^{0.5s}}{s+1}\right] = \frac{z-1}{z^3}\,\mathcal{Z}\left[(t+0.5)1(t+0.5) - 1(t+0.5) + e^{-(t+0.5)}1(t+0.5)\right]$$

$$= \frac{z-1}{z^3}\cdot\left[\sum_{k=0}^{\infty}(k+0.5)\cdot z^{-k} - \frac{z}{z-1} + \sum_{k=0}^{\infty} e^{-(k+0.5)} z^{-k}\right]$$

$$= \frac{z-1}{z^3}\left[\frac{z}{(z-1)^2} + \frac{0.5\,z}{z-1} - \frac{z}{z-1} + \frac{e^{-0.5}\cdot z}{z-e^{-1}}\right] \qquad ?$$

$$= \frac{0.1065(z+0.1191)(z+4.3034)}{z^2(z-0.3679)(z-1)} \quad \underline{\underline{\text{QED.}}} \qquad -1$$

(e) $G_8(s) = 1/(s^2-1) = 1/(s+1)(s-1)$

$$G_8(z) = \mathcal{Z}[H_0(s)G_8(s)] = \mathcal{Z}\left[\frac{1-e^{-sT}}{s(s+1)(s-1)}\right] = (1-z^{-1})\cdot\mathcal{Z}\left[\frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s+1} - \frac{1}{s}\right]$$

$$= (1-z^{-1})\cdot\left[\frac{\frac{1}{2}z}{z-e^{T}} + \frac{\frac{1}{2}z}{z-e^{-T}} - \frac{z}{z-1}\right] = (e^{T}+e^{-T}-2)\cdot\frac{z+1}{2(z-e^{T})(z-e^{-T})}, \quad \underline{\underline{\text{QED!}}}$$

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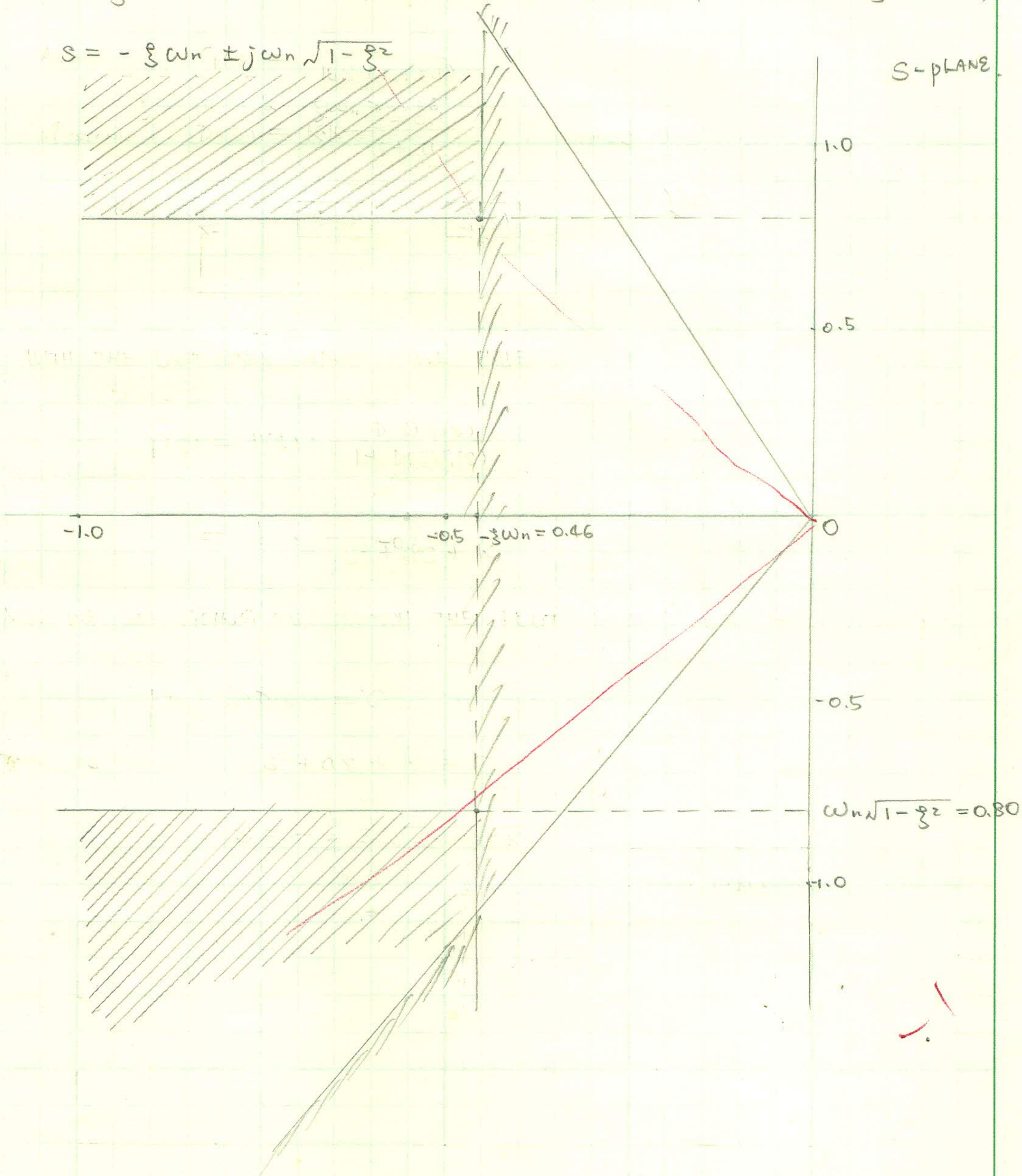
5.2. Sketch the acceptable region in the s-plane for the specification on the antenna given before (5.15) and sketch the s-plane root locus corresponding to the controller (5.15)

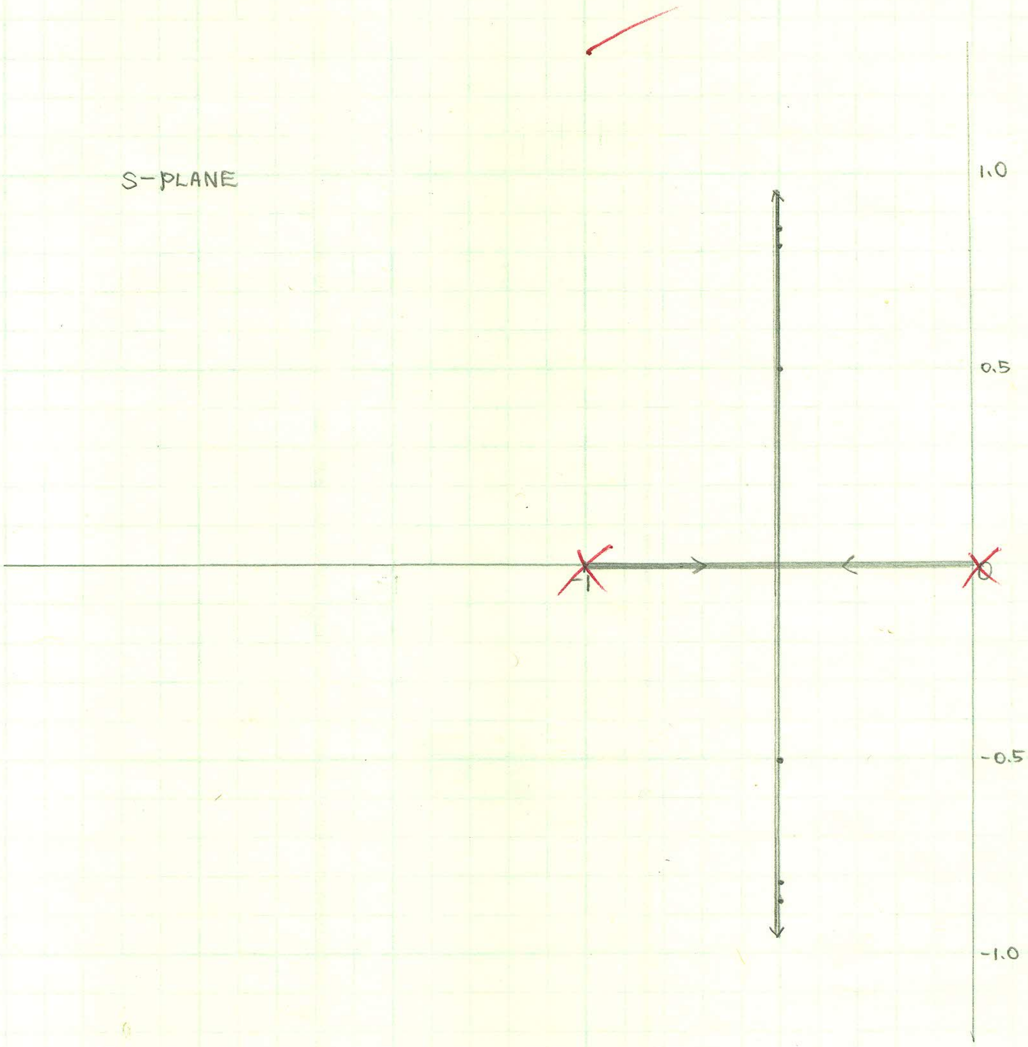
FIRST PART : THE SPECIFICATION ON THE ANTENNA GIVEN BEFORE (5.15) IS :

$\zeta = 0.5$ (P.O. = 17%) AND SETTING TIME OF 10 SEC. ($10\zeta\omega_n = 4.6$)



The following is the root locus on s-plane corresponding to the controller. (5.15)

$$1 + G(s)D(s) = 1 + \frac{K}{s(s+1)} = \frac{s^2 + s + K}{s(s+1)}$$



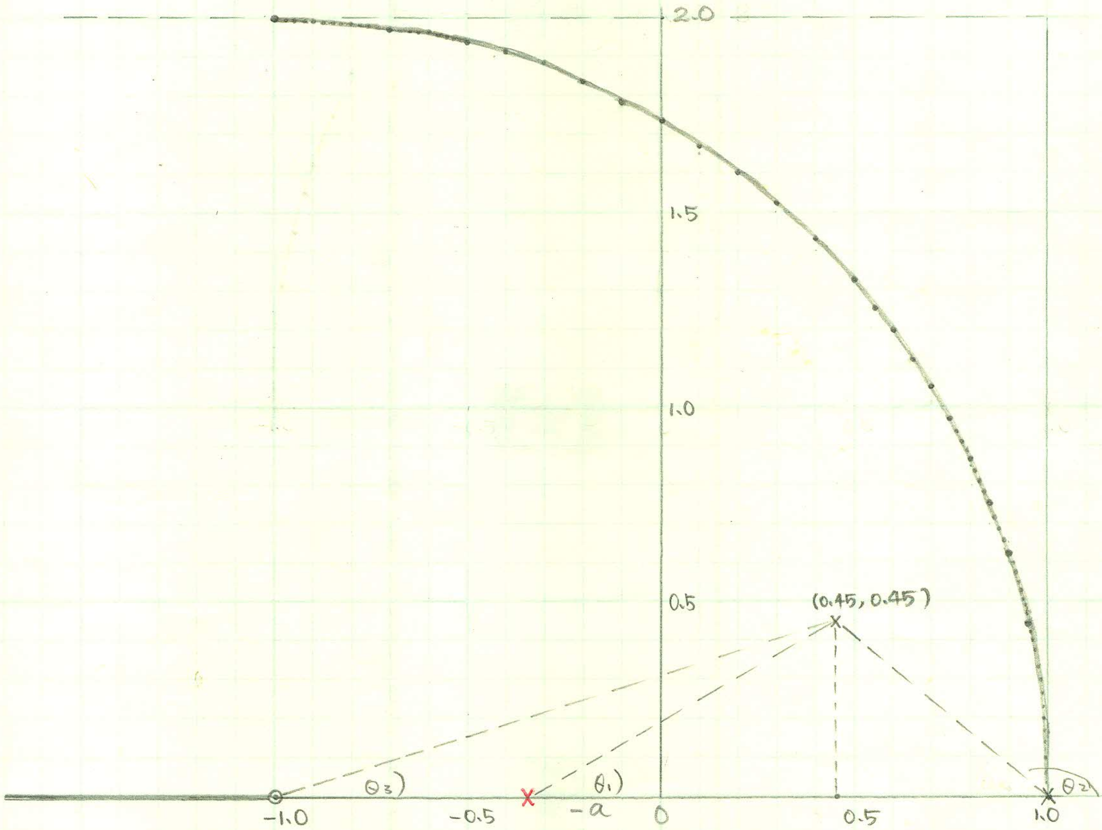
5.4 Appendix A gives the transfer function of a satellite attitude control as

$$G_1(z) = K \cdot \frac{z+1}{(z-1)^2}$$

a) Sketch the root locus of this system as a function of K with unity feedback. What is the type of the uncompensated system.

$$1 + G_1(z) = 1 + K \cdot \frac{z+1}{(z-1)^2}$$

$$= \frac{z^2 + (K-2)z + (1+K)}{(z-1)^2} = 0$$



And this system is a Type II one, because $G_1(z)$ CAN BE REWRITE AS

$$G_1(z) = \frac{G_1'(z)}{(z-1)^2}$$

b) Add a lead network so that the domiant poles are at $\xi = 0.5$ and $\theta = 45°$. Plot the closed-loop step response.

$\xi = 0.5$ and $\omega_n\sqrt{1-\xi^2} = 0.866\,\omega_n$

$$z = e^{TS} = e^{T(-0.5\omega_n \pm j0.866\omega_n)} = e^{0.5T\omega_n} \cdot (\cos 0.866\omega_n T \pm j \sin 0.866\omega_n T)$$

SO, $0.866\omega_n T = 45^\circ = \frac{\pi}{4} = 0.7854$

$$\omega_n T = 0.9069$$

$$\therefore \quad z = 0.4493 \pm j0.4493$$

SO. WE HAVE TO DESIGN A $D(z)$ WHICH SHALL HAVE A ZERO AT $z=1$ AND A POLE SOMEWHERE ON THE LEFT OF $z = 1.0$. SEE THE FIG.

$$\theta_1 + \theta_2 - \theta_3 = 180^\circ$$

$$\arctan\frac{0.4493}{0.4493+a} + \left(180^\circ - \arctan\frac{0.4493}{1-0.4493}\right) - \arctan^{-1}\frac{0.4493}{1+0.4493} = 180^\circ$$

$$\therefore \arctan\frac{0.4493}{0.4493+a} = 17.22^\circ + 39.21^\circ = 56.43^\circ$$

$$\frac{0.4493}{0.4493+a} = 1.50691$$

$$a = -0.15115$$

$$\therefore \quad D(z) \triangleq \frac{z-1}{z-0.15115}$$

AND $1 + D(z)G_1(z) = 1 + \frac{K(z+1)}{(z-0.15115)(z-1)}$

$$= \frac{z^2 - (1.15115 - K)z + (0.15115 + K)}{(z-0.15115)(z-1)}$$

SO, $K = 0.25253$.

$$D(z)G_1(z) = \frac{0.25253\,(z+1)}{(z-0.15115)(z-1)}$$

2

$$E(z)H(z) = \frac{D(z)\cdot G_1(z)}{1 + D(z)G_1(z)} \cdot E(z)$$

$$= \frac{0.25253\,(z+1)}{z^2 - 0.8986\,z + 0.40368} \cdot \frac{z}{z-1}$$

$$0.25253z^{-1} + 0.73z^{-2} + 1.06z^{-3} + 1.16z^{-4} + 1.11z^{-5} + 1.03z^{-6} + \cdots$$

$z^3 - 1.8986z^2 + 1.3023z - 0.4037 \,\big)\, 0.25253z^2 + 0.25253z$

$0.25253z^2 - 0.47945z + 0.3289z^{-0} - 0.10z^{-2}$

$0.73z - 0.33z^{-0} + 0.10z^{-1}$

$0.73z - 1.39z^{-0} + 0.95z^{-1} - 0.29z^{-2}$

$1.06z^{-0} - 0.85z^{-1} + 0.29z^{-2}$

$1.06z^{-0} - 2.01z^{-1} + 1.38z^{-2} - 0.43z^{-3}$

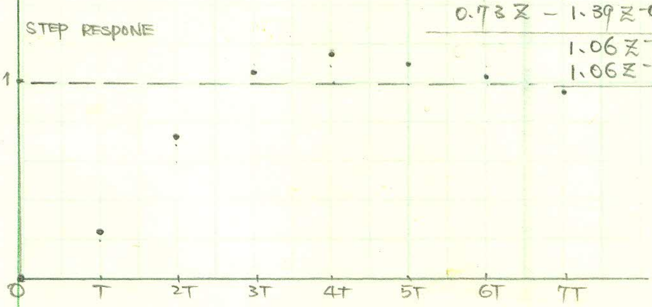
$1.16z^{-1} - 1.09z^{-2} + 0.43z^{-3}$

$1.16z^{-1} - 2.20z^{-2} + 1.51z^{-3} - 0.46z^{-4}$

$1.11z^{-2} - 1.08z^{-3} + 0.46z^{-4}$

$1.11z^{-2} - 2.11z^{-3} + 1.44z^{-4} - 0.44z^{-5}$

$1.03z^{-3} - 0.98z^{-4} + 0.44z^{-5}$



AND OVERSHOOT IS ABOUT 16%.

5.6. Repeat the design of the antenna by pole-zero mapping of D(s) of (5.15) but use sample period $T = 0.2$.

SOLUTION:

$$D(s) = \frac{10s + 1}{s + 1} \quad \cdots\cdots \quad (5.15)$$

LET $T = 0.2$,

ZERO AT $z = e^{-0.02} = 0.9802$

POLE AT $z = e^{-0.2} = 0.8187$

GAIN AT $z = 1 = K \frac{1-0.9802}{1-0.8187}$

THEREFORE, $K = 9.1550$

AND WE HAVE THE COMPENSATION

$$D(z) = 9.1550 \cdot \frac{z - 0.9802}{z - 0.8187}$$

$$G(z) = \frac{Az + B}{a(z-1)(z-e^{aT})}$$

$$A = e^{-aT} + aT - 1 , \quad B = 1 - e^{-aT} - aTe^{-aT}$$

FOR $T = 0.2$ and $a = 0.1$ as in this case, WE HAVE

$$G(z) = 0.0020 \cdot \frac{(z + 0.9934)}{(z-1)(z-0.9802)}$$

$$H(z) = \frac{G(z)D(z)}{1 + G(z)D(z)} = \frac{0.0182\,(z + 0.9934)}{z^2 - 1.8005z + 0.8368}$$

UINT STEP RESPONE

$$H(z) \cdot 1(z) = \frac{0.0182z^2 + 0.0181 \cdot z}{z^3 - 2.8005z^2 + 2.6373z - 0.8368}$$

TO BE CONTINUED ON NEXT PAGE

$$0.0182\, z^{-1} + 0.0691\, z^{-2} + 0.1455\, z^{-3} + 0.2405\, z^{-4} + 0.3476\, z^{-5} + 0.4610\, z^{-6}$$

$$z^3 - 2.8005\, z^2 + 2.6373\, z - 0.8368 \;\big)\; 0.0182\, z^2 + 0.0181\, z$$

$$0.0182\, z^2 - 0.0510\, z + 0.0480 - 0.0152\, z^{-1}$$

$$0.0691\, z - 0.0480 + 0.0152\, z^{-1}$$

$$0.0691\, z - 0.1935 + 0.1822\, z^{-1} - 0.0578\, z^{-2}$$

$$0.1455 - 0.1670\, z^{-1} + 0.0578\, z^{-2}$$

$$0.1455 - 0.4075\, z^{-1} + 0.3837\, z^{-2} - 0.1218\, z^{-3}$$

$$0.2405\, z^{-1} - 0.3259\, z^{-2} + 0.1218\, z^{-3}$$

$$+ 0.5756\, z^{-7} + 0.6871\, z^{-8} + 0.7920\, z^{-9} + 0.8876\, z^{-10}$$

$$0.2405\, z^{-1} - 0.3259\, z^{-2} + 0.1218\, z^{-3}$$

$$0.2405\, z^{-1} - 0.6735\, z^{-2} + 0.6343\, z^{-3} - 0.2013\, z^{-4}$$

$$0.3476\, z^{-2} - 0.5125\, z^{-3} + 0.2013\, z^{-4}$$

$$0.3476\, z^{-2} - 0.9735\, z^{-3} + 0.9167\, z^{-4} - 0.2909\, z^{-5}$$

$$0.4610\, z^{-3} - 0.7154\, z^{-4} + 0.2909\, z^{-5}$$

$$0.4610\, z^{-3} - 1.2910\, z^{-4} + 1.2158\, z^{-5} - 0.3858\, z^{-6}$$

$$0.5756\, z^{-4} - 0.9249\, z^{-5} + 0.3858\, z^{-6}$$

$$+ 0.9720\, z^{-11} + 1.0439\, z^{-12} + 1.1026\, z^{-13}$$

$$0.5756\, z^{-4} - 1.6120\, z^{-5} + 1.5180\, z^{-6} - 0.4817\, z^{-7}$$

$$0.6871\, z^{-5} - 1.1322\, z^{-6} + 0.4817\, z^{-7}$$

$$0.6871\, z^{-5} - 1.9242\, z^{-6} + 1.8121\, z^{-7} - 0.5750\, z^{-8}$$

$$0.7920\, z^{-6} - 1.3304\, z^{-7} + 0.5750\, z^{-8}$$

$$0.7920\, z^{-6} - 2.2180\, z^{-7} + 2.0887\, z^{-8} - 0.6627\, z^{-9}$$

$$0.8876\, z^{-7} - 1.5137\, z^{-8} + 0.6627\, z^{-9}$$

$$0.8876\, z^{-7} - 2.4857\, z^{-8} + 2.3409\, z^{-9} - 0.7427\, z^{-10}$$

$$0.9720\, z^{-8} - 1.6782\, z^{-9} + 0.7427\, z^{-10}$$

$$+ 1.1481\, z^{-14} + 1.1809\, z^{-15} + 1.2019\, z^{-16}$$

$$0.9720\, z^{-8} - 2.7221\, z^{-9} + 2.5635\, z^{-10} - 0.8134\, z^{-11}$$

$$1.0439\, z^{-9} - 1.8208\, z^{-10} + 0.8134\, z^{-11}$$

$$1.0439\, z^{-9} - 2.9234\, z^{-10} + 2.7531\, z^{-11} - 0.8735\, z^{-12}$$

$$1.1026\, z^{-10} - 1.9397\, z^{-11} + 0.8735\, z^{-12}$$

$$1.1026\, z^{-10} - 3.0878\, z^{-11} + 2.9079\, z^{-12} - 0.9227\, z^{-13}$$

$$1.1481\, z^{-11} - 2.0344\, z^{-12} + 0.9227\, z^{-13}$$

$$1.1481\, z^{-11} - 3.2153\, z^{-12} + 3.0279\, z^{-13} \ldots$$

$$+ 1.2122\, z^{-17} + 1.2132\, z^{-18} + 1.2064\, z^{-19} + 1.1933\, z^{-20} + \ldots$$

$$1.1809\, z^{-12} - 2.1052\, z^{-13} + 0.9607\, z^{-14}$$

$$1.1809\, z^{-12} - 3.3071\, z^{-13} + 3.1144\, z^{-14}$$

$$1.2019\, z^{-13} - 2.1537\, z^{-14} + 0.9882$$

$$1.2019\, z^{-13} - 3.3659\, z^{-14} + 3.1698\,\cdot$$

$$1.2122\, z^{-14} - 2.1816\,\cdot + 1.0057\,\cdot$$

$$1.2122\, z^{-14} - 3.3948 + 3.1969$$

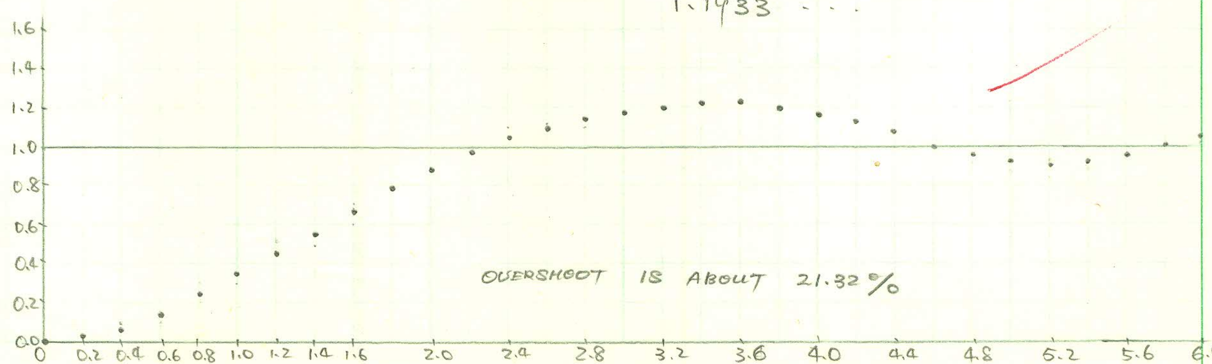
$$1.2132 - 2.1912 + 1.0144$$

$$1.2132 - 3.3976 + 3.1996$$

$$1.2064 - 2.1852 + 1.0152$$

$$1.2064 - 3.3785$$

$$1.1933 \ldots$$



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1. Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

SLN:

$$\det(\lambda I - A) = 0$$

$$\det\left[\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{pmatrix}\right] = 0$$

$$\det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2 & 4 & \lambda+3 \end{bmatrix} = 0$$

THAT IS,

$$\lambda^3 + 3\lambda^2 + 4\lambda + 2 = 0$$

EIGENVALUES: $\lambda_1 = -1$ $\quad (\lambda_{2,3} = -1 \pm j)$

2. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$$

Solution: $\det[\lambda I - A] = 0$

$$\det\left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}\right] = \det\begin{bmatrix} \lambda & -1 \\ -6 & \lambda-1 \end{bmatrix} = 0$$

THAT IS,

$$\lambda^2 - \lambda - 6 = 0$$

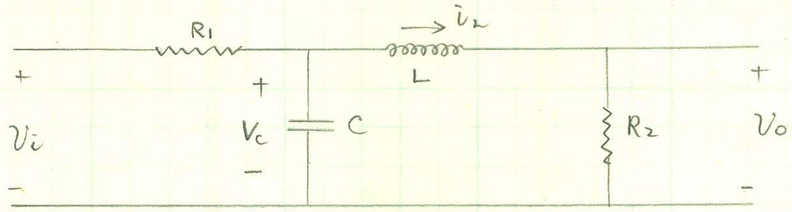
EIGENVALUES: $\lambda_1 = 3$, $\lambda_2 = -2$

LET EIGENVACTOR BE $V = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix} \Rightarrow \begin{cases} y = 3x \\ 2y = 6x \end{cases}$$ Choose $x=1$, $y=3$ $\Rightarrow V = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \end{bmatrix} \Rightarrow \begin{cases} y = -2x \\ 6x = -3y \end{cases}$$ choose $x=1$, $y=-2$ $\Rightarrow V = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

3. Write the state variable equations for



in term of R_1, R_2, L, C for the state variables V_c, i_L. Find the eigenvalue of the F matrix if

$R_1 = 2\Omega$; $R_2 = 4\Omega$; $L = 4H$; $C = \frac{1}{2}F$

Find the s-domain transfer function for $\frac{V_o}{V_i}$ and compare the poles of this with the eigenvalues.

Solution:

$$\dot{i}_L = \frac{V_L}{L} = \frac{1}{L}\left(V_c - i_L R_2\right)$$

$$\dot{V}_c = \frac{i_c}{C} = \frac{1}{C}\left(\frac{V_i}{R_1} - \frac{V_c}{R_1} - i_L\right)$$

$$\therefore \begin{pmatrix} \dot{V}_c \\ \dot{i}_L \end{pmatrix} = \begin{bmatrix} -1/CR_1 & -1/C \\ 1/L & -R_2/L \end{bmatrix} \begin{pmatrix} V_c \\ i_L \end{pmatrix} + \begin{bmatrix} 1/CR_1 \\ 0 \end{bmatrix} V_i$$

$$V_o = i_L R_2$$

$$V_o = \begin{bmatrix} 0 & R_2 \end{bmatrix} \begin{pmatrix} V_c \\ i_L \end{pmatrix}$$

$$F = \begin{bmatrix} -1/CR_1 & -1/C \\ 1/L & -R_2/L \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1/4 & -1 \end{bmatrix}$$

$$\det(\lambda I - F) = \det \begin{bmatrix} \lambda+1 & 2 \\ -\frac{1}{4} & \lambda+1 \end{bmatrix} = 0 \Rightarrow \lambda^2 + 2\lambda + 1.5 = 0 \quad \Rightarrow \lambda_{1,2} = -1 \pm j0.707$$

According to the notes: IF $i_L(0) = 0$ and $V_c(0) = 0$, THEN

$$\frac{V_o(s)}{V_i(s)} = H(sI - F)^{-1} G = [0, 4] \cdot \begin{bmatrix} s+1 & 2 \\ -0.25 & s+1 \end{bmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\cancel{-8}\ 1}{s^2 + 2s + 1.5}$$

−2

POLES : $s_{1,2} = -1 \pm j0.707$ SAME AS EIGENVALUES.

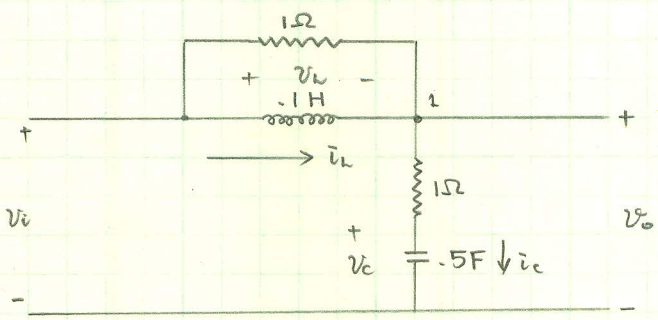
$\frac{20}{20}$

For the given circuit

a) Find the state variable equations with i_L and v_c, the state varibles.

b) Find the controllable canonic form for the state equations

c) Compute the transfer function



SOLUTION:

(a) $$\dot{i}_L = \frac{v_L}{L} = 10 \cdot v_L = 10 \cdot v_L$$

FROM NODE 1:

$$i_L + \frac{v_L}{1\Omega} = i_c$$

$$v_L + \left(i_L + \frac{v_L}{1\Omega}\right) \cdot 1\Omega + v_c = v_i$$

$$v_L = \frac{v_i}{2} - \frac{i_L}{2} - \frac{v_c}{2}$$

$$\therefore \quad \dot{i}_L = 5v_i - 5i_L - 5v_c \quad \cdots\cdots \quad (1)$$

$$\dot{v}_c = \frac{i_c}{C} = 2i_c = 2(i_L + v_L)$$

$$= 2i_L + v_i - i_L - v_c = v_i + i_L - v_c \quad \cdots\cdots \quad (2)$$

FROM EQ (1) and (2)

$$\begin{pmatrix} \dot{i}_L \\ \dot{v}_c \end{pmatrix} = \begin{pmatrix} -5 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} i_L \\ v_c \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} v_i$$

$$v_o = v_c + i_c \cdot 1\Omega = \tfrac{1}{2}(v_i + i_L + v_c) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \begin{pmatrix} i_L \\ v_c \end{pmatrix} + \tfrac{1}{2} v_i$$

(b) $$\frac{V_o(s)}{V_i(s)} = H(sI-F)^{-1}G + J$$

$$= \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\begin{pmatrix} s+5 & 5 \\ -1 & s+1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \frac{1}{2}$$

$$= \frac{3s+5}{s^2+6s+10} + \frac{1}{2}$$

$$= \frac{1}{2} \cdot \frac{s^2+12s+20}{s^2+6s+10}$$

FROM THE TRANSFER FUNCTION ABOVE, WE HAVE THE CANONIC FORM

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -10 & -6 \end{bmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} V_i$$

$$V_o = (5, 3)\, x + (.5)\, V_i$$

(c) THE TRANSFER FUNCTION IS

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{2} \cdot \frac{s^2+12s+20}{s^2+6s+10}$$

18/20

6.1 Compute Φ by changing states so that the system matrix is diagonal.

a) Compute e^{AT} where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

using the infinite series of (6.14)

From the (6.14),

$$e^{AT} = I + AT + \frac{A^2T^2}{2!} + \frac{A^3T^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -T & 0 \\ 0 & -2T \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2!} & 0 \\ 0 & 4T^2/2! \end{bmatrix} + \begin{bmatrix} -\frac{T^3}{3!} & 0 \\ 0 & -\frac{8T^3}{3!} \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 - T + \frac{T^2}{2!} - \frac{T^3}{3!} + \frac{T^4}{4!} - \cdots & 0 \\ 0 & 1 - 2T + \frac{(2T)^2}{2!} - \frac{(2T)^3}{3!} + \frac{(2T)^4}{4!} - \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix}$$

(b) Show that if $F = T^{-1}AT$ for some nonsigular transformation matrix T, then $e^{FT} = T^{-1}e^{AT}T$.

from the (6.14)

$$Te^{FT} = T\cdot[I + FT + \frac{F^2T^2}{2!} + \frac{F^3T^3}{3!} + \cdots]$$

$$= T + TF\cdot T + TF^2\cdot T^2/2! + TF^3\cdot T^3/3! + \cdots\cdots$$

$$e^{AT}\cdot T = [I + A\cdot T + \frac{A^2T^2}{2!} + \frac{A^3T^3}{3!} + \cdots]\cdot T$$

$$= T + A\cdot T\cdot T + A^2\cdot T\cdot T/2! + A^3T\cdot T^3/3! + \cdots\cdots$$

Because

$$TF^k = T\cdot T^{-1}A\cdot T\cdot T^{-1}\cdot A\cdot T \cdots T^{-1}\cdot A\cdot T = A^k\cdot T$$

SO, WE HAVE

$$Te^{FT} = e^{AT}T \quad \text{THEN} \quad e^{FT} = T^{-1}e^{AT}T$$ QED

(c) Show that if

$$F = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

there exists a T so that $T^{-1}AT = F$

SHOW: LET $T = \begin{bmatrix} x & y \\ z & k \end{bmatrix}$

$$TF = \begin{bmatrix} x & y \\ z & k \end{bmatrix} \cdot \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3x+y & -2x \\ -3z+k & -2z \end{bmatrix}$$

$$AT = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & k \end{bmatrix} = \begin{bmatrix} -x & -y \\ -2z & -2k \end{bmatrix}$$

SO,
$$\begin{cases} -3x+y = -x \\ -2x = -y \\ -3z+k = -2z \\ -2z = -2k \end{cases} \Rightarrow \begin{cases} x=1 \\ y=2 \\ z=1 \\ k=1 \end{cases}$$

$$T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$ will satisfy the condition.

(d) Compute e^{FT}.

According to results in part (b)

$$e^{FT} = T^{-1} e^{AT} \cdot T$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-T} & 2e^{-2T} \\ e^{-T} & -e^{-2T} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2T} - e^{-T} & 2e^{-2T} - 2e^{-T} \\ e^{-T} - e^{-2T} & 2e^{-T} - e^{-2T} \end{bmatrix}$$

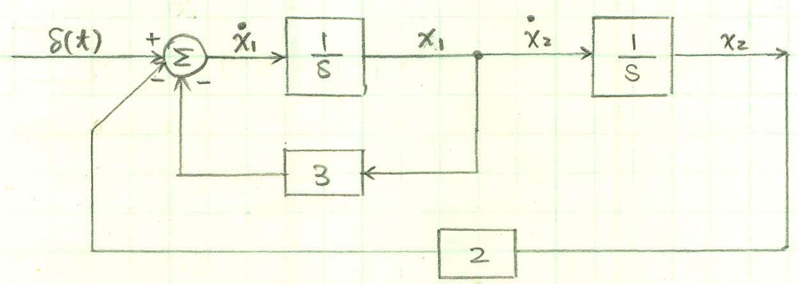
6.2 Compute Φ by Laplace transforms. we have shown in (6.4) that the solution to $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x}$ is $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$. Thus, if $\mathbf{x}_0 = [1\ 0\ 0]^T$, then the solution will be the first column of $\Phi(t)$; if $\mathbf{x}_0 = [0\ 1\ 0]^T$, we obtain the second column and so on. Furthermore, if we envision an analog computer or block-diagram realization of $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{x}_0\delta(t)$, then we can compute $\mathbf{x}(s)$, the Laplace transform of the solution to an impulse input which sets up $\mathbf{x}_0$.

a) Draw a block-diagram realization of $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{x}_0\delta(t)$ for

$$\mathbf{F} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = (1\ \ 0)^T$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta(t)$$

$$= \begin{bmatrix} -3x_1 - 2x_2 + \delta(t) \\ x_1 \end{bmatrix}$$



Block-diagram Realization

(b) Compute $\mathbf{x}(s)$, the solution for the system of part (a).

From $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{x}_0\delta(t)$,

$$s\mathbf{X}(s) - \mathbf{x}_0 = \mathbf{F}\mathbf{X}(s) + \mathbf{x}_0$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{F})^{-1} \cdot \cancel{2} \cdot \mathbf{x}_0 = \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}^{-1} \cdot \cancel{2} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \left[\frac{\cancel{2}s}{(s+1)(s+2)} \,,\ \frac{\cancel{2}}{(s+1)(s+2)} \right]^T$$

(c) *Repeat* (b) *for* $X_0 = (0 \;\; 1)^T$ *and write* $\Phi(s)$

$$X(s) = \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix} \cdot \cancel{2} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

−1

$$= \left[\frac{-\cancel{4}\; 2}{(s+1)(s+2)} \;,\; \frac{\cancel{2}(s+3)}{(s+1)(s+2)} \right]^T$$

$$\Phi(s) = \begin{bmatrix} \dfrac{s}{(s+1)(s+2)} & \dfrac{-2}{(s+1)(s+2)} \\[2ex] \dfrac{1}{(s+1)(s+2)} & \dfrac{(s+3)}{(s+1)(s+2)} \end{bmatrix}$$

(d) *Solve by inverse transforms for* $\Phi(t)$ *and verify that this solution is the same as that given in Problem 6.1 (d) if we let* $t = T$

$$\Phi(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \mathcal{L}^{-1}\left\{ \begin{pmatrix} \dfrac{2}{s+2} - \dfrac{1}{s+1} & \dfrac{2}{s+2} - \dfrac{2}{s+1} \\[2ex] \dfrac{1}{s+1} - \dfrac{1}{s+2} & \dfrac{2}{s+1} - \dfrac{1}{s+2} \end{pmatrix} \right\}$$

$$= \begin{bmatrix} 2e^{-2T} - e^{-T} & 2e^{-2T} - 2e^{-T} \\ e^{-T} - e^{-2T} & 2e^{-T} - e^{-2T} \end{bmatrix}$$

<u>QED!</u>

1. Give: $\mathbf{X}_{n+1} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \mathbf{X}_n + \begin{pmatrix} 0 \\ 1 \end{pmatrix} U_n$

$$y_n = (1 \quad 2)\, \mathbf{X}_n$$

Find: $\frac{Y(Z)}{U(Z)}$, THE Z-TRANSFER FUNCTION.

Solution: FROM THE GIVENS, WE HAVE

$$\Phi = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad H = (1 \quad 2)$$

THE TRANSFER FUNCTION IS GIVEN AS

$$\frac{Y(Z)}{U(Z)} = H(ZI - \Phi)^{-1} \cdot \Gamma$$

$$= (1 \quad 2) \begin{pmatrix} Z-1 & -T \\ 0 & Z-1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{2Z - (2-T)}{Z^2 - 2Z + 1}$$

2. Give: $\frac{Y(Z)}{U(Z)} = \frac{5(Z + .6)}{Z^2 - .8Z + .25}$

FIND: State variable representation in controllable canonic form.

Solution:

$$\mathbf{X}_{n+1} = \begin{pmatrix} 0 & 1 \\ -.25 & .8 \end{pmatrix} \mathbf{X}_n + \begin{pmatrix} 0 \\ 1 \end{pmatrix} U_n$$

$$y_n = (3 \quad 5)\, \mathbf{X}_n$$

3. Give: $\frac{W(S)}{V(S)} = \frac{S+3}{(S+.2)(S+.5)}$ $\quad T = .5$

FIND: a) State variable form for continuous system.

b) Equiv. Discrete-time

1. By series.

2. Convert G(S) TO G(Z) by hold equivalance.

<u>Solution to problem 3:</u>

$$\frac{W(S)}{V(S)} = \frac{S+3}{(S+.2)(S+.5)} = \frac{S+3}{S^2+.7S+.1}$$

a) $$\dot{\mathbf{X}} = \begin{pmatrix} 0 & 1 \\ -.1 & -.7 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} V$$

$$W = (3 \quad 1)\,\mathbf{X} \qquad , \text{ where } \quad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , \ \dot{\mathbf{X}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

(b) 1. *According the results from the class.*

$$\Phi(t) = \mathcal{L}^{-1}\{(SI-F)^{-1}\}$$

$$= \mathcal{L}^{-1}\left\{ \begin{pmatrix} S & -1 \\ .1 & S+.7 \end{pmatrix}^{-1} \right\} = \mathcal{L}^{-1}\left\{ \begin{pmatrix} \frac{S+.7}{(S+.2)(S+.5)} & \frac{+1}{(S+.2)(S+.5)} \\ \frac{-.1}{(S+.2)(S+.5)} & \frac{S+0}{(S+.2)(S+.5)} \end{pmatrix} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{5/3}{S+.2} - \frac{2/3}{S+.5} , & \frac{10/3}{S+.2} - \frac{10/3}{S+.5} \\ \frac{1/3}{S+.5} - \frac{1/3}{S+.2} , & \frac{5/3}{S+.5} - \frac{2/3}{S+.2} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \frac{5}{3}e^{-.2t} - \frac{2}{3}e^{-.5t} & \frac{10}{3}e^{-.2t} - \frac{10}{3}e^{-.5t} \\ \frac{1}{3}e^{-.5t} - \frac{1}{3}e^{-.2t} & \frac{5}{3}e^{-.5t} - \frac{2}{3}e^{-.2t} \end{bmatrix}$$

$$\Phi = \Phi(t)\Big|_{t=T=.5} = \begin{bmatrix} 0.989 & 0.420 \\ -0.042 & 0.695 \end{bmatrix}$$

$$\Gamma = \mathcal{L}^{-1}\left[\tfrac{1}{S}(SI-F)^{-1}G\right]_{t=T=0.5}$$

$$= \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{S+.7}{S(S+.2)(S+.5)} & \frac{1}{S(S+.2)(S+.5)} \\ \frac{-.1}{S(S+.2)(S+.5)} & \frac{1}{(S+.2)(S+.5)} \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \Bigg|_{t=T=.5}$$

$$= \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{S} - \frac{5/3}{S+.2} + \frac{2/3}{S+.5} \\ \frac{10/3}{S+.2} - \frac{10/3}{S+.5} \end{bmatrix} \right\} \Bigg|_{t=0.5}$$

$$= \begin{bmatrix} 1 - \frac{5}{3}e^{-.2t} + \frac{2}{3}e^{-.5t} \\ \frac{10}{3}e^{-.2t} - \frac{10}{3}e^{-.5t} \end{bmatrix}_{t=0.5} = \begin{bmatrix} 0.011 \\ 0.420 \end{bmatrix}$$

$$\therefore \quad \mathbf{X}_{n+1} = \begin{bmatrix} 0.989 & 0.420 \\ -0.042 & 0.695 \end{bmatrix} \mathbf{X}_n + \begin{bmatrix} 0.011 \\ 0.420 \end{bmatrix} u_n$$

$$\underline{W_n = (3 \quad 1)\,\mathbf{X}_n}$$

$$\frac{W(z)}{U(z)} = (3 \quad 1)\begin{pmatrix} z-0.989 & -0.420 \\ 0.042 & z-0.695 \end{pmatrix}^{-1}\begin{pmatrix} 0.011 \\ 0.420 \end{pmatrix}$$

$.754\,z - .123$

$$= \frac{0.453z + 0.091}{z^2 - 1.684z + 0.705}$$

-2

2. $$G(s) = \frac{s+3}{(s+.2)(s+.5)}$$

$$G(z) = \frac{z-1}{z}\,\mathcal{Z}\left[\frac{s+3}{s(s+.2)(s+.5)}\right]$$

$$= \frac{z-1}{z}\,\mathcal{Z}\left[\frac{3}{s} - \frac{5/3}{s+.2} - \frac{4/3}{s+.5}\right]$$

$$= \frac{z-1}{z}\cdot\left[3\cdot\frac{z}{z-1} - \frac{5}{3}\cdot\frac{z}{z-0.905} - \frac{4}{3}\cdot\frac{z}{z-.779}\right]$$

$$= \frac{0.453z + 0.091}{z^2 - 1.684z + 0.705}$$

-1

$$\mathbf{x}_{n+1} = \begin{pmatrix} 0 & 1 \\ -.705 & 1.684 \end{pmatrix}\mathbf{x}_n + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_n$$

$$y_n = (0.091 \quad 0.453)\,\mathbf{x}_n$$

Given: $G(s) = \frac{0.1}{s(s+0.1)}$

Specifications: $T_s = 4$ sec. (2%) and P.O. = 5 % $T = 0.2$ sec.?

Find: a) Control gains K & K'

K' for discrete time system in control cannoic form.

K for system in form of $\Phi(t)$ found from continuous system.

b) Transformation T such that $K = TK'$

c) Diagram systems

d) Design observer gain so that observer poles are at $z=0$, use original form.

Solution:

From the specifications, we have

$$\zeta \doteq (0.6)\cdot\left(1 - \frac{\%\text{ OVERSHOOT}}{100}\right) = 0.57$$

$$\zeta\omega_n \doteq 4.6/t_s = 1.15 \qquad \omega_n \doteq 2.0175$$

$$S_{1,2} = -1.15 \pm 1.66$$

$$Z_{1,2} = e^{TS} = e^{-1.15T}(\cos(1.66T) \pm j\sin(1.66T))$$

LET $T = .2$ sec. THEN $Z_{1,2} = 0.75 \pm j0.26$ (DESERED POLES)

(a) FROM $G(s) = \frac{0.1}{s(s+0.1)}$

$$G(z) = \frac{z-1}{z}\,\mathcal{Z}\left[\frac{0.1}{s^2(s+0.1)}\right] = \frac{z-1}{z}\,\mathcal{Z}\left[\frac{-10}{s} + \frac{1}{s^2} + \frac{10}{s+0.1}\right]$$

$$= \frac{z-1}{z}\left[\frac{-10z}{z-1} + \frac{0.2z}{(z-1)^2} + \frac{10z}{(z-0.98)}\right]$$

$$= \frac{0.004}{z^2 - 1.98z + 0.98}$$

Controllable canonic form given by

$$X_{n+1} = \begin{pmatrix} 0 & 1 \\ -.98 & 1.98 \end{pmatrix} X_n + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y_n = (0.004,\ 0)\,X_n$$

$\det\ (zI - \Phi + \Gamma' K)$

$$= \det\begin{bmatrix} z & -1 \\ .98+K_1 & z-1.98+K_2 \end{bmatrix} = z^2+1.5z+0.63$$

$$z^2 + (K_2 - 1.98)z + (.98+K_1) = z^2 + 1.5z + 0.63$$

WE HAVE

$$K_1 = -0.35 \quad \text{and} \quad K_2 = 3.48$$

$$\underline{K' = (-0.35 \quad 3.48)}$$

FROM $\quad G(S) = \dfrac{0.1}{S(S+0.1)} = \dfrac{0.1}{S^2+0.1S+0}$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -.1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = (0.1 \quad 0)\, x$$

$$\Phi(t) = \mathcal{L}^{-1}(sI-F)^{-1} = \mathcal{L}^{-1}\begin{bmatrix} s & -1 \\ 0 & s+0.1 \end{bmatrix}^{-1}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} s+0.1 & 1 \\ 0 & s \end{bmatrix} \Big/ s(s+0.1)$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s} & \frac{10}{s} - \frac{10}{s+.1} \\ 0 & \frac{1}{s+.1} \end{bmatrix}$$

error

$$= \begin{bmatrix} 1 & 10 - {}^{10}e^{-.1t} \\ 0 & e^{-.1t} \end{bmatrix}$$

-3

$$\Phi = \begin{bmatrix} 1 & 10 - e^{-.1t} \\ 0 & e^{-.1t} \end{bmatrix}_{t=T=.2} = \begin{bmatrix} 1 & 9.02 \\ 0 & 0.98 \end{bmatrix}$$

$$\Gamma = \mathcal{L}^{-1}\left[\tfrac{1}{s}(sI-F)^{-1}G\right]_{t=T}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s^2} & \frac{10}{s^2} - \frac{10}{s(s+.1)} \\ 0 & \frac{1}{s(s+.1)} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{t=T=.2}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{10}{s^2} - \frac{100}{s} + \frac{100}{s+.1} \\ \frac{10}{s} - \frac{10}{s+.1} \end{bmatrix}_{t=T=.2}$$

$$= \begin{bmatrix} 10t - 100 + 100e^{-.1t} \\ 10 - 10e^{-.1t} \end{bmatrix}_{t=.2} = \begin{bmatrix} 0.0199 \\ 0.1980 \end{bmatrix}$$

$$X'_{n+1} = \begin{pmatrix} 1 & 9.02 \\ 0 & 0.98 \end{pmatrix} X'_n + \begin{pmatrix} 0.0199 \\ 0.1980 \end{pmatrix} u$$

$$y_n = (0.1 \quad 0)\, X'_n$$

$$\det(zI - \Phi + \Gamma K) = z^2 + 1.5z + .63$$

$$= \det \begin{bmatrix} z - 1 + 0.0199k_1 & -9.02 + 0.0199k_2 \\ 0.1980k_1 & z - 0.98 + 0.1980k_2 \end{bmatrix} = z^2 + 1.5z + .63$$

$$z^2 + (0.0199k_1 - 1 + 0.1980k_2 - 0.98)z$$

$$+ (0.0199k_1 - 1)(0.1980k_2 - .98) - .1980k_1(.0199k_2 - 9.02)$$

$$= z^2 + 1.5z + .63$$

$$z^2 + (0.0199k_1 + 0.1980k_2 - 1.98)z$$

$$+ (1.7666k_1 - 0.1980k_2 + .98) = z^2 + 1.5z + .63$$

$$\begin{cases} 0.0199k_1 + 0.1980k_2 = 3.48 \\ 1.7666k_1 - 0.1980k_2 = -.35 \end{cases}$$

$$k_1 = 1.752 \qquad k_2 = 17.399$$

$$K = (1.752 \quad 17.399)$$

b) LET $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$K'T = (-.35 \quad 3.48)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= (-.35a + 3.48c \quad -.35b + 3.48d) = (1.752 \quad 17.399)$$

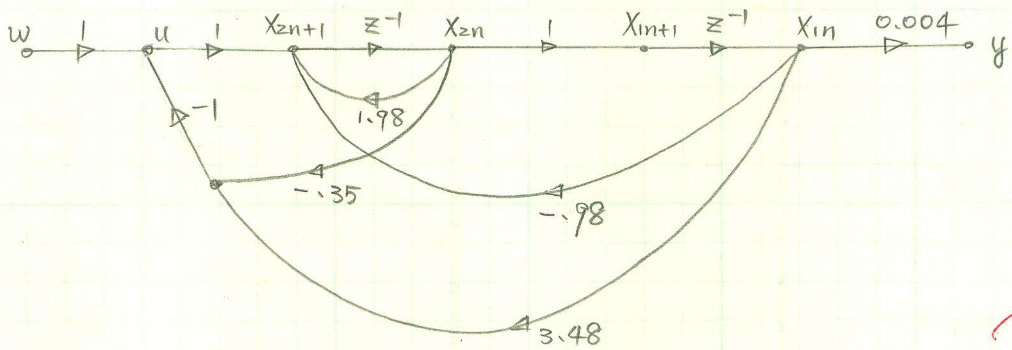
$$\begin{cases} -.35a + 3.48c = 1.752 \\ -.35b + 3.48d = 17.399 \end{cases}$$

LET $a = 1 \Rightarrow c = 0.604$

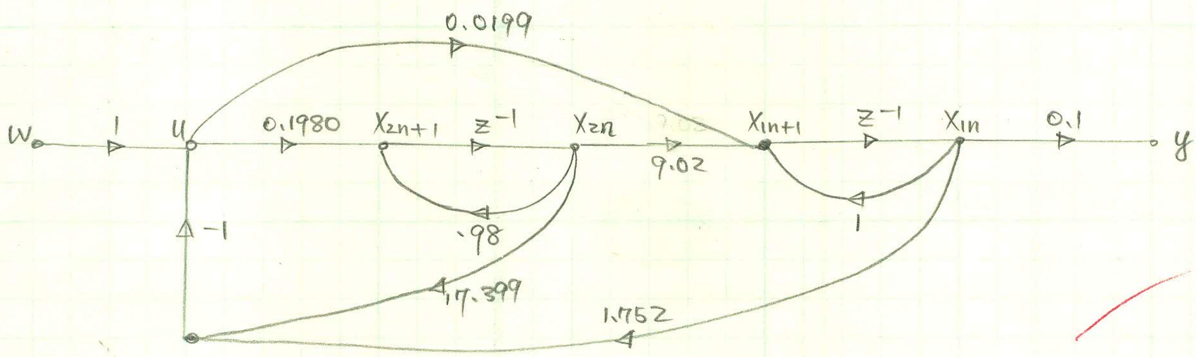
$b = -10 \Rightarrow d = 3.994$ $\qquad \Rightarrow T = \begin{pmatrix} 1 & -10 \\ .604 & 3.994 \end{pmatrix}$

$$\det T = 10.034 \neq 0$$

c) FOR THE CONTROLLABLE CONONIC FORM



FOR THE OTHER FORM:



d) $\det(zI - \Phi + LH) = z^2$

$$= \det\left[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 1 & 9.02 \\ 0 & 0.98 \end{pmatrix} + \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}(0.1 \;\; 0)\right]$$

$$= \det\begin{bmatrix} z-1+.1\ell_1 & -9.02 \\ .1\ell_2 & z-.98 \end{bmatrix} = z^2$$

$$z^2 + (.1\ell_1 - 1.98)z - .98(.1\ell_1 - 1) + .902\ell_2 = z^2$$

$$\begin{cases} .1\ell - 1.98 = 0 \\ -.098\ell_1 + .98 + .902\ell_2 = 0 \end{cases}$$

$$\begin{cases} \ell_1 = 19.8 \\ \ell_2 = 1.065 \end{cases}$$