E-E 428 Discrete Signal Processing
One Hour 15 minutes Examination
February 25, 1988
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148/150

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One Sheet of Notes

30 pts each

1. A discrete time LII system is characterized by

$$y(k+2) + \frac{1}{4}y(k+1) - \frac{1}{8}y(k) = x(k)$$

Find (a) the transfer function $H(z)$; (b) $y(k)$ for $x(k) = ku(k)$; (c) identify the forced and natural response found in (b).

2. A discrete time filter is characterized by

$$H(z) = \frac{1}{1 - 0.5z^{-1} + 0.5z^{-2}}$$

Find (a) the magnitude-square frequency response function , $A^2(\omega)$, (b) the phase frequency response function $\phi(\omega)$.

3. Suppose the input to the filter in problem 2 is

$$x(k) = 2\cos \frac{\pi}{4}k$$

Find the steady state output $y(k)$.

4. (a) Which of the following represent the transfer function of a stable system? Why?

$$H_1(z) = \frac{z^3 + .5z^2 - 2.5z + 1}{z^3 + 2.8z^2 + 2.6z + .8}$$

$$H_2(z) = \frac{z^3 + 1.5z^2 - 1.5z - 1}{z^3 - .7z^2 - .7z + .4}$$

(b) Find the unit-sample response of the stable system(s).

5. A second order system is characterized by

$$y(k) + 1.7321y(k-1) + y(k-2) = 2x(k)$$

Find the unit step response. Express as a real function of k.

1.

(a) $$y(k+2) + \frac{1}{4}y(k+1) - \frac{1}{8}y(k) = x(k)$$

For Linear system, we have

$$z^2 Y(z) + \frac{1}{4} z Y(z) - \frac{1}{8} Y(z) = X(z)$$

$$(z^2 + 0.25z - 0.125)\, Y(z) = X(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^2 + 0.25z - 0.125}$$

(b) For $x(k) = k\,u(k) \quad \longleftrightarrow \quad X(z) = \frac{z}{(z-1)^2}$

$$Y(z) = H(z)X(z)$$

$$= \frac{1}{z^2 + 0.25z - 0.125} \cdot \frac{z}{(z-1)^2}$$

$$= \frac{z}{(z-0.25)(z+0.5)(z-1)^2}$$

OK

$$\frac{Y(z)}{z} = \frac{A}{z-0.25} + \frac{B}{z+0.5} + \frac{C}{z-1} + \frac{Dz}{(z-1)^2}$$

$$A = (z-0.25)\cdot\frac{Y(z)}{z}\bigg|_{z=0.25} = \frac{1}{0.75\,(0.25-1)^2} = 2.3704$$

$$B = (z+0.5)\,\frac{Y(z)}{z}\bigg|_{z=-0.5} = \frac{1}{-.75\,(-.5-1)^2} = -0.5926$$

$$D = (z-1)^2\cdot\frac{Y(z)}{z}\bigg|_{z=1} = \frac{1}{0.75\times 1.5} = 0.8889$$

Let $z = 0$ We have

$$\frac{Y(z)}{z}\bigg|_{z=0} = \frac{1}{-.25\times 0.5\times 1} = -8.$$

$$\frac{2.3704}{-0.25} + \frac{-0.5926}{0.5} - C = -8$$

$$\Rightarrow \quad C = -2.6668$$

$$\therefore \quad Y(z) = \frac{2.3704z}{z-0.25} - \frac{0.5926z}{z+0.5} - \frac{2.6668z}{z-1} + z\,\frac{0.8889z}{(z-1)^2}$$

$$\therefore\quad y(k) = [\underbrace{2.3704\,(0.25)^k - 0.5926\,(-0.5)^k}_{\text{Natural Response}} \underbrace{-2.6668 + 0.8889(k+1)}_{\text{Forced Response}}]\,u(k)$$

(c)

2.

(a)

$$H(z) = \frac{1}{1 - 0.5z^{-1} + 0.5z^{-2}}$$

$$a_0 = 1 \;,\quad a_1 = -0.5 \;,\quad a_2 = 0.5 \;,\quad a_n = 0 \;,\; n > 2$$

$$\therefore\quad c_0 = a_0^2 + a_1^2 + a_2^2 = 1.5$$

$$c_1 = a_0 a_1 + a_1 a_2 = -0.5 - 0.25 = -0.75$$

$$c_2 = a_0 a_2 = 0.5 .$$

$$b_0 = 1 \;,\quad b_n = 0 \;,\quad n > 0$$

$$\Rightarrow d_0 = 1$$

$$A^2(\omega) = \frac{1}{1.5 - 1.5\cos\omega T + \cos 2\omega T}$$

(b)

$$H(z) = \frac{z^2}{z^2 - 0.5z + 0.5} = \frac{z^2}{(z - 0.25 + j0.6614)(z - 0.25 - j0.6614)}$$

$$H(e^{j\omega T}) = \frac{e^{j2\omega T}}{[(\cos\omega T - 0.25) + j(\sin\omega T + 0.6614)]} \cdot \frac{1}{(\cos\omega T - 0.25) + j(\sin\omega T - 0.6614)}$$

$$\phi(\omega) = -\angle H(e^{j\omega T}) = -2\omega T + \arctan\left(\frac{\sin\omega T + 0.6614}{\cos\omega T - 0.25}\right) + \arctan\left(\frac{\sin\omega T - 0.6614}{\cos\omega T - 0.25}\right)$$

3. $x(k) = 2\cos\frac{\pi}{4}k$, $\omega_1 T = \frac{\pi}{4}$, $\theta_1 = 0$

$$H(z) = \frac{1}{1-0.5z^{-1}+0.5z^{-2}}$$

$$|H(e^{j\omega_1 T})| = \sqrt{A^2(\omega_1)} = 1.50869$$ ← From pro. 2

$$\phi(\omega) = -2\,\frac{\pi}{4} + \arctan\left(\frac{\sin\frac{\pi}{4}+0.6614}{\cos\frac{\pi}{4}-0.25}\right) + \arctan\left(\frac{\sin\frac{\pi}{4}-0.6614}{\cos\frac{\pi}{4}-0.25}\right)$$

$$= -2\times45^\circ + 5.71^\circ + 71.53^\circ = -12.76^\circ$$

$\angle H(e^{j\omega T}) = -\phi(\omega)$

∴ s.s. response

-2

$$y(k) = 2\times1.50869\cos\left(\frac{\pi}{4}k + 12.76^\circ\right)$$

$$= 3.01738\cos\left(\frac{\pi}{4}k + 12.76^\circ\right)$$

4. (a)

$$H_1(z) = \frac{z^3+0.5z^2-2.5z+1}{z^3+2.8z^2+2.6z+0.8}$$

$$= \frac{(z+2)(z-1)(z-0.5)}{(z+1)(z+0.8)(z+1)}$$

This system is unstable, because it has two poles at $z=-1$ which is outside unit circle.

$$H_2(z) = \frac{z^3+1.5z^2-1.5z-1}{z^3-0.7z^2-0.7z+0.4}$$

$$= \frac{(z-1)(z+0.5)(z+2)}{(z+0.8)(z-1)(z-0.5)}$$

$$= \frac{(z+0.5)(z+2)}{(z+0.8)(z-0.5)}$$

This system is stable, because all the poles of system are inside unit circle.

4 (b) For the stable system,

$$H_2(z) = \frac{(z+0.5)(z+2)}{(z+0.8)(z-0.5)}$$

And we know that the z-transform for unit sample is 1.

$$Y(z) = H_2(z)\cdot 1$$

$$= \frac{(z+0.5)(z+2)}{(z+0.8)(z-0.5)} = \frac{z^2+2.5z+1}{(z+0.8)(z-0.5)}$$

For $k<0$ $\quad y(k) = 0$

For $k=0$,

$$y(0) = \lim_{z\to\infty} Y(z) = 1$$

For $k>0$:

$$F(z) = \frac{z^2+2.5z+1}{(z+0.8)(z-0.5)}\cdot z^{k-1}$$

$$\xi_1 = (z+0.8)F(z)\Big|_{z=-0.8}$$

$$= \frac{(-.8)^2-2.5\times 0.8+1}{-1.3}\cdot(-0.8)^{k-1}$$

$$= 0.27692\cdot(-0.8)^{k-1} = -0.34615\cdot(-.8)^k$$

$$\xi_2 = (z-0.5)F(z)\Big|_{z=0.5}$$

$$= \frac{0.25+2.5\times 0.5+1}{1.3}(0.5)^{k-1} = 3.84615(0.5)^k$$

$$\therefore\ y(k) = h(k) = 3.84615(.5)^k u(k) - 0.34615(-.8)^k u(k) - 2.5\,\delta(k)$$

5.

$$y(k) + 1.7321\, y(k-1) + y(k-2) = 2x(k)$$

$$Y(z) + 1.7321\, z^{-1} Y(z) + z^{-2} Y(z) = 2\, X(z)$$

$$(1 + 1.7321 z^{-1} + z^{-2})\, Y(z) = 2\, X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 + 1.7321 z^{-1} + z^{-2}}$$

$$= \frac{2z^2}{z^2 + 1.7321 z + 1}$$

For the unit step signal to the input, we have

$$Y(z) = H(z)U(z) = \frac{2z^2}{z^2 + 1.7321z + 1} \cdot \frac{z}{z-1}$$

$$\frac{Y(z)}{z} = \frac{2z^2}{(z-1)(z^2 + 1.7321z + 1)}$$

$$= \frac{A}{z-1} + \frac{Dz + E}{z^2 + 1.7321z + 1}$$

$$A = (z-1)\frac{Y(z)}{z}\Big|_{z=1} = \frac{2}{1 + 1.7321 + 1} = 0.53589$$

Let $z = 0$, we have

$$E = A = 0.53589$$

Let $z = 2$, we have

$$\frac{8}{2^2 + 1.7321 \times 2 + 1} = \frac{A}{2-1} + \frac{2D + E}{2^2 + 1.7321 \times 2 + 1}$$

$$0.94516 = A + 0.23629\, D + 0.11814\, E$$

$$\Rightarrow D = (0.94516 - A - 0.11814E)/0.23629$$

$$= 1.46411$$

5 (CONT.)

$$\therefore \quad Y(z) = \frac{0.53589z}{z-1} + \frac{z(1.46411z + 0.53589)}{z^2 + 1.73212z + 1}.$$

Use my own formula.

$$\frac{z(Dz+E)}{Az^2+Bz+C} \xleftrightarrow{B^2-4AC<0} \frac{D}{A}e^{-aTk}\cos k\omega T + \frac{2AE-BD}{2A^2\sin\omega T}\sqrt{\frac{A}{C}}\,e^{-aTk}\sin k\omega T$$

$$aT = \frac{1}{2}\ln(A/C) \quad \text{and} \quad \omega T = \arccos\left(-\frac{B}{2A}\sqrt{\frac{A}{C}}\right)$$

we have.

$$A=1,\quad B=1.73 21,\quad C=1,\quad D=1.46411,\quad E=0.53589$$

$$\Rightarrow \quad aT = \frac{1}{2}\ln(1/1) = 0. \quad \Rightarrow \quad e^{-aTk} = 1$$

$$\omega T = \arccos\left(-\frac{1.7321}{2}\cdot 1\right) = \frac{5\pi}{6}$$

$$\frac{D}{A} = 1.46411$$

$$\frac{2AE-BD}{2A^2\sin\omega T}\sqrt{\frac{A}{C}} = \frac{2\times0.53589 - 1.7321\times1.46411}{2\times\sin\left(\frac{5}{6}\pi\right)}\cdot 1 = -1.46420$$

$$\therefore \quad y(k) = 0.53589u(k) + 1.46411\cos\left(\frac{5\pi}{6}k\right)u(k)$$
$$- 1.46411\sin\left(\frac{5\pi}{6}k\right)u(k)$$

High

$\frac{142}{150}$ Ave. $\frac{123}{150}$ 82.%

E-E 428 Discrete Signal Processing
One Hour 15 minutes Examination
April 19, 1988
Dr. R. A. Birgenheier

Ben Chen

25 pts.

1. a) Obtain a **closed form** expression for the DFT of the following sequence:

$$x_k = \begin{cases} 1 & \text{for } 0 \le k \le 7 \\ 0 & \text{for } 8 \le k \le 31 \end{cases}$$

b) Find $|X_n|^2$ (simplify as much as possible).

10 pts.

$-\frac{8}{32}$ ✗

2. The DFT of the sequence

$$x_k = \begin{cases} e^{-0.2k} & \text{for } 0 \le k \le 7 \\ 0 & \text{for } 8 \le k \le 31 \end{cases}$$

is

$$X_n = \frac{1-e^{-1.6}[\cos(\frac{n\pi}{2}) - j\sin(\frac{n\pi}{2})]}{1-e^{-0.2}[\cos(\frac{n\pi}{16}) - j\sin(\frac{n\pi}{16})]}$$

Use the modulation property to find the DFT of

$$x_k = \begin{cases} e^{-(0.2-j\frac{\pi}{8})k}, & \text{for } 0 \le k \le 7 \\ 0, & \text{for } 8 \le k \le 31 \end{cases}$$

25 pts.

3. We wish to obtain an estimate of the power spectral density of a signal by averaging 100 periodograms obtained by utilizing an FFT proussor. We will use a Hanning window and wish to have frequency samples separated by Δf=8Hz. Furthermore, we know that the signal is bandlimited by an antialiasing filter to 25 KHz. Determine the following:

a) Maximum sampling interval, T.

b) Minimum number of samples, N. (use the smallest power of 2).

c) Approximate number of complex multiplications and additions required for an estimate of the power spectral density.

30 pts.

4. a) Find the lowest order transfer function, $H(s)$, of a low-pass Chebychev filter to satisfy the following specifications:

D.C. Gain:	0dB
Passband:	< 0.5dB peak-to-peak variation for $0 \leq f \leq 2KHz$
Stopband:	> 40dB attenuation for $f > 8.5KHz$

b) Calculate the exact attenuation at $f = 3KHz$

30 pts.

5. Repeat problem 4 for a Butterworth filter.

30 pts.

6. Find the lowest-order transfer function, $H(s)$, of a bandpass Butterworth filter that satisfies the following specifications:

-3dB frequencies: $f_1 = 400$ Hz

$f_2 = 1200$ Hz

Attenuation: > 20dB for $f > 3000$Hz.

1. (a) $\text{DFT}[x_k] = X_n = \sum_{k=0}^{31} x_k e^{-j\frac{\pi}{16}nk}$

$$= \sum_{k=0}^{7} e^{-j\frac{\pi}{16}nk}$$

$$= \frac{1-e^{-j\frac{\pi}{16}n\cdot 8}}{1-e^{-j\frac{\pi}{16}n}} = \frac{1-e^{-j\frac{\pi}{2}n}}{1-e^{-j\frac{\pi}{16}n}}$$

$$= \frac{e^{-j\frac{\pi}{4}n}\left(e^{j\frac{\pi}{4}n}-e^{-j\frac{\pi}{4}n}\right)}{e^{-j\frac{\pi}{32}n}\left(e^{j\frac{\pi}{32}n}-e^{-j\frac{\pi}{32}n}\right)}$$

$$= e^{-j\frac{7}{32}\pi n}\cdot\frac{\sin(\frac{\pi}{4}n)}{\sin(\frac{\pi}{32}n)} \qquad n=0,1,\cdots,31$$

(b) $$|X_n|^2 = \frac{\sin^2(\frac{\pi}{4}n)}{\sin^2(\frac{\pi}{32}n)} = \frac{1-\cos(\frac{n\pi}{2})}{1-\cos(\frac{n\pi}{16})}$$

2. $$j\frac{\pi}{8}k = j\frac{2\pi}{32}\cdot 2\cdot k \quad \Rightarrow \quad m=2.$$

Using modulation property $\text{DFT}[e^{j\frac{2\pi}{32}\cdot m\cdot k}\cdot x_k] = X_{n-m}$

$$X_n = X'_{n-2}$$

$$= \frac{1-e^{-1.6}\left[\cos\frac{(n-2)\pi}{2} - j\sin\frac{(n-2)\pi}{2}\right]}{1-e^{-0.2}\left[\cos\frac{(n-2)\pi}{2} - j\sin\frac{(n-2)\pi}{2}\right]}, \quad n=0,1,\cdots,31$$

3. (a) $T_1 = \frac{1}{2B} = \frac{1}{2\times 25\times 1000} = 0.00002$ sec. $= 20\ \mu s$.

for real signal.

$T_2 = \frac{1}{B} = 40\ \mu s$ for complex signal.

(b) $N_1 \geq 1/\Delta f\cdot T_1 \geq \frac{1}{8\times 0.00002} \geq 6250$

$\underline{N_1 = 8192}$ (2^{13}) <u>For real signal</u>

$N_2 = \underline{4096}$ (2^{12}) <u>For complex signal</u>

(c) Complex mutiplications $\frac{N_1}{2}\log_2 N_1 \times 100 \doteq \underline{5324800}$ <u>for real signal</u>

3 (c) (CONT.)

complex additions $= N_1 \log_2 N_1 \times 100 = 10649600$

for real signal.

For complex signal:

complex multiplications $= \frac{N_2}{2} \log_2 N_2 \times 100$

$= 2457600$

complex additions $= 4915200$.

4. (a) $\omega_c = 2\pi \times 2000$ rad/sec.

$$H_{NLp}(p) \longrightarrow H_{Lp,\omega_c}(s)$$

$$p = s/\omega_c = s/2\pi \times 2000$$

① $s = j\,8500 \times 2\pi \Rightarrow p = j\,4.25 \Rightarrow \omega_p = 4.25$ rad/sec.

$A = 40$ dB , $\gamma = 0.5$ dB .

$$n \geq \cosh^{-1}\left[\sqrt{(10^{A/10}-1)/(10^{\gamma/10}-1)}\right] / \cosh^{-1}(\omega_p)$$

$$= \cosh^{-1}\left[\sqrt{(10^{4}-1)/(10^{0.05}-1)}\right] / \cosh^{-1}(4.25)$$

$$= 2.9869$$

$\therefore\ n = 3$

$$H_{NLp}(p) = \frac{0.7157}{0.7157 + 1.5349 \cdot p + 1.2529 \cdot p^2 + p^3}$$

Scale to $\omega_c = 2\pi \times 2000$

$$H(s) = \frac{0.7157}{0.7157 + 1.5349\left(\frac{s}{2\pi \times 2000}\right) + 1.2529\left(\frac{s}{2\pi \times 2000}\right)^2 + \left(\frac{s}{2\pi \times 2000}\right)^3}$$

$$= \frac{1.42024 \times 10^{12}}{1.42024 \times 10^{12} + 2.42382 \times 10^{8}\, s + 1.57444 \times 10^{4} s^2 + s^3}$$

(b) $s = j\,3000 \times 2\pi \Rightarrow p = j1.5$, $\omega_p = 1.5$ rad/sec.

$$|H(\omega_p)|^2 = \frac{1}{1 + \varepsilon^2 \cosh^2(n \cosh^{-1}(\omega_p))}$$

$$= \frac{1}{1 + (10^{0.05}-1)\cosh^2(3 \cdot \cosh^{-1}(1.5))} = (10.88349)^{-1}$$

Exact attenuation = $\underline{10.36768 \text{ dB}}$

5. (a) $\omega_c = 2\pi \times 2000$ rad/sec. Note: at ω_c only have 0.5 dB

$H_{NLP}(p) \longrightarrow H_{LP,\omega_c}(s)$

$p = s/2\pi \times 2000$

$s = j\,8500 \times 2\pi \Rightarrow \omega_p = 4.25$ rad/sec. $A = 40$ dB

For Butterworth (normalized LP),

$n \geq \ln(10^{A/10} - 1)/2\ln(\omega_p)$

$= \ln(10^4 - 1)/2\ln(4.25)$

$= 3.18271$

$f_c = \frac{f_{c_1} + f_{c_2}}{2} = 2.65\text{ KHz}$

$f_{c_1} = 2601.5$ Hz
$f_{c_2} = 2,688$ Hz

$n = 4$

$$H_{NLP}(p) = \frac{1}{p^4 + 2.6131p^3 + 3.4142p^2 + 2.6131p + 1}$$

Scale to $\omega_c = 2\pi \times 2000$ $= 2\pi \times 2.65 \times 10^3$ — 8

$$H_{LP}(s) = \frac{1}{\left(\frac{s}{2\pi \times 2000}\right)^4 + 2.6131\left(\frac{s}{2\pi \times 2000}\right)^3 + 3.4142\left(\frac{s}{2\pi \times 2000}\right)^2 + 2.6131\left(\frac{s}{2\pi \times 2000}\right) + 1}$$

$$= \frac{\overset{7.6860 \times 10^{16}}{2.49367 \times 10^{16}}}{s^4 + \underset{4.3509 \times 10^4}{3.28372 \times 10^4}\, s^3 + \underset{+9.4654 \times 10^8}{5.39149 \times 10^8}\, s^2 + \underset{1.2062 \times 10^{13}}{5.18544 \times 10^{12}}\, s + 2.49367 \times 10^{16}}$$

(b) $f = 3$ KHz $\Rightarrow \omega_p' = 1.5$ rad/sec.

$$|H_n(\omega)|^2 = \frac{1}{1 + \omega^{2n}} = \frac{1}{1 + 1.5^8}$$

$= (2.66289 \times 10)^{-1}$

Exact attenuation $\stackrel{?}{=}$ 14.2535 dB = 5.679 dB

6. $\omega_1 = 2\pi \times 400$ rad/sec. $\omega_2 = 2\pi \times 1200$ rad/sec.

$\omega_0 = \sqrt{\omega_1 \omega_2} = 2\pi \times 692.82032$ rad/sec.

$B = \omega_2 - \omega_1 = 2\pi \times 800$ rad/sec.

$$H_{LP}(P) \longrightarrow H_{BP}(S)$$

$$P = \frac{S^2 + \omega_0^2}{BS} = \frac{S^2 + (2\pi \times 692.82)^2}{2\pi \times 800 \cdot S}$$

$S = j\,3000 \times 2\pi \Rightarrow P = j\,3.55 \rightarrow \omega_P = 3.55$ rad/sec.

$A = 20$ dB.

For Butterworth LP (N),

$$n \geq \ln(10^{20/10} - 1) / 2 \ln(3.55)$$

$$= 1.81346.$$

$\therefore n = 2.$

$$H_{LP,N}(P) = \frac{1}{P^2 + 1.4142P + 1}$$

$$H(S) = \frac{1}{\left(\frac{S^2 + (2\pi \times 692.82)^2}{2\pi \times 800 \cdot S}\right)^2 + 1.4142\left(\frac{S^2 + (2\pi \times 692.82)^2}{2\pi \times 800\, S}\right) + 1}$$

$$= \frac{2.52662 \times 10^7 S^2}{S^4 + 3.78992 \times 10^7 S^2 + 3.59088 \times 10^{14} + 7.10854 \times 10^3 S^3 + 1.34704 \times 10^{11} S + 2.52662 \times 10^7 S^2}$$

$$= \frac{2.52662 \times 10^7 \cdot S^2}{S^4 + 7.10854 \times 10^3 \cdot S^3 + 6.31654 \times 10^7 S^2 + 1.34704 \times 10^{11} S + 3.59088 \times 10^{14}}$$