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E-E 428 Z-TRANSFORMS
Homework - Spring 1988
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Find the z-transforms and ROC for the following sequences. Express all summations in closed formand simplify when you can.

1. $x(k) = ku(k)$

2. $x(k) = k^n u(k)$

3. $x(k) = e^{-\alpha k} u(k)$

4. $x(k) = \sin \beta k \; u(k)$

5. $y(k) = e^{-\alpha k} x(k)$

6. $x(k) = \cancel{-\alpha} e^{-\alpha k} \sin\beta k \; u(k)$

7. $x(k) = \begin{cases} 1, & k = 0,1 \\ -1, & k = 2,3 \\ 0 & \text{otherwise} \end{cases}$

8. $x(k) = \begin{cases} 1, & 0 \le k \le 5 \\ 0, & k > 5 \end{cases}$

9. $x(k) = e^{-\alpha |k|}$

10. $x(k) = u(k-5T) - u(k-10T)$

Find $X(0)$ and $X(\infty)$ if

11. $X_+(z) = \exp(1/z)$

12. $X_+(z) = \dfrac{2z^2 + 1}{2z^2 + 3z + 1}$

Find the z-transforms and ROC for the following sequences. Express all summations in closed formand simplify when you can. (Assume sample period is T)

1. $x(k) = k\,u(k)$

Considering $x(kT) - x(kT-T) = kT - kT + T = T$ for $k > 0$

$= 0$ for $k \leq 0$

$$\mathcal{Z}\{x(kT)-x(kT-T)\} = \mathcal{Z}\{x(k)\} - \mathcal{Z}\{x(k-1)\}$$

$$= X(z) - z^{-1}X(z) = (1-z^{-1})\cdot X(z) \quad \cdots\cdots \quad (1)$$

$$\mathcal{Z}\{x(kT)-x(kT-T)\} = \sum_{k=1}^{\infty} Tz^{-k} = \sum_{k=0}^{\infty} Tz^{-k} - T = \frac{T}{1-z^{-1}} - T \quad |z|>1$$

$$= \frac{Tz^{-1}}{1-z^{-1}}, \quad |z|>1 \quad \cdots\cdots \quad (2)$$

(1) = (2), So that we have

$$(1-z^{-1})X(z) = \frac{Tz^{-1}}{1-z^{-1}}, \quad |z|>1$$

$$\therefore \quad X(z) = \frac{Tz^{-1}}{(1-z^{-1})^2} = \frac{Tz}{(z-1)^2}, \quad |z|>1 \quad \text{(ROC)}$$ ✓ ✓

2. $x(k) = k^n u(k)$

Considering $e^{-akT} = 1 - akT + \frac{a^2(kT)^2}{2!} \cdots + (-1)^n \cdot \frac{a^n(kT)^n}{n!} + \cdots$

$$\frac{\partial^n e^{-akT}}{\partial a^n} = (-1)^n \cdot \frac{n!}{n!}(kT)^n + (-1)^{n+1} \cdot \frac{(n+1)n(n-1)\cdots 2}{(n+1)!} \cdot a(kT)^{n+1} + \cdots$$

$$\lim_{a\to 0} \frac{\partial^n e^{-akT}}{\partial a^n} = (-1)^n \cdot (kT)^n \Rightarrow (kT)^n = (-1)^n \cdot \lim_{a\to 0} \frac{\partial^n e^{-akT}}{\partial a^n}$$

$$\mathcal{Z}\{x(kT)\} = \mathcal{Z}\{(kT)^n u(kT)\} = \sum_{k=0}^{\infty} (kT)^n z^{-k} = \sum_{k=0}^{\infty} (-1)^n \cdot \lim_{a\to 0} \frac{\partial^n e^{-akT}}{\partial a^n} \cdot z^{-k}$$

$$= (-1)^n \cdot \lim_{a\to 0} \frac{\partial^n}{\partial a^n}\left(\sum_{k=0}^{\infty} e^{-akT} z^{-k}\right)$$

$$= (-1)^n \cdot \lim_{a\to 0} \frac{\partial^n}{\partial a^n}\left(\sum_{k=0}^{\infty} (e^{aT}z)^{-k}\right)$$

$$= (-1)^n \cdot \lim_{a\to 0} \frac{\partial^n}{\partial a^n}\left(\frac{1}{1-e^{-aT}z^{-1}}\right) \quad |e^{aT}z| > 1$$

$$= (-1)^n \cdot \lim_{a\to 0} \frac{\partial^n}{\partial a^n}\left(\frac{z}{z-e^{-aT}}\right) \quad |z| > |e^{-aT}| \longrightarrow 1, \quad a \to 0$$

$$\therefore \quad X(z) = (-1)^n \cdot \lim_{a\to 0} \frac{\partial^n}{\partial a^n}\left(\frac{z}{z-e^{-aT}}\right), \quad |z| > 1 \quad \text{(ROC)}$$ ✓

3. $x(k) = e^{-\alpha k} u(k)$

$$\mathcal{Z}\{x(kT)\} = \sum_{k=0}^{\infty} e^{-\alpha kT} z^{-k} = \sum_{k=0}^{\infty} (e^{\alpha T} z)^{-k} = \frac{1}{1-e^{-\alpha T} z^{-1}} \quad , \quad |e^{\alpha T} z| > 1$$

$$X(z) = \frac{z}{z - e^{-\alpha T}} \quad , \quad |z| > |e^{-\alpha T}| \quad \text{(ROC)}$$

✓ ✓

4. $x(k) = \sin \beta k \; u(k)$

Considering $e^{j\beta kT} = \cos\beta kT + j\sin\beta kT$, $\mathcal{Z}\{e^{j\beta kT} u(k)\} = \mathcal{Z}\{\cos\beta kT u(k)\} + j\,\mathcal{Z}\{\sin\beta kT u(k)\}$

Thus, $\mathcal{Z}\{\sin\beta kT u(kT)\} = \mathrm{Im}\{\mathcal{Z}\{e^{j\beta kT} u(kT)\}\}$

$$\mathcal{Z}\{e^{j\beta kT} u(kT)\} = \sum_{k=0}^{\infty} e^{j\beta kT} z^{-k} = \sum_{k=0}^{\infty} (e^{-j\beta T} z)^{-k}$$

$$= \frac{1}{1-e^{j\beta T} z^{-1}} = \frac{z}{z - e^{j\beta T}} \quad , \quad |e^{-j\beta T} z| > 1 \text{ OR } |z| > |e^{j\beta T}| = 1$$

$$= \frac{z(z - \cos\beta T + j\sin\beta T)}{(z - \cos\beta T - j\sin\beta T)(z - \cos\beta T + j\sin\beta T)} \quad , \quad |z| > 1$$

$$= \frac{z(z - \cos\beta T) + j(\sin\beta T) z}{z^2 - (2\cos\beta T)\cdot z + 1} \qquad |z| > 1$$

$$\therefore \quad \mathcal{Z}\{\sin\beta kT u(kT)\} = \frac{z \sin\beta T}{z^2 - (2\cos\beta T) z + 1} \quad , \quad |z| > 1 \quad \text{(ROC)}$$

✓ ✓

and $\mathcal{Z}\{\cos\beta kT u(kT)\} = \dfrac{z^2 - z\cos\beta T}{z^2 - (2\cos\beta T) z + 1}$, $|z| > 1$ (ROC)

5. $y(k) = e^{-ak} x(k)$

$$\mathcal{Z}\{y(kT)\} = \sum_{k=-\infty}^{\infty} e^{-akT} x(k) \cdot z^{-k} = \sum_{k=-\infty}^{\infty} x(k) \cdot (e^{aT} z)^{-k}$$

$$= X(e^{aT} z)$$

$Y(z) = X(e^{aT} z)$ ✓ and ROC of $Y(z)$ is depended on ROC of $X(z)$.

6. $x(k) = e^{-\alpha k} \sin\beta k \; u(k)$

Because, $e^{-\alpha kT + j\beta kT} = e^{-\alpha kT} \cos\beta kT + j e^{-\alpha kT} \sin\beta kT$, and

$$\mathcal{Z}\{e^{-\alpha kT + j\beta kT} u(kT)\} = \sum_{k=0}^{\infty} e^{-\alpha kT + j\beta kT} z^{-k} = \sum_{k=0}^{\infty} (e^{(\alpha - j\beta)T} z)^{-k}$$

$$= \frac{1}{1 - e^{-(\alpha + j\beta)T} z^{-1}} = \frac{z}{z - e^{-(\alpha + j\beta)T}} \quad , \quad |e^{(\alpha - j\beta)T} z| > 1 \text{ OR } |z| > |e^{-\alpha T}|$$

$$= \frac{z(z - e^{-\alpha}\cos\beta + j e^{-\alpha}\sin\beta)}{(z - e^{-\alpha}\cos\beta - j e^{-\alpha}\sin\beta)(z - e^{-\alpha}\cos\beta + j e^{-\alpha}\sin\beta)} \quad , \quad |z| > |e^{-\alpha T}|$$

$$X(Z) = \mathcal{Z}\{e^{-\alpha kT} \sin\beta kT u(kT)\} = \frac{z e^{-\alpha T} \sin\beta T}{z^2 - (2e^{-\alpha T}\cos\beta T) z + e^{-2\alpha T}} \quad , \quad |z| > |e^{-\alpha T}| \quad \text{(ROC)}$$

✓ ✓

7. $x(k) = \begin{cases} 1, & k=0,1 \\ -1, & k=2,3 \\ 0, & \text{otherwise} \end{cases}$

$$\mathcal{Z}\{x(kT)\} = \sum_{k=-\infty}^{\infty} x(kT) z^{-k} = \sum_{k=0}^{1} z^{-k} + \sum_{k=2}^{3} -1 \cdot z^{-k} = 1 + z^{-1} - z^{-2} - z^{-3}$$

$$X(z) = \frac{z^3 + z^2 - z - 1}{z^3}$$

ROC = whole z - plane except $z = (0,0)$

8. $x(k) = \begin{cases} 1, & 0 \leq k \leq 5 \\ 0, & k > 5 \end{cases}$

$$X_+(z) = \sum_{k=0}^{\infty} x(kT) z^{-k} = \sum_{k=0}^{5} z^{-k} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$= \frac{1 - z^{-6}}{1 - z^{-1}} = \frac{z^6 - 1}{z^5(z-1)}$$

ROC is same as that in 7.

~~$z \neq 1$~~

9. $x(k) = e^{-\alpha|k|}$

$$X(z) = \sum_{k=-\infty}^{\infty} x(kT) z^{-k} = \sum_{k=-\infty}^{-1} e^{\alpha kT} z^{-k} + \sum_{k=0}^{\infty} e^{-\alpha kT} z^{-k}$$

$$= \sum_{k=0}^{\infty} e^{-\alpha kT} z^{k} + \sum_{k=0}^{\infty} e^{-\alpha kT} z^{-k} - 1$$

$$= \sum_{k=0}^{\infty} (e^{-\alpha T} z)^{k} + \sum_{k=0}^{\infty} (e^{\alpha T} z)^{-k} - 1$$

$$= \frac{1}{1 - e^{-\alpha T} z} + \frac{z}{z - e^{-\alpha T}} - 1, \quad |e^{-\alpha T} z| < 1 \ \& \ |e^{\alpha T} z| > 1$$

$$= \frac{z(e^{-\alpha T} - e^{\alpha T})}{z^2 - z(e^{-\alpha T} + e^{\alpha T}) + 1}, \quad |e^{-\alpha T}| < |z| < |e^{\alpha T}| \ \text{(ROC)}$$

10. $x(k) = u(k - 5T) - u(k - 10T)$

$$X(z) = \sum_{k=5}^{9} z^{-k} = z^{-5} + z^{-6} + z^{-7} + z^{-8} + z^{-9} = z^{-9}(z^4 + z^3 + z^2 + z + 1)$$

$$= z^{-9} \cdot \frac{z^5 - 1}{z - 1} = \frac{z^5 - 1}{z^9 (z - 1)}$$

ROC : $|z| > 0$

Find $X(0)$ and $X(\infty)$ if (I think they shall be $x(0)$ and $x(\infty)$.)

11. $X_+(z) = \exp(1/z)$

$$x(0) = \lim_{z\to\infty} X_+(z) = \lim_{z\to\infty} \exp(1/z) = 1$$ ✓

$$x(\infty) = \lim_{z\to1} (z-1) X_+(z) = \lim_{z\to1} (z-1)\exp(1/z) = 0$$ ✓

By the way,

$X(0)$ is not defined.

$X(\infty) = 1$

12. $X_+(z) = \dfrac{2z^2+1}{2z^2+3z+1}$

$$x(0) = \lim_{z\to\infty} X_+(z) = \lim_{z\to\infty} \frac{2z^2+1}{2z^2+3z+1} = \lim_{z\to\infty} \frac{2+z^{-2}}{2+3z^{-1}+z^{-2}} = 1$$ ✓

$$x(\infty) = \lim_{z\to1} (z-1)\cdot X_+(z) = \lim_{z\to1} (z-1)\cdot \frac{2z^2+1}{2z^2+3z+1} = 0.$$ ✓

$X(0) = 1$

$X(\infty) = x(0) = 1$

E-E 428 Discrete Signal Processes
Homework

Benmei Chen $\frac{89}{90}$

z-Transforms

1. Find the inverse z-transform of the following function by the partial fraction method.

a) $X(z) = \dfrac{3z + 1}{(z + 0.5)(z - 0.8)}$, $|z|>0.8$

b) $X(z) = \dfrac{2z^3 + z^2 + z}{(z - 0.6)(z - 0.4)(z - 1)^2}$, $|z|>1.0$

c) $X(z) = \dfrac{z^3 - 3.1z^2 + 1.8z - 0.1}{(z - 1)(z - 0.5)(z - 0.2)}$, $0.5<|z|<1$

d) $X(z) = \dfrac{2z^3 - 2z^2 + 1.25z}{(z - 0.5)(z^2 - z + 1)}$, $|z|>1$

2. Repeat problem 1 using the residue mehod.

3. Find the inverse z-transform of the following functions using any method you wish.

a) $X(z) = \dfrac{z}{z^2 + 1}$, $|z|>1$

b) $X(z) = \dfrac{z^2 + 1}{z^2 - 1}$, $|z|>1$

c) $X(z) = \dfrac{e^{-\alpha}z^{-1}}{(1 - e^{-\alpha}z^{-1})^2}$, $|z|>e^{-\alpha}$

d) $X(z) = \dfrac{z^2 + 1}{(z - 1)^3}$, $|z|>1$

4. Solve each of the following difference equations using z-transforms.

a) $y(n+2) - 2y(n) = 0$; $y(0) = 0$, $y(1) = -1$

b) $y(n+1) - y(n) = 2n + 1$; $y(0) = K$

c) $y(n+2) + 3y(n+1) + 2y(n) = n$; $y(0) = y(1) = 1$

Z – TRANSFORMS

1. Find the inverse z-transform of the following function by the partial fraction method.

a) $X(z) = \dfrac{3z+1}{(z+0.5)(z-0.8)}$, $|z| > 0.8$

$$\frac{X(z)}{z} = \frac{3z+1}{z\,(z+0.5)(z-0.8)} = \frac{A}{z} + \frac{B}{z+0.5} + \frac{C}{z-0.8}$$

$$A = \left.\frac{3z+1}{(z+0.5)(z-0.8)}\right|_{z=0} = \frac{1}{0.5 \times (-0.8)} = -2.5$$

$$B = \left.\frac{3z+1}{z\,(z-0.8)}\right|_{z=-0.5} = \frac{3(-.5)+1}{-0.5(-.5-.8)} = -\frac{10}{13} = -0.76923$$

$$C = \left.\frac{3z+1}{z\,(z+0.5)}\right|_{z=0.8} = \frac{3\times.8+1}{.8\,(.8+.5)} = \frac{85}{26} = 3.26923$$

$$X(z) = -2.5 - \frac{0.76923\,z}{z+0.5} + 3.26923 \cdot \frac{z}{z-0.8}$$

$$= -2.5 - 0.76923 \cdot \frac{1}{1-(-.5z^{-1})} + 3.26923 \cdot \frac{1}{1-(.8z^{-1})}$$

$$\therefore \quad \underline{x(k) = -2.5\,\delta(k) - \tfrac{10}{13}(-.5)^k u(k) + \tfrac{85}{26}(.8)^k u(k)}$$ ✓

b) $X(z) = \dfrac{2z^3+z^2+z}{(z-0.6)(z-.4)(z-1)^2}$, $|z| > 1.0$

$$\frac{X(z)}{z} = \frac{2z^2+z+1}{(z-.6)(z-.4)(z-1)^2} = \frac{A}{z-.6} + \frac{B}{z-.4} + \frac{C}{(z-1)} + \frac{Dz+E}{(z-1)^2} \quad \cdots \quad (1)$$

$$A = \frac{2\cdot(.6)^2+.6+1}{(.6-.4)(.6-1)^2} = \underline{72.5} \qquad B = \frac{2\times(.4)^2+.4+1}{(.4-.6)(.4-1)^2} = \underline{-23.88889}$$

$$D+E = Dz+E\Big|_{z=1} = (z-1)^2 \cdot \frac{X(z)}{z}\Big|_{z=1} = \frac{2\cdot(1)^2+1+1}{(1-0.6)(1-.4)} = 16.66667 \quad \cdots \quad (a)$$

Let $z=0$ in eq. (1) , we have $C+E = -65.27777$ ··· (b)

Let $z=2$ in eq (1) , we have $C+E+2D = -31.94444$ ···· (c)

Solve (a), (b) & (c), we obtain $\underline{C = -65.27777}$, $\underline{D = 16.66667}$, $\underline{E=0}$

$$\therefore \quad X(z) = 72.5\,\frac{1}{1-(.6z^{-1})} - 23.88889\,\frac{1}{1-(.4z^{-1})} - 65.27777\,\frac{1}{1-z^{-1}} + 16.66667\,\frac{z^2}{(z-1)^2}$$

$$\underline{x(k) = 72.5\cdot(0.6)^k u(k) - 23.88889\cdot(0.4)^k u(k) - 65.27777\,u(k) + 16.66667\,(k+1)\,u(k)}$$ ✓

c) $$X(z) = \frac{z^3 - 3.1z^2 + 1.8z - 0.1}{(z-1)(z-0.5)(z-0.2)} \quad , \quad 0.5 < |z| < 1$$

$$X(z) = X_+(z) + \frac{A}{z-1}$$

$$A = (z-1)X(z)\Big|_{z=1} = \frac{1-3.1+1.8-0.1}{(1-0.5)(1-0.2)} = -1$$

$$X_+(z) = X(z) + \frac{1}{z-1} = \frac{z(z-1.1)}{(z-0.5)(z-0.2)}$$

$$\frac{X_+(z)}{z} = \frac{z-1.1}{(z-.5)(z-.2)} = \frac{B}{z-.5} + \frac{C}{z-.2}$$

$$B = \frac{.5-1.1}{.5-.2} = -2 \qquad C = \frac{.2-1.1}{.2-.5} = 3$$

$$X_+(z) = -2\frac{z}{z-.5} + \frac{3z}{z-.2}$$

$$\therefore \quad X(z) = -2\cdot\frac{1}{1-(.5z^{-1})} + 3\cdot\frac{1}{1-(.2z^{-1})} + \frac{1}{1-z}$$

$$\underline{x(k) = u(-k) - 2\cdot(0.5)^k u(k) + 3\cdot(0.2)^k u(k)}$$

✓

d) $$X(z) = \frac{2z^3 - 2z^2 + 1.25z}{(z-.5)(z^2-z+1)} \quad , \quad |z| > 1$$

$$= \frac{2z^3 - 2z^2 + 1.25z}{(z-.5)(z-.5+j0.86603)(z-0.5-j0.86603)}$$

$$\frac{X(z)}{z} = \frac{2z^2 - 2z + 1.25}{(z-.5)(z-.5+j\frac{\sqrt{3}}{2})(z-.5-j\frac{\sqrt{3}}{2})} = \frac{A}{z-0.5} + \frac{B}{z-0.5+j\frac{\sqrt{3}}{2}} + \frac{C}{z-0.5-j\frac{\sqrt{3}}{2}}$$

$$A = \frac{2\times(.5)^2 - 2\times.5 + 1.25}{.5^2 - .5 + 1} = 1$$

$$B = \frac{2\times(0.5-j\frac{\sqrt{3}}{2})^2 - 2(0.5-j\frac{\sqrt{3}}{2}) + 1.25}{(0.5-j\frac{\sqrt{3}}{2}-.5)(-j\sqrt{3})} = +0.5$$

$$C = \frac{2\times(0.5+j\frac{\sqrt{3}}{2})^2 - 2(0.5+j\frac{\sqrt{3}}{2}) + 1.25}{j\frac{\sqrt{3}}{2}\cdot j\sqrt{3}} = +0.5$$

$$X(z) = \frac{z}{z-0.5} + \frac{0.5z}{z-0.5+j\frac{\sqrt{3}}{2}} + \frac{0.5z}{z-0.5-j\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{1-(.5z^{-1})} + 0.5\frac{1}{1-(e^{-j\frac{\pi}{3}}z^{-1})} + 0.5\frac{1}{1-(e^{j\frac{\pi}{3}}z^{-1})}$$

$$x(k) = 0.5^k u(k) + 0.5e^{-j\frac{\pi k}{3}}u(k) - 0.5e^{j\frac{\pi k}{3}}u(k)$$

$$\underline{= 0.5^k u(k) + (\cos\frac{\pi k}{3})\,u(k)}$$

✓

2. Repeat problem 1 using the residue method.

a) $X(z) = \dfrac{3z + 1}{(z + 0.5)(z - 0.8)}$, $|z| > 0.8$

$$F(z) = z^{k-1} X(z) = \frac{3z+1}{(z+0.5)(z-0.8)} \cdot z^{k-1}$$

for $k = 0$,

$$z^{k-1} X(z) = \frac{3z + 1}{z\,(z+0.5)(z-0.8)}$$

$$\xi_0 = zF(z)\Big|_{z=0} = \frac{1}{0.5\cdot(-.8)} = -2.5$$

$$\xi_1 = (z+0.5)F(z)\Big|_{z=-.5} = \frac{3(-.5)+1}{-.5(-.5-.8)} = -\frac{10}{13} = -0.76923$$

$$\xi_2 = (z-0.8)F(z)\Big|_{z=0.8} = \frac{3\times.8+1}{.8(.8+.5)} = 3.26923$$

$$x(0) = -2.5 - 0.76923 + 3.26923 = 0$$

for $k < 0$, $x(k) = 0$

for $k > 0$,

$$\xi_1 = (z+0.5)F(z)\Big|_{z=-.5} = \frac{3\times(-.5)+1}{-.5-.8}(-.5)^{k-1} = 0.38462\cdot(-.5)^{k-1}$$

$$\xi_2 = (z-0.8)F(z)\Big|_{z=0.8} = \frac{3\times.8+1}{0.8+0.5}\cdot(.8)^{k-1} = 2.61538\,(.8)^{k-1}$$

$$x(k) = \xi_1 + \xi_2 = 0.38462\,(-.5)^{k-1} + 2.61538\,(.8)^{k-1}$$

$$= -0.76923\cdot(-.5)^k + 3.26923\cdot(.8)^k$$

$$\therefore\ x(k) = -2.5\,\delta(k) - \left[0.76923\cdot(-.5)^k - 3.26923\,(.8)^k\right]u(k)$$

$$= -2.5\,\delta(k) - \frac{10}{13}(-.5)^k u(k) + \frac{85}{26}(.8)^k u(k)$$

b) $$X(z) = \frac{2z^3 + z^2 + z}{(z-0.6)(z-0.4)(z-1)^2} \quad , \quad |z| > 1.0$$

$$F(z) = z^{k-1} \cdot X(z) = \frac{2z^2 + z + 1}{(z-0.6)(z-0.4)(z-1)^2} \cdot z^k$$

For $k < 0$, $x(k) = 0$

For $k \geq 0$,

$$\xi_1 = \frac{2 \times .6^2 + .6 + 1}{.2 \cdot (.6-1)^2} \cdot (0.6)^k = 72.5 \cdot (0.6)^k$$

$$\xi_2 = \frac{2 \times .4^2 + .4 + 1}{-.2 \times (.4-1)^2} \cdot (.4)^k = -23.88889\,(0.4)^k$$

$$(z-1)^2 \cdot F(z) = \frac{2z^2 + z + 1}{(z-.6)(z-.4)} \cdot z^k$$

$$\frac{d}{dz}(z-1)^2 F(z) = \frac{[2(k+2)z^{k+1} + (k+1)z^k + kz^{k-1}](z-.6)(z-.4) - (2z^2+z+1)z^k(z-1)}{(z-.6)^2(z-.4)^2}$$

$$\xi_3 = \frac{d}{dz}(z-1)^2 F(z)\Big|_{z=1} = \frac{0.24\,[2(k+2)+k+1+k]}{0.24^2} - \frac{4}{0.24^2}$$

$$= 16.66667(k+1) - 65.27777$$

$$x(k) \underline{= \xi_1 + \xi_2 + \xi_3} = [72.5 \cdot (0.6)^k - 23.88889\,(0.4)^k \underline{- 65.27777 + 16.66667}(k+1)]\,u(k)$$

C) $X(z) = \dfrac{z^3 - 3.1z^2 + 1.8z - 0.1}{(z-1)(z-0.5)(z-0.2)}$, $0.5 < |z| < 1$

$$F(z) = z^{k-1}X(z) = \frac{z^3 - 3.1z^2 + 1.8z - 0.1}{(z-1)(z-0.5)(z-0.2)}\, z^{k-1}$$

For $k < 0$,

$$\xi = (z-1)\cdot F(z)\Big|_{z=1} = \frac{1-3.1+1.8-0.1}{(1-0.5)(1-.2)}\cdot(1)^{k-1} = -1$$

$x(k) = -\xi = 1$ for $k<0$

For $k=0$,

$$\xi_0 = zF(z)\Big|_{z=0} = \frac{-0.1}{-1\times(-.5)\cdot(-.2)} = 1$$

$$\xi_1 = (z-0.5)F(z)\Big|_{z=0.5} = \frac{(.5)^3-3.1(.5)^2+1.8(.5)-.1}{.5(.5-1)(.5-.2)} = -2$$

$$\xi_2 = (z-0.2)F(z)\Big|_{z=0.2} = \frac{(.2)^3-3.1(.2)^2+1.8(.2)-.1}{.2(.2-1)(.2-.5)} = 3$$

$$x(0) = \xi_0 + \xi_1 + \xi_2 = 1-2+3 = 2$$

For $k>0$,

$$\xi_1 = (z-0.5)F(z)\Big|_{z=0.5} = \frac{(.5)^3-3.1(.5)^2+1.8(.5)-.1}{(.5-1)(.5-.2)}\cdot(.5)^{k-1} = -2\cdot(.5)^k$$

$$\xi_2 = (z-0.2)F(z)\Big|_{z=0.2} = 3(.2)^k$$

$$x(k) = -2(.5)^k + 3(.2)^k$$

Thus, the close-form for $x(k)$

$$\underline{x(k) = u(-k) - 2(.5)^k u(k) + 3(.2)^k u(k)}$$ Checked.

d) $X(z) = \dfrac{2z^3-2z^2+1.25z}{(z-0.5)(z^2-z+1)}$, $|z|>1$

$$F(z) = \frac{2z^3-2z^2+1.25z}{(z-0.5)(z-e^{j\frac{\pi}{3}})(z-e^{-j\frac{\pi}{3}})} z^{k-1} = \frac{(2z^2-2z+1.25)\,z^k}{(z-.5)(z-e^{j\frac{\pi}{3}})(z-e^{-j\frac{\pi}{3}})}$$

For $k<0$, $x(k)=0$

For $k \geq 0$,

$$\xi_1 = (z-0.5)F(z)\Big|_{z=0.5} = \frac{(2(.5)^2-2\times.5+1.25)(.5)^k}{(.5)^2-.5+1} = (.5)^k$$

$$\xi_2 = (z-e^{j\frac{\pi}{3}})F(z)\Big|_{z=e^{j\frac{\pi}{3}}} = \frac{2e^{j\frac{2\pi}{3}}-2e^{j\frac{\pi}{3}}+1.25}{(e^{j\frac{\pi}{3}}-0.5)(e^{j\frac{\pi}{3}}-e^{-j\frac{\pi}{3}})} e^{j\frac{k\pi}{3}} = 0.5e^{j\frac{k\pi}{3}}$$

$$\xi_3 = (z-e^{-j\frac{\pi}{3}})F(z)\Big|_{z=e^{-j\pi/3}} = 0.5\,e^{-jk\pi/3}$$

$$x(k) = (0.5)^k u(k) + 0.5e^{jk\pi/3}u(k) + 0.5e^{-jk\pi/3}u(k)$$

$$\underline{= (0.5)^k u(k) + (\cos\tfrac{\pi k}{3})\,u(k)}$$ ✓

3. <u>Find the inverse z-transform of the following functions using any method.</u>

a) $X(z) = \dfrac{z}{z^2+1}$, $|z|>1$

$$F(z) = z^{k-1}\cdot X(z) = \frac{1}{z^2+1}\cdot z^k = \frac{1}{(z+j)(z-j)} z^k$$

For $k<0$, $x(k)=0$

For $k\geq 0$,

$$\xi_1 = (z+j)F(z)\Big|_{z=-j} = \frac{1}{-2j}(-j)^k = 0.5j(-j)^k = 0.5je^{-jk\pi/2}$$

$$\xi_2 = (z-j)F(z)\Big|_{z=j} = \frac{1}{2j}(j)^k = -0.5j(j)^k = -0.5je^{jk\pi/2}$$

$$x(k) = 0.5j\,e^{-jk\pi/2} - 0.5je^{jk\pi/2} = \sin(k\pi/2)$$

$$\therefore \quad x(k) = \sin(k\pi/2)\,u(k)$$ ✓

b) $X(z) = \frac{z^2+1}{z^2-1} \quad , \quad |z| > 1$

$$\frac{X(z)}{z} = \frac{z^2+1}{z(z+1)(z-1)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-1}$$

$$A = z \cdot \frac{X(z)}{z}\Big|_{z=0} = -1 .$$

$$B = (z+1)\frac{X(z)}{z}\Big|_{z=-1} = \frac{(-1)^2+1}{-1(-2)} = 1$$

$$C = (z-1)\frac{X(z)}{z}\Big|_{z=1} = \frac{1+1}{1\cdot(2)} = 1$$

$$X(z) = -1 + \frac{z}{z+1} + \frac{z}{z-1}$$

$$= -1 + \frac{1}{1-(-z^{-1})} + \frac{1}{1-z^{-1}}$$

$$\underline{x(k) = -\delta(k) + u(k) + (-1)^k u(k)}$$

c) $X(z) = \frac{e^{-\alpha} z^{-1}}{(1 - e^{-\alpha} z^{-1})^2} \quad , \quad |z| > e^{-\alpha}$

$$F(z) = z^{k-1} \cdot \frac{e^{-\alpha} \cdot z}{(z - e^{-\alpha})^2} = \frac{e^{-\alpha} \cdot z^k}{(z - e^{-\alpha})^2}$$

For $k < 0$, $x(k) = 0$

For $k \geq 0$,

$$\xi = \textcircled{\tfrac{1}{2}} \frac{d(e^{-\alpha} z^k)}{dz}\Big|_{z=e^{-\alpha}} = \frac{e^{-\alpha}}{2} \cdot k z^{k-1}\Big|_{z=e^{-\alpha}} = \frac{k}{\cancel{2}} e^{-\alpha k}$$

$$\underline{x(k) = \frac{k}{\cancel{2}} e^{-\alpha k} u(k)}$$

-1

d) $X(z) = \frac{z^2+1}{(z-1)^3} \quad , \quad |z| > 1$

$$F(z) = \frac{z^2+1}{(z-1)^3} \cdot z^{k-1}$$

For $k < 0$, $x(k) = 0$

For $k = 0$, $x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{z^2+1}{(z-1)^3} = 0$

For $k > 0$,

$$\xi = \frac{1}{2}\frac{d^2}{dz^2} z^{k-1}(z^2+1)\Big|_{z=1} = \frac{1}{2}\frac{d}{dz}\left((k+1)z^k + (k-1)z^{k-2}\right)\Big|_{z=1} = k^2 - k + 1$$

$$\therefore \underline{x(k) = -\delta(k) + (k^2 - k + 1) u(k)}$$

✓

4. Solve each of the following difference equations using z-transforms.

a) $y(n+2) - 2y(n) = 0$; $y(0) = 0$, $y(1) = -1$

$$Z^2 Y(Z) - Z^2 y(0) - Z y(1) - 2Y(Z) = 0$$

$$Z^2 Y(Z) + Z - 2Y(Z) = 0 \quad , \quad (Z^2 - 2) Y(Z) = -Z$$

$$Y(Z) = \frac{-Z}{Z^2 - 2} = \frac{-Z}{(Z+\sqrt{2})(Z-\sqrt{2})}$$

$$F(Z) = Z^{n-1} \cdot Y(Z) = -\frac{Z^n}{(Z+\sqrt{2})(Z-\sqrt{2})}$$

For $n < 0$, $y(n) = 0$

For $n \geq 0$,

$$\xi_1 = (Z+\sqrt{2})F(Z)\Big|_{Z=-\sqrt{2}} = \frac{(-\sqrt{2})^n}{+2\sqrt{2}}$$

$$\xi_2 = (Z-\sqrt{2})F(Z)\Big|_{Z=\sqrt{2}} = \frac{-(\sqrt{2})^n}{2\sqrt{2}}$$

$$y(n) = \frac{\sqrt{2}}{4}(\sqrt{2})^n + \frac{-\sqrt{2}}{4}(-\sqrt{2})^n = \frac{-1}{4}\left[(\sqrt{2})^{n+1} + (-\sqrt{2})^{n+1}\right]$$

CLOSE-FORM: $y(n) = -\frac{1}{4}(\sqrt{2})^{n+1}\left[1 + (-1)^{n+1}\right]u(n)$ ✓

(b) $y(n+1) - y(n) = 2n + 1$; $y(0) = K$

$$Z\,Y(Z) - Z\,y(0) - Y(Z) = \frac{2Z}{(Z-1)^2} + \frac{Z}{Z-1}$$

$$(Z-1)\,Y(Z) = \frac{Z^2 + Z}{(Z-1)^2} + KZ$$

$$Y(Z) = \frac{KZ^3 - (2K-1)Z^2 + (K+1)Z}{(Z-1)^3}$$

$$F(Z) = Z^{n-1} \cdot Y(Z) = \frac{KZ^{n+2} - (2K-1)Z^{n+1} + (K+1)Z^n}{(Z-1)^3}$$

$$\frac{d}{dZ}(Z-1)^3 F(Z) = K(n+2)Z^{n+1} - (2K-1)(n+1)Z^n + (K+1)nZ^{n-1}$$

$$\frac{d^2}{dZ^2}(Z-1)^3 F(Z) = K(n+2)(n+1)Z^n - (2K-1)(n+1)nZ^{n-1} + (K+1)n(n-1)Z^{n-2}$$

$$\xi = \frac{1}{2}\cdot\frac{d^2}{dZ^2}(Z-1)^3 F(Z)\Big|_{Z=1} = \frac{1}{2}\left[K(n+2)(n+1) - (2K-1)(n+1)n + (K+1)n(n-1)\right]$$

$$= n^2 + K$$

Thus,

$y(n) = (n^2 + K)\,u(n)$ ✓

C) $y(n+2)+3y(n+1)+2y(n)=n \; ; \; y(0)=y(1)=1$

$$z^2 Y(z)-z^2y(0)-zy(1)+3zY(z)-3zy(0)+2Y(z)=\frac{z}{(z-1)^2}$$

$$(z^2+3z+2)Y(z) = \frac{z}{(z-1)^2}+z^2+4z$$

$$Y(z) = \frac{z^4+2z^3-7z^2+5z}{(z-1)^2(z+1)(z+2)}$$

$$F(z)= \frac{z^3+2z^2-7z+5}{(z-1)^2(z+1)(z+2)}\cdot z^n$$

$$\xi_1=(z+1)F(z)\Big|_{z=-1} = \frac{-1+2+7+5}{4\times(+1)}\cdot(-1)^n=+\frac{13}{4}(-1)^n$$

$$\xi_2=(z+2)F(z)\Big|_{z=-2} = \frac{-8+8+14+5}{9\times(-1)}(-2)^n=-\frac{19}{9}(-2)^n$$

$$\frac{d}{dz}(z-1)^2F(z) = \frac{[(n+3)z^{n+2}+2(n+2)z^{n-1}-7(n+1)z^n+5nz^{n-1}](z+1)(z+2)}{(z+1)^2(z+2)^2}$$

$$-\frac{[z^{n+3}+2z^{n+2}-7z^{n+1}+5z^n](2z+3)}{(z+1)^2(z+2)^2}$$

$$\xi_3=\frac{d}{dz}(z-1)^2F(z)\Big|_{z=1}$$

$$=\frac{[(n+3)+2(n+2)-7(n+1)+5n]\cdot 6-[1+2-7+5]\cdot 5}{2^2\cdot 3^2}$$

$$=\frac{1}{36}(6n-5)$$

Thus,

$$y(n)=\frac{1}{36}(6n-5)u(n)+\frac{13}{4}(-1)^n u(n)-\frac{19}{9}(-2)^n u(n)$$

✓

E-E 428 DISCRETE SIGNAL PROCESSING
HOMEWORK
Dr. R. A. Birgenheier

$\frac{85}{90}$

Homework no: 3

Ben Chen

1. $$H_1(z) = \frac{8z^3 + 24z^2 + 24z + 8}{25z^3 + 21z^2 + 11z + 7}$$

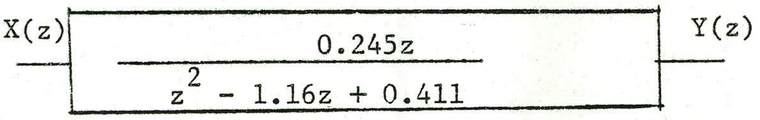
$$H_2(z) = \frac{z^3 + 0.5z^2 + 0.02z - 0.008}{z^4 + 0.3z^3 - 0.63z^2 + 0.085z + 0.015}$$

$$H_3(z) = \frac{z^2 + 2z + 3}{z^3 + 3z^2 + 3z + 2}$$

a) Determine which of the systems are BIBO-stable.

b) For the stable systems, find the response to the unit step input. Identify the (i) forced response, (ii) natural response, and (iii) steady-state response.

2.



$$\frac{0.245z}{z^2 - 1.16z + 0.411}$$

a) Find the steady state response to the input signal $x(k) = 2\cos(0.2k\pi)$

b) Determine and plot the magnitude and phase-response functions.

3. Determine and plot the magnitude frequency response of the following digital filter

$$\begin{aligned} y(k) &= 0.00808\, x(k-1) + 0.01245\, x(k-2) \\ &+ 0.00479\, x(k-3) + 2.058y(k-1) \\ &- 1.4534y(k-2) + 0.3701y(k-3) \end{aligned}$$

4. Compare the magnitude-frequency responses of the following two low-pass digital filters by plotting their response curves in dB.

$$H_1(z) = \frac{0.02721z^2 + 0.02177z}{z^3 - 1.958z^2 + 1.3376z - 0.32896}$$

$$H_2(z) = \frac{0.01615z^2 + 0.0249z + 0.00959}{z^3 - 1.958z^2 + 1.3376z - 0.32896}$$

1.

$$H_1(z) = \frac{8z^3 + 24z^2 + 24z + 8}{25z^3 + 21z^2 + 11z + 7}$$

$$H_2(z) = \frac{z^3 + 0.5z^2 + 0.02z - 0.008}{z^4 + 0.3z^3 - 0.63z^2 + 0.085z + 0.015}$$

$$H_3(z) = \frac{z^2 + 2z + 3}{z^3 + 3z^2 + 3z + 2}$$

a) Determine which of the systems are BIBO-stable.

b) For the stable systems, find the response to the unit step input. Identify the (i) forced response, (ii) natural response, and (iii) steady-state response.

Solution:
$$H_1(z) = \frac{8z^3+24z^2+24z+8}{25z^3+21z^2+11z+7} = \frac{8(z+1)^3}{25(z+0.75072)(z+0.04464-j0.60908)(z+0.04464+j0.60908)}$$

a) This system is BIBO stable because all the poles of the system are inside unit circle. ✓

b) Apply the unit step to the input, we have the response

$$Y(z) = H(z)U(z)$$

$$= \frac{8(z+1)^3}{25(z+0.75072)(z+0.04464-j0.60908)(z+0.04464+j0.60908)} \cdot \frac{z}{z-1}$$

$$F(z) \triangleq z^{k-1} \cdot Y(z) = \frac{8}{25} \cdot \frac{(z+1)^3 z^k}{(z-1)(z+0.75072)(z+0.04464-j0.60908)(z+0.04464+j0.60908)}$$

$$\xi_1 = (z-1)F(z)\Big|_{z=1} = 1$$

$$\xi_2 = (z+0.75072)F(z)\Big|_{z=-.75072} = -0.00326 \cdot (-.75072)^k$$

$$\xi_3 = (z+0.04464-j0.60908)F(z)\Big|_{z=-0.04464+j0.60908} = .33883\underline{/-182.98^\circ} \cdot (.61071)^k \cdot e^{jk0.52329\pi}$$

$$\xi_4 = (z+0.04464+j0.60908)F(z)\Big|_{z=-0.04464-j0.60908}$$

$$= 0.33883\underline{/-182.98^\circ}\ (.61071)^k \cdot e^{-jk0.52329\pi}$$

$$y(k) = 1 - 0.00326(-.75072)^k + 0.33883\underline{/-182.98^\circ}(.61071)^k \cdot e^{jk(0.52329\pi)}$$

$$+ 0.33883\underline{/182.98^\circ}(.61071)^k \cdot e^{-jk0.52329\pi}$$

$$= \underbrace{1}_{\text{forced \& steady response}} \underbrace{-0.00326(-.75072)^k - 0.67674(.61071)^k\cos(0.52329k\pi) - 0.03522(.61071)^k\sin(0.52329k\pi)}_{\text{natural response.}}, \quad k \geq 0$$

✓

4 OK

1 (CONT.)

$$H_2(Z) = \frac{Z^3 + 0.5Z^2 + 0.02Z - 0.008}{Z^4 + 0.3Z^3 - 0.63Z^2 + 0.085Z + 0.015} = \frac{(Z-0.1)(Z+0.2)(Z+0.4)}{(Z+0.1)\ (Z-.5)(Z-0.3)(Z+1)}$$

a) This system isn't BIBO-stable, because there is a pole, −1, on unit circle.

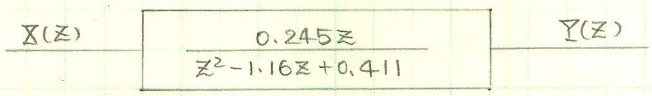
b) None.

$$H_3(Z) = \frac{Z^2 + 2Z + 3}{Z^3 + 3Z^2 + 3Z + 2} = \frac{(Z+1+j\sqrt{2})(Z+1-j\sqrt{2})}{(Z+2)(Z+0.5+j0.86603)(Z+0.5-j0.86603)}$$

a) This system is not BIBO-stable, because all the poles are outside or on the unit circle of Z-plane.

b) none.

2.



a) Find the steady state response to the input signal $x(k) = 2\cos(0.2k\pi)$

b) Determine and plot the magnitude and phase-response functions.

Solution: a) According the leture, we have the steady response to $x(k) = A\cos(\omega_1 kT + \theta_1)$

$$y(k) = A \cdot |H(e^{j\omega_1 T})| \cdot \cos(\omega_1 kT + \theta_1 + \angle H(e^{j\omega_1 T}))$$

From the givens, we have

$\omega_1 T = 0.2\pi$, $\theta_1 = 0°$, $A = 2$, and

$$H(e^{j0.2\pi}) = \frac{0.245\,e^{j0.2\pi}}{e^{j0.4\pi} - 1.16\,e^{j0.2\pi} + 0.411} = 0.70666\angle{-93.05°}$$

Thus,

$$y(k) = 2 \times 0.70666\ \cos(0.2k\pi - 93.05°)$$

$$= 1.41332\cos(0.2k\pi - 93.05°)$$

part (b) is continued on next page.

2. (CONT.)

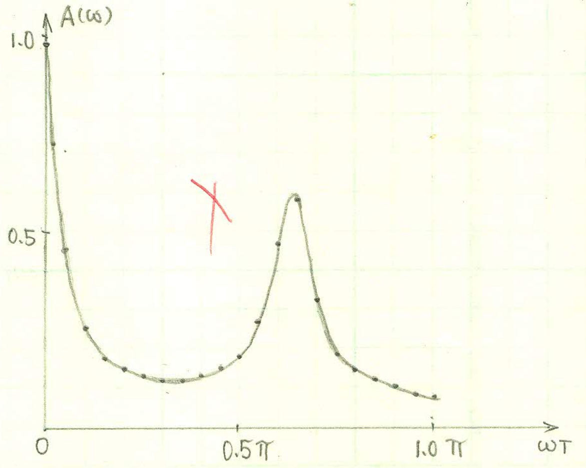
b) $$H(e^{j\omega T}) = \frac{0.245\,(\cos\omega T + j\sin\omega T)}{(\cos 2\omega T - 1.16\cos\omega T + 0.411) + j(\sin 2\omega T - 1.16\sin\omega T)}$$

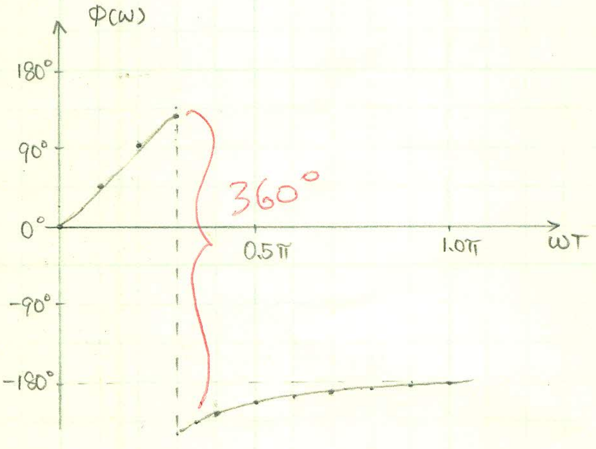
$$A(\omega) = \frac{0.245}{\sqrt{2.51452 - 0.95352\cos\omega T + 0.822\cos 2\omega T - 2.32\cos 3\omega T}}$$

-5

$$\phi(\omega) = -\omega T + \arctan\frac{\sin 2\omega T - 1.16\sin\omega T}{\cos 2\omega T - 1.16\cos\omega T + .411}$$

$\frac{\omega T}{\pi}$	$A(\omega)$	$\phi(\omega)$
0.00	0.97610	0.00°
0.05	0.45703	21.52°
0.10	0.25697	45.01°
0.15	0.18337	70.02°
0.20	0.14872	93.05°
0.25	0.13132	111.29°
0.30	0.12396	124.76°
0.35	0.12422	-225.30°
0.40	0.13225	-217.73°
0.45	0.15088	-211.77°
0.50	0.18832	-206.92°
0.55	0.26914	-202.85°
0.60	0.47393	-199.34°
0.65	0.58936	-196.25°
0.70	0.31253	-193.47°
0.75	0.19690	-190.92°
0.80	0.14582	-188.55°
0.85	0.11940	-186.31°
0.90	0.10495	-184.16°
0.95	0.09757	-182.07°
1.00	0.09529	-180.00





3. Determine and plot the magnitude frequency response of the following digital filter

$$y(k) = 0.00808\,x(k-1) + 0.01245\,x(k-2) + 0.00479\,x(k-3)$$

$$+ 2.058\,y(k-1) - 1.453\,y(k-2) + 0.3701\,y(k-3)$$

- -

Solution:

$$y(k) - 2.058\,y(k-1) + 1.453\,y(k-2) - 0.3701\,y(k-3)$$

$$= 0.00808\,x(k-1) + 0.01245\,x(k-2) + 0.00479\,x(k-3)$$

$$(1 - 2.058\,Z^{-1} + 1.453\,Z^{-2} - 0.3701\,Z^{-3})\,Y(Z)$$

$$= (0.00808\,Z^{-1} + 0.01245\,Z^{-2} + 0.00479\,Z^{-3})\,X(Z)$$

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{0.00808\,Z^{-1} + 0.01245\,Z^{-2} + 0.00479\,Z^{-3}}{1 - 2.058\,Z^{-1} + 1.453\,Z^{-2} - 0.3701\,Z^{-3}}$$

We are going to find $A^2(\omega)$ first,

$b_0 = 0$, $b_1 = 0.00808$, $b_2 = 0.01245$, $b_3 = 0.00479$, $b_n = 0$, $n > 3$

$a_0 = 1$, $a_1 = -2.058$, $a_2 = 1.453$, $a_3 = -0.3701$, $a_n = 0$, $n > 3$

$d_0 = 0.00808^2 + 0.01245^2 + 0.00479^2 = .000243$

$d_1 = b_1 b_2 + b_2 b_3 = 0.00808 \times 0.01245 + 0.01245 \times 0.00479 = 0.000160$

$d_2 = b_1 b_3 = 0.00808 \times 0.00479 = 0.000039$, $d_3 = b_0 b_3 = 0$

$C_0 = a_0^2 + a_1^2 + a_2^2 + a_3^2 = 7.483547$

$C_1 = a_0 a_1 + a_1 a_2 + a_2 a_3 = -5.586029$

$C_2 = a_0 a_2 + a_1 a_3 = 1.453 + 2.058 \times 0.3701 = 2.214665$

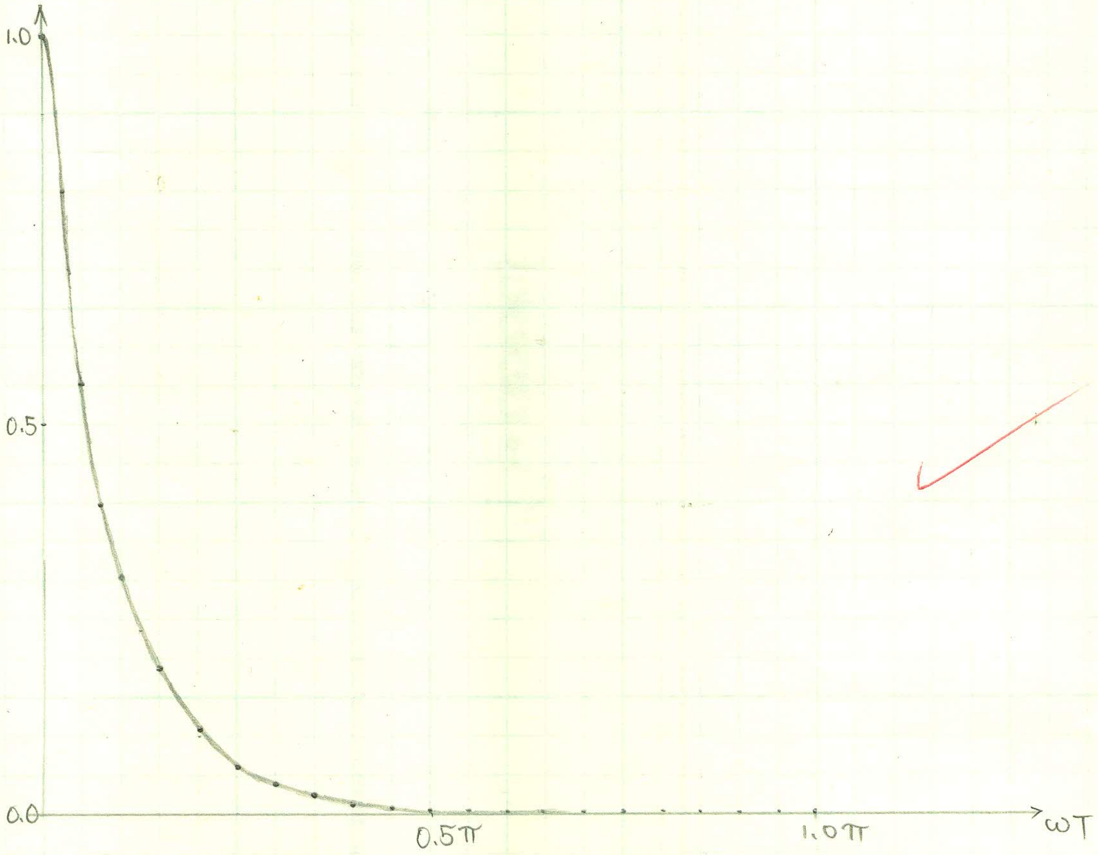
$C_3 = a_0 a_3 = -0.3701$

$$\therefore \quad A^2(\omega) = \frac{0.000243 + 0.00032\cos\omega T + 0.000078\cos 2\omega T}{7.483547 - 11.172058\cos\omega T + 4.42933\cos 2\omega T - 0.7402\cos 3\omega T}$$

$$A(\omega) = \sqrt{A^2(\omega)} = \sqrt{\frac{0.000243 + 0.00032\cos\omega T + 0.000078\cos 2\omega T}{7.48354 - 11.172058\cos\omega T + 4.42933\cos 2\omega T - 0.7402\cos 3\omega T}}$$

3. (CONT.)

$\frac{\omega T}{\pi}$	$A(\omega)$
0.00	0.9992
0.05	0.5519
0.10	0.3033
0.15	0.1843
0.20	0.1110
0.25	0.0662
0.30	0.0402
0.35	0.0252
0.40	0.0163
0.45	0.0108
0.50	0.0074
0.55	0.0050
0.60	0.0035
0.65	0.0024
0.70	0.0016
0.75	0.0011
0.80	0.0007
0.85	0.0004
0.90	0.0003
0.95	0.0002
1.00	0.0001



4. Compare the magnitude-frequency responses of the following two low-pass digital filter by plotting their response curves in dB

$$H_1(Z) = \frac{0.02721 Z^2 + 0.02177 Z}{Z^3 - 1.958 Z^2 + 1.3376 Z - 0.32896}$$

$$H_2(Z) = \frac{0.01615 Z^2 + 0.0249 Z + 0.00959}{Z^3 - 1.958 Z^2 + 1.3376 Z - 0.32896}$$

- -

Solution: For $H_1(Z)$, we rewite

$$H_1(Z) = Z^{-1} \cdot \frac{0.02721 + 0.02177 Z^{-1}}{1 - 1.958 Z^{-1} + 1.3376 Z^{-2} - 0.32896 Z^{-3}}$$

For $H_2(Z)$,

$$H_2(Z) = Z^{-1} \cdot \frac{0.01615 + 0.0249 Z^{-1} + 0.00959 Z^{-2}}{1 - 1.958 Z^{-1} + 1.3376 Z^{-2} - 0.32896 Z^{-3}}$$

Then, for low-pass $H_1(Z)$

$b_0 = 0.02721$, $b_1 = 0.02177$, $b_n = 0$, $n > 1$

$d_0 = b_0^2 + b_1^2 = 0.001214317$; $d_1 = b_0 b_1 = 0.000592362$

For low-pass $H_2(Z)$

$b_0 = 0.01615$, $b_1 = 0.0249$, $b_2 = 0.00959$, $b_n = 0$, $n > 2$

$d_0 = b_0^2 + b_1^2 + b_2^2 = 0.000972801$, $d_1 = b_0 b_1 + b_1 b_2 = 0.000640926$

$d_2 = b_0 b_2 = 0.000154879$

For both $H_1(Z)$ and $H_2(Z)$

$a_0 = 1$, $a_1 = -1.958$, $a_2 = 1.3376$, $a_3 = -0.32896$, $a_n = 0$, $n > 3$

$C_0 = a_0^2 + a_1^2 + a_2^2 + a_3^2 = 6.731152442$, $C_1 = a_0 a_1 + a_1 a_2 + a_2 a_3 = -5.017037696$

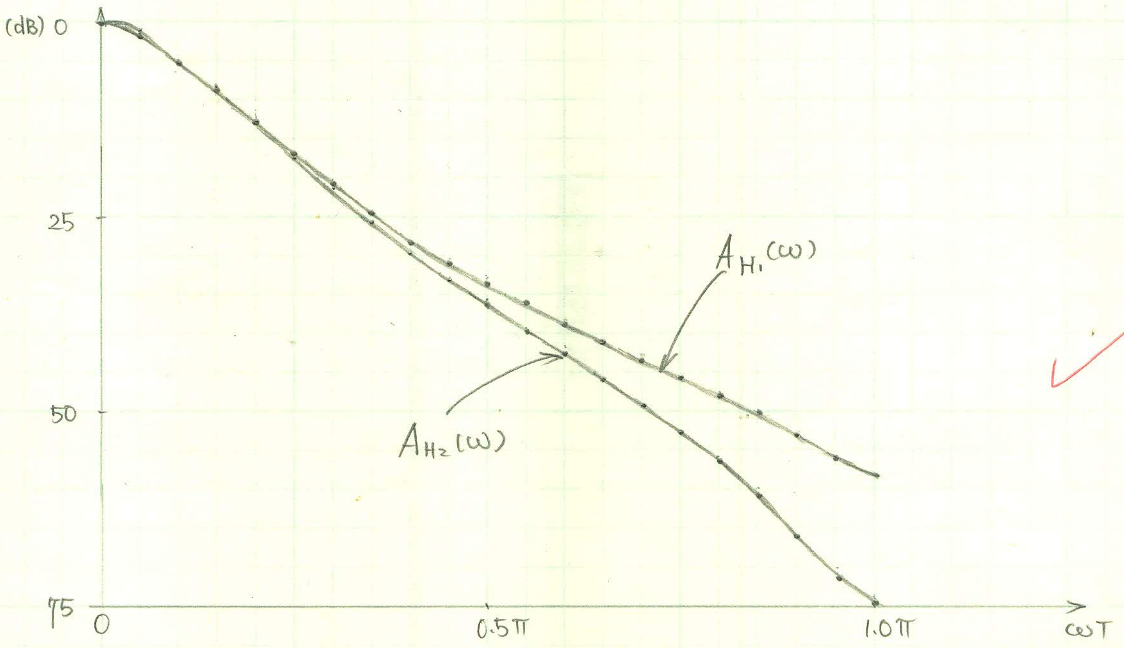
$C_2 = a_0 a_2 + a_1 a_3 = 1.981703680$, $C_3 = a_0 a_3 = -0.32896$

$$\therefore \quad A_{H_1}^2(\omega) = \frac{0.001214317 + 0.001184724 \cos \omega T}{6.731152442 - 10.03407539 \cos \omega T + 3.963407360 \cos 2\omega T - 0.65792 \cos 3\omega T}$$

$$A_{H_2}^2(\omega) = \frac{0.000972801 + 0.001281852 \cos \omega T + 0.000309758 \cos 2\omega T}{6.731152442 - 10.03407539 \cos \omega T + 3.963407360 \cos 2\omega T - 0.65792 \cos 3\omega T}$$

4 (CONT.)

$\frac{\omega T}{\pi}$	$A_{H_1}(\omega)$ in dB	$A_{H_2}(\omega)$ in dB
0.00	−0.29	−0.00
0.05	−2.05	−1.79
0.10	−5.30	−5.11
0.15	−8.85	−8.80
0.20	−12.81	−12.95
0.25	−16.93	−17.31
0.30	−20.88	−21.57
0.35	−24.52	−25.58
0.40	−27.82	−29.32
0.45	−30.82	−32.84
0.50	−33.58	−36.21
0.55	−36.14	−39.47
0.60	−38.56	−42.72
0.65	−40.89	−46.00
0.70	−43.18	−49.40
0.75	−45.50	−53.04
0.80	−47.93	−57.03
0.85	−50.57	−61.53
0.90	−53.56	−66.66
0.95	−56.77	−71.99
1.00	−58.59	−74.81



From Lecture

10/10

Using the fact that $x(t_{K+N}) = x(t_K)$ and

$$X[j(\omega_{n+N})] = X(j\omega_n)$$

Show that we can replace limits from $-N/2$ to $N/2-1$ to 0 to $N-1$

- -

SHOW From Lectur

$$x(t_K) = \frac{\Delta\omega}{2\pi}\sum_{n=-N/2}^{N/2-1} X(j\omega_n)e^{j\omega_n t_k} \qquad k=0,1,2,\cdots,N-1$$

$$= \frac{\Delta\omega}{2\pi}\sum_{n=-N/2}^{-1} X(j\omega_n)e^{j\omega_n t_k} + \frac{\Delta\omega}{2\pi}\sum_{n=0}^{N/2-1} X(j\omega_n)e^{j\omega_n t_k}$$

$$= \frac{\Delta\omega}{2\pi}\sum_{n=N/2}^{N-1} X(j\omega_{n+N})e^{j\omega_{n+N} t_k} + \frac{\Delta\omega}{2\pi}\sum_{n=0}^{N/2-1} X(j\omega_n)e^{j\omega_n t_k}$$

$\because$ $X(j\omega_{n+N}) = X(j\omega_n)$ &

$\omega_{n+N} = (n+N)\Delta\omega = n\Delta\omega + N\Delta\omega = \omega_n + N\Delta\omega$

$$\therefore \quad X(t_K) = \frac{\Delta\omega}{2\pi}\cdot\sum_{n=N/2}^{N-1} X(j\omega_n)e^{j\omega_n t_k}\cdot e^{jN\Delta\omega t_k} + \frac{\Delta\omega}{2\pi}\sum_{n=0}^{N/2-1} X(j\omega_n)e^{j\omega_n t_k}$$

But $\Delta\omega = \frac{2\pi}{NT}$ and $t_k = kT$

$$e^{jN\Delta\omega t_k} = e^{jN\cdot\frac{2\pi}{NT}kT} = e^{j2k\pi} = 1 \quad . \text{ for } k=0,1,2,\cdots,N-1$$

$$\therefore X(t_k) = \frac{\Delta\omega}{2\pi}\cdot\sum_{n=0}^{N/2-1} X(j\omega_n)e^{j\omega_n t_k} + \frac{\Delta\omega}{2\pi}\cdot\sum_{n=N/2}^{N-1} X(j\omega_n)e^{j\omega_n t_k}$$

$$= \frac{\Delta(\omega)}{2\pi}\sum_{n=0}^{N-1} X(j\omega_n)e^{j\omega_n t_k}$$

Showed !

7-2 Find a closed-form expression for the DFT of the following discrete signal:

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$$x_K = \begin{cases} e^{-0.2k} & \text{for } 0 \le k \le 7 \\ 0 & \text{for } 8 \le k \le 31 \end{cases}$$

Solution:

$$\text{DFT}[x_K] \triangleq X_n = \sum_{k=0}^{31} x_K e^{-j\frac{2\pi}{32}kn}$$

$$= \sum_{k=0}^{7} e^{-(0.2+j\frac{\pi n}{16})k} \qquad \text{for } n=0,1,\cdots,31$$

$$= \frac{1-e^{-(0.2+j\frac{\pi n}{16})(7+1)}}{1-e^{-(0.2+j\frac{\pi n}{16})}}$$

$$= \frac{1-e^{-1.6}\cdot e^{-j\frac{n\pi}{2}}}{1-e^{-0.2}\cdot e^{-j\frac{\pi n}{16}}} \qquad \text{FOR } n=0,1,\cdots,31$$ ⟸

End.

7-3 Find a closed-form expression for the DFT of the 16-point sequence

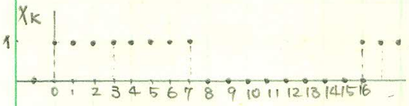
$$x_K = \begin{cases} 1 & \text{for } 0 \le k \le 7 \\ 0 & \text{for } 8 \le k \le 15 \end{cases}$$

Solution:

$$X_n = \text{DFT}[x_K] = \sum_{k=0}^{15} x_K e^{-j\frac{2\pi}{16}kn}$$

$$= \sum_{k=0}^{7} e^{-j\frac{n\pi}{8}k}$$

$$= \frac{1-e^{j\frac{n\pi}{8}(7+1)}}{1-e^{j\frac{n\pi}{8}}}$$



$$= \frac{1-e^{jn\pi}}{1-e^{jn\pi/8}} = e^{-j\frac{n\pi}{2}+j\frac{n\pi}{16}} \frac{e^{+j\frac{n\pi}{2}}-e^{-j\frac{n\pi}{2}}}{e^{+j\frac{n\pi}{16}}-e^{-j\frac{n\pi}{16}}}$$

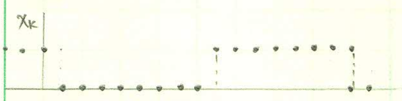
$$= e^{-j\frac{7}{16}n\pi} \frac{\sin(\frac{n\pi}{2})}{\sin(\frac{n\pi}{16})} \qquad \text{FOR } n=0,1,\cdots,16$$ ⟸

End.

7-4 Use the result obtained in Prob. 7-3 to find the DFT of the following 16-point sequence:

$$x_K = \begin{cases} 1 & \text{for } k=0 \\ 0 & \text{for } 1 \le k \le 8 \\ 1 & \text{for } 9 \le k \le 15 \end{cases}$$

Sltn:



Compare this sequence to the sequence in Prob. 7-3 (assuming it is x'_K now), we see that $x_K = x'_{-K}$.

$$X'_n = e^{-j\frac{7n\pi}{16}} \cdot \frac{\sin(n\pi/2)}{\sin(n\pi/16)}$$

By property 3, we have

$$X_n = X'_{-n} = e^{j\frac{7n\pi}{16}} \cdot \frac{\sin(n\pi/2)}{\sin(n\pi/16)} \qquad \text{FOR } n=0,1,\cdots,15$$ ⟸

7-5 Determine the DFT of the following 16-point sequence:

$$x_K = \begin{cases} 0 & \text{for } k = 0.1.2 \\ 1 & \text{for } 3 \le k \le 10 \\ 0 & \text{for } 11 \le k \le 15 \end{cases}$$

Solution: Assuming once again that the sequence in Prob. 7-3 is x'_K, then

$$x_K = x'_{K-3}$$

By property 7

$$X_n = \text{DFT}[x_K] = \text{DFT}[x'_{K-3}]$$

$$= e^{-j\frac{2\pi}{16}3\cdot n} \cdot X'_n$$

$$= e^{-j\frac{6n\pi}{16}} \cdot e^{-j\frac{7n\pi}{16}} \cdot \frac{\sin(n\pi/2)}{\sin(n\pi/16)}$$

$$= e^{-j\frac{13n\pi}{16}} \cdot \frac{\sin(n\pi/2)}{\sin(n\pi/16)} \quad \text{FOR } n = 0, 1, \dots, 15 \qquad \longleftarrow$$

7-6 Consider the following 16-point signal:

$$x_K = \begin{cases} e^{-0.5K} & \text{for } 0 \le k \le 5 \\ 0 & \text{for } 6 \le k \le 15 \end{cases}$$

(a) Determine a closed-form expression for the 16-point DFT of the signal.

(b) Display the time signal x_K and its DFT $= X_n$ in magnitude for $-16 \le n, k \le 31$

(c) Use the result found in part a to obtain the DFT of the complex signal

$$x_K = \begin{cases} e^{-(0.5 - j\pi/4)k} & \text{for } 0 \le k \le 5 \\ 0 & \text{for } 6 \le k \le 15 \end{cases}$$

Solution: (a)

$$X_n = \sum_{k=0}^{5} e^{-0.5k} \cdot e^{-j\frac{2\pi}{16}nk} = \frac{1 - e^{-3} \cdot e^{-j\frac{3n\pi}{4}}}{1 - e^{-0.5} \cdot e^{-j\frac{n\pi}{8}}} \qquad \longleftarrow$$

(b)

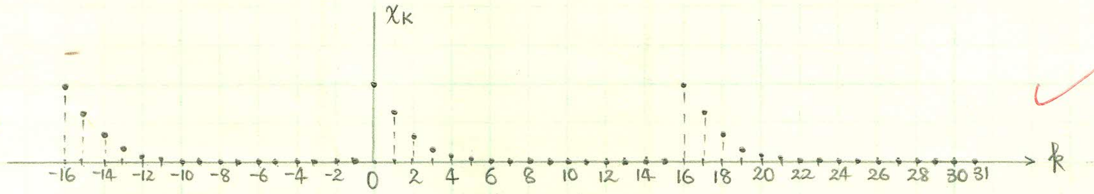
$$|X_n|^2 = \frac{(1 - e^{-3}\cos\frac{3n\pi}{4})^2 + (e^{-3}\sin\frac{3n\pi}{4})^2}{(1 - e^{-0.5}\cos\frac{n\pi}{8})^2 + (e^{-0.5}\sin\frac{n\pi}{8})^2}$$

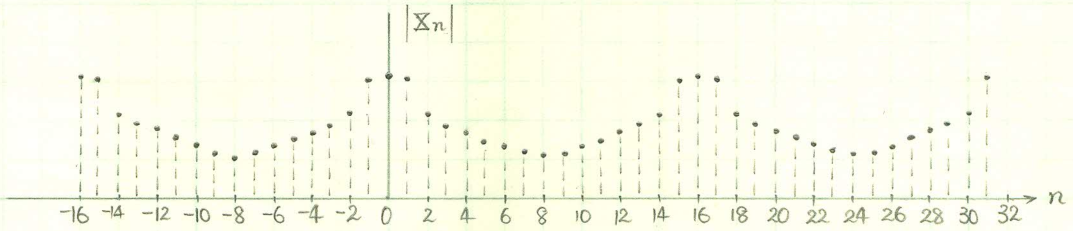
$$= \frac{1 + e^{-3} - 2e^{-3}\cos\frac{3n\pi}{4}}{1 + e^{-.5} - 2e^{-.5}\cos\frac{n\pi}{8}} = \frac{1.04979 - 0.09957\cos(0.75n\pi)}{1.60653 - 1.21306\cos(0.125n\pi)}$$

And the following data for $|X_n|$ are obtained from my HP-41 CX

$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\|X_n\| =$	1.554	1.518	1.184	0.926	0.846	0.688	0.653	0.641	0.581	0.641	0.653	0.688	0.846	0.926	1.184	1.518

7-6 (CONT.) (b)





(c) $e^{j\frac{\pi}{4}k} = e^{j\frac{2\pi}{16}2\cdot k} \Rightarrow m=2$

From property 8,

$$X_n = \frac{1-e^{-3}\cdot e^{-j\frac{3\pi}{4}(n-2)}}{1-e^{-5}\,e^{-j\frac{\pi}{8}(n-2)}} \qquad \text{FOR } n=0,1,2,\cdots,15$$

7-7 Prove the following relationship:

$$\sum_{K=0}^{N-1} e^{j(2\pi/N)(\gamma-n)k} = \begin{cases} N & \text{for } \gamma-N=mN, \ m=0,1,2,\cdots \\ 0 & \text{elsewhere} \end{cases}$$

PROVE:

$$\sum_{K=0}^{N-1} e^{j(2\pi/N)(\gamma-n)k}$$

$$= \frac{1-e^{j(2\pi/N)(\gamma-n)\cdot N}}{1-e^{j(2\pi/N)(\gamma-n)}} \qquad \text{FOR } \gamma-N\neq mN, \ m=0,1,2,\cdots$$

$$= \frac{1-e^{j2\pi(\gamma-n)}}{1-e^{j(2\pi/N)(\gamma-n)}} = 0 \qquad \text{FOR } \gamma-N\neq mN, \ m=0,1,2,\cdots$$

For $\gamma-N=mN$, $m=0,1,2,\cdots$

$$e^{j(2\pi/N)(\gamma-n)k} = e^{j(2\pi/N)\cdot m\cdot N\cdot k} = e^{j2mk\pi} = 1$$

$$\sum_{K=0}^{N-1} e^{j(2\pi/N)(\gamma-n)k} = N$$

PROVED!

7-8 *Use the relationship in Prob. 7-7 to show that the DFT of the N-point sequence*

$$x_k = 0.5\left(1 - \cos\frac{2k\pi}{N}\right) \qquad k = 0, 1, \cdots, N-1$$

has three nonzero samples and that the other N-3 samples are all zeros. What are the three nonzero samples of the DFT? This sequence is called the Hanning window.

SHOW:

$$x_k = 0.5\ (1 - \cos\frac{2k\pi}{N})$$

$$= 0.5 - 0.5\cos\frac{2k\pi}{N} = 0.5 - 0.5\cdot\frac{1}{2}(e^{j\frac{2k\pi}{N}} + e^{-j\frac{2k\pi}{N}})$$

$$= 0.5e^{j2k\pi} - 0.25e^{j\frac{2k\pi}{N}} - 0.25e^{-j\frac{2k\pi}{N}}$$

$$\mathrm{DFT}[x_k] = X_n = \sum_{k=0}^{N-1}(0.5e^{j2k\pi} - 0.25e^{j\frac{2k\pi}{N}} - 0.25e^{-j\frac{2k\pi}{N}})e^{-j\frac{2k\pi}{N}\cdot n}$$

$$= \sum_{k=0}^{N-1}0.5\,e^{j(2\pi/N)(N-n)k} - 0.25\sum_{k=0}^{N-1}e^{j(2\pi/N)(1-n)k} - 0.25\sum_{k=0}^{N-1}e^{j(2\pi/N)(-1-n)k}$$

Applying the relationship in Prob. 7-7, we have that

X_n is nonzero iff $n=0$, $n=1$ and $n=-1$, or $N-1$

For $n=0$, $X_0 = 0.5N$ ✓

$n=1$, $X_1 = -0.25N$ ✓

$n=N-1$, $X_{N-1} = -0.25N$ ✓

7-9 *Obtain the DFT of the following Blackman window sequence:*

$$x_k = 0.42 - 0.5\cos\frac{2k\pi}{N} + 0.08\cos\frac{4k\pi}{N} \qquad k = 0, 1, \cdots, N-1$$

Solution:

$$x_k = 0.42e^{j2k\pi} - 0.25e^{j\frac{2\pi}{N}k} - 0.25e^{-j\frac{2\pi}{N}k} + 0.04e^{j\frac{2\pi}{N}2\cdot k} + 0.04e^{-j\frac{2\pi}{N}\cdot 2\cdot k}$$

$$\mathrm{DFT}[x_k] = \sum_{k=0}^{N-1}(0.42e^{j\frac{2\pi}{N}\cdot Nk} - 0.25e^{j\frac{2\pi}{N}k} - 0.25e^{-j\frac{2\pi}{N}k} + 0.04e^{j\frac{2\pi}{N}\cdot 2\cdot k} + 0.04e^{-j\frac{2\pi}{N}\cdot 2\cdot k})e^{-j\frac{2\pi}{N}n\cdot k}$$

$$= 0.42\sum_{k=0}^{N-1}e^{j(2\pi/N)(N-n)k} - 0.25\sum_{k=0}^{N-1}e^{j(2\pi/N)(1-n)k} - 0.25\sum_{k=0}^{N-1}e^{j(2\pi/N)(-1-n)k}$$

$$+ 0.04\sum_{k=0}^{N-1}e^{j(2\pi/N)(2-n)k} + 0.04\sum_{k=0}^{N-1}e^{j(2\pi/N)(-2-n)k}$$

Thus

$$X_n = \begin{cases} 0.42N \checkmark & \text{for } n=0 \\ -0.25N \checkmark & \text{for } n=1 \\ -0.25N \checkmark & \text{for } n=N-1 \\ 0.04N \checkmark & \text{for } n=2 \\ 0.04N \checkmark & \text{for } n=N-2 \\ 0 \checkmark & \text{eleswhere} \end{cases}$$

7-10 Find the DFT of these sequences:

(a) $x_K = e^{j(2\pi/N)mk}$ $k=0,1,\cdots,N-1$

(b) $x_K = \cos\frac{2\pi}{N}k$ $k=0,1,\cdots,N-1$

(c) $x_K = \sin\frac{2\pi}{N}k$ $k=0,1,\cdots,N-1$

Solution: (a)

$$\text{DFT}[x_K] = \sum_{k=0}^{N-1} e^{j(2\pi/N)mk}\cdot e^{-j(2\pi/N)\cdot n\cdot k} \qquad n=0,1,\cdots,N-1$$

$$= \sum_{k=0}^{N-1} e^{j(2\pi/N)(m-n)k}$$

✓

$$= \begin{cases} N & \text{for } n=m \\ 0 & \text{eleswhere} \end{cases} \qquad (\text{Assume } m<N \text{ otherwise } n=m\cdot\text{mod}\cdot N)$$

(b)

$$x_K = \cos\frac{2\pi}{N}k = 0.5e^{j(2\pi/N)k} + 0.5e^{-j(2\pi/N)k}$$

$$\text{DFT}[x_K] = 0.5\sum_{k=0}^{N-1} e^{j(2\pi/N)(1-n)k} + 0.5\sum_{k=0}^{N-1} e^{j(2\pi/N)(-1-n)k}$$

✓

$$= \begin{cases} 0.5N & \text{for } n=1 \text{ and } n=N-1 \\ 0 & \text{eleswhere} \end{cases}$$

(c)

$$x_K = \sin\frac{2\pi}{N}k = 0.5j\,e^{-j(2\pi/N)k} - 0.5j\,e^{j(2\pi/N)k}$$

$$\text{DFT}[x_K] = 0.5j\sum_{k=0}^{N-1} e^{j(2\pi/N)(-1-n)k} - 0.5j\sum_{k=0}^{N-1} e^{j(2\pi/N)(1-n)k}$$

$$= \begin{cases} 0.5N\cdot j & \text{for } n=N-1 \\ -0.5N\cdot j & \text{for } n=1 \\ 0 & \text{eleswhere} \end{cases}$$

✓

7-15 Find the circular convolution of the following two eight-point sequences:

$x_K = \{0,1,2,3,4,2,0,-1\}$ and $y_K = \{2,4,3,1,0,-1,1,2\}$

-1

Solution:

0 1 2 3 4 2 0 -1 0 1 2 3 4 2 0 -1 0 1 2 3 4

$k=0$ 2 1 -1 0 1 3 4 2 = -1 ✓

$k=1$ 2 1 -1 0 1 3 4 2 = 2 ✓

$k=2$ 2 1 -1 0 1 3 4 2 = 15 15

$k=3$ 2 1 -1 0 1 3 4 2 = 27 ✓

$k=4$ 2 1 -1 0 1 3 4 2 = 32 ✓

$k=5$ 2 1 -1 0 1 3 4 2 = 30 ✓

$k=6$ 2 1 -1 0 1 3 4 2 = 20 ✓

$k=7$ 2 1 -1 0 1 3 4 2 = 7 ✓

7-16 Use the circular convolution to obtain the linear convolution of the following 2-sequences:

$x_K = \{1, 2, 3, 2, 1, -1\}$ 6

$y_K = \{1, 1, 1, 1, -1, -1, -1, -1\}$ 8.

Solution:

	0	0	0	0	0	0	0	1	2	3	2	1	-1	0	0	0	0	0	0	0	1	2			
$k=0$	-1	-1	-1	-1	1	1	1	1															=	1	✓
$k=1$		-1	-1	-1	-1	1	1	1	1														=	3	✓
$k=2$			-1	-1	-1	-1	1	1	1	1													=	6	✓
$k=3$				-1	-1	-1	-1	1	1	1	1												=	8	✓
$k=4$					-1	-1	-1	-1	1	1	1	1											=	7	✓
$k=5$						-1	-1	-1	-1	1	1	1	1										=	2	✓
$k=6$							-1	-1	-1	-1	1	1	1	1									=	-4	✓
$k=7$								-1	-1	-1	-1	1	1	1	1								=	-8	-8
$k=8$									-1	-1	-1	-1	1	1	1	1							=	-9	✓
$k=9$										-1	-1	-1	-1	1	1	1	1						=	-5	✓
$k=10$											-1	-1	-1	-1	1	1	1	1					=	-2	✓
$k=11$												-1	-1	-1	-1	1	1	1	1				=	0	✓
$k=12$													-1	-1	-1	-1	1	1	1	1			=	1	✓

LINEAR CONVOLUTION

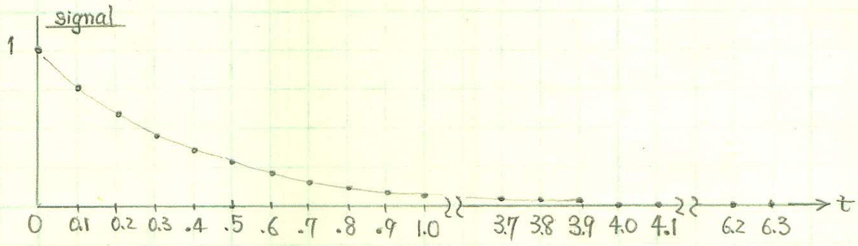
SEQUENCE

7-17. Obtain the circular correlation of the two sequences in Prob. 7-15

x_K :		0	1	2	3	4	2	0	-1	0	1	2	3	4	2	0	-1			
	$m=0$									2	4	3	1	0	-1	1	2	=	9	✓
$\gamma_{xy}(m)$ =	$m=1$								2	4	3	1	0	-1	1	2		=	1	✓
	$m=2$							2	4	3	1	0	-1	1	2			=	2	✓
	$m=3$						2	4	3	1	0	-1	1	2				=	10	✓
	$m=4$					2	4	3	1	0	-1	1	2					=	22	✓
	$m=5$				2	4	3	1	0	-1	1	2						=	33	✓
	$m=6$			2	4	3	1	0	-1	1	2							=	33	✓
	$m=7$		2	4	3	1	0	-1	1	2								=	22	✓

7-31 Use the zero-padding technique to evaluate the 64-point DFT from 40 values of the exponential function e^{-t} sampled at $T = 0.1$. Compare the result with direct calculation from the Fourier transform function $1/(1+j\omega)$ at those frequencies.

Solution: a) signal

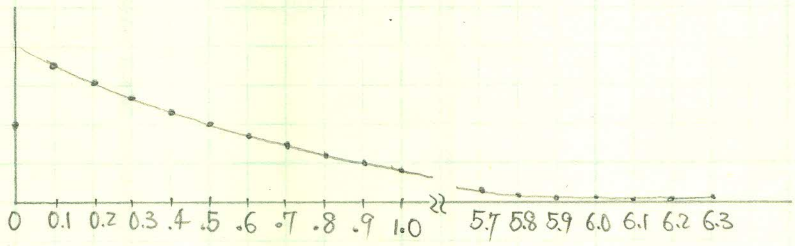


Theoretical values

$$F(\omega) = \frac{1}{1+j\omega} = \frac{1-j\omega}{1+\omega^2} = \frac{1}{1+\omega^2} - j\frac{\omega}{1+\omega^2}$$

b) Additonal problem

signal:



Theoretical values

$$F(\omega) = \frac{1}{1+\omega^2} - j\cdot\frac{\omega}{1+\omega^2}$$

```
C A PROGRAM FOR PROBLEM 7-31, HOMEWORK NO:SIX, BEN M. CHEN
        COMPLEX F[64],CMPLX,BENC,CHEN
        N=64
        DT=.1
        DO 5 I=2,64
5       F[I]=CMPLX[EXP[-.1*[I-1]],0.]
10      F[1]=CMPLX[0.5,0.]
        DF=1./[N*DT]
        CALL FFT[1,N,DT,F,BAD]
        WRITE [6,107]
107     FORMAT['1',7X,'COMPRISSION BETWEEN THE COMPUTED FFT AND THEORETICAL
     1  FOURIER TRANSFORM'//5X, 'FREQUENCY[HZ]'/]
        DO 20 I=1,N
        J=I-1
        DFN=J*DF
        DPI=2*3.14159265*DFN
        BENC=CMPLX[1./[1.+DPI*DPI],-DPI/[1.+DPI*DPI]]
        CHEN=F[I]-BENC
        AMAGF=CABS[F[I]]
        FRF=REAL[F[I]]
        FIF=AIMAG[F[I]]
        IF [FRF .EQ. 0. .AND. FIF .EQ. 0.] GO TO 12
        ANGF=57.2959*ATAN2[FIF,FRF]
        GO TO 13
12      ANGF=0.
13      FRB=REAL[BENC]
        FIB=AIMAG[BENC]
        AMAGB=CABS[BENC]
        IF [FRB .EQ. 0. .AND. FIB .EQ. 0.] GO TO 14
        ANGB=57.2959*ATAN2[FIB,FRB]
        GO TO 15
14      ANGB=0.
15      DMAG=AMAGF-AMAGB
        DPHS=ANGF-ANGB
20      WRITE [6,108] J,DFN,F[I],BENC
108     FORMAT[I3,X,1PE13.6,3X,1PE13.6,3X,1PE13.6,3X,1PE13.6,2X,1PE
     1  13.6]
        STOP
        END
```

```
     COMPRISSION BETWEEN THE COMPUTED FFT AND THEORETICAL  FOURIER TRANSFORM
     ***********************************************************************

[ PROBLEM 7-31: FFT 40 VALUES FROM exp[-t], 24 VALUES FROM ZERO-PADDING;
                      THEORETICAL VALUES FROM 1/[1+jw] ]
                ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

                              COMPUTED FFT                        THERETICAL VALUES
     FREQUENCY[HZ]        REAL PART     IMAGINARY            REAL PART        IMAGINARY

 0   0.000000E+00     1.031587E+00    0.000000E+00       1.000000E+00    0.000000E+00
 1   1.562500E-01     5.608324E-01   -5.128145E-01       5.092094E-01   -4.999152E-01
 2   3.125000E-01     2.641716E-01   -3.980615E-01       2.059599E-01   -4.044013E-01
 3   4.687500E-01     1.482938E-01   -3.000644E-01       1.033650E-01   -3.044350E-01
 4   6.250000E-01     1.137829E-01   -2.401814E-01       6.089667E-02   -2.391407E-01
 5   7.812500E-01     9.199633E-02   -1.878410E-01       3.984744E-02   -1.956007E-01
 6   9.375000E-01     7.592889E-02   -1.615192E-01       2.801291E-02   -1.650097E-01
 7   1.093750E+00     7.428616E-02   -1.375700E-01       2.073501E-02   -1.424959E-01
 8   1.250000E+00     6.558861E-02   -1.165092E-01       1.595277E-02   -1.252928E-01
 9   1.406250E+00     6.298568E-02   -1.064619E-01       1.264700E-02   -1.117455E-01
10   1.562500E+00     6.283789E-02   -9.138200E-02       1.026875E-02   -1.008132E-01
11   1.718750E+00     5.754646E-02   -8.233652E-02       8.501720E-03   -9.181198E-02
12   1.875000E+00     5.911101E-02   -7.558810E-02       7.153521E-03   -8.427543E-02
13   2.031250E+00     5.713614E-02   -6.533986E-02       6.101764E-03   -7.787511E-02
14   2.187500E+00     5.507482E-02   -6.157779E-02       5.265641E-03   -7.237343E-02
15   2.343750E+00     5.695153E-02   -5.485366E-02       4.590073E-03   -6.759441E-02
16   2.500000E+00     5.397635E-02   -4.883983E-02       4.036488E-03   -6.340501E-02
     2.656250E+00     5.466266E-02   -4.640236E-02       3.577223E-03   -5.970282E-02
18   2.812500E+00     5.492172E-02   -3.987061E-02       3.192028E-03   -5.640779E-02
19   2.968750E+00     5.269056E-02   -3.715849E-02       2.865804E-03   -5.345644E-02
20   3.125000E+00     5.459101E-02   -3.389815E-02       2.587112E-03   -5.079782E-02
21   3.281250E+00     5.308759E-02   -2.878896E-02       2.347151E-03   -4.839052E-02
22   3.437500E+00     5.272136E-02   -2.761442E-02       2.139069E-03   -4.620058E-02
23   3.593750E+00     5.404744E-02   -2.319427E-02       1.957463E-03   -4.419990E-02
24   3.750000E+00     5.195590E-02   -2.027190E-02       1.798027E-03   -4.236501E-02
25   3.906250E+00     5.326983E-02   -1.875500E-02       1.657295E-03   -4.067614E-02
26   4.062500E+00     5.300422E-02   -1.416026E-02       1.532454E-03   -3.911657E-02
27   4.218750E+00     5.181054E-02   -1.301129E-02       1.421199E-03   -3.767200E-02
28   4.375000E+00     5.355986E-02   -1.010151E-02       1.321630E-03   -3.633019E-02
29   4.531250E+00     5.196795E-02   -6.621444E-03       1.232165E-03   -3.508057E-02
30   4.687500E+00     5.243218E-02   -5.874181E-03       1.151482E-03   -3.391396E-02
31   4.843750E+00     5.321569E-02   -1.801991E-03       1.078470E-03   -3.282236E-02
32   5.000000E+00     5.153637E-02    0.000000E+00       1.012186E-03   -3.179877E-02
33   5.156250E+00     5.321565E-02    1.801920E-03       9.518286E-04   -3.083703E-02
34   5.312500E+00     5.243213E-02    5.874145E-03       8.967115E-04   -2.993171E-02
35   5.468750E+00     5.196794E-02    6.621468E-03       8.462456E-04   -2.907799E-02
36   5.625000E+00     5.355985E-02    1.010152E-02       7.999220E-04   -2.827158E-02
37   5.781250E+00     5.181054E-02    1.301128E-02       7.572996E-04   -2.750866E-02
38   5.937500E+00     5.300420E-02    1.416027E-02       7.179943E-04   -2.678579E-02
39   6.093750E+00     5.326980E-02    1.875502E-02       6.816710E-04   -2.609993E-02
40   6.250000E+00     5.195590E-02    2.027191E-02       6.480353E-04   -2.544829E-02
41   6.406250E+00     5.404745E-02    2.319427E-02       6.168286E-04   -2.482837E-02
42   6.562500E+00     5.272136E-02    2.761441E-02       5.878225E-04   -2.423793E-02
     6.718750E+00     5.308758E-02    2.878896E-02       5.608150E-04   -2.367489E-02
44   6.875000E+00     5.459100E-02    3.389814E-02       5.356266E-04   -2.313741E-02
45   7.031250E+00     5.269055E-02    3.715850E-02       5.120976E-04   -2.262378E-02
46   7.187500E+00     5.492173E-02    3.987062E-02       4.900852E-04   -2.213244E-02
47   7.343750E+00     5.466274E-02    4.640236E-02       4.694621E-04   -2.166199E-02
48   7.500000E+00     5.397634E-02    4.883983E-02       4.501137E-04   -2.121111E-02
49   7.656250E+00     5.695152E-02    5.485361E-02       4.319370E-04   -2.077860E-02
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50   7.812500E+00      5.507480E-02      6.157780E-02      4.148394E-04   -2.036338E-02
51   7.968750E+00      5.713613E-02      6.533986E-02      3.987371E-04   -1.996442E-02
     8.125000E+00      5.911101E-02      7.558811E-02      3.835543E-04   -1.958079E-02
53   8.281250E+00      5.754644E-02      8.233652E-02      3.692224E-04   -1.921161E-02
54   8.437500E+00      6.283788E-02      9.138200E-02      3.556790E-04   -1.885610E-02
55   8.593750E+00      6.298567E-02      1.064619E-01      3.428672E-04   -1.851350E-02
56   8.750000E+00      6.558859E-02      1.165092E-01      3.307352E-04   -1.818312E-02
57   8.906250E+00      7.428613E-02      1.375700E-01      3.192360E-04   -1.786432E-02
58   9.062500E+00      7.592886E-02      1.615192E-01      3.083261E-04   -1.755651E-02
59   9.218750E+00      9.199629E-02      1.878410E-01      2.979660E-04   -1.725912E-02
60   9.375000E+00      1.137828E-01      2.401814E-01      2.881194E-04   -1.697163E-02
61   9.531250E+00      1.482937E-01      3.000644E-01      2.787529E-04   -1.669357E-02
62   9.687500E+00      2.641716E-01      3.980615E-01      2.698358E-04   -1.642446E-02
63   9.843750E+00      5.608323E-01      5.128146E-01      2.613398E-04   -1.616389E-02
FORTRAN STOP
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      COMPRISSION BETWEEN THE COMPUTED FFT AND THEORETICAL  FOURIER TRANSFORM
      -----------------------------------------------------------------------
    PROBLEM 7-31: DEFFERENCES BETWEEN COMPUTED RESULTS AND THEORETICAL RESULTS.
           FOR FFT 40 VALUES FROM EXP[-T], 20 VALUES FROM ZERO-PADDING.]
     ******************************************************************

                              DEFFRENCES IN                          DEFFERENCES IN
      FREQUENCY[HZ]       REAL PART       IMAGINARY            MAGNITUDE        PHASE[DEG]

 0   0.000000E+00      3.158653E-02     0.000000E+00        3.158653E-02     0.000000E+00
 1   1.562500E-01      5.162305E-02    -1.289928E-02        4.635286E-02     2.033115E+00
 2   3.125000E-01      5.821176E-02     6.339729E-03        2.391630E-02     6.580509E+00
 3   4.687500E-01      4.492878E-02     4.370540E-03        1.320416E-02     7.544968E+00
 4   6.250000E-01      5.288621E-02    -1.040727E-03        1.899743E-02     1.106207E+01
 5   7.812500E-01      5.214889E-02     7.759675E-03        9.540945E-03     1.457899E+01
 6   9.375000E-01      4.791598E-02     3.490448E-03        1.110536E-02     1.554290E+01
 7   1.093750E+00      5.355114E-02     4.925847E-03        1.234902E-02     2.008938E+01
 8   1.250000E+00      4.963584E-02     8.783557E-03        7.397875E-03     2.212120E+01
 9   1.406250E+00      5.033868E-02     5.283579E-03        1.123968E-02     2.415265E+01
10   1.562500E+00      5.256914E-02     9.431198E-03        9.567246E-03     2.869798E+01
11   1.718750E+00      4.904474E-02     9.475462E-03        8.248694E-03     2.965998E+01
12   1.875000E+00      5.195749E-02     8.687332E-03        1.137812E-02     3.317422E+01
13   2.031250E+00      5.103438E-02     1.253525E-02        8.683875E-03     3.668784E+01
14   2.187500E+00      4.980918E-02     1.079565E-02        1.004919E-02     3.764803E+01
15   2.343750E+00      5.236146E-02     1.274075E-02        1.132204E-02     4.219027E+01
16   2.500000E+00      4.993987E-02     1.456517E-02        9.259328E-03     4.421747E+01
17   2.656250E+00      5.108544E-02     1.330045E-02        1.189217E-02     4.624373E+01
     2.812500E+00      5.172970E-02     1.653719E-02        1.136994E-02     5.078338E+01
19   2.968750E+00      4.982476E-02     1.629796E-02        1.094197E-02     5.173912E+01
20   3.125000E+00      5.200390E-02     1.689967E-02        1.339569E-02     5.524648E+01
21   3.281250E+00      5.074044E-02     1.960155E-02        1.194380E-02     5.875259E+01
22   3.437500E+00      5.058229E-02     1.858616E-02        1.326545E-02     5.970457E+01
23   3.593750E+00      5.208998E-02     2.100563E-02        1.457089E-02     6.423793E+01
24   3.750000E+00      5.015787E-02     2.209311E-02        1.336749E-02     6.625539E+01
25   3.906250E+00      5.161253E-02     2.192114E-02        1.576511E-02     6.827107E+01
26   4.062500E+00      5.147176E-02     2.495630E-02        1.571653E-02     7.279919E+01
27   4.218750E+00      5.038935E-02     2.466071E-02        1.572054E-02     7.374238E+01
28   4.375000E+00      5.223823E-02     2.622868E-02        1.814989E-02     7.723611E+01
29   4.531250E+00      5.073578E-02     2.845913E-02        1.728588E-02     8.072739E+01
30   4.687500E+00      5.128070E-02     2.803977E-02        1.882671E-02     8.166314E+01
31   4.843750E+00      5.213722E-02     3.102037E-02        2.040612E-02     8.617883E+01
32   5.000000E+00      5.052418E-02     3.179877E-02        1.972150E-02     8.817702E+01
33   5.156250E+00      5.226382E-02     3.263895E-02        2.239444E-02     9.017157E+01
34   5.312500E+00      5.153542E-02     3.580586E-02        2.281501E-02     9.467659E+01
35   5.468750E+00      5.112169E-02     3.569946E-02        2.329777E-02     9.559440E+01
36   5.625000E+00      5.275993E-02     3.837310E-02        2.622122E-02     9.906017E+01
37   5.781250E+00      5.105324E-02     4.051993E-02        2.590026E-02     1.025205E+02
38   5.937500E+00      5.228621E-02     4.094607E-02        2.806767E-02     1.034222E+02
39   6.093750E+00      5.258813E-02     4.485495E-02        3.036614E-02     1.079001E+02
40   6.250000E+00      5.130786E-02     4.572020E-02        3.031411E-02     1.098560E+02
41   6.406250E+00      5.343062E-02     4.802264E-02        3.397808E-02     1.118035E+02
42   6.562500E+00      5.213354E-02     5.185234E-02        3.527047E-02     1.162556E+02
43   6.718750E+00      5.252676E-02     5.246386E-02        3.670966E-02     1.171139E+02
     6.875000E+00      5.405537E-02     5.703555E-02        4.111572E-02     1.205122E+02
45   7.031250E+00      5.217846E-02     5.978227E-02        4.184560E-02     1.238959E+02
46   7.187500E+00      5.443164E-02     6.200306E-02        4.573011E-02     1.247097E+02
47   7.343750E+00      5.419328E-02     6.806435E-02        5.003506E-02     1.290862E+02
48   7.500000E+00      5.352623E-02     7.005094E-02        5.157680E-02     1.309246E+02
49   7.656250E+00      5.651958E-02     7.563222E-02        5.828899E-02     1.327344E+02
50   7.812500E+00      5.465996E-02     8.194118E-02        6.224631E-02     1.370240E+02
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51   7.968750E+00    5.673739E-02    8.530428E-02    6.682926E-02    1.376882E+02
52   8.125000E+00    5.872745E-02    9.516890E-02    7.637208E-02    1.408522E+02
     8.281250E+00    5.717722E-02    1.015481E-01    8.123828E-02    1.439489E+02
54   8.437500E+00    6.248220E-02    1.102381E-01    9.204262E-02    1.444057E+02
55   8.593750E+00    6.264281E-02    1.249754E-01    1.051819E-01    1.483297E+02
56   8.750000E+00    6.525785E-02    1.346923E-01    1.155160E-01    1.495810E+02
57   8.906250E+00    7.396689E-02    1.554343E-01    1.384784E-01    1.506080E+02
58   9.062500E+00    7.562053E-02    1.790757E-01    1.609167E-01    1.538164E+02
59   9.218750E+00    9.169833E-02    2.051001E-01    1.918975E-01    1.529177E+02
60   9.375000E+00    1.134947E-01    2.571530E-01    2.487958E-01    1.536791E+02
61   9.531250E+00    1.480150E-01    3.167580E-01    3.180125E-01    1.527448E+02
62   9.687500E+00    2.639017E-01    4.144860E-01    4.613176E-01    1.454890E+02
63   9.843750E+00    5.605709E-01    5.289785E-01    7.437758E-01    1.315132E+02
FORTRAN STOP
```

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   COMPRISSION BETWEEN THE COMPUTED FFT AND THEORETICAL  FOURIER TRANSFORM
   ------------------------------------------------------------------------
 ROBLEM 7-31: FFT ALL 64 VALUES ARE FROM EXP[-T], THEORETICAL VALUES FROM
                                1/[1+jw]]
               ****************************************

                           COMPUTED  FFT                   THEORETICAL RESULTS
    FREQUENCY[HZ]      REAL PART       IMAGINARY         REAL PART      IMAGINARY

 0  0.000000E+00     9.990873E-01    0.000000E+00     1.000000E+00   0.000000E+00
 1  1.562500E-01     5.091125E-01   -4.982681E-01     5.092094E-01  -4.999152E-01
 2  3.125000E-01     2.063680E-01   -4.020956E-01     2.059599E-01  -4.044013E-01
 3  4.687500E-01     1.039456E-01   -3.014766E-01     1.033650E-01  -3.044350E-01
 4  6.250000E-01     6.155071E-02   -2.354696E-01     6.089667E-02  -2.391407E-01
 5  7.812500E-01     4.054010E-02   -1.911775E-01     3.984744E-02  -1.956007E-01
 6  9.375000E-01     2.872973E-02   -1.598089E-01     2.801291E-02  -1.650097E-01
 7  1.093750E+00     2.146930E-02   -1.364992E-01     2.073501E-02  -1.424959E-01
 8  1.250000E+00     1.670129E-02   -1.184858E-01     1.595277E-02  -1.252928E-01
 9  1.406250E+00     1.340824E-02   -1.041154E-01     1.264700E-02  -1.117455E-01
10  1.562500E+00     1.104206E-02   -9.234818E-02     1.026875E-02  -1.008132E-01
11  1.718750E+00     9.287140E-03   -8.250036E-02     8.501720E-03  -9.181198E-02
12  1.875000E+00     7.951373E-03   -7.410523E-02     7.153521E-03  -8.427543E-02
13  2.031250E+00     6.912719E-03   -6.683396E-02     6.101764E-03  -7.787511E-02
14  2.187500E+00     6.090441E-03   -6.044815E-02     5.265641E-03  -7.237343E-02
15  2.343750E+00     5.429696E-03   -5.477102E-02     4.590073E-03  -6.759441E-02
16  2.500000E+00     4.892043E-03   -4.966836E-02     4.036488E-03  -6.340501E-02
17  2.656250E+00     4.449936E-03   -4.503682E-02     3.577223E-03  -5.970282E-02
    2.812500E+00     4.083281E-03   -4.079494E-02     3.192028E-03  -5.640779E-02
19  2.968750E+00     3.777081E-03   -3.687787E-02     2.865804E-03  -5.345644E-02
20  3.125000E+00     3.520039E-03   -3.323314E-02     2.587112E-03  -5.079782E-02
21  3.281250E+00     3.303546E-03   -2.981764E-02     2.347151E-03  -4.839052E-02
22  3.437500E+00     3.120872E-03   -2.659559E-02     2.139069E-03  -4.620058E-02
23  3.593750E+00     2.966835E-03   -2.353689E-02     1.957463E-03  -4.419990E-02
24  3.750000E+00     2.837315E-03   -2.061582E-02     1.798027E-03  -4.236501E-02
25  3.906250E+00     2.729065E-03   -1.781027E-02     1.657295E-03  -4.067614E-02
26  4.062500E+00     2.639551E-03   -1.510088E-02     1.532454E-03  -3.911657E-02
27  4.218750E+00     2.566762E-03   -1.247037E-02     1.421199E-03  -3.767200E-02
28  4.375000E+00     2.509132E-03   -9.903348E-03     1.321630E-03  -3.633019E-02
29  4.531250E+00     2.465421E-03   -7.385600E-03     1.232165E-03  -3.508057E-02
30  4.687500E+00     2.434862E-03   -4.904020E-03     1.151482E-03  -3.391396E-02
31  4.843750E+00     2.416706E-03   -2.446151E-03     1.078470E-03  -3.282236E-02
32  5.000000E+00     2.410698E-03    0.000000E+00     1.012186E-03  -3.179877E-02
33  5.156250E+00     2.416730E-03    2.446103E-03     9.518286E-04  -3.083703E-02
34  5.312500E+00     2.434826E-03    4.903996E-03     8.967115E-04  -2.993171E-02
35  5.468750E+00     2.465427E-03    7.385588E-03     8.462456E-04  -2.907799E-02
36  5.625000E+00     2.509111E-03    9.903336E-03     7.999220E-04  -2.827158E-02
37  5.781250E+00     2.566749E-03    1.247038E-02     7.572996E-04  -2.750866E-02
38  5.937500E+00     2.639547E-03    1.510087E-02     7.179943E-04  -2.678579E-02
39  6.093750E+00     2.729050E-03    1.781029E-02     6.816710E-04  -2.609993E-02
40  6.250000E+00     2.837310E-03    2.061583E-02     6.480353E-04  -2.544829E-02
41  6.406250E+00     2.966838E-03    2.353689E-02     6.168286E-04  -2.482837E-02
42  6.562500E+00     3.120884E-03    2.659561E-02     5.878225E-04  -2.423793E-02
43  6.718750E+00     3.303534E-03    2.981764E-02     5.608150E-04  -2.367489E-02
    6.875000E+00     3.520036E-03    3.323314E-02     5.356266E-04  -2.313741E-02
45  7.031250E+00     3.777082E-03    3.687790E-02     5.120976E-04  -2.262378E-02
46  7.187500E+00     4.083286E-03    4.079494E-02     4.900852E-04  -2.213244E-02
47  7.343750E+00     4.449988E-03    4.503680E-02     4.694621E-04  -2.166199E-02
48  7.500000E+00     4.892036E-03    4.966836E-02     4.501137E-04  -2.121111E-02
49  7.656250E+00     5.429677E-03    5.477099E-02     4.319370E-04  -2.077860E-02
50  7.812500E+00     6.090421E-03    6.044815E-02     4.148394E-04  -2.036338E-02
```

```
51   7.968750E+00     6.912691E-03     6.683395E-02     3.987371E-04   -1.996442E-02
     8.125000E+00     7.951376E-03     7.410526E-02     3.835543E-04   -1.958079E-02
     8.281250E+00     9.287119E-03     8.250035E-02     3.692224E-04   -1.921161E-02
54   8.437500E+00     1.104204E-02     9.234820E-02     3.556790E-04   -1.885610E-02
55   8.593750E+00     1.340822E-02     1.041154E-01     3.428672E-04   -1.851350E-02
56   8.750000E+00     1.670127E-02     1.184857E-01     3.307352E-04   -1.818312E-02
57   8.906250E+00     2.146927E-02     1.364992E-01     3.192360E-04   -1.786432E-02
58   9.062500E+00     2.872971E-02     1.598089E-01     3.083261E-04   -1.755651E-02
59   9.218750E+00     4.054007E-02     1.911775E-01     2.979660E-04   -1.725912E-02
60   9.375000E+00     6.155067E-02     2.354696E-01     2.881194E-04   -1.697163E-02
61   9.531250E+00     1.039456E-01     3.014766E-01     2.787529E-04   -1.669357E-02
62   9.687500E+00     2.063680E-01     4.020956E-01     2.698358E-04   -1.642446E-02
63   9.843750E+00     5.091124E-01     4.982682E-01     2.613398E-04   -1.616389E-02
FORTRAN STOP
```

```
    COMPRISSION BETWEEN THE COMPUTED FFT AND THEORETICAL  FOURIER TRANSFORM
    ----------------------------------------------------------------------
 ROBLEM 7-31: DEFERENCES BETWEEN COMPUTED RESULTS AND THEORETICAL RESULTS.
                FOR FFT ALL 64 VALUES FROM EXP[-T]]
             **************************************************

                            DEFFERENCES IN                        DEFFERENCES   IN
     FREQUENCY[HZ]       REAL PART        IMAGINARY          MAGNITUDE        PHASE[DEG]

 0   0.000000E+00     -9.127259E-04     0.000000E+00      -9.127259E-04     0.000000E+00
 1   1.562500E-01     -9.685755E-05     1.647055E-03      -1.222074E-03     8.907318E-02
 2   3.125000E-01      4.081726E-04     2.305627E-03      -1.867145E-03     1.787605E-01
 3   4.687500E-01      5.806088E-04     2.958417E-03      -2.611160E-03     2.696686E-01
 4   6.250000E-01      6.540343E-04     3.671110E-03      -3.391296E-03     3.624802E-01
 5   7.812500E-01      6.926544E-04     4.423216E-03      -4.189670E-03     4.578552E-01
 6   9.375000E-01      7.168166E-04     5.200803E-03      -4.999802E-03     5.565338E-01
 7   1.093750E+00      7.342882E-04     5.996630E-03      -5.819231E-03     6.593704E-01
 8   1.250000E+00      7.485207E-04     6.807014E-03      -6.647237E-03     7.672501E-01
 9   1.406250E+00      7.612351E-04     7.630080E-03      -7.483661E-03     8.812180E-01
10   1.562500E+00      7.733172E-04     8.465014E-03      -8.328848E-03     1.002441E+00
11   1.718750E+00      7.854206E-04     9.311624E-03      -9.183325E-03     1.132347E+00
12   1.875000E+00      7.978524E-04     1.017020E-02      -1.004789E-02     1.272530E+00
13   2.031250E+00      8.109547E-04     1.104115E-02      -1.092329E-02     1.425018E+00
14   2.187500E+00      8.248002E-04     1.192528E-02      -1.181054E-02     1.592102E+00
15   2.343750E+00      8.396232E-04     1.282338E-02      -1.271058E-02     1.776718E+00
16   2.500000E+00      8.555553E-04     1.373665E-02      -1.362467E-02     1.982521E+00
17   2.656250E+00      8.727130E-04     1.466600E-02      -1.455376E-02     2.213989E+00
     2.812500E+00      8.912529E-04     1.561286E-02      -1.549925E-02     2.477043E+00
19   2.968750E+00      9.112766E-04     1.657857E-02      -1.646242E-02     2.779228E+00
20   3.125000E+00      9.329270E-04     1.756468E-02      -1.744461E-02     3.130669E+00
21   3.281250E+00      9.563947E-04     1.857288E-02      -1.844732E-02     3.545197E+00
22   3.437500E+00      9.818031E-04     1.960499E-02      -1.947199E-02     4.041916E+00
23   3.593750E+00      1.009373E-03     2.066302E-02      -2.052009E-02     4.648499E+00
24   3.750000E+00      1.039288E-03     2.174919E-02      -2.159300E-02     5.406044E+00
25   3.906250E+00      1.071770E-03     2.286587E-02      -2.269174E-02     6.378525E+00
26   4.062500E+00      1.107097E-03     2.401569E-02      -2.381674E-02     7.671333E+00
27   4.218750E+00      1.145563E-03     2.520163E-02      -2.496701E-02     9.470215E+00
28   4.375000E+00      1.187502E-03     2.642685E-02      -2.613796E-02     1.213402E+01
29   4.531250E+00      1.233256E-03     2.769497E-02      -2.731597E-02     1.644818E+01
30   4.687500E+00      1.283379E-03     2.900993E-02      -2.845828E-02     2.445998E+01
31   4.843750E+00      1.338236E-03     3.037621E-02      -2.940145E-02     4.277123E+01
32   5.000000E+00      1.398512E-03     3.179877E-02      -2.940417E-02     8.817702E+01
33   5.156250E+00      1.464901E-03     3.328313E-02      -2.741311E-02     1.335784E+02
34   5.312500E+00      1.538115E-03     3.483571E-02      -2.446997E-02     1.518800E+02
35   5.468750E+00      1.619181E-03     3.646358E-02      -2.130408E-02     1.598735E+02
36   5.625000E+00      1.709189E-03     3.817492E-02      -1.806665E-02     1.641623E+02
37   5.781250E+00      1.809449E-03     3.997903E-02      -1.478729E-02     1.667928E+02
38   5.937500E+00      1.921553E-03     4.188666E-02      -1.146560E-02     1.685501E+02
39   6.093750E+00      2.047379E-03     4.391022E-02      -8.090675E-03     1.697926E+02
40   6.250000E+00      2.189275E-03     4.606412E-02      -4.646381E-03     1.707054E+02
41   6.406250E+00      2.350010E-03     4.836527E-02      -1.112901E-03     1.713929E+02
42   6.562500E+00      2.533062E-03     5.083353E-02       2.533039E-03     1.719183E+02
43   6.718750E+00      2.742719E-03     5.349253E-02       6.318552E-03     1.723213E+02
     6.875000E+00      2.984409E-03     5.637055E-02       1.027543E-02     1.726280E+02
45   7.031250E+00      3.264985E-03     5.950168E-02       1.444125E-02     1.728558E+02
46   7.187500E+00      3.593200E-03     6.292738E-02       1.886092E-02     1.730160E+02
47   7.343750E+00      3.980526E-03     6.669878E-02       2.358904E-02     1.731159E+02
48   7.500000E+00      4.441922E-03     7.087947E-02       2.869281E-02     1.731595E+02
49   7.656250E+00      4.997740E-03     7.554960E-02       3.425638E-02     1.731480E+02
50   7.812500E+00      5.675582E-03     8.081153E-02       4.038659E-02     1.730799E+02
```

```
51   7.968750E+00     6.513954E-03     8.679837E-02     4.722209E-02    1.729510E+02
     8.125000E+00     7.567822E-03     9.368604E-02     5.494608E-02    1.727539E+02
     8.281250E+00     8.917896E-03     1.017120E-01     6.380627E-02    1.724766E+02
54   8.437500E+00     1.068636E-02     1.112043E-01     7.414655E-02    1.721013E+02
55   8.593750E+00     1.306536E-02     1.226289E-01     8.645857E-02    1.716011E+02
56   8.750000E+00     1.637054E-02     1.366689E-01     1.014709E-01    1.709350E+02
57   8.906250E+00     2.115003E-02     1.543635E-01     1.203101E-01    1.700381E+02
58   9.062500E+00     2.842138E-02     1.773654E-01     1.448116E-01    1.688027E+02
59   9.218750E+00     4.024211E-02     2.084366E-01     1.781669E-01    1.670388E+02
60   9.375000E+00     6.126256E-02     2.524412E-01     2.264071E-01    1.643787E+02
61   9.531250E+00     1.036668E-01     3.181701E-01     3.021972E-01    1.600201E+02
62   9.687500E+00     2.060981E-01     4.185201E-01     4.355342E-01    1.518908E+02
63   9.843750E+00     5.088511E-01     5.144321E-01     6.962009E-01    1.334572E+02
FORTRAN STOP
```

7-34. *Use the FFT algorithm to evaluate the complex Fourier-series coeffients for the periodic signal shown in Fig. P7-34. Sample the waveform for one cycle at $T = 0.1$. Compare the results with the theoreticall values.*

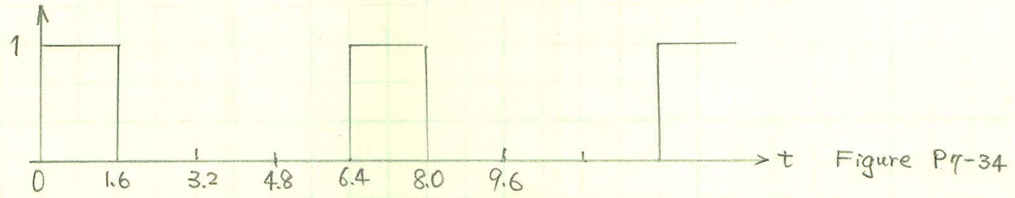


Figure P7-34

Solution: From the properties of Fourier-series and Fourier transform,

$$F_n = \frac{1}{\text{PERIOD}} \int_0^{\text{PERIOD}} f(t) e^{-j\omega_o n t} \, dt \qquad n \neq 0$$

$$\omega_o = \frac{2\pi}{\text{PERIOD}} = \frac{2\pi}{6.4}$$

$$= \frac{1}{6.4} \int_0^{6.4} f(t) e^{-j\omega_o n t} \, dt$$

$$= 0.15625 \int_0^{1.6} e^{-j\omega_o n t} \, dt = 0.15625 \cdot j \left. \frac{e^{-jn\omega_o t}}{n\omega_o} \right|_0^{1.6}$$

$$= 0.15625 \cdot j \left(e^{-jn \cdot \frac{2\pi}{6.4} \times 1.6} - 1 \right) / \left(n \cdot \frac{2\pi}{6.4} \right)$$

$$= \frac{1}{2n\pi} \sin(0.5n\pi) + j \frac{1}{2n\pi} [\cos(0.5n\pi) - 1] \qquad n \neq 0$$

$$F_0 = 0.25$$

And

$$F_n = \frac{1}{\text{PERIOD}} F(\omega) \Big|_{\omega = n\omega_o} = \frac{1}{6.4} F(\omega) \Big|_{\omega = 0.15625 n \cdot 2\pi}$$

$$= 0.15625 \, F(0.15625 n \cdot 2\pi)$$

If we compute the DFT of the signal with $T=0.1$, we have

$$X_N \doteq F(N \cdot \Delta\omega) \doteq \sum_{k=0}^{63} x_k e^{-j(\frac{2\pi}{64}) \cdot N \cdot k} \qquad (N=0,1,\cdots,63)$$

$$\Delta\omega = \frac{2\pi}{64 \times 0.1} = 0.15625 \cdot 2\pi$$

So that.

$$F_n = 0.15625 \cdot X_n \quad , \qquad n = 0, 1, \cdots, 63$$

The following program and data are based on this fact.

```
C A PROGRAM FOR PROBLEM 7-34, HOMEWORK NO:SIX, BEN M. CHEN
      COMPLEX F[64],CMPLX,BENC,CHEN
      N=64
      DT=.1
      DO 5 I=1,16
5     F[I]=CMPLX[1.,0.]
      DO 10 I=17,64
10    F[1]=CMPLX[0.,0.]
      DF=1./[N*DT]
      CALL FFT[1,N,DT,F,BAD]
      WRITE [6,107]
107   FORMAT['1',7X,'COMPRISSION BETWEEN THE COMPUTED FFT AND THEORETICAL
     1 FOURIER TRANSFORM'//5X, 'FREQUENCY[HZ]'/]
      DO 20 I=1,N
      F[I]=0.15625*F[I]
      J=I-1
      DFN=J*DF
      DPI=2*3.14159265*DFN
      IF [J .EQ. 0] GO TO 3
      AA=J*3.14159265
      BENC=CMPLX[SIN[.5*AA]/[2*AA],[COS[.5*AA]-1]/[2*AA]]
      GO TO 4
 3    BENC=[0.25,0]
 4    CHEN=F[I]-BENC
      AMAGF=CABS[F[I]]
      FRF=REAL[F[I]]
      FIF=AIMAG[F[I]]
      IF [FRF .EQ. 0. .AND. FIF .EQ. 0.] GO TO 12
      ANGF=57.2959*ATAN2[FIF,FRF]
      GO TO 13
 12   ANGF=0.
 13   FRB=REAL[BENC]
      FIB=AIMAG[BENC]
      AMAGB=CABS[BENC]
      IF [FRB .EQ. 0. .AND. FIB .EQ. 0.] GO TO 14
      ANGB=57.2959*ATAN2[FIB,FRB]
      GO TO 15
 14   ANGB=0.
 15   DMAG=AMAGF-AMAGB
      DPHS=ANGF-ANGB
 20   WRITE [6,108] J,DFN,CHEN,DMAG,DPHS
 108  FORMAT[I3,X,1PE13.6,3X,1PE13.6,3X,1PE13.6,3X,1PE13.6,2X,1PE
     1 13.6]
      STOP
      END
```

```
  COMPRISSION BETWEEN THE COMPUTED FFT AND THEORETICAL  FOURIER TRANSFORM
  ------------------------------------------------------------------------
 [PROBLEM 7-34: FOURIER SERIERS COFFIENTS FROM FFT & THEORETICAL RESULTS.]
                 ****************************************

                            FROM FFT RESULTS                   THEORETICAL RESULTS
  FREQUENCY[HZ]         REAL PART        IMAGINARY            REAL PART     IMAGINARY

0  0.000000E+00      2.343750E-01     0.000000E+00        2.500000E-01   0.000000E+00
1  1.562500E-01      1.512146E-01    -1.512146E-01        1.591549E-01  -1.591549E-01
2  3.125000E-01      1.421817E-08    -1.586433E-01       -6.956884E-09  -1.591549E-01
3  4.687500E-01     -6.048010E-02    -6.048010E-02       -5.305165E-02  -5.305165E-02
4  6.250000E-01     -1.562500E-02     0.000000E+00        6.956884E-09   0.000000E+00
5  7.812500E-01      2.337675E-02    -2.337675E-02        3.183099E-02  -3.183100E-02
6  9.375000E-01      2.173292E-10    -5.150872E-02       -6.326346E-10  -5.305165E-02
7  1.093750E+00     -2.964698E-02    -2.964699E-02       -2.273642E-02  -2.273640E-02
8  1.250000E+00     -1.562500E-02     0.000000E+00        6.956884E-09   0.000000E+00
9  1.406250E+00      8.705649E-03    -8.705642E-03        1.768388E-02  -1.768388E-02
10 1.562500E+00      3.507961E-09    -2.923232E-02       -1.075143E-08  -3.183099E-02
11 1.718750E+00     -2.084687E-02    -2.084687E-02       -1.446863E-02  -1.446861E-02
12 1.875000E+00     -1.562500E-02     0.000000E+00        6.326346E-10   0.000000E+00
13 2.031250E+00      2.721442E-03    -2.721434E-03        1.224269E-02  -1.224270E-02
14 2.187500E+00      3.709789E-09    -1.903912E-02       -1.508806E-08  -2.273642E-02
15 2.343750E+00     -1.643226E-02    -1.643229E-02       -1.061033E-02  -1.061033E-02
16 2.500000E+00     -1.562500E-02     0.000000E+00        6.956884E-09   0.000000E+00
17 2.656250E+00     -7.316629E-04     7.316666E-04        9.362055E-03  -9.362067E-03
18 2.812500E+00      5.137776E-09    -1.282310E-02       -6.326346E-10  -1.768388E-02
   2.968750E+00     -1.360664E-02    -1.360665E-02       -8.376576E-03  -8.376569E-03
20 3.125000E+00     -1.562500E-02     0.000000E+00        1.075143E-08   0.000000E+00
21 3.281250E+00     -3.129866E-03     3.129870E-03        7.578806E-03  -7.578822E-03
22 3.437500E+00      1.148652E-09    -8.351736E-03       -1.903045E-08  -1.446863E-02
23 3.593750E+00     -1.150754E-02    -1.150754E-02       -6.919780E-03  -6.919784E-03
24 3.750000E+00     -1.562500E-02     0.000000E+00        6.326346E-10   0.000000E+00
25 3.906250E+00     -5.017141E-03     5.017148E-03        6.366197E-03  -6.366202E-03
26 4.062500E+00      4.439284E-09    -4.739791E-03       -8.416325E-09  -1.224269E-02
27 4.218750E+00     -9.769429E-03    -9.769432E-03       -5.894627E-03  -5.894615E-03
28 4.375000E+00     -1.562500E-02     0.000000E+00        1.508806E-08   0.000000E+00
29 4.531250E+00     -6.653626E-03     6.653626E-03        5.488101E-03  -5.488099E-03
30 4.687500E+00      1.930944E-08    -1.538925E-03       -6.326346E-10  -1.061033E-02
31 4.843750E+00     -8.196294E-03    -8.196317E-03       -5.134030E-03  -5.134027E-03
32 5.000000E+00     -1.562500E-02     0.000000E+00        6.956884E-09   0.000000E+00
33 5.156250E+00     -8.196309E-03     8.196302E-03        4.822877E-03  -4.822887E-03
34 5.312500E+00     -6.829901E-10     1.538932E-03       -1.253710E-08  -9.362055E-03
35 5.468750E+00     -6.653620E-03    -6.653626E-03       -4.547284E-03  -4.547269E-03
36 5.625000E+00     -1.562500E-02     0.000000E+00        6.326346E-10   0.000000E+00
37 5.781250E+00     -9.769432E-03     9.769432E-03        4.301485E-03  -4.301488E-03
38 5.937500E+00      2.173292E-10     4.739791E-03       -5.958318E-09  -8.376576E-03
39 6.093750E+00     -5.017142E-03    -5.017142E-03       -4.080896E-03  -4.080887E-03
40 6.250000E+00     -1.562500E-02     0.000000E+00        1.075143E-08   0.000000E+00
41 6.406250E+00     -1.150754E-02     1.150754E-02        3.881828E-03  -3.881826E-03
42 6.562500E+00     -2.173292E-10     8.351739E-03       -1.508806E-08  -7.578806E-03
43 6.718750E+00     -3.129868E-03    -3.129867E-03       -3.701278E-03  -3.701275E-03
   6.875000E+00     -1.562500E-02     0.000000E+00        1.903045E-08   0.000000E+00
   7.031250E+00     -1.360665E-02     1.360665E-02        3.536776E-03  -3.536784E-03
46 7.187500E+00      1.381482E-09     1.282311E-02        3.766843E-09  -6.919780E-03
47 7.343750E+00     -7.316461E-04    -7.316610E-04       -3.386275E-03  -3.386264E-03
48 7.500000E+00     -1.562500E-02     0.000000E+00        6.326346E-10   0.000000E+00
49 7.656250E+00     -1.643227E-02     1.643227E-02        3.248060E-03  -3.248075E-03
50 7.812500E+00     -6.503757E-09     1.903912E-02       -4.680154E-09  -6.366197E-03
51 7.968750E+00      2.721436E-03     2.721438E-03       -3.120685E-03  -3.120667E-03
```

```
52   8.125000E+00    -1.562500E-02     0.000000E+00     8.416325E-09     0.000000E+00
     8.281250E+00    -2.084687E-02     2.084687E-02     3.002923E-03    -3.002922E-03
     8.437500E+00    -2.576639E-09     2.923232E-02    -1.187574E-08    -5.894627E-03
55   8.593750E+00     8.705645E-03     8.705648E-03    -2.893726E-03    -2.893724E-03
56   8.750000E+00    -1.562500E-02     0.000000E+00     1.508806E-08     0.000000E+00
57   8.906250E+00    -2.964699E-02     2.964698E-02     2.792192E-03    -2.792198E-03
58   9.062500E+00    -1.046188E-08     5.150873E-02     2.856607E-09    -5.488101E-03
59   9.218750E+00     2.337674E-02     2.337675E-02    -2.697541E-03    -2.697532E-03
60   9.375000E+00    -1.562500E-02     0.000000E+00     6.326346E-10     0.000000E+00
61   9.531250E+00    -6.048011E-02     6.048008E-02     2.609097E-03    -2.609110E-03
62   9.687500E+00    -3.284462E-08     1.586433E-01    -3.896763E-09    -5.134030E-03
63   9.843750E+00     1.512146E-01     1.512146E-01    -2.526269E-03    -2.526254E-03
FORTRAN STOP
```

```
  COMPRISSION BETWEEN THE COMPUTED FFT AND THEORETICAL  FOURIER TRANSFORM
 ---------------------------------------------------------------------------
[ PROBLEM 7-34: DEFFERENCES OF RESULTS FROM FFT AND THEORETICAL METHOD. ]
                 *********************************

                            DEFFERENCES IN                        DEFFERENCES IN
     FREQUENCY[HZ]       REAL PART        IMAGINARY             MAGNITUDE        PHASE[DEG]

 0   0.000000E+00     -1.562500E-02     0.000000E+00      -1.562500E-02      0.000000E+00
 1   1.562500E-01     -7.940322E-03     7.940337E-03      -1.122932E-02      3.814697E-06
 2   3.125000E-01      2.117506E-08     5.116165E-04      -5.116165E-04      1.525879E-05
 3   4.687500E-01     -7.428456E-03    -7.428456E-03       1.050542E-02      0.000000E+00
 4   6.250000E-01     -1.562501E-02     0.000000E+00       1.562499E-02      1.800004E+02
 5   7.812500E-01     -8.454235E-03     8.454248E-03      -1.195610E-02      1.144409E-05
 6   9.375000E-01      8.499637E-10     1.542922E-03      -1.542922E-03      0.000000E+00
 7   1.093750E+00     -6.910557E-03    -6.910581E-03       9.773023E-03      3.051758E-05
 8   1.250000E+00     -1.562501E-02     0.000000E+00       1.562499E-02      1.800004E+02
 9   1.406250E+00     -8.978234E-03     8.978240E-03      -1.269714E-02      2.288818E-05
10   1.562500E+00      1.425939E-08     2.598668E-03      -2.598668E-03      2.288818E-05
11   1.718750E+00     -6.378240E-03    -6.378260E-03       9.020211E-03      4.577637E-05
12   1.875000E+00     -1.562500E-02     0.000000E+00       1.562500E-02      1.800004E+02
13   2.031250E+00     -9.521246E-03     9.521263E-03      -1.346509E-02      1.068115E-04
14   2.187500E+00      1.879785E-08     3.697298E-03      -3.697298E-03      5.340576E-05
15   2.343750E+00     -5.821935E-03    -5.821958E-03       8.233475E-03      3.051758E-05
16   2.500000E+00     -1.562501E-02     0.000000E+00       1.562499E-02      1.800004E+02
17   2.656250E+00     -1.009372E-02     1.009373E-02      -1.220522E-02      1.800003E+02
18   2.812500E+00      5.770411E-09     4.860778E-03      -4.860778E-03      3.814697E-05
     2.968750E+00     -5.230068E-03    -5.230076E-03       7.396439E-03      3.051758E-05
20   3.125000E+00     -1.562501E-02     0.000000E+00       1.562499E-02      1.800004E+02
21   3.281250E+00     -1.070867E-02     1.070869E-02      -6.291760E-03      1.800004E+02
22   3.437500E+00      2.017910E-08     6.116894E-03      -6.116894E-03      9.155273E-05
23   3.593750E+00     -4.587757E-03    -4.587755E-03       6.488067E-03     -1.525879E-05
24   3.750000E+00     -1.562500E-02     0.000000E+00       1.562500E-02      1.800004E+02
25   3.906250E+00     -1.138334E-02     1.138335E-02      -1.907852E-03      1.800004E+02
26   4.062500E+00      1.285561E-08     7.502897E-03      -7.502897E-03      9.155273E-05
27   4.218750E+00     -3.874801E-03    -3.874817E-03       5.479807E-03      7.629395E-05
28   4.375000E+00     -1.562501E-02     0.000000E+00       1.562499E-02      1.800004E+02
29   4.531250E+00     -1.214173E-02     1.214173E-02       1.648303E-03      1.800004E+02
30   4.687500E+00      1.994208E-08     9.071405E-03      -9.071405E-03      7.324219E-04
31   4.843750E+00     -3.062264E-03    -3.062290E-03       4.330714E-03      9.155273E-05
32   5.000000E+00     -1.562501E-02     0.000000E+00       1.562499E-02      1.800004E+02
33   5.156250E+00     -1.301919E-02     1.301919E-02       4.770742E-03      1.800005E+02
34   5.312500E+00      1.185411E-08     1.090099E-02      -7.823123E-03      1.800005E+02
35   5.468750E+00     -2.106336E-03    -2.106357E-03       2.978824E-03      1.373291E-04
36   5.625000E+00     -1.562500E-02     0.000000E+00       1.562500E-02      1.800004E+02
37   5.781250E+00     -1.407092E-02     1.407092E-02       7.732843E-03      1.800004E+02
38   5.937500E+00      6.175647E-09     1.311637E-02      -3.636785E-03      1.800004E+02
39   6.093750E+00     -9.362460E-04    -9.362544E-04       1.324058E-03      7.629395E-05
40   6.250000E+00     -1.562501E-02     0.000000E+00       1.562499E-02      1.800004E+02
41   6.406250E+00     -1.538937E-02     1.538937E-02       1.078438E-02      1.800004E+02
42   6.562500E+00      1.487073E-08     1.593055E-02       7.729330E-04      1.800005E+02
43   6.718750E+00      5.714099E-04     5.714076E-04      -8.080946E-04      1.525879E-05
     6.875000E+00     -1.562502E-02     0.000000E+00       1.562498E-02      1.800004E+02
     7.031250E+00     -1.714343E-02     1.714343E-02       1.424095E-02      1.800004E+02
46   7.187500E+00     -2.385361E-09     1.974289E-02       5.903329E-03      1.800003E+02
47   7.343750E+00      2.654629E-03     2.654603E-03      -3.754194E-03      6.713867E-04
48   7.500000E+00     -1.562500E-02     0.000000E+00       1.562500E-02      1.800004E+02
49   7.656250E+00     -1.968033E-02     1.968034E-02       1.864528E-02      1.800005E+02
50   7.812500E+00     -1.823603E-09     2.540532E-02       1.267293E-02      1.800004E+02
51   7.968750E+00      5.842121E-03     5.842105E-03      -5.646087E-04      1.800006E+02
```

```
52   8.125000E+00    -1.562501E-02    0.000000E+00    1.562499E-02    1.800004E+02
     8.281250E+00    -2.384979E-02    2.384979E-02    2.523516E-02    1.800004E+02
54   8.437500E+00     9.299105E-09    3.512695E-02    2.333769E-02    1.800005E+02
55   8.593750E+00     1.159937E-02    1.159937E-02    8.219298E-03    1.800004E+02
56   8.750000E+00    -1.562501E-02    0.000000E+00    1.562499E-02    1.800004E+02
57   8.906250E+00    -3.243918E-02    3.243917E-02    3.797840E-02    1.800004E+02
58   9.062500E+00    -1.331848E-08    5.699683E-02    4.602063E-02    1.800004E+02
59   9.218750E+00     2.607429E-02    2.607429E-02    2.924482E-02    1.800005E+02
60   9.375000E+00    -1.562500E-02    0.000000E+00    1.562500E-02    1.800004E+02
61   9.531250E+00    -6.308921E-02    6.308919E-02    8.184194E-02    1.800005E+02
62   9.687500E+00    -2.894786E-08    1.637774E-01    1.535093E-01    1.800004E+02
63   9.843750E+00     1.537408E-01    1.537409E-01    2.102771E-01    1.800006E+02
FORTRAN STOP
```

$\frac{50}{50}$

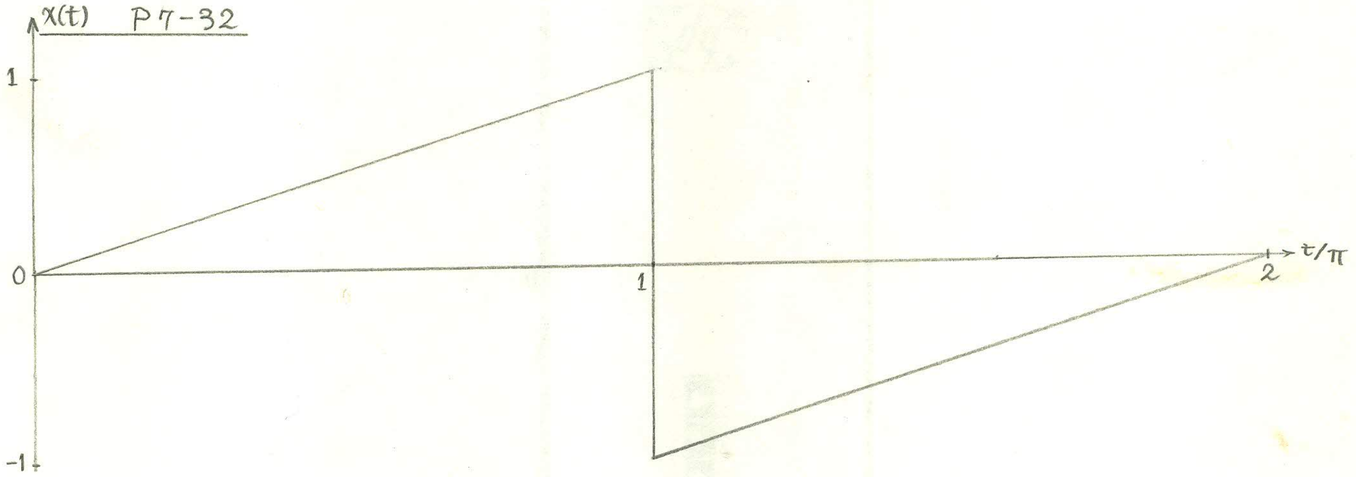
HOMEWORK ASSIGNMENT NO:7 DISCRETE SIGNAL & SYSTEM BEN M. CHEN

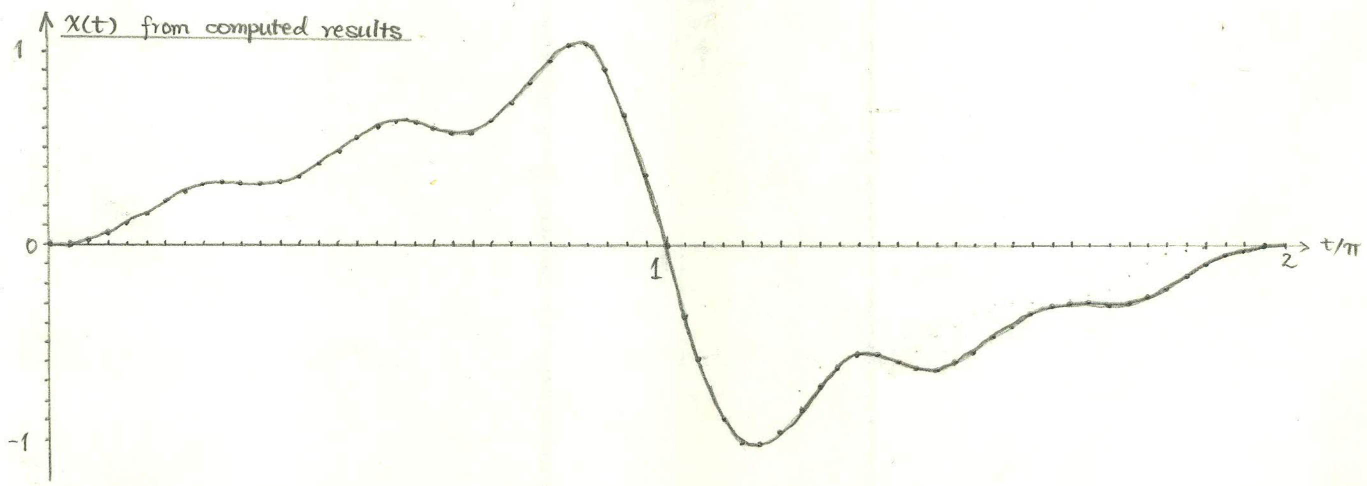
PROBLEM 7-31: It is known from Fourier series analysis that the following sawtooth waveform has the infinite-series representation

$$x[t] = \frac{2}{\pi} [\sin t - \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t - \frac{1}{4} \sin 4t + \ldots]$$

Write a computer program to generate the spectrum of the signal by using the first six frequency components of the above series, and call the inverse FFT subroutine to output the time samples. Plot the signal from the program output and compare with the sawtooth waveforms shown in Fig. P7-32. Use the 64-point FFT.

SOLUTION: PROGRAM AND DATA ARE IN FOLLOWING PAGES.





HOMEWORK ASSIGNMENT NO:7   DISCRETE SIGNAL & SYSTEM   BEN M. CHEN

```
C A PROGRAM FOR PROBLEM 7-32, HOMEWORK NO:7, BEN M. CHEN
        COMPLEX F[64],CMPLX,BEN[64]
        N=64
        DT=2*3.14159265/64.
        DO 5 I=1,64
        TD=[I-1]*DT
        XT=SIN[TD]-.5*SIN[2*TD]+SIN[3*TD]/3-SIN[4*TD]/4+SIN[5*TD]/5
        XT=XT-SIN[6*TD]/6
        XT=2*XT/3.1415926535
5       F[I]=CMPLX[XT,0.]
        DF=1./[N*DT]
        CALL FFT[1,N,DT,F,BAD]
        DO 10 I=1,N
10      BEN[I]=F[I]
        CALL FFT[-1,N,DF,BEN,BAD]
        DO 20 I=1,N
        J=I-1
        DFN=J*DF
        DTN=J*DT*3.14159265
20      WRITE [6,108] J,DTN,BEN[I]
108     FORMAT[5X,I6,5X,1PE13.6,10X,1PE13.6,7X,1PE13.6]
        STOP
        END
```

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HOMEWORK ASSIGNMENT NO:7       DISCRETE SIGNAL & SYSTEM          BEN M. CHEN
~~~~~~~~~~~~~~~~~~~~~~~~       ************************          ~~~~~~~~~~~~~

                         THE SPECTRUM OF THE SIGNAL
                         --------------------------

    POINT     FREQUENCY(HZ)                REAL   PART          IMAGINARY PART

      0       0.000000E+00              -2.582967E-08            0.000000E+00
      1       1.591549E-01               3.382956E-07           -2.000000E+00
      2       3.183099E-01              -2.661881E-07            1.000000E+00
      3       4.774648E-01               2.692726E-07           -6.666667E-01
      4       6.366197E-01              -1.640037E-07            5.000002E-01
      5       7.957747E-01               2.534464E-07           -4.000000E-01
      6       9.549296E-01              -3.033443E-07            3.333335E-01
      7       1.114085E+00              -5.832263E-08           -2.083049E-07
      8       1.273239E+00              -3.781937E-09            8.077240E-08
      9       1.432394E+00               2.871972E-08           -5.304951E-08
     10       1.591549E+00               3.712337E-08            8.744792E-09
     11       1.750704E+00              -8.937992E-08           -6.923726E-08
     12       1.909859E+00              -4.942418E-09            5.670447E-08
     13       2.069014E+00              -6.408752E-08           -1.320038E-07
     14       2.228169E+00               2.430015E-08            1.265128E-08
     15       2.387324E+00               2.097249E-07           -1.996847E-07
     16       2.546479E+00               2.390954E-08            5.559089E-08
     17       2.705634E+00              -5.331564E-08            5.599040E-08
     18       2.864789E+00               6.150560E-08            2.340669E-08
     19       3.023944E+00              -7.886977E-08            3.033120E-08
     20       3.183099E+00              -2.964404E-08           -9.185121E-08
     21       3.342254E+00               2.108536E-08            3.026263E-08
     22       3.501409E+00               5.686978E-08           -2.140283E-09
     23       3.660563E+00               1.196495E-08            2.368867E-08
     24       3.819719E+00              -2.447071E-08           -1.870604E-08
     25       3.978873E+00               3.455064E-08           -4.474457E-08
     26       4.138028E+00               3.517307E-08            3.511003E-08
     27       4.297184E+00               2.350241E-08            0.000000E+00
     28       4.456338E+00              -5.735414E-08            9.362676E-08
     29       4.615493E+00               4.750248E-09           -2.340669E-08
     30       4.774648E+00              -1.014616E-07            4.681338E-08
     31       4.933803E+00               1.229808E-07           -9.362676E-08
     32       5.092958E+00              -3.168134E-08            0.000000E+00
     33       5.252113E+00              -7.132149E-08            0.000000E+00
     34       5.411268E+00               1.469222E-08           -9.362676E-08
     35       5.570423E+00               9.571827E-11            4.681338E-08
     36       5.729578E+00              -5.867360E-08           -1.170335E-07
     37       5.888732E+00              -4.027255E-09            0.000000E+00
     38       6.047888E+00               7.116275E-08           -5.851673E-08
     39       6.207043E+00               5.626846E-08            3.847354E-08
     40       6.366198E+00              -2.447072E-08            1.870603E-08
     41       6.525352E+00              -1.954412E-08           -7.086801E-08
     42       6.684507E+00               5.662946E-08           -2.044814E-08
     43       6.843662E+00               1.935128E-08           -2.438950E-08
     44       7.002817E+00              -1.613911E-08            8.373567E-08
     45       7.161972E+00              -6.685215E-08            3.837705E-08
     46       7.321127E+00               7.706365E-08           -1.265128E-08
     47       7.480282E+00               9.475350E-08            1.243113E-08
     48       7.639437E+00               2.390955E-08           -5.559089E-08
     49       7.798592E+00               1.330362E-07            3.763636E-08
     50       7.957747E+00               6.150560E-08            2.340669E-08
     51       8.116901E+00              -9.648898E-08            1.335156E-07
```

```
        52        8.276056E+00            -5.779737E-09         -9.540231E-08
        53        8.435211E+00            -4.084322E-08          6.336413E-08
        54        8.594367E+00             5.034568E-08         -3.296975E-08
        55        8.753522E+00             1.387875E-08          1.461427E-07
        56        8.912677E+00            -3.781923E-09         -8.077239E-08
        57        9.071832E+00            -7.153863E-08          1.686621E-07
        58        9.230987E+00            -2.105972E-07         -3.333336E-01
        59        9.390141E+00             1.873492E-07          4.000000E-01
        60        9.549296E+00            -1.041675E-07         -5.000001E-01
        61        9.708451E+00             1.217837E-07          6.666669E-01
        62        9.867606E+00             1.557186E-08         -1.000000E+00
        63        1.002676E+01            -4.036697E-07          2.000000E+00
FORTRAN STOP
```

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HOMEWORK ASSIGNMENT NO:7    DISCRETE SIGNAL & SYSTEM     BEN M. CHEN
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                  TIME SAMPLES FROM INVERSE FFT
                  -----------------------------

  POINT      TIME/PI[SEC]               REAL   PART         IMAGINARY PART

     0       0.000000E+00              1.809382E-14         -1.422956E-08
     1       3.084252E-01              2.067119E-03          1.778695E-08
     2       6.168503E-01              1.558186E-02          8.537736E-08
     3       9.252754E-01              4.760164E-02          7.589099E-08
     4       1.233701E+00              9.804881E-02          2.845912E-08
     5       1.542126E+00              1.596540E-01         -1.897275E-08
     6       1.850551E+00              2.206402E-01          1.517820E-07
     7       2.158976E+00              2.691787E-01          6.403302E-08
     8       2.467401E+00              2.979728E-01         -9.012055E-08
     9       2.775826E+00              3.072399E-01         -3.794549E-08
    10       3.084252E+00              3.049155E-01          4.268868E-08
    11       3.392677E+00              3.039011E-01          2.845912E-08
    12       3.701102E+00              3.172768E-01         -3.794550E-08
    13       4.009527E+00              3.531967E-01         -8.537736E-08
    14       4.317952E+00              4.113825E-01          7.589099E-08
    15       4.626377E+00              4.826340E-01         -2.371591E-09
    16       4.934803E+00              5.517371E-01          2.371593E-08
    17       5.243228E+00              6.029161E-01          2.134434E-08
    18       5.551652E+00              6.259806E-01          6.166142E-08
    19       5.860078E+00              6.209326E-01          9.486374E-08
    20       6.168503E+00              5.991779E-01          2.845912E-08
    21       6.476928E+00              5.805508E-01          1.897275E-08
    22       6.785354E+00              5.867887E-01          5.691824E-08
    23       7.093778E+00              6.333720E-01          1.019785E-07
    24       7.402204E+00              7.223860E-01          4.268868E-08
    25       7.710629E+00              8.389349E-01         -1.138365E-07
    26       8.019054E+00              9.526717E-01         -1.470388E-07
    27       8.327479E+00              1.024464E+00          4.743187E-08
    28       8.635904E+00              1.016569E+00          1.897274E-08
    29       8.944329E+00              9.034978E-01          4.743187E-08
    30       9.252754E+00              6.803382E-01         -9.012055E-08
    31       9.561180E+00              3.659730E-01         -3.438810E-08
    32       9.869605E+00             -2.630989E-07          2.371593E-08
    33       1.017803E+01             -3.659737E-01          4.387448E-08
    34       1.048646E+01             -6.803388E-01         -9.486374E-08
    35       1.079488E+01             -9.034982E-01         -7.589099E-08
    36       1.110330E+01             -1.016570E+00         -9.486370E-09
    37       1.141173E+01             -1.024464E+00          1.897275E-08
    38       1.172016E+01             -9.526712E-01         -1.897275E-07
    39       1.202858E+01             -8.389344E-01         -1.256944E-07
    40       1.233701E+01             -7.223858E-01          6.166143E-08
    41       1.264543E+01             -6.333720E-01          1.130864E-15
    42       1.295386E+01             -5.867885E-01         -7.114780E-08
    43       1.326228E+01             -5.805507E-01         -1.612683E-07
    44       1.357071E+01             -5.991780E-01          3.794549E-08
    45       1.387913E+01             -6.209329E-01          4.743187E-08
    46       1.418756E+01             -6.259803E-01         -3.794549E-08
    47       1.449598E+01             -6.029159E-01         -4.980346E-08
    48       1.480441E+01             -5.517372E-01          2.371594E-08
    49       1.511283E+01             -4.826337E-01         -4.506028E-08
    50       1.542126E+01             -4.113823E-01          4.268868E-08
```

```
        51      1.572968E+01          -3.531965E-01       -5.691824E-08
        52      1.603811E+01          -3.172767E-01       -4.743186E-08
        53      1.634653E+01          -3.039015E-01       -5.691825E-08
        54      1.665496E+01          -3.049158E-01       -9.486374E-08
        55      1.696338E+01          -3.072399E-01       -8.774895E-08
        56      1.727181E+01          -2.979726E-01       -7.114780E-08
        57      1.758023E+01          -2.691787E-01        1.138365E-07
        58      1.788866E+01          -2.206403E-01        8.063417E-08
        59      1.819708E+01          -1.596538E-01        4.743187E-08
        60      1.850551E+01          -9.804865E-02       -1.897275E-08
        61      1.881393E+01          -4.760141E-02        2.845912E-08
        62      1.912236E+01          -1.558205E-02        1.280660E-07
        63      1.943079E+01          -2.067081E-03        1.339950E-07
FORTRAN STOP
```

$\frac{54}{60}$

8-1 Find the transfer function of a low-pass Butterworth analog filter which meets the following specifications:

(a) The attenuation should be no more than 3dB in freq. range from dc to 4KHz.

(b) The attenuation should be at least 35 dB at frequency of 16 KHz.

Solution: For the normalized Butterworth Low-pass

$$|H(j\omega)|^2 = \frac{1}{1+\omega^{2N}}$$

Then, the B.W. Low-pass for the cut-off freq. at 4KHz $= 8\pi\times10^3$ rad/s

$$|H(j\omega)|^2 = \frac{1}{1+(\omega/8\pi\times10^3)^{2N}}$$

$$\left|H(j\cdot 2\pi\times16\times10^3)\right|^2 = \frac{1}{1+\left[\frac{2\pi\times16\times10^3}{2\pi\times4\times10^3}\right]^{2N}}$$

$$10\times\log_{10}[1+4^{2N}] \geq 35$$

$$\log_{10}[1+4^{2N}] \geq 3.5$$

$$1+4^{2N} \geq 3162.27766$$

$$N > 2.90657 \qquad \therefore \quad N \geq 3$$

From table 8-1 on page 314, we have transfer function for $N=3$ normalized Butterworth Low-pass:

$$H_{NORMAL}(S) = \frac{1}{S^3+2S^2+2S+1}$$

Then, scale the cut-off freq. to $8\pi\times10^3$ rad/s

$$H(S) = \frac{1}{\left(\frac{S}{8\pi\times10^3}\right)^3+2\left(\frac{S}{8\pi\times10^3}\right)^2+2\left(\frac{S}{8\pi\times10^3}\right)+1}$$

$$= \frac{1.58752\times10^{13}}{S^3+50265.48246\,S^2+1263309363\,S+1.58752\times10^{13}}$$

8-2 Find a low-pass Chebyshev analog filter which meets the following specifications:

(a) The ripple in the passband is 0.5 dB

(b) The cut-off frequency is 5 KHz

(c) The attenuation at 15 KHz should be at least 30 dB

(d) The dc gain should be such that the max. magnitude in the passband is unity.

Solution: For normalized Low-pass chebyshev filter,

$$|H(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 T_n^2(\omega)}$$

Then Chebyshev Low-pass for cut-off frequency at $5000 \times 2\pi$ rad/s

$$|H(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 T_n^2(\omega/2\pi \times 5000)}$$

$$H \triangleq -|H(j\omega)|^2 \text{ in dB} = 10 \log[1 + \varepsilon^2 T_n^2(\omega/2\pi \times 5000)]$$

$$T_n^2(\omega/2\pi \times 5000) = [10^{\frac{H}{10}} - 1]/\varepsilon^2$$

And

$$T_n(\omega/2\pi \times 5000) = \cosh[n \cosh^{-1}(\omega/2\pi \times 5000)] \qquad \text{for } |\omega/2\pi \times 5000| > 1$$

$$\therefore \quad n \geq \cosh^{-1}(\sqrt{(10^{H/10} - 1)/\varepsilon^2}) / \cosh^{-1}(\omega/2\pi \times 5000)$$

For specification (a), $A = 0.5$, $\varepsilon = \sqrt{10^{0.05} - 1} = 0.34931$

For specification (C), $n \geq \cosh^{-1}(\sqrt{(10^{30/10} - 1)/.12202}) / \cosh^{-1}(15000 \times 2\pi / 2\pi \times 5000) = 2.94896$

For specification (d), we see that $n = 3$. the dc gain is unity.

From Table 8-4, $r = .5$dB. $n = 3$, we have normalized Low-pass Chevbyshev,

$$H_N(S) = \frac{0.7157}{0.7157 + 1.5349 S + 1.2529 S^2 + S^3}$$

The transfer for this problem is

$$H(S) = \frac{0.7157}{\left(\frac{S}{10000\pi}\right)^3 + 1.2529\left(\frac{S}{10000\pi}\right)^2 + 1.5349\left(\frac{S}{10000\pi}\right) + 0.7157}$$

$$= \frac{2.21912 \times 10^{13}}{S^3 + 3.93610 \times 10^4 S^2 + 1.51489 \times 10^9 S + 2.21912 \times 10^{13}}$$

8-3 It is desired to have a low-pass analog filter with its magnitude response staying within 1 dB from dc to 700 Hz, while providing an attenuation of at least 20 dB at 4 KHz. Find a Butterworth filter which meets these specifications.

Solution: For Low-pass, Butterworth, with cut-off frequency at ω_c

$$|H(j\omega)|^2 = \frac{1}{1+(\omega/\omega_c)^{2n}}$$

$$A \triangleq -|H(j\omega)|^2 \text{ dB} = 10\log[1+(\omega/\omega_c)^{2n}]$$

$$(\omega/\omega_c)^{2n} = 10^{A/10} - 1$$

$$n = \ln[10^{A/10}-1] / 2\ln(\omega/\omega_c)$$

For $\omega = 2\pi \times 700$, $A = 1$

$$n = -1.35122 / 2\ln(2\pi \times 700/\omega_c) \quad \cdots\cdots \quad (1)$$

For $\omega = 2\pi \times 4000$, $A = 20$

$$n = 4.59512 / 2\ln(2\pi \times 4000/\omega_c) \quad \cdots\cdots \quad (2)$$

Solve eq. (1) and (2), we have

$$\omega_c = 6535.6170 \text{ rad/s} = \underline{1040.17575 \text{ Hz}}$$

$$n \geq 1.70581, \quad \underline{n=2}$$

Then, check if these meet the specifications

$\omega = 2\pi \times 700$ rad/s, $A = 10\log[1+(2\pi\times 700/6535.6170)^4] = \underline{0.81 \text{ dB} < 1 \text{ dB}}$

$\omega = 2\pi \times 4000$ rad/s, $A = 10\log[1+(2\pi\times 4000/6535.6170)^4] = \underline{23.41795 \text{ dB} > 20 \text{ dB}}$

For $n=2$, normalized B.W. Low-pass

$$H(s) = \frac{1}{1+\sqrt{2}s+s^2}$$

The transfer function for this problem,

$$H(s) = \frac{1}{\left(\frac{s}{6535.62}\right)^2 + \sqrt{2}\left(\frac{s}{6535.62}\right) + 1} = \frac{4.27143 \times 10^7}{s^2 + 9242.7582s + 4.27143 \times 10^7}$$

8-4 Find a chebyshev filter which meets the specifications in Prob. 8-3.

Solution: From Problem. 8-2, in page 2, we have

$$n \geq \cos^{-1}\left(\sqrt{(10^{H/10}-1)/\varepsilon^2}\right) / \cos^{-1}(\omega/2\pi\times700) \qquad \text{for } |\omega| \leq 2\pi\times700 \text{ rad/s}$$

$$n \geq \cosh^{-1}\left[\sqrt{(10^{H/10}-1)/\varepsilon^2}\right] / \cosh^{-1}(\omega/2\pi\times700) \qquad \text{for } |\omega| > 2\pi\times700 \text{ rps}$$

$$\gamma = 1 \text{ dB}$$

$$\varepsilon = \sqrt{10^{0.1}-1} = 0.50885$$

For $\omega = 2\pi\times4000$ rad/sec. $H = 20$ dB

$$n \geq \cosh^{-1}\left[\sqrt{(10^2-1)/0.50885^2}\right] / \cosh^{-1}(2\pi\times4000/2\pi\times700)$$

$$= 1.50951$$

$$\therefore \quad n = 2$$

Normalized Chebyshev Lowpass for $n=2$, $\gamma=1$

$$H_N(S) = \frac{1.1025}{1.1025 + 1.0977S + S^2}.$$

Scale the cut-off frequency to $2\pi\times700$ rad/sec.

$$H(S) = \frac{2.13272\times10^7}{S^2 + 4827.93676S + \boxed{2.13272}\times10^7}$$

8-6 Find a Chebyshev low-pass filter which meets the following specifications:

(a) Passband edge frequency = 2000 rad/s

(b) Passband ripple size = $\overset{.5}{\boxed{1}}$ dB

(c) The attenuation at 8000 rad/sec. is at least 80 dB

(d) Maximum magnitude in passband = 0 dB

Solution: $\omega_c = 2000$ rad/s

$\gamma = \overset{.5}{\boxed{1}}$ dB , $\varepsilon = 0.50885$

Using the same equation as that in Prob. 8-2 & 8-4

$$n \geq \cosh^{-1}\left[\sqrt{(10^{H/10}-1)/\varepsilon^2}\right] / \cosh^{-1}(\omega/\omega_c)$$

For $\omega = 8000$ rad/s $H = 80$ dB

-3

$$n \geq \cosh^{-1}\left[\sqrt{(10^{80/10}-1)/0.50885^2}\right] / \cosh^{-1}(8000/2000)$$

$$= 5.12693$$

$\therefore$ $n = 6$. ✓

From table 8-4 , $\gamma = 1$ dB , $n = 6$.

$$H_N(s) = 0.0689 / (0.0689 + 0.3071 s + 0.9393 s^2 + 1.2021 s^3 + 1.9308 s^4 + 0.9283 s^5 + s^6)$$

Scale the passband edge frequency = 2000 rad/sec.

$$H(s) = \frac{4.4096 \times 10^{18}}{s^6 + 1856.6 \cdot s^5 + 7.7232 \times 10^6 \cdot s^4 + 9.6168 \times 10^9 s^3 + 1.50288 \times 10^{13} s^2 + 9.8272 \times 10^{15} s + 4.4096 \times 10^{18}}$$

8-7 Find an elliptic low-pass analog filter which meets the specifications in Prob. 8-6.

Solution: Assume ω_c, ω_r are the cut-off frequency and stopband edge frequency, resp.

$$\omega_0 = \sqrt{\omega_c \omega_r} = 1 \qquad (\text{normalized}) \quad \cdots\cdots \quad (1)$$

Then scale frequency by ω_s, we have

$$\frac{2000}{\omega_s} = \omega_c \qquad \cdots\cdots \quad (2)$$

$$\frac{8000}{\omega_s} = \omega_r \qquad \cdots\cdots \quad (3)$$

Solve eq. (1), (2) and (3) above, we have

$$\omega_s = 4000 \text{ rad/s}, \quad \omega_c = 0.5 \text{ rad/sec}, \quad \omega_r = 2 \text{ rad/s}$$

Using FORTRAN program in text, we get the normalized elliptic filter

$$N = 5$$

$$H_n(S) = \frac{0.799314\times10^{-4}\cdot(S^2+11.3374)(S^2+4.40884)}{(S+0.147291)\cdot(S^2+0.233910S+0.109689)(S^2+.086698\,9S+0.247295)}$$

Scale the frequency by $\omega_s = 4000$ rad/s.

-3

✓ ✓ ✗

$$H(S) = \frac{0.319730 \times (S^2+1.81398\times10^8)(S^2+7.05414\times10^7)}{(S+589.164)\cdot(S^2+935.640S+1.75502\times10^6)(S^2+346.795S+3.95672\times10^6)}$$

Appendix:

```
DESIGN PARAMETERS:  [ Ben Chen , April 11, 1988 ]

PASSBAND EDGE FREQUENCY =    0.500000E+00 RAD/SEC
STOPBAND EDGE FREQUENCY =    0.200000E+01 RAD/SEC
PASSBAND RIPPLIE =        1.00 DB
STOPBAN D MINIMUM ATTENUATION = 80.0 DB

ELLIPTIC LOW-PASS FILTER ORDER =  5

   I        A[I]           B[I]           C[I]

   1    0.113374E+02   0.233910E+00   0.109689E+00
   2    0.440884E+01   0.866989E-01   0.247295E+00

ADDITIONAL PARAMETER FOR ODD N: So =    0.147291E+00
CONSTANT MULTIPLIER Ho =    0.799314E-04
FORTRAN STOP
```

Homework no: 9 | Discrete Signal and Systems | $\frac{32}{40}$ | Ben Chen | 01

8-11 Find a Chebyshev high-pass analog filter to meet these specifications:

(a) The attenuation should be no more than 1 dB at frequencies from 1 KHz and up.

(b) The ripples in the passband should be such that the max. magnitude is unity.

(c) At frequency below 100 Hz, the attenuation should be at least 90 dB.

Solution: Ripple $\gamma = 1$ dB. $\omega_{oh} = 2\pi \times 10^{3}$ rad/sec.

According 8-49 in text, on page 329. $H_{HP}(S)$ can be obtained from $H_{LP}(P)$ through the frequency transformation

$$P = \frac{\omega_{oh}}{S}$$

When $S = j100 \times 2\pi$ ran/sec. $P = -j10$ and $A = 90$ dB, $\omega_P = 10$ rad/s

Using the formula I derived for Prob. 8-2

$$n \geqslant \cosh^{-1}[(10^{\frac{A}{10}}-1)/(10^{\frac{\gamma}{10}}-1)]/\cosh^{-1}(\omega_P)$$

$$= \cosh^{-1}[\sqrt{(10^{9}-1)/(10^{0.1}-1)}]/\cosh^{-1}(10)]$$

$$= 3.91898$$ ✓

$n = 4$

$$H_{LP}(P) = \frac{0.2756}{0.2756 + 0.7426P + 1.4539P^{2} + 0.9528P^{3} + P^{4}}$$

-1

Replace P with $2000\pi/S$, we get

.2456

$$H_{HP}(S) = \frac{0.2756}{0.2756 + 0.7426\left(\frac{2000\pi}{S}\right) + 1.4539\left(\frac{2000\pi}{S}\right)^{2} + 0.9528\left(\frac{2000\pi}{S}\right)^{3} + \left(\frac{2000\pi}{S}\right)^{4}}$$

$$= \frac{0.2756S^{4}}{0.2756S^{4} + 4.66589\times10^{3}S^{3} + 5.73977\times10^{7}S^{2} + 2.36342\times10^{11}S + 1.55855\times10^{15}}$$

$$= \frac{S^{4}}{S^{4} + 1.69299\times10^{4}S^{3} + 2.08264\times10^{8}S^{2} + 8.57555\times10^{11}S + 5.65510\times10^{15}}$$

8-12 Find the transfer function of a Butterworth bandpass analog filter which meets the following specifications:

(a) The attenuation should be no more than 3 dB from 1 to 2 KHz

(b) The attenuation at frequencies above 4 KHz should be at least 30 dB.

(c) The attenuation at frequencies below 200 Hz should be at least 50 dB.

Solution: $H_{LP}(P) \longrightarrow H_{BP}(S)$

$$P = \frac{\omega_{ob}}{B}\left(\frac{S}{\omega_{ob}} + \frac{\omega_{ob}}{S}\right) = \frac{S^2+\omega_{ob}^2}{BS}$$

$$\omega_{ob} = \sqrt{\omega_1\omega_2} = \sqrt{2000\pi \times 4000\pi} = 1414.21 \times 2\pi \quad rad/s$$

$$B = \omega_2 - \omega_1 = 1000 \times 2\pi \quad rad/s$$

When $S = j\,4000 \times 2\pi$ $\quad P_1 = j3.5 \quad \omega_{P_1} = 3.5 \quad rad/s$

$$|H_{LP}(j\omega_{P_1})|^2 = \frac{1}{1+\omega_{P_1}^{2N}}$$

$$A \triangleq |H_{LP}(j\omega_{P_1})|^2 \text{ in dB} = 10\log_{10}(1+\omega_{P_1}^{2N}) = 30dB$$

$$N_1 \geq \ln(10^{A/10}-1)/2\ln(\omega_{P_1})$$

$$= \ln(10^3-1)/2\ln(3.5) = 2.75661$$

For $S = j200 \times 2\pi$ $\quad P_2 = -j9.79995 \quad \omega_{P_2} = 9.79995 \quad rad/s$

$$N_2 \geq \ln(10^5-1)/2\ln(9.79995) = 2.50218$$

So.

$$n = 3 \quad ✓$$

$$H_{LP}(P) = \frac{1}{1+2\cdot P+2P^2+P^3}$$

Then,
$$H_{HP}(S) = \frac{1}{1+2\cdot\frac{S^2+8\times10^6\cdot\pi^2}{2000\pi\ S}+2\left(\frac{S^2+8\times10^6\pi^2}{2000\pi\ S}\right)^2+\left(\frac{S^2+8\times10^6\pi^2}{2000\pi\ S}\right)^3}$$

$$= \frac{2.48050\times10^{11}\ S^3}{2.4805\times10^{11}S^3+2(S^2+8\times10^6\pi^2)\cdot3.9478\times10^7S^2+2(S^2+8\times10^6\pi^2)^2\cdot 6.2832\times10^3S+(S^2+8\times10^6\pi^2)^3}$$

-1

$$= \frac{2.4805\times10^{11}\ S^3}{S^6+1.2566\times10^4\cdot S^5+3.1583\times10^8\cdot S^4+8.7971\times10^{11}\cdot S^3+2.4937\times10^{16}\cdot S^2+7.8341\times10^{19}S+4.9223\times10^{23}}$$

2.2325×10^{12}

8-13 Find a bandpass Chebyshev analog filter which statisfies the following requirements:

(a) The attenuation should vary between 0 and 1 dB from 3 to 4 KHz.

(b) The attenuation at frequencies below 500Hz should be at least 80 dB

(c) The attenuation at frequencies above 10 KHz should be at least 60 dB

Solution: $H_{LP}(P) \longrightarrow H_{BP}(S)$

$$P = \frac{S^2 + \omega_{ob}^2}{BS}$$

$B = 4000 - 3000 = 1000\,Hz = 2000\pi$ rad/sec.

$\omega_{ob}^2 = 3000 \times 2\pi \times 4000 \times 2\pi = 4.8 \times 10^7 \pi^2$ $(rad/sec)^2$

For $s = j500 \times 2\pi = j1000\pi$ $\quad P_1 = -j23.5$, $\omega_{p_1} = 23.5$, $A = 80\,dB$

$$n_1 \geq \cosh^{-1}\left[\sqrt{(10^{A/10}-1)/(10^{\gamma/10}-1)}\right] / \cosh^{-1}(\omega_{p_1})$$

$$= \cosh^{-1}\left[\sqrt{(10^8-1)/(10^{0.1}-1)}\right] / \cosh^{-1}(23.5) = 2.748$$

For $s = j10^4 \cdot 2\pi$, $P_1 = j8.8$, $\omega_{p_2} = 8.8$ rad/sec. $A = 60\,dB$

$$n_2 \geq \cosh^{-1}\left[\sqrt{(10^6-1)/(10^{.1}-1)}\right] / \cosh^{-1}(8.8) = 2.8892$$

So that, we pick $n = 3$

$$H_{LP}(P) = \frac{0.4913}{0.4913 + 1.2384\,P + 0.9883\,P^2 + P^3}$$

$$H_{BP}(S) = \frac{0.4913}{0.4913 + 1.2384\left(\frac{S^2+4.8\times10^7\pi^2}{2000\pi S}\right) + 0.9883\left(\frac{S^2+4.8\times10^7\pi^2}{2000\pi S}\right)^2 + \left(\frac{S^2+4.8\times10^7\pi^2}{2000\pi S}\right)^3}$$

multiply out

−3

8-14 Find a bandstop Butterworth analog filter to meet the following specifications:

(a) The attenuation should be no more than 3dB at 2 and 5 KHz.

(b) The attenuation should be at least 20 dB from 2.5 to 4 KHz.

Solution: $H_{LP}(P) \longrightarrow H_{BS}(S)$

$$P = \frac{BS}{S^2 + \omega_o^2}$$

$$\omega_o^2 = 2000 \times 2\pi \times 5000 \times 2\pi = 4 \times 10^7 \pi^2 \quad (rad/sec)^2$$

$$B = 5KHz - 2KHz = 6000\pi \quad rad/sec.$$

For $S = j2.5 \times 10^3 \cdot 2\pi \qquad P_1 = j2 \qquad \omega_{P_1} = 2 rad/s \ , \ A = 20 dB$

$$n_1 \geqslant \ln(10^{A/10} - 1)/2\ln(\omega_{P_1})$$

$$= \ln(10^2 - 1)/2\ln(2) = 3.3147$$

For $S = j4 \times 10^3 \cdot 2\pi \qquad P_2 = -j2.0 \qquad \omega_{P_2} = 2 rad/s$

$\therefore \quad n = 4$

$$H_{LP}(P) = \frac{1}{1 + 2.6131 \cdot P + 3.4142 P^2 + 2.6131 P^3 + P^4}$$

$$H_{BS}(S) = \frac{1}{1 + 2.6131 \cdot \left(\frac{6000\pi S}{S^2 + 4 \times 10^7 \pi^2}\right) + 3.4142\left(\frac{6000\pi S}{S^2 + 4 \times 10^7 \pi^2}\right)^2 + 2.6131\left(\frac{6000\pi S}{S^2 + 4 \times 10^7 \pi^2}\right)^3 + \left(\frac{6000\pi S}{S^2 + 4 \times 10^7 \pi^2}\right)^4}$$

multiply out

-3

E-E 428 Discrete Signal Processing
Homework - 8% of Course Grade
April 19, 1988
Dr. R. A. Birgenheier Due Next Thur. OR. Final week

Each student is expected to design a low-pass filter and a bandpass filter according to the specifications given below. Use the invariant impulse response method for the low-pass filter and the bilinear tranformation for the bandpass filter. Each student should turn in the following during the finals week:

a) Derivation of the transfer function $H(z)$.

b) Magnitude frequency response $|H(e^{j\omega T})|^2$ in dB. Neatly draw the response curves and clearly show that the specifications are met.

1. Chen

   i) Cheby-LP (r=0.5)

   f_c = 1.2 MHz , A > 60 dB for f > 3.0 MHz

   ii) BW-BP

   f_1 = 65 KHz , f_2 = 75 KHz

   A > 30 dB for f < 55 KHz and f > 90 KHz

2. Kiley

   i) BW-LP

   f_c = 1.2 KHz , A > 60 dB for f > 4.0 KHz

   ii) Cheby-BP (r=1)

   f_1 = 9 KHz , f_2 = 11 KHz

   A > 20 dB for f < 7 KHz and f > 14 KHz

3. Lincoln

   i) BW-LP

   f_c = 15 KHz , A > 60 dB for f > 50 KHz

   ii) Cheby-BP (r=1)

   f_1 = 10.2 MHz , f_2 = 11.2 MHz

   A > 20 dB for f < 9 MHz and f > 13.0 MHz

4. Michels

   i) BW-LP

   f_c = 2 MHz, A > 50 dB for f > 6.5 MHz

   ii) Cheby-BP (r=1)

   f_1 = 95 KHz , f_2 = 105 KHz

   A > 30 dB for f < 85 KHz and f > 115 KHz

5. Noble

   i) BW-LP

   f_c = 3.3 KHz , A > 50 dB for f > 11 KHz

   ii) Cheby-BP (r=1)

   f_1 = 24 KHz , f_2 = 28 KHz

   A > 30 dB for f < 21 KHz and f > 32 KHz

6. Reitcheck

   i) Cheby-LP

   f_c = 20 KHz , A > 50 dB for f > 50 KHz

   ii) BW-BP (r=0.5)

   f_1 = 4 MHz , f_2 = 6 MHz

   A > 30 dB for f < 2.5 MHz and f > 9.5 MHz

7. Shepard

   i) Cheby-LP

   f_c = 1.0 MHz , A > 50 dB for f > 2.5 MHz

   ii) BW-LP (r=0.5)

   f_1 = 600 Hz , f_2 = 700 Hz

   A > 30 dB for f < 500 Hz and f > 850 Hz

8. Upchurch

   i) Cheby-LP

   f_c = 120 KHz , A > 60 dB for f > 280 KHz

   ii) BW-LP (r=0.5)

   f_1 = 95 Hz , f_2 = 105 Hz

   A > 20 dB for f < 80 Hz and f > 120 Hz

9. Vlahovich

   i) Cheby-LP

   f_c = 18 KHz , A > 60 dB for f > 42 KHz

   ii) BW-BP (r=0.5)

   f_1 = 12 KHz , f_2 = 15 KHz

   A > 20 dB for f < 9 KHz and f > 20 kHz

1. For Cheby - Low-pass $\gamma = 0.5$ dB

$f_c = 1.2$ MHz , $A > 60$ dB for $f > 3.0$ MHz

Design Procedure

Step 1 . Find the normalized analog low-pass transfer function.

$\omega_c = 1.2 \times 2\pi \times 10^6$ rad/second.

ω_c is corresponding to 1 rad/sec. in NLP and $f = 3.0$ MHz is corresponding to $\omega_p = 2.5$ rad/sec. $A = 60$ dB

$$n \geqslant \cosh^{-1}[\sqrt{(10^{A/10}-1)/(10^{\gamma/10}-1)}\,]/\cosh^{-1}(\omega_p)$$

$$= \cosh^{-1}[\sqrt{(10^6-1)/(10^{0.05}-1)}\,]/\cosh^{-1}(2.5)$$

$$= 5.52253 \qquad \therefore \quad \underline{\underline{n=6}}$$

Then, from Table 8-5 a., we have the transfer function for NLP

$$H(S) = \frac{1.}{(S^2+0.1553\,S+1.0230)(S^2+0.4243\,S+0.5900)(S^2+0.5796\,S+0.1570)}$$

Step 2 .

$$H(S) = \frac{AS+B}{S^2+0.1553\,S+1.0230} + \frac{Cs+D}{S^2+0.4243\,S+0.5900} + \frac{Es+F}{S^2+0.5796\,S+0.1570}$$

We can find the A, B, C, D, E, F by picking some particular values of s. (It can be shown this method is OK. if the equations are linear indep.)

(i) Let $s=0$, we have

$$0.9775171\,B + 1.6949153\,D + 6.3694268\,F = 10.5529214 \quad \cdots (1)$$

Let $s=1$, we have

$$0.4590736\,(A+B) + 0.4964504\,(C+D) + 0.5758378\,(E+F) = 0.1312376 \quad \cdots (2)$$

$s=-1$: $0.5354179\,(B-A) + 0.8578537\,(D-C) + 1.7319016\,(F-E) = 0.7954801 \quad \cdots (3)$

$S=0.5$: $0.7403843\,(0.5A+B) + 0.9504348\,(0.5C+D) + 1.435132\,(0.5E+F) = 1.0098837 \quad \cdots (4)$

$S=-0.5$: $0.8365751\,(B-0.5A) + 1.5927371\,(D-0.5C) + 8.5324232\,(F-0.5E) = 11.3689774$

$S=2$: $0.1874906\,(2A+B) + 0.1838708\,(2C+D) + 0.1881043\,(2E+F) = 0.00648471\,9$

1. (CONT.)

$$\begin{bmatrix} 0.006484719 \\ 11.3689774 \\ 1.0098837 \\ 0.7954801 \\ 0.1312376 \\ 10.5529214 \end{bmatrix} = \begin{bmatrix} 0.3749812 & 0.1874906 & 0.3677416 & 0.1838708 & 0.3762086 & 0.1881043 \\ -0.4182876 & 0.8365751 & -0.7963686 & 1.5927371 & -4.2662116 & 8.5324232 \\ 0.3701922 & 0.7403843 & 0.4752174 & 0.9504348 & 0.7175660 & 1.435132 \\ -0.5354179 & 0.5354179 & -0.8578537 & 0.8578537 & -1.7319016 & 1.7319016 \\ 0.4590736 & 0.4590736 & 0.4964504 & 0.4964504 & 0.5758378 & 0.5758378 \\ 0.0 & 0.9775171 & 0.0 & 1.6949153 & 0.0 & 6.3694268 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix}$$

Solve the equations. I got

$A=1.566242 \quad B=1.178862 \quad C=-3.1288395 \quad D=-3.860105 \quad E=1.561786 \quad F=2.503069$

Then,

$$H(S)=\frac{1.566242S+1.178862}{S^2+0.1553S+1.0230}-\frac{3.1288395S+3.860105}{S^2+0.4243S+0.5900}+\frac{1.561786S+2.503069}{S^2+0.5796S+0.1570}$$

$$=\frac{1.566242(S+0.07765)}{(S+0.07765)^2+(1.00885)^2}+\frac{1.0396008(1.00885)^2}{(S+0.07765)^2+(1.00885)^2}$$

$$-\frac{3.1288395(S+0.21215)}{(S+0.21215)^2+(0.738236)^2}-\frac{5.8648925(0.738236)^2}{(S+0.21215)^2+(0.738236)^2}$$

$$+\frac{1.561786(S+0.2898)}{(S+0.2898)^2+(0.2702147)^2}+\frac{28.0824003(0.2702147)^2}{(S+0.2898)^2+(0.2702147)^2}$$

pick $K_s=\frac{\omega_s}{\omega_n}=10$, $T_n=\frac{2\pi}{10}=0.6283185$

$$H_n(z)=\frac{1.566242-1.2018850z^{-2}}{1-1.5347372\cdot z^{-1}+0.9070317z^{-2}}+\frac{0.5915993z^{-1}}{1-1.5347372\cdot z^{-1}+0.9070317z^{-2}}$$

$$-\frac{3.1288395-2.4490317z^{-1}+1.6953261z^{-1}}{1-1.5654569z^{-1}+0.7659828z^{-1}}$$

$$+\frac{1.561786-1.2830766z^{-1}+1.0687209z^{-1}}{1-1.6430888z^{-1}+0.6947707z^{-2}}$$

$$=\frac{1.566242-0.6102857z^{-1}}{1-1.5347372\cdot z^{-1}+0.9070317z^{-2}}-\frac{3.1288395-0.7537056z^{-1}}{1-1.5654569z^{-1}+0.7659828z^{-2}}+\frac{1.561786-0.2143557z^{-1}}{1-1.6430888z^{-1}+0.6947707z^{-2}}$$

$$=\frac{10^{-3}(-0.7175207+4.648209z^{-1}+7.797241z^{-2}+43.74171z^{-3}+8.207321z^{-4}+1.258105z^{-5}}{1-4.7432829z^{-1}+9.8642443\cdot z^{-2}-11.4459642z^{-3}+7.791003z^{-4}-2.9448482z^{-5}+0.4827063z^{-6}}$$

Step 3.

$$H(z)=\frac{H_n(z)}{H_n(1)}=\frac{10^{-4}(-0.4263353+2.7618644z^{-1}+4.6329506z^{-2}+25.9903704z^{-3}+4.8766112z^{-4}+0.7475386z^{-5})}{1-4.7432829z^{-1}+9.8642443z^{-2}-11.4459642z^{-3}+7.791003z^{-4}-2.9448482z^{-5}+0.4827063z^{-6}}$$

Step 4.

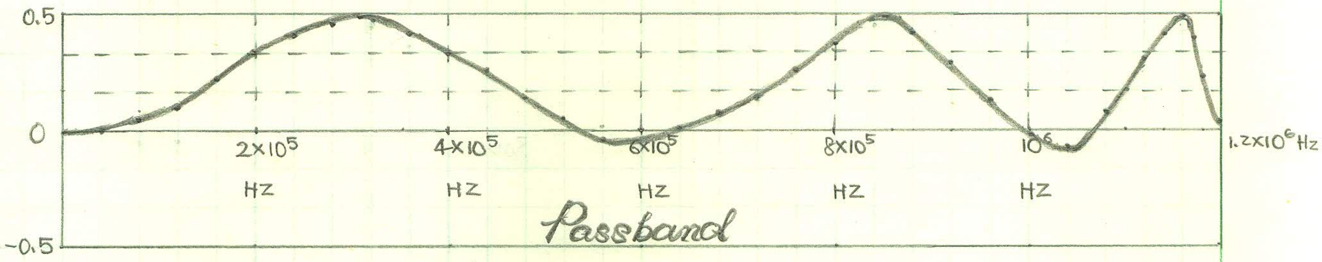
$$-2.8942504\times10^{-4}\cos(4\omega T)-6.3740419\times10^{-3}\cos(5\omega T)]$$

$$A^2(\omega)=\frac{10^{-6}[7.2911339+5.2484118\cos\omega T+2.2365718\cos(2\omega T)+.1170247\cos(3\omega T)}{321.4170089-555.9566614\cos\omega T+356.9508816\cos(2\omega T)-165.9492732\cos(3\omega T)+53.04156796\cos(4\omega T)-10.46892148\cos(5\omega T)+0.9654126\cos(6\omega T)}$$

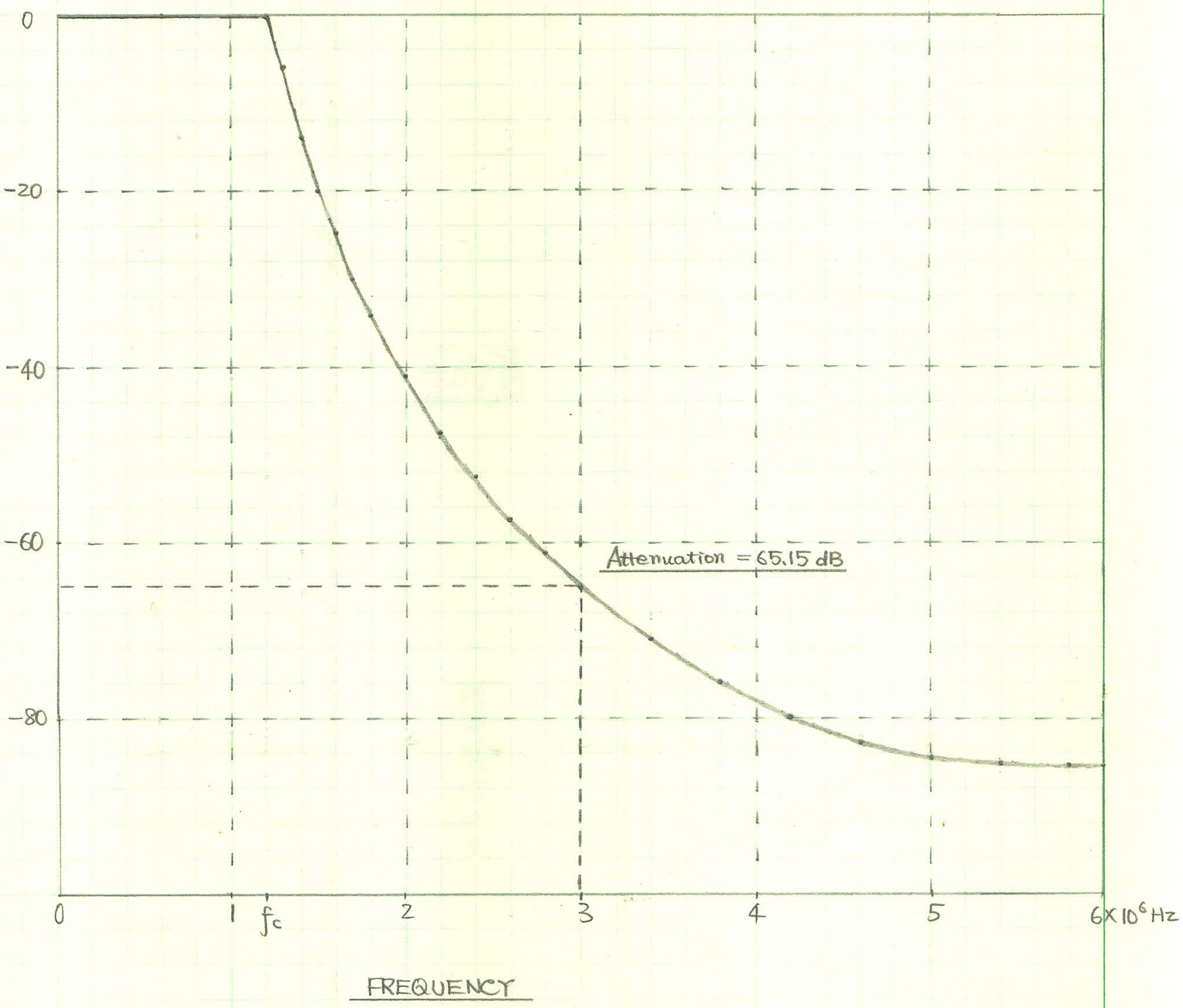
Step 5. plot

$f\left(=\frac{\omega}{2\pi}\right)$ Hz	$A^2(\omega)$ in dB	
0	−0.03	
10	−0.02	
100	−0.01	
1000	−0.02	
2000	−0.00	
3000	−0.03	
4000	−0.02	
5000	−0.03	
6000	−0.02	
7000	−0.01	
10^4	−0.02	
2×10^4	0.00	
3×10^4	−0.01	
4×10^4	−0.02	
10^5	0.09	
1.5×10^5	0.20	
2×10^5	0.33	
2.5×10^5	0.39	
3×10^5	0.47	FIRST RIPPLE
3.5×10^5	0.44	
4×10^5	0.33	
5×10^5	0.08	
6×10^5	−0.01	
7×10^5	0.12	
8×10^5	0.37	
8.5×10^5	0.44	SECOND RIPPLE
9×10^5	0.40	
10^6	−0.04	
1.1×10^6	0.17	
1.15×10^6	0.44	THIRD RIPPLE
1.2×10^6	0.04	
1.5×10^6	−20.47	
2×10^6	−41.50	
2.5×10^6	−55.11	
3×10^6	−65.15	specification is met.
4×10^6	−78.06	
6×10^6	−85.46	

$A^2(\omega)$ dB FREQUENCY RANGE FROM 0 — $f_c = 1.2$ MHz



$A^2(\omega)$ dB FREQUENCY RANGE FROM $f_c = 1.2$ MHz TO $\frac{\omega_s}{4\pi} = 6$ MHz



2. <u>Butterworth Bandpass</u> Specifications:

$f_1 = 65$ KHz , $f_2 = 75$ KHz

$A > 30$ dB for $f < 55$ KHz and $f > 90$ KHz

<u>Design Procedure:</u>

$$P = \frac{S^2 + \omega_0^2}{BS} = \frac{S^2 + (65 \times 2\pi \times 10^3 \times 75 \times 2\pi \times 10^3)}{10 \times 2\pi \times 10^3 \; S}$$

$$= \frac{S^2 + 1.9245728 \times 10^{11}}{62831.85308 \; S}$$

$S_1 = j\,55 \times 2\pi \times 10^3$ rad/s $\Rightarrow$ $P_1 = j\text{-}3.3636364$ $A = 30$ dB

$$n_1 \geqslant \ln(10^{30/10} - 1)/2\ln(3.3636364) = 2.84692$$

$S_2 = j\,90 \times 2\pi \times 10^3$ rad/s $\Rightarrow$ $P_2 = j3.583333$ $A = 30$ dB

$$n_2 \geqslant \ln(10^{30/10} - 1)/2\ln(3.583333) = 2.705786$$

$\therefore \; n = 3$

$$H_n(p) = \frac{1}{(p+1)(p^2+p+1)}$$

$$H_{BP}(S) = \frac{1}{\left(\frac{S^2+\omega_0^2}{BS} + 1\right)\left(\frac{S^4 + 2\omega_0^2 S^2 + \omega_0^4}{B^2 S^2} + \frac{S^2 + \omega_0^2}{BS} + 1\right)}$$

$$= \frac{B^3 S^3}{(S^2 + BS + \omega_0^2)(S^4 + 2\omega_0^2 S^2 + \omega_0^4 + BS^3 + B\omega_0^2 S + B^2 S^2)}$$

$$= \frac{B^3 S^3}{(S^2 + BS + \omega_0^2) \cdot [S^4 + BS^3 + (B^2 + 2\omega_0^2) S^2 + B\omega_0^2 S + \omega_0^4]}$$

$$= \frac{2.4805021 \times 10^{14} S^3}{(S^2 + 6.2831853 \times 10^4 S + 1.9245729 \times 10^{11}) \cdot (S^4 + 6.2831853 \times 10^4 S^3 + 3.8886241 \times 10^{11} S^2 + 1.2092448 \times 10^{16} S + 3.7039807 \times 10^{22})}$$

Pick $\omega_s = 1000000$ Hz $= 6.2831853 \times 10^6$ rad/s . $\omega_p = \omega_0 = 4.3869953 \times 10^5$ rad/s

$$K = \omega_p / \tan(\omega_p \pi / \omega_s) = 1967820.423$$

$$S^2 + 6.2831853\times10^4 S + 1.9245729\times10^{11}$$

$$\longrightarrow \frac{4.1884163\times10^{12}(z^2-1.7571605z+0.9409601)}{(z+1)^2}$$

$$S^4 + 6.2831853\times10^4 S^3 + 3.8886241\times10^{11} S^2 + 1.2092447\times10^{16} S + 3.7039806\times10^{22}$$

$$\longrightarrow \frac{1.7040255(z^4-3.564569421z^3+5.116102122z^2-3.457767142z+0.941013083)\times10^{25}}{(z+1)^4}$$

$$S \longrightarrow \frac{1.9678204\times10^6(z-1)}{(z+1)}$$

$$S^2 \longrightarrow \frac{3.8723172\times10^{12}(z^2-2z+1)}{(z+1)^2}$$

$$H_{BP}(z) = \frac{2.6483176\times10^{-5}(z-1)^3(z+1)^3}{(z^2-1.7571605z+0.9409601)(z^4-3.564569421z^3+5.116102122z^2-3.457767142z+0.941013083)}$$

$$A^2(\omega) = \frac{7.0135859\times10^{-10}[2-2\cos(\omega T)]^3\cdot[2+2\cos(\omega T)]^3}{[4.9730189-6.8211568\cos(\omega T)+1.8819202\cos(2\omega T)]\cdot[52.72231531-85.490729060\cos(\omega T)+44.51174434\cos(2\omega T)-13.6241472\cos(3\omega T)+1.882026165\cos(4\omega T)]}$$

$T = 1\times10^{-6}$ seconds = 1 μs.

	f (Hz)	$A^2(\omega)$ dB	
	0	$-\infty$	
	5K	−119.63	
	10K	−101.17	
	15K	−89.92	
	20K	−81.44	
	25K	−74.30	
	30K	−67.83	
	35K	−61.62	
	40K	−55.35	
	45K	−48.69	
	50K	−41.24	
O.K.	55K	−32.29	
	60K	−20.42	
	65K	−3.43	not O.K.
	65.1K	−3.06	Actural cut-off frequencies.
Cetral freq	6.98K (f_0)	−0.02	
	70K	−0.07	
	74.8K	−2.96	
	75K	−3.42	not O.K.

f (Hz)	$A^2(\omega)$ dB	
80K	−17.86	
85K	−27.55	
90K	−34.38	O.K.
95K	−39.63	
100K	−43.87	
200K	−78.85	
300K	−97.38	
400K	−118.90	
500K	$-\infty$	

From the data we obtained above, we see that the actural cut-off frequencies are different from that in analog filter.

$$B_{analog} = 10 \text{ KHz}$$

$$B_{DIGITAL} \doteq 74.8\text{KHz} - 65.1\text{KHz} = 9.70 \text{ KHz}$$

$$\omega_{o,a} = 69.821 \text{ KHz}$$

Now, we try to rebuild the filter by select

$$f_1 = 64.85 \text{ KHz} \qquad f_2 = 75.2 \text{ KHz}$$

$$B_{new} = 65030.968 \text{ rad/second}$$

$$\omega_{o,new} = 69.834 \text{ rad/second}$$

$$H_{BP,new}(S) = \frac{2.7501771\times10^{14} S^3}{(S^2 + 6.5030968\times10^4 S + 1.9252519\times10^{11})\cdot(S^4 + 6.5030968\times10^4 \cdot S^3 + 3.8927940\times10^{11} S^2 + 1.2520099\times10^{16} S + 3.7065948\times10^{22})}$$

Pick $\omega_s = 1\text{MHz} = 6.2831853\times10^6$ rad/sec. $\omega_p = \omega_o = 4.3877692\times10^5$ rad

$$K = \omega_p / \tan(\omega_p \pi/\omega_s) = 1.9678090\times10^6$$

$$T = 1\,\mu s.$$

$$S^2 + 6.5030968\times10^4 S + 1.9252519\times10^{11}$$

$$\longrightarrow \frac{4.1927661\times10^{12}(z^2 - 1.755283795 z + 0.93895747)}{(z+1)^2}$$

$$S^4 + 6.5030968\times10^{14} S^3 + 3.8927940\times10^{11} S^2 + 1.2520099\times10^{16} S + 3.7065948\times10^{22}$$

$$\longrightarrow \frac{1.7059121\times10^{25}(z^4 - 3.562404657 z^3 + 5.110143856 z^2 - 3.451990571 z + 0.939016068}{(z+1)^4}$$

$$S \longrightarrow \frac{1.9678090\times10^6 (z-1)}{z+1}$$

$$S^2 \longrightarrow \frac{3.8722724\times10^{12} (z-1)^2}{(z+1)^2}$$

$$H_{BP}(z) = \frac{2.9298961\times10^{-5} (z-1)^3 \cdot (z+1)^3}{(z^2 - 1.755283795 z + 0.93895747)(z^4 - 3.562404657 z^3 + 5.110143856 z^2 - 3.451990571 z + 0.939016068)}$$

$$A^2(\omega) = \frac{(2.9298961\times10^{-5})^2 [2 - 2\cos\omega T]^3 \cdot [2 + 2\cos\omega T]^3}{[4.962662332 - 6.806841252\cos\omega T + 1.87791494 \cdot \cos(2\omega T)] \cdot [52.60228725 - 85.2968959\cos\omega T + 44.41207668\cos(2\omega T) - 13.59429157\cos(3\omega T) + 1.878032136\cos(4\omega T)}$$

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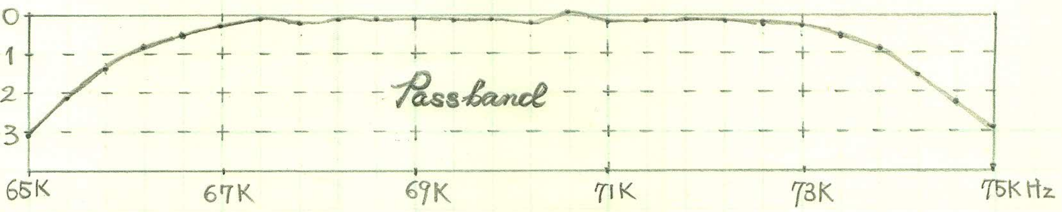
DATA:

f (Hz)	$A^2(\omega)$ dB	f(Hz)	$A^2(\omega)$ dB	f(Hz)	$A^2(\omega)$ dB
0	−∞	60K	−19.57	95K	−38.72
5K	−118.74	65K	−3.08	100K	−42.97
10K	−100.28	67K	−0.23	150K	−65.74
15K	−89.04	68K	−0.12	200K	−77.95
20K	−80.56	68.6K	−0.04	250K	−87.52
25K	−73.42	70K	−0.05	300K	−96.48
30K	−66.95	72K	−0.17	350K	−106.08
35K	−60.74	74K	−1.14	400K	−118.01
40K	−54.47	75K	−2.97	450K	−136.84
45K	−47.81	80K	−16.96	500K	−∞
50K	−40.36	85K	−26.64		
55K	−31.42	90K	−33.47		

PLOT: $A^2(\omega)$ in dB, PASSBAND FREQUENCY 65K ~ 75K



$A^2(\omega)$ in dB, FREQUENCIES FROM 0 ~ 500KHz