HOMEWORK #1 EE 425 | DIGITAL COMMUNICATON SYSTEM | Ben Chen

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Derive equations A-42 and A-43 :

A-42 : Given the joint probability density function of N jointly Gaussian random variables

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = (2\pi)^{-N/2} \cdot |\det \mathbb{C}|^{-1/2} \exp\left[-\tfrac{1}{2}(\underset{\sim}{x}-\underset{\sim}{m})^t \mathbb{C}^{-1}(\underset{\sim}{x}-\underset{\sim}{m})\right]$$

and the definition of joint characteristic function

$$M_{\underset{\sim}{X}}(\underset{\sim}{v}) = E[\exp(j\underset{\sim}{v}^t\underset{\sim}{X})] .$$

Show that the joint characteristic function of the Gaussian R.V.'s $X_1, X_2, \cdots X_N$ is

$$M_{\underset{\sim}{X}}(\underset{\sim}{v}) = \exp(j\underset{\sim}{m}^t\underset{\sim}{v} - \tfrac{1}{2}\underset{\sim}{v}^t\mathbb{C}\underset{\sim}{v})$$

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SHOW: Because $\underset{\sim}{v}^t\underset{\sim}{x}$, $\underset{\sim}{x}^t\underset{\sim}{v}$, $\underset{\sim}{x}^t\underset{\sim}{m}$ and $\underset{\sim}{m}^t\underset{\sim}{x}$ are all scalars, we have

$$j\underset{\sim}{v}^t\underset{\sim}{x} = \tfrac{1}{2}j\underset{\sim}{v}^t\underset{\sim}{x} + \tfrac{1}{2}j\underset{\sim}{x}^t\underset{\sim}{v} \quad \& \quad j\underset{\sim}{m}^t\underset{\sim}{v} = \tfrac{1}{2}j\underset{\sim}{m}^t\underset{\sim}{v} + \tfrac{1}{2}\cdot j\cdot\underset{\sim}{v}^t\underset{\sim}{m} \quad \cdots\cdots (1)$$

$$\begin{aligned}
M_{\underset{\sim}{X}}(\underset{\sim}{v}) &= E[\exp(j\cdot\underset{\sim}{v}^t\underset{\sim}{X})] \\
&= \int_{-\infty}^{\infty} \exp(j\cdot\underset{\sim}{v}^t\underset{\sim}{X})\cdot f_{\underset{\sim}{X}}(\underset{\sim}{x})\,d\underset{\sim}{x} \\
&= (2\pi)^{-N/2}\cdot|\det\mathbb{C}|^{-1/2}\cdot\int_{-\infty}^{\infty}\exp[j\cdot\underset{\sim}{v}^t\underset{\sim}{x} - \tfrac{1}{2}(\underset{\sim}{x}-\underset{\sim}{m})^t\mathbb{C}^{-1}(\underset{\sim}{x}-\underset{\sim}{m})]\,d\underset{\sim}{x}
\end{aligned}$$

Use eq. (1) above ; $\longrightarrow$

$$= (2\pi)^{-N/2}\cdot|\det\mathbb{C}|^{-1/2}\cdot\int_{-\infty}^{\infty}\exp(j\underset{\sim}{m}^t\underset{\sim}{v} - \tfrac{1}{2}\underset{\sim}{v}^t\mathbb{C}\underset{\sim}{v})\cdot\exp[-\tfrac{1}{2}(\underset{\sim}{x}-\underset{\sim}{m})^t\mathbb{C}^{-1}(\underset{\sim}{x}-\underset{\sim}{m}) + \tfrac{1}{2}j\underset{\sim}{v}^t\underset{\sim}{x}$$
$$- \tfrac{1}{2}j\underset{\sim}{v}^t\underset{\sim}{m} + \tfrac{1}{2}j\underset{\sim}{x}^t\underset{\sim}{v} - \tfrac{1}{2}j\underset{\sim}{m}^t\underset{\sim}{v} + \tfrac{1}{2}\underset{\sim}{v}^t\mathbb{C}\underset{\sim}{v}]\,d\underset{\sim}{x}$$

$\mathbb{C}$ is p.d., $\mathbb{C}^t = \mathbb{C}$; $\longrightarrow$

$$= \exp(j\underset{\sim}{m}^t\underset{\sim}{v} - \tfrac{1}{2}\underset{\sim}{v}^t\mathbb{C}\underset{\sim}{v})\cdot\int_{-\infty}^{\infty}(2\pi)^{-N/2}\cdot|\det\mathbb{C}|^{-1/2}\cdot\exp[-\tfrac{1}{2}(\underset{\sim}{x}-\underset{\sim}{m}-j\mathbb{C}\underset{\sim}{v})^t\mathbb{C}^{-1}(\underset{\sim}{x}-\underset{\sim}{m}-j\mathbb{C}\underset{\sim}{v})]\,d\underset{\sim}{x}$$

Let $\underset{\sim}{u} = (\underset{\sim}{x}-\underset{\sim}{m}-j\mathbb{C}\underset{\sim}{v})$; $\longrightarrow$

$$= \exp(j\underset{\sim}{m}^t\underset{\sim}{v} - \tfrac{1}{2}\underset{\sim}{v}^t\mathbb{C}\underset{\sim}{v})\cdot\int_{-\infty}^{\infty *}\underline{(2\pi)^{-N/2}\cdot|\det\mathbb{C}|^{-1/2}\cdot\exp[-\tfrac{1}{2}\underset{\sim}{u}^t\mathbb{C}^{-1}\underset{\sim}{u}]}\,d\underset{\sim}{u}$$

notice that $(2\pi)^{-N/2}\cdot|\det\mathbb{C}|^{-1/2}\cdot\exp(-\tfrac{1}{2}\underset{\sim}{u}^t\mathbb{C}^{-1}\underset{\sim}{u})$ is a joint p.d.f. of new Gaussian variables $\underset{\sim}{u}$. So, the integral in eq. above is 1. Thus, we have

$$M_{\underset{\sim}{X}}(\underset{\sim}{v}) = \exp(j\underset{\sim}{m}^t\underset{\sim}{v} - \tfrac{1}{2}\underset{\sim}{v}^t\mathbb{C}\underset{\sim}{v}) \qquad \underline{\text{Q.E.D.}}$$

Eq. A-34 on page 658 shall be : $M_{\underset{\sim}{X}}(\underset{\sim}{v}) = E[\exp(j\underset{\sim}{v}^t\underset{\sim}{X})]$

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* Actually, this is an integral with complex variables (or vector $\underset{\sim}{u}$). According to Cauthy theorem, and because exp(·) is differentiable in whole complex plane, we get the same result by taking the integral along the real axis (from $-\infty$ to ∞).

A-43: Show that for any four zero-mean Gaussian random variables

$$E(X_1X_2X_3X_4) = E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3)$$

SHOW: With $\underset{\sim}{m} = \underset{\sim}{0}$, the equation A-42 can be written as

$$M_{\underset{\sim}{X}}(\underset{\sim}{V}) = \exp(-\tfrac{1}{2}\underset{\sim}{V}^t \mathbb{C}\, \underset{\sim}{V}) \quad \text{and} \quad C_{ik} = R_{X_iX_k}$$

Use the equation on top of page 6 in Lecture notes NO: 2, we have

$$\underset{\sim}{V}^t \mathbb{C}\, \underset{\sim}{V} = \sum_{i=1}^{4}\sum_{k=1}^{4} V_i V_k R_{X_iX_k} \qquad \text{(Assume we only have 4 RV's)}$$

Then expand $M_{\underset{\sim}{X}}(\underset{\sim}{V}) = \exp(-\frac{1}{2}\underset{\sim}{V}^t \mathbb{C}\, \underset{\sim}{V}) = \exp(-\frac{1}{2}\sum_{i=1}^{4}\sum_{k=1}^{4} V_i V_k R_{X_iX_k})$.

a. $$M_{\underset{\sim}{X}}(\underset{\sim}{V}) = \exp(-\tfrac{1}{2}\sum_{i=1}^{4}\sum_{k=1}^{4} V_i V_k R_{X_iX_k})$$

$$= 1 - \frac{1}{2}\sum_{i=1}^{4}\sum_{k=1}^{4} V_i V_k R_{X_iX_k} + \frac{1}{2!}\left(-\frac{1}{2}\sum_{i=1}^{4}\sum_{k=1}^{4} V_i V_k R_{X_iX_k}\right)^2 + \frac{1}{3!}\left(-\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} V_i V_k R_{X_iX_k}\right)^3 + \cdots$$

$$= 1 - \frac{1}{2}\sum_{i=1}^{4}\sum_{k=1}^{4} V_i V_k R_{X_iX_k} + \underbrace{\frac{1}{8}\sum_{i=1}^{4}\sum_{k=1}^{4}\sum_{m=1}^{4}\sum_{n=1}^{4} V_i V_k V_m V_n \cdot R_{X_iX_k} \cdot R_{X_mX_n}}$$

$$- \frac{1}{48}\sum_{i=1}^{4}\sum_{k=1}^{4}\sum_{m=1}^{4}\sum_{n=1}^{4}\sum_{r=1}^{4}\sum_{s=1}^{4} V_i V_k V_m V_n V_r V_s R_{X_iX_k} R_{X_mX_n} R_{X_rX_s} + \cdots$$

From A-36 in text book, we see that

b. $$E(X_1X_2X_3X_4) = (-j)^4 \left.\frac{\partial^4 M_{\underset{\sim}{X}}(\underset{\sim}{V})}{\partial V_1 \partial V_2 \partial V_3 \partial V_4}\right|_{V=0}$$

From a. and b. above, we see that if we take partial diff. to $M_{\underset{\sim}{X}}(\underset{\sim}{V})$ in a. and let $\underset{\sim}{V} = 0$, then all terms in a. must be zero, except the terms have the form like

$$V_1V_2V_3V_4 R_{X_1X_2}R_{X_3X_4} \quad \text{OR} \quad V_1V_2V_3V_4 R_{X_1X_3}R_{X_2X_4} \quad \text{OR} \quad V_1V_2V_3V_4 R_{X_1X_4}R_{X_2X_3} \quad \cdots \quad (1)$$

$$\left.\frac{\partial^4}{\partial V_1 \partial V_2 \partial V_3 \partial V_4} V_1V_2V_3V_4 R_{X_1X_2}R_{X_3X_4}\right|_{\underset{\sim}{V}=0} = R_{X_1X_2} R_{X_3X_4} = E(X_1X_2)E(X_3X_4)$$

$$\left.\frac{\partial^4}{\partial \underset{\sim}{V}} V_1V_2V_3V_4 R_{X_1X_3}R_{X_2X_4}\right|_{\underset{\sim}{V}=0} = E(X_1X_3)E(X_2X_4) \quad \& \quad \left.\frac{\partial^4}{\partial \underset{\sim}{V}} V_1V_2V_3V_4 R_{X_1X_4}R_{X_2X_3}\right|_{\underset{\sim}{V}=0} = E(X_1X_4)E(X_2X_3)$$

And from page 7 in Lecture note 2, we know each of the forms in (1) has $4!/3 = 2 \times 4 = 8$ terms. So, we have

$$E(X_1X_2X_3X_4) = \frac{1}{8}\left(8E(X_1X_2)E(X_3X_4) + 8E(X_1X_3)E(X_2X_4) + 8E(X_1X_4)E(X_2X_3)\right]$$

$$= E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3) \qquad \underline{\text{Q.E.D}}$$

A-3. *The Rayleigh probability density function is given by*

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} & 0 \le r < \infty \\ 0 & r < 0 \end{cases}$$

(a) *Find the distribution function of a Rayleigh random variable.*

(b) *Find* $Pr(0.1\sigma < r \le 0.2\sigma)$ *for a Rayleigh random variable.*

(c) *Show that the mean and mean square of a Rayleigh random variable are given by* $\sqrt{\pi/2}\,\sigma$ *and* $2\sigma^2$, *respectively.*

(d) *Find the variance of a Rayleigh random variable.*

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Solution: (a) Distribution function:

$$F_R(r) = \int_{-\infty}^{r} f_R(x)\,dx = \int_0^r \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} dx \qquad (\text{If } 0 \le r < \infty)$$

$$= \int_0^r e^{-x^2/2\sigma^2} \cdot d\left(\frac{x^2}{2\sigma^2}\right)$$

$$(\text{Let } y = x^2/2\sigma^2) \qquad = \int_0^{r^2/2\sigma^2} e^{-y}\,dy$$

$$= e^{-y}\Big|_{r^2/2\sigma^2}^{0} = 1 - e^{-r^2/2\sigma^2} \qquad 0 \le r < \infty$$

$$\therefore \quad F_R(r) = \begin{cases} 1 - e^{-r^2/2\sigma^2} & 0 \le r < \infty \\ 0 & r < 0 \end{cases}$$

(b)

$$Pr(0.1\sigma < r \le 0.2\sigma) = F_R(0.2\sigma) - F_R(0.1\sigma)$$

$$= e^{-0.01\sigma^2/2\sigma^2} - e^{-0.04\sigma^2/2\sigma^2}$$

$$= 0.0148$$

(c)

$$m_R = \int_{-\infty}^{\infty} r \cdot f_R(r)\,dr = \int_0^{\infty} r \cdot \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} dr$$

$$= \frac{1}{\sigma^2} \cdot \frac{2\sigma^2}{4} \cdot \sqrt{\pi \cdot 2\sigma^2} = \sqrt{\pi/2} \cdot \sigma \qquad \text{Q.E.D}$$

$$m_2 = \int_{-\infty}^{\infty} r^2 f_R(r)\,dr = \int_0^{\infty} \frac{1}{\sigma^2} r^3 \cdot e^{-r^2/2\sigma^2} dr$$

$$= -\int_0^{\infty} r^2 \cdot d e^{-r^2/2\sigma^2} = -r^2 \cdot e^{-r^2/2\sigma^2}\Big|_0^{\infty} + \int_0^{\infty} e^{-r^2/2\sigma^2} \cdot 2 \cdot r\,dr$$

$$= 2\int_0^{\infty} r \cdot e^{-r^2/2\sigma^2} dr = 2 \cdot \frac{2\sigma^2}{2} = 2\sigma^2 \qquad \text{Q.E.D.}$$

(A-3. CONT.) (d) Variance:

$$\underline{\sigma_R^2 = E\{(R-m_R)^2\} = m_2 - m_R^2 = 2\sigma^2 - \frac{\pi}{2}\sigma^2 = 0.4292\,\sigma^2}$$

<u>A-4.</u> Two random variables, R and Φ, have the joint probability density function

$$f_{R\Phi}(r,\varphi) = \begin{cases} \frac{r}{2\pi\sigma^2}\exp\left[\frac{-(r^2-2Ar\cos\varphi+A^2)}{2\sigma^2}\right] & 0\le r<\infty;\ -\pi<\varphi\le\pi \\ 0 & \text{Otherwise} \end{cases}$$

Show that the marginal probability density function of R is

$$f_R(r) = \frac{r}{\sigma^2}\cdot\exp\left[\frac{-(A^2+r^2)}{2\sigma^2}\right]\cdot I_0\left(\frac{Ar}{\sigma}\right) \qquad r\ge 0$$

where $I_0(u)$, the modified Bessel function of the first kind and order 0, is given by

$$I_0(u) = \frac{1}{2\pi}\int_0^{2\pi} e^{u\cos\varphi}\,d\varphi$$

This probability density function is known as the Rice-Nakagami, or simply Rician, density fun.

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<u>SHOW:</u> The marginal probability density function of R:

$$f_R(r) = \int_{-\infty}^{\infty} f_{R\Phi}(r,\varphi)\,d\varphi$$

If $r\ge 0$ →

$$= \int_{-\pi}^{\pi}\frac{r}{2\pi\sigma^2}\cdot\exp\left[\frac{-(r^2-2Ar\cos\varphi+A^2)}{2\sigma^2}\right]\cdot d\varphi$$

$$= \frac{r}{2\pi\sigma^2}\cdot\exp\left(\frac{-r^2-A^2}{2\sigma^2}\right)\cdot\int_{-\pi}^{\pi} e^{2Ar\cos\varphi/2\sigma^2}\,d\varphi$$

$$= \frac{r}{\sigma^2}\cdot\exp\left[\frac{-(r^2+A^2)}{2\sigma^2}\right]\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\frac{Ar}{\sigma^2}\cos\varphi}\,d\varphi$$

Let $\theta = \varphi + 2\pi$

$$= \frac{r}{\sigma^2}\cdot\exp\left[\frac{-(r^2+A^2)}{2\sigma^2}\right]\cdot\frac{1}{2\pi}\cdot\left[\underbrace{\int_{-\pi}^{0} e^{\frac{Ar}{\sigma^2}\cos\varphi}\,d\varphi}_{} + \int_0^{\pi} e^{\frac{Ar}{\sigma^2}\cos\varphi}\,d\varphi\right]$$

$d\theta = d\varphi$

$\cos\theta = \cos(2\pi+\varphi)$

$= \cos\varphi$

$$= \frac{r}{\sigma^2}\cdot\exp\left[\frac{-(A^2+r^2)}{2\sigma^2}\right]\cdot\frac{1}{2\pi}\cdot\left[\int_{\pi}^{2\pi} e^{\frac{Ar}{\sigma^2}\cos\theta}\,d\theta + \int_0^{\pi} e^{\frac{Ar}{\sigma^2}\cos\varphi}\,d\varphi\right]$$

$$= \frac{r}{\sigma^2}\cdot\exp\left[\frac{-(A^2+r^2)}{2\sigma^2}\right]\cdot\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{Ar}{\sigma^2}\cos\varphi}\,d\varphi$$

$$\underline{F_R(r) = \frac{r}{\sigma^2}\cdot\exp\left[\frac{-(A^2+r^2)}{2\sigma^2}\right]\cdot I_0\left(\frac{Ar}{\sigma^2}\right)} \qquad r\ge 0$$

It is close to the equation in statement. But, we have $I_0(\frac{Ar}{\sigma^2})$ instead of $I_0(\frac{Ar}{\sigma})$.

A-23 OR P10 IN L:2 : Perform the appropriate matrix operations to show that for $N=2$, the jointly Gaussian, they have the joint probability density function

$$f_{X_1X_2}(x_1,x_2)=\frac{\exp\left[-\frac{\left(\frac{x_1-m_{X_1}}{\sigma_{X_1}}\right)^2-2\rho\left(\frac{x_1-m_{X_1}}{\sigma_{X_1}}\right)\left(\frac{x_2-m_{X_2}}{\sigma_{X_2}}\right)+\left(\frac{x_2-m_{X_2}}{\sigma_{X_2}}\right)^2}{2(1-\rho^2)}\right]}{2\pi\,\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}}$$

SHOW : From the definition of joint Gaussian p.d.f., we know

$$f_{X_1X_2}(x_1,x_2)=(2\pi)^{-2/2}\cdot|\det\mathbb{C}|^{-1/2}\exp\left[-\tfrac{1}{2}(\underset{\sim}{x}-\underset{\sim}{m})^t\,\mathbb{C}^{-1}(\underset{\sim}{x}-\underset{\sim}{m})\right]\quad\cdots\cdots(1)$$

where

$$\underset{\sim}{x}-\underset{\sim}{m}=\begin{pmatrix}x_1\\x_2\end{pmatrix}-\begin{pmatrix}m_{X_1}\\m_{X_2}\end{pmatrix}=\begin{pmatrix}x_1-m_{X_1}\\x_2-m_{X_2}\end{pmatrix}$$

$$\mathbb{C}=\begin{bmatrix}E\{(X_1-m_{X_1})^2\} & E\{(X_1-m_{X_1})(X_2-m_{X_2})\}\\ E\{(X_2-m_{X_2})(X_1-m_{X_1})\} & E\{(X_2-m_{X_2})^2\}\end{bmatrix}$$

$$=\begin{bmatrix}\sigma^2_{X_1} & \rho\cdot\sigma_{X_1}\sigma_{X_2}\\ \rho\sigma_{X_1}\sigma_{X_2} & \sigma^2_{X_2}\end{bmatrix}$$

$$\mathbb{C}^{-1}=\frac{\begin{bmatrix}\sigma^2_{X_2} & -\rho\sigma_{X_1}\sigma_{X_2}\\ -\rho\sigma_{X_1}\sigma_{X_2} & \sigma^2_{X_1}\end{bmatrix}}{\sigma^2_{X_1}\sigma^2_{X_2}(1-\rho^2)}=\frac{1}{1-\rho^2}\cdot\begin{bmatrix}\frac{1}{\sigma^2_{X_1}} & -\frac{\rho}{\sigma_{X_1}\sigma_{X_2}}\\ -\frac{\rho}{\sigma_{X_1}\sigma_{X_2}} & \frac{1}{\sigma^2_{X_2}}\end{bmatrix}\quad\cdots\cdots(2)$$

$$\det\mathbb{C}=\sigma^2_{X_1}\sigma^2_{X_2}(1-\rho^2)\qquad|\det\mathbb{C}|^{-1/2}=\frac{1}{\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}}\quad\cdots\cdots(3)$$

$$(\underset{\sim}{x}-\underset{\sim}{m})^t\,\mathbb{C}^{-1}(\underset{\sim}{x}-\underset{\sim}{m})=[x_1-m_{X_1}\quad x_2-m_{X_2}]\cdot\frac{1}{1-\rho^2}\cdot\begin{bmatrix}\frac{1}{\sigma^2_{X_1}} & -\frac{\rho}{\sigma_{X_1}\sigma_{X_2}}\\ -\frac{\rho}{\sigma_{X_1}\sigma_{X_2}} & \frac{1}{\sigma^2_{X_2}}\end{bmatrix}\cdot\begin{bmatrix}x_1-m_{X_1}\\ x_2-m_{X_2}\end{bmatrix}$$

$$=\frac{\left[\frac{1}{\sigma^2_{X_1}}(x_1-m_{X_1})-\frac{\rho}{\sigma_{X_1}\sigma_{X_2}}(x_2-m_{X_2}),\ \frac{1}{\sigma^2_{X_2}}(x_2-m_{X_2})-\frac{\rho}{\sigma_{X_1}\sigma_{X_2}}(x_1-m_{X_1})\right]\cdot\begin{bmatrix}x_1-m_{X_1}\\ x_2-m_{X_2}\end{bmatrix}}{1-\rho^2}$$

$$=\frac{\left(\frac{x_1-m_{X_1}}{\sigma_{X_1}}\right)^2-2\rho\left(\frac{x_1-m_{X_1}}{\sigma_{X_1}}\right)\left(\frac{x_2-m_{X_2}}{\sigma_{X_2}}\right)+\left(\frac{x_2-m_{X_2}}{\sigma_{X_2}}\right)^2}{1-\rho^2}\quad\cdots(4)$$

Substitute equation (3) and (4) into eq. (1), we have

$$f_{X_1X_2}(x_1,x_2)=\frac{\exp\left[-\frac{\left(\frac{x_1-m_{X_1}}{\sigma_{X_1}}\right)^2-2\rho\left(\frac{x_1-m_{X_1}}{\sigma_{X_1}}\right)\left(\frac{x_2-m_{X_2}}{\sigma_{X_2}}\right)+\left(\frac{x_2-m_{X_2}}{\sigma_{X_2}}\right)^2}{2(1-\rho^2)}\right]}{2\pi\,\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}}\qquad\text{Q.E.D.}$$

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1-1. A modulation scheme is employed wherein a signal whose amplitude can assume one of 64 different levels is transmitted each 10^{-4} s trough a channel

(a) What is the baud rate ?

(b) How many binary digits are necessary to specify a particular level ?

(c) What is the bit rate through the channel ?

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Solution: $T_s = 10^{-4}$ seconds $q = 64$ symbols

(a) Baud rate or symbol rate : $R_s = \frac{1}{T_s} = \frac{1}{10^{-4}} = \underline{10^4 = 10000}$ symbols per second.

(b) $k = \log_2 q = \log_2 64 = \log_2 2^6 = \underline{6}$ (binary digits)

(c) Bit rate : $R_b = k/T_s = 6/10^{-4} = \underline{60000}$ bits per second.

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1-6. Consider a zero-memory source with q symbols, $x_1, x_2, \cdots, x_q$ with corresponding probability $P_1, P_2, \cdots, P_q$. The nth extension of this source is a source whose symbols are all possible n-symbol sequences of symbols from the original source, and whose symbol probabilities are the products of the corresponding symbol probabilities from the original source. For example, one possible symbol for the third extension of a four-symbol source is $x_1 x_4 x_2$ with probability $P_1 P_2 P_4$.

Denote the original source by X and its nth extension by X^n. Show that

$$H(X^n) = n\, H(X)$$

where the left-hand side represents the entropy of the nth extension and $H(X)$ is the entropy of the source X.

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To be showed on next page.

1-6 (CONT.)

SHOW: Firstmost, we see that nth extension of q symbols has q^n nth extension symbols. for example, in octal scale, we have $q=10$ (0,1,2,3,4,5,6,7,8,9), if $n=1$, we have 10 detterent numbers; $n=2$, we have 100 detterent numbers; ...

For this problem, we can use the mathematical inductive method to prove it.

Step I: For $n=1$, it is easy to see that

$$H(X^1) = H(X) = 1\cdot H(X)$$

Step II: Assuming that for $n=k$, we have

$$H(X^k) = k\,H(X) \quad \text{------} \quad (1)$$

and the kth extension symbols and probabilities are:

$$\begin{array}{lll}
 & \underline{\text{symbols}} & \underline{\text{probabilities}} \\
X_1^k : & \underbrace{x_1x_1\cdots x_1}_{k} \text{ ------ } & P(X_1^k) \\
X_2^k : & x_1x_1\cdots x_2 \text{ ------ } & P(X_2^k) \\
\vdots & \quad\vdots \qquad\quad \vdots & \quad\vdots \\
X_{q^k}^k : & x_qx_q\cdots x_q \text{ ------ } & P(X_{q^k}^k)
\end{array}$$

Step III: For $n=k+1$, according the definition of $(k+1)$th extension, we have

$$\begin{array}{lll}
\multicolumn{2}{l}{\underline{(k+1)\text{th extension symbols}}} & \underline{\text{probabilities}} \\
X_1^{k+1} : & x_1\, X_1^k \text{ ---------- } & P_1\cdot P(X_1^k) \\
X_2^{k+1} : & x_1\, X_2^k \text{ ---------- } & P_1\cdot P(X_2^k) \\
\vdots & \vdots \qquad\quad \vdots & \vdots \\
X_{q^k}^{k+1} : & x_1\, X_{q^k}^k \text{ ---------- } & P_1\cdot P(X_{q^k}^k) \\
X_{q^k+1}^{k+1} : & x_2\, X_1^k \text{ ---------- } & P_2\cdot P(X_1^k) \\
\vdots & \vdots \qquad\quad \vdots & \vdots \\
X_{2q^k}^{k+1} : & x_2\, X_{q^k}^k \text{ ---------- } & P_2\cdot P(X_{q^k}^k) \\
\vdots & \qquad\quad \vdots & \vdots \\
X_{q^{k+1}-q^k+1}^{k+1} : & x_q\, X_1^k \text{ ---------- } & P_q\cdot P(X_1^k) \\
\vdots & \vdots \qquad\quad \vdots & \vdots \\
X_{q^{k+1}}^{k+1} : & x_q\, X_q^k \text{ ---------- } & P_q\cdot P(X_{q^k}^k)
\end{array}$$

(To be continued on next page.)

1-6 (CONT.)

Now, we can use the definition of entropy to find $H(\mathbb{X}^{k+1})$

$$H(\mathbb{X}^{k+1}) = -\sum_{i=1}^{q^{k+1}} P(x_i^{k+1}) \log_2 P(x_i^{k+1})$$

$$= -\sum_{i=1}^{q^k} P(x_i^{k+1}) \log_2 P(x_i^{k+1}) - \sum_{i=q^k+1}^{2q^k} P(x_i^{k+1}) \log_2 P(x_i^{k+1}) - \cdots$$

$$- \sum_{i=(q-1)q^k+1}^{q^{k+1}} P(x_i^{k+1}) \log_2 P(x_i^{k+1})$$

$$= -\sum_{i=1}^{q^k} P_1 \cdot P(x_i^k) \log_2 [P_1 \cdot P(x_i^k)] - \sum_{i=1}^{q^k} P_2 \cdot P(x_i^k) \log_2 [P_2 \cdot P(x_i^k)]$$

$$- \cdot \cdot \cdot - \sum_{i=1}^{q^k} P_q P(x_i^k) \log [P_q \cdot P(x_i^k)]$$

$$= -\sum_{j=1}^{q} \sum_{i=1}^{q^k} P_j P(x_i^k) \log_2 [P_j P(x_i^k)]$$

$$= -\sum_{j=1}^{q} P_j \cdot \sum_{i=1}^{q^k} P(x_i^k) \cdot [\log_2 P_j + \log_2 P(x_i^k)]$$

$$= \underbrace{-\sum_{j=1}^{q} P_j \log_2 P_j}_{= H(\mathbb{X})} \underbrace{\left(\sum_{i=1}^{q^k} P(x_i^k)\right]}_{=1} - \underbrace{\sum_{j=1}^{q} P_j}_{=1} \cdot \underbrace{\sum_{i=1}^{q^k} P(x_i^k) \log_2 P(x_i^k)}_{-H(\mathbb{X}^k)}$$

$$= H(\mathbb{X}) \cdot 1 - 1 \cdot [-H(\mathbb{X}^k)]$$

$$= H(\mathbb{X}) + k H(\mathbb{X}) \quad \leftarrow - - - \quad \text{eq. (1)}$$

$$\therefore \quad H(\mathbb{X}^{k+1}) = (k+1) H(\mathbb{X})$$

Thus, we proved that

$$H(\mathbb{X}^n) = n H(\mathbb{X})$$

✓

for $n = 1, 2, \cdots, k, k+1, \cdots$

Q.E.D.

1-7. Referring to Problem 1-6, consider a three-symbol source, $X = \{x_1, x_2, x_3\}$, with symbol probability $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{4}$.

(a) List all symbols of the second extension and their corresponding probab.

(b) Find the entropies of the original source and of the second extension, thus verifying that $H(X^2) = 2H(X)$.

- - - - - - - - - -

Solution: (a)

second extension symbols	corresponding probabilities
$x_1 x_1$	(1/4) 0.25
$x_1 x_2$	(1/8) 0.125
$x_1 x_3$	(1/8) 0.125
$x_2 x_1$	(1/8) 0.125
$x_2 x_2$	(1/16) 0.0625
$x_2 x_3$	(1/16) 0.0625
$x_3 x_1$	(1/8) 0.125
$x_3 x_2$	(1/16) 0.0625
$x_3 x_3$	(1/16) 0.0625

(b)

$$H(X) = -\left(\tfrac{1}{2}\log_2 2^{-1} + \tfrac{1}{4}\log_2 2^{-2} + \tfrac{1}{4}\log_2 2^{-2}\right)$$

$$= \frac{1}{2} + \frac{2}{4} + \frac{2}{4} = 1.5 \quad \text{bits}$$

$$H(X^2) = -(0.25\log_2 0.25 + 0.125\log_2 0.125 + 0.125\log_2 0.125$$

$$+ 0.125\log_2 0.125 + 0.0625\log_2 0.0625 + 0.0625\log_2 0.0625$$

$$+ 0.125\log_2 0.125 + 0.0625\log_2 0.0625 + 0.0625\log_2 0.0625)$$

$$= \frac{1}{0.30103}\left(0.25 \times 0.60206 + 4 \times 0.125 \times 0.90309 + 4 \times 0.0625 \times 1.20412\right)$$

$$= 3 \quad \text{bits}$$

Hence,

$$H(X^2) = 3$$

$$2H(X) = 2 \times 1.5 = 3$$

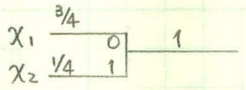
$$H(X^2) = 2H(X)$$ VERIFIED!

- -

1-10. (a) Obtain Huffman codes for a binary source with symbol probabilities $\frac{3}{4}$ and $\frac{1}{4}$ and its second and third extensions.

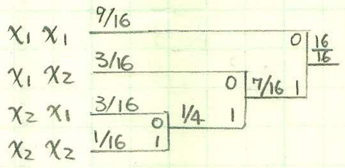
(b) Compare the average lengths of the codes found in (a) with lower bound.

Solution : (a) i) Original source



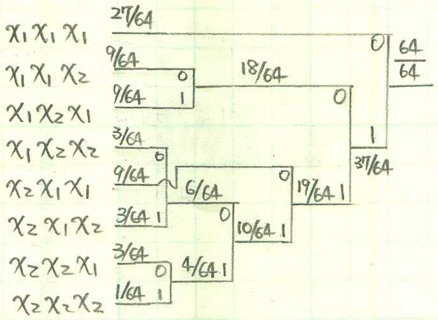
symbols	codes
x_1	0
x_2	1

ii) Second extension



symbols	codes
$x_1 x_1$	0
$x_1 x_2$	10
$x_2 x_1$	110
$x_2 x_2$	111

iii) Third extension



symbols	codes
$x_1x_1x_1$	0
$x_1x_1x_2$	100
$x_1x_2x_1$	101
$x_1x_2x_2$	11100
$x_2x_1x_1$	110
$x_2x_1x_2$	11101
$x_2x_2x_1$	11110
$x_2x_2x_2$	11111

(b) $H(X) = -\frac{3}{4}\cdot\log_2\frac{3}{4} - \frac{1}{4}\log_2\frac{1}{4} = 0.81128$ bits

According to Shannon's first theorem, we know that the lower bound of the averge length is 0.81128 bits.

Comparision : For i) $L = 1\times\frac{3}{4} + 1\times\frac{1}{4} = 1$ bit

$\text{Effic.} = \frac{0.81128}{1} = 81.128\%$, Redun. = 18.872%

For ii) $L = \frac{1}{2}(1\times\frac{9}{16} + 2\times\frac{3}{16} + \frac{3}{16}\times3 + \frac{1}{16}\times3) = 0.84375$ bits

$\text{Effic.} = \frac{0.81128}{0.84375} = 96.152\%$ Redun. = 3.848 %

For iii) $L = \frac{1}{3}(\frac{27}{64} + 3\times\frac{9}{64} + 3\times\frac{9}{64} + 5\times\frac{3}{64} + 3\times\frac{9}{64} + 5\times\frac{3}{64} + 5\times\frac{3}{64} + \frac{5}{64})$

$= 0.82292$

$\text{Effic.} = \frac{0.81128}{0.82292} = 98.586\%$ Redun. = 1.414 %

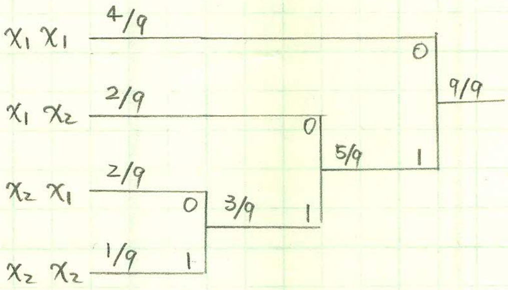
1-11 A binary source has symbol probabilities $\frac{2}{3}$ and $\frac{1}{3}$. Symbols are emitted at a rate of 1000 per second. The channel can accept binary symbols at a rate of 950 /s

(a) Is it possible, by suitable encoding, to transmit the source trough the channel?

(b) If the answer to part (a) is yes, find a code that will allow it.

Solution: (a) Yes, it is possible.

(b) We can find the second extension Huffman code as below:



second extension symbols	codes
$x_1\ x_1$	0
$x_1\ x_2$	10
$x_2\ x_1$	110
$x_2\ x_2$	111

average $L = 1\times\frac{4}{9} + 2\times\frac{2}{9} + 3\times\frac{2}{9} + 3\times\frac{1}{9} = \frac{17}{9} = 1.88889$ bits

Obviously, the second extension symbols are emitted at a rate of 500 per second and we only have to emit $1.88889\times 500 =$ 944.44 bits per second by using this code. Since

$$944.44 < 950$$

Thus, it is O.K. to transmit the source output trough this channel.

Homework assignment no:3 Digital Communications 50/50 Ben Chen

1. HOMEWORK P15 LECTURE 7:

a) Find a linear (5,2) for which $d_{min} = 3$ (Show linearity).

b) Demonstrate that one error can be corrected for this code.

Solution: a) We can show that following code is linear (5,2) with $d_{min} = 3$

$$\underset{\sim}{S}_0 = (0\ 0\ 0\ 0\ 0)$$

$$\underset{\sim}{S}_1 = (0\ 1\ 1\ 0\ 1)$$

$$\underset{\sim}{S}_2 = (1\ 0\ 1\ 1\ 0)$$

$$\underset{\sim}{S}_3 = (1\ 1\ 0\ 1\ 1)$$

Linearity: $\underset{\sim}{S}_0 \oplus \underset{\sim}{S}_i = \underset{\sim}{S}_i$; $i = 1, 2, 3$

$$\underset{\sim}{S}_1 \oplus \underset{\sim}{S}_2 = (1\ 1\ 0\ 1\ 1) = S_3$$

$$\underset{\sim}{S}_1 \oplus \underset{\sim}{S}_3 = (1\ 0\ 1\ 1\ 0) = S_2$$

$$\underset{\sim}{S}_2 \oplus \underset{\sim}{S}_3 = (0\ 1\ 1\ 0\ 1) = S_1$$ CHECKED! ✓

∴ This code is linear.

$$W(\underset{\sim}{S}_1) = 3,\ W(\underset{\sim}{S}_2) = 3,\ W(\underset{\sim}{S}_3) = 4.$$

$$d_{min} = \min_{i=1,2,3} W(\underset{\sim}{S}_i) = 3$$ CHECKED! ✓

b) Assume that we have error vector $\underset{\sim}{e} = (0\ 0\ 1\ 0\ 0)$ and codeword $\underset{\sim}{S}_2 = (1\ 0\ 1\ 1\ 0)$ is transmitted. Then, the received code is

$$\underset{\sim}{r} = \underset{\sim}{S}_2 \oplus \underset{\sim}{e} = (1\ 0\ 0\ 1\ 0)$$

$$d_{\underset{\sim}{r},\underset{\sim}{S}_0} = W(\underset{\sim}{r} \oplus \underset{\sim}{S}_0) = W[(1\ 0\ 0\ 1\ 0)] = 2$$

$$d_{\underset{\sim}{r},\underset{\sim}{S}_1} = W(\underset{\sim}{r} \oplus \underset{\sim}{S}_1) = W[(1\ 1\ 1\ 1\ 1)] = 5$$

$$d_{\underset{\sim}{r},\underset{\sim}{S}_2} = W(\underset{\sim}{r} \oplus \underset{\sim}{S}_2) = W[(0\ 0\ 1\ 0\ 0)] = 1$$

$$d_{\underset{\sim}{r},\underset{\sim}{S}_3} = W(\underset{\sim}{r} \oplus \underset{\sim}{S}_3) = W[(0\ 1\ 0\ 0\ 1)] = 2$$ ✓

Thus, $\underset{\sim}{r}$ is closer to $\underset{\sim}{S}_2$. The error can be corrected.

2. HOMEWORK FROM P12 LECTURE 8

a) Find the generator matrix corresponding to the parity check matrix given by (17).

b) Construct a table of information words and corresponding codewords.

C) Is this code (i) linear (ii) systematic ? Prove !

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Solution: a) Equation (17) gives the parity check matrix as

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Thus, we have

$$H_p = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad n = 6, \quad k = r = 3$$

The generator matrix is

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and codeword vector } \underset{\sim}{T} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

b). According to the equation above, we can construct the table now.

INFORMATION WORD		CODEWORD
0 0 0	$\underset{\sim}{S_0}$	0 0 0 0 0 0
0 0 1	$\underset{\sim}{S_1}$	0 0 1 0 1 1
0 1 0	$\underset{\sim}{S_2}$	0 1 0 1 1 0
0 1 1	$\underset{\sim}{S_3}$	0 1 1 1 0 1
1 0 0	$\underset{\sim}{S_4}$	1 0 0 1 0 1
1 0 1	$\underset{\sim}{S_5}$	1 0 1 1 1 0
1 1 0	$\underset{\sim}{S_6}$	1 1 0 0 1 1
1 1 1	$\underset{\sim}{S_7}$	1 1 1 0 0 0

C) This code is both linear and systematic.

It is easy to see from the table above that the first three bits of codewords are the same as the corresponding information words. Thus it is systematic code. Linearity is to be proved on next page

2. HOMEWORK (CONT.)

c) Linearity

From the table in part b. we have

$\underset{\sim}{S}_1 \oplus \underset{\sim}{S}_2 = (0\ 1\ 1\ 1\ 0\ 1) = \underset{\sim}{S}_3$ $\qquad$ $\underset{\sim}{S}_2 \oplus \underset{\sim}{S}_7 = (1\ 0\ 1\ 1\ 1\ 0) = \underset{\sim}{S}_5$

$\underset{\sim}{S}_3 \oplus \underset{\sim}{S}_4 = (1\ 1\ 1\ 0\ 0\ 0) = \underset{\sim}{S}_7$ $\qquad$ $\underset{\sim}{S}_2 \oplus \underset{\sim}{S}_4 = (1\ 1\ 0\ 0\ 1\ 1) = \underset{\sim}{S}_6$

$\underset{\sim}{S}_5 \oplus \underset{\sim}{S}_6 = (0\ 1\ 1\ 1\ 0\ 1) = \underset{\sim}{S}_3$

Then, we check

1) $\underset{\sim}{S}_0 \oplus \underset{\sim}{S}_i = \underset{\sim}{S}_i \quad ; \quad i = 1, 2, 3, 4, 5, 6, 7$

2) $\underset{\sim}{S}_1 \oplus \underset{\sim}{S}_2 = \underset{\sim}{S}_3 \Rightarrow \underset{\sim}{S}_1 \oplus \underset{\sim}{S}_3 = \underset{\sim}{S}_2$

$\underset{\sim}{S}_1 \oplus \underset{\sim}{S}_4 = \underset{\sim}{S}_1 \oplus \underset{\sim}{S}_3 \oplus \underset{\sim}{S}_7 = \underset{\sim}{S}_2 \oplus \underset{\sim}{S}_7 = \underset{\sim}{S}_5 \Rightarrow \underset{\sim}{S}_1 \oplus \underset{\sim}{S}_5 = \underset{\sim}{S}_4$

$\underset{\sim}{S}_1 \oplus \underset{\sim}{S}_6 = \underset{\sim}{S}_4 \oplus \underset{\sim}{S}_5 \oplus \underset{\sim}{S}_6 = \underset{\sim}{S}_4 \oplus \underset{\sim}{S}_3 = \underset{\sim}{S}_7 \Rightarrow \underset{\sim}{S}_1 \oplus \underset{\sim}{S}_7 = \underset{\sim}{S}_6$

3) $\underset{\sim}{S}_2 \oplus \underset{\sim}{S}_3 = \underset{\sim}{S}_1$

$\underset{\sim}{S}_2 \oplus \underset{\sim}{S}_4 = \underset{\sim}{S}_6 \Rightarrow \underset{\sim}{S}_2 \oplus \underset{\sim}{S}_6 = \underset{\sim}{S}_4$

$\underset{\sim}{S}_2 \oplus \underset{\sim}{S}_5 = \underset{\sim}{S}_4 \oplus \underset{\sim}{S}_6 \oplus \underset{\sim}{S}_5 = \underset{\sim}{S}_4 \oplus \underset{\sim}{S}_3 = \underset{\sim}{S}_7 \Rightarrow \underset{\sim}{S}_2 \oplus \underset{\sim}{S}_7 = \underset{\sim}{S}_5$

4) $\underset{\sim}{S}_3 \oplus \underset{\sim}{S}_4 = \underset{\sim}{S}_7 \Rightarrow \underset{\sim}{S}_3 \oplus \underset{\sim}{S}_7 = \underset{\sim}{S}_4$

$\underset{\sim}{S}_3 \oplus \underset{\sim}{S}_5 = \underset{\sim}{S}_6 \Rightarrow \underset{\sim}{S}_3 \oplus \underset{\sim}{S}_6 = \underset{\sim}{S}_5$

5) $\underset{\sim}{S}_4 \oplus \underset{\sim}{S}_5 = \underset{\sim}{S}_2 \oplus \underset{\sim}{S}_6 \oplus \underset{\sim}{S}_5 = \underset{\sim}{S}_2 \oplus \underset{\sim}{S}_3 = \underset{\sim}{S}_1$

$\underset{\sim}{S}_4 \oplus \underset{\sim}{S}_6 = \underset{\sim}{S}_2$

$\underset{\sim}{S}_4 \oplus \underset{\sim}{S}_7 = \underset{\sim}{S}_3$

6) $\underset{\sim}{S}_5 \oplus \underset{\sim}{S}_6 = \underset{\sim}{S}_3$

$\underset{\sim}{S}_5 \oplus \underset{\sim}{S}_7 = \underset{\sim}{S}_2$

7) $\underset{\sim}{S}_6 \oplus \underset{\sim}{S}_7 = \underset{\sim}{S}_1$

Thus, $S_i \oplus S_j$, for any i and j, is also a codeword. ✓

Therefore, this code is linear.

(I think it is easy to show the linearity by using computER)

1-18 A channel has transition matrix

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix}$$

Let the input symbols have probabilities P_1 and P_2, where $P_1 + P_2 = 1$.

(a) Find the probabilities, Q_1 and Q_2, of the output symbols in terms of P_1 & P_2.

(b) Solve the equations obtained in part (a) for P_1 and P_2.

(c) Obtain expressions for $P(x_m | y_n)$ for $m, n = 1, 2$

(d) If $P_1 = P_2 = 0.5$, calculate Q_1, Q_2, and $P(x_m | y_n)$ for $m, n = 1, 2$

Solution: (a)

$$[Q_1 \quad Q_2] = [P_1 \quad P_2] \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix}$$

$$\begin{cases} Q_1 = 0.9 P_1 + 0.3 P_2 \\ Q_2 = 0.1 P_1 + 0.7 P_2 \end{cases} \quad \cdots\cdots \quad (1)$$

(b)

$$[P_1 \quad P_2] = [Q_1 \quad Q_2] \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix}^{-1} = [Q_1 \quad Q_2] \begin{bmatrix} 1.16667 & -0.16667 \\ -0.50000 & 1.50000 \end{bmatrix}$$

$$\begin{cases} P_1 = 1.16667\, Q_1 - 0.5 Q_2 \\ P_2 = -0.16667 Q_1 + 1.5\, Q_2 \end{cases} \quad \cdots\cdots \quad (2)$$

(c)

$$\begin{bmatrix} P(x_1|y_1) & P(x_1|y_2) \\ P(x_2|y_1) & P(x_2|y_2) \end{bmatrix} = \begin{bmatrix} \dfrac{P(y_1|x_1)P_1}{Q_1} & \dfrac{P(y_2|x_1)P_1}{Q_2} \\ \dfrac{P(y_1|x_2)P_2}{Q_1} & \dfrac{P(y_2|x_2)P_2}{Q_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 P_1/Q_1 & 0.1\, P_1/Q_2 \\ 0.3 P_2/Q_1 & 0.7\, P_2/Q_2 \end{bmatrix} \quad \cdots\cdots \quad (3)$$

(d) Let $P_1 = P_2 = 0.5$, from equations (1) and (3), we have

$Q_1 = 0.6$

$Q_2 = 0.4$

$$\begin{bmatrix} P(x_1|y_1) & P(x_1|y_2) \\ P(x_2|y_1) & P(x_2|y_2) \end{bmatrix} = \begin{bmatrix} 0.75 & 0.125 \\ 0.25 & 0.875 \end{bmatrix}$$

1-20 An M-ary symmetric channel is one with transition matrix

$$\begin{bmatrix} p & \frac{1-p}{M-1} & \frac{1-p}{M-1} & \cdots & \frac{1-p}{M-1} \\ \frac{1-p}{M-1} & p & \frac{1-p}{M-1} & \cdots & \frac{1-p}{M-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1-p}{M-1} & \frac{1-p}{M-1} & \frac{1-p}{M-1} & \cdots & p \end{bmatrix} = [p(\underline{y}|\underline{x})]$$

Show that its capacity is

$$C = \log_2 M - (1-p)\log_2(M-1) - \mathcal{H}(p)$$

SHOW: From the transition matrix, we know that the channel has M-inputs and M-outputs. Assuming M input symbols are $x_1, x_2, \cdots, x_M$ with probabilities $P_1, P_2, \cdots, P_M$ and output $y_1, y_2, \cdots, y_M$.

$$[P(\underline{x}, \underline{y})] = \begin{bmatrix} P_1 & 0 & 0 & \cdots & 0 \\ 0 & P_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & P_M \end{bmatrix} \cdot [p(\underline{y}|\underline{x})]$$

$$= \begin{bmatrix} P_1 p & \frac{P_1(1-p)}{M-1} & \frac{P_1(1-p)}{M-1} & \cdots & \frac{P_1(1-p)}{M-1} \\ \frac{P_2(1-p)}{M-1} & P_2 p & \frac{P_2(1-p)}{M-1} & \cdots & \frac{P_2(1-p)}{M-1} \\ \vdots & & & & \\ \frac{P_M(1-p)}{M-1} & \frac{P_M(1-p)}{M-1} & \frac{P_M(1-p)}{M-1} & \cdots & P_M p \end{bmatrix}$$

Then,

$$-H(Y|X) = \sum_{i=1}^{M}\sum_{j=1}^{M} P(x_i, y_j)\log_2 P(y_j|x_i)$$

$$= \sum_{i=1}^{M} P_i \cdot \sum_{j=1}^{M} P(y_j|x_i)\log_2 P(y_i|x_i)$$

$$= \underbrace{\sum_{i=1}^{M} P_i}_{=1}\left[(M-1)\cdot\frac{1-p}{M-1}\log_2\frac{1-p}{M-1} + p\cdot\log_2 p\right]$$

$$= (1-p)\log_2(1-p) + p\log_2 p - (1-p)\log_2(M-1)$$

$$= -(1-p)\log_2(M-1) - \mathcal{H}(p)$$

$$I(X;Y) = H(Y) - (1-p)\log_2(M-1) - \mathcal{H}(p)$$

In this problem, we can pick $P_1, P_2, \cdots, P_M$ such that $P(y_1) = P(y_2) = \cdots = P(y_M) = \frac{1}{M}$

Then $H_{max}(Y) = \log_2 M$ and $\underline{C = \log_2 M - (1-p)\log_2(M-1) - \mathcal{H}(p)}$ Q.E.D.

1-21. Obtain the capacity of the symmetric binary erasure channel, which has two inputs ±1 and three outputs −1, 0, and 1. The channel is defined by the transition probabilities

$$P(0|+1)=P(0|-1)=P$$

$$P(+1|+1)=P(-1|-1)=q$$

with $p+q=1$. At the output, 0 corresponds to an erasure with p being the erasure probability.

Solution: From the givens, we can write the transition matrix as

$$\begin{bmatrix} q & P & 0 \\ 0 & P & q \end{bmatrix}$$

And assuming $x_1=-1$, $x_2=1$, $y_1=-1$, $y_2=0$, and $y_3=1$.

$$[p(x,y)]=\begin{bmatrix} P(x_1) & 0 \\ 0 & P(x_2) \end{bmatrix}\cdot\begin{bmatrix} q & P & 0 \\ 0 & P & q \end{bmatrix}$$

$$=\begin{bmatrix} P(x_1)q & P(x_1)P & 0 \\ 0 & P(x_2)P & P(x_2)q \end{bmatrix}$$

$$-H(Y|X)=\sum_{i=1}^{2}\sum_{j=1}^{3}P(x_i,y_j)\log_2 P(y_j|x_i)$$

$$=\sum_{i=1}^{2}P(x_i)\cdot\sum_{j=1}^{3}P(y_j|x_i)\log_2 P(y_i|x_i)$$

$$=p(x_1)[q\log_2 q+p\log_2 P+\cancelto{0}{0\cdot\log_2 0}]+P(x_2)[q\log_2 q+P\log_2 P]$$

$$=q\log_2 q+p\log_2 P$$

$$I(X;Y)=H(Y)+q\log_2 q+P\log_2 P$$

But in this problem, we could not pick $p(y_1)=p(y_2)=p(y_3)=\frac{1}{3}$ if $p\neq\frac{1}{3}$, $q\neq\frac{2}{3}$. So, we have to find the maximum value of $H(Y)$ as below.

(TO BE CONT. ON NEXT PAGE)

1-21 (CONT.)

$$[P(y_1),\ P(y_2),\ P(y_3)] = [\,P(x_1),\ P(x_2)] \cdot \begin{bmatrix} \varepsilon & P & 0 \\ 0 & P & \varepsilon \end{bmatrix}$$

$$= [\,\varepsilon P(x_1),\ P[P(x_1)+P(x_2)]\ ;\ \varepsilon P(x_2)]$$

$$= [\,\varepsilon P(x_1),\ P\ ,\ \varepsilon P(x_2)]$$

Then,

$$H(Y) = -\sum_{i=1}^{3} P(y_i)\log_2 P(y_i)$$

$$= -\varepsilon P(x_1)\cdot[\log_2 \varepsilon + \log_2 P(x_1)] - p\log_2 P$$

$$- \varepsilon P(x_2)\cdot[\log_2 \varepsilon + \log_2 P(x_2)]$$

$$= -\varepsilon\log_2 \varepsilon\cdot[P(x_1)+P(x_2)] - \varepsilon[P(x_1)\log_2 P(x_1) + P(x_2)\log_2 P(x_2)]$$

$$-p\log_2 P$$

$$= -\varepsilon\log_2\varepsilon - p\log_2 P + \varepsilon H(X)$$

Only when $p(x_1) = p(x_2) = 0.5$, $H(X)$ *reaches the maximum value* $H_{max}(X) = \log_2 2 = 1$. *So does* $H(Y)$ *with*

$$H_{max}(Y) = -\varepsilon\log_2\varepsilon - p\log_2 P + \varepsilon$$

Finally, we have the capacity

$$C = \max_{P(x_i)} I(X;Y)$$

$$= -\varepsilon\log_2\varepsilon - p\log_2 P + \varepsilon + \varepsilon\log_2\varepsilon + p\log_2 P$$

$$= \varepsilon$$

✓

55/55

HOMEWORK 1 (P19 LECTURE 8)

(10)

a) Construct a table using (40) giving all 4-bit information words and the corresponding codewords.

b) From the table, determine d_{min}.

C) Pick two codewords at random and show that the sum of two codewords is a codeword.

d) Show that all columns of G given by (40) are orthogonal to all rows of H given by (38).

- - - - - - - - - - - - - - -

Solution: a)

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{T} = G\underset{\sim}{X}$$

INFORMATION WORD ($\underset{\sim}{X}$)		CORRESPONDING CODEWORD ($\underset{\sim}{T}$)	
0 0 0 0	$\underset{\sim}{S}_0$	0 0 0 0 0 0 0	W = 0
0 0 0 1	$\underset{\sim}{S}_1$	0 0 0 1 1 1 1	W = 4
0 0 1 0	$\underset{\sim}{S}_2$	0 0 1 0 1 1 0	W = 3
0 0 1 1	$\underset{\sim}{S}_3$	0 0 1 1 0 0 1	W = 3
0 1 0 0	$\underset{\sim}{S}_4$	0 1 0 0 1 0 1	W = 3
0 1 0 1	$\underset{\sim}{S}_5$	0 1 0 1 0 1 0	W = 3
0 1 1 0	$\underset{\sim}{S}_6$	0 1 1 0 0 1 1	W = 4
0 1 1 1	$\underset{\sim}{S}_7$	0 1 1 1 1 0 0	W = 4
1 0 0 0	$\underset{\sim}{S}_8$	1 0 0 0 0 1 1	W = 3
1 0 0 1	$\underset{\sim}{S}_9$	1 0 0 1 1 0 0	W = 3
1 0 1 0	$\underset{\sim}{S}_{10}$	1 0 1 0 1 0 1	W = 4
1 0 1 1	$\underset{\sim}{S}_{11}$	1 0 1 1 0 1 0	W = 4
1 1 0 0	$\underset{\sim}{S}_{12}$	1 1 0 0 1 1 0	W = 4
1 1 0 1	$\underset{\sim}{S}_{13}$	1 1 0 1 0 0 1	W = 4
1 1 1 0	$\underset{\sim}{S}_{14}$	1 1 1 0 0 0 0	W = 3
1 1 1 1	$\underset{\sim}{S}_{15}$	1 1 1 1 1 1 1	W = 7

b) From the weights associated with the codewords in the table above. We see that $d_{min} = \min_{r \neq 0} W(\underset{\sim}{S}_r) = 3$, because this code is linear.

HOMEWORK 1 (CONT.)

c) We pick $\underset{\sim}{S}_5 = (0\,1\,0\,1\,0\,1\,0)$ and $\underset{\sim}{S}_{11} = (1\,0\,1\,1\,0\,1\,0)$, then ✓

$$\underset{\sim}{S}_5 \oplus \underset{\sim}{S}_{11} = (1\,1\,1\,0\,0\,0\,0) = \underset{\sim}{S}_{14}$$

SHOWED

d) In order to show that all collums of G given by (40) are orthogonal to all rows of H given by (38), we only have to show ✓

$$HG = \underset{\sim}{0}$$

$$HG = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$ ✓

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q.E.D ✓

HOMEWORK 2 (P26 Lecture 8)

(10)

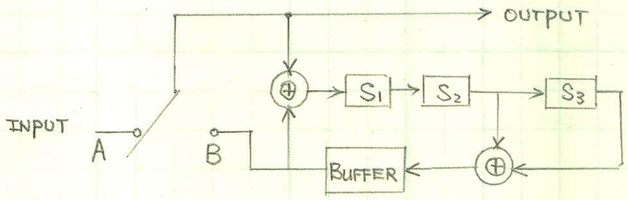
For the encoder / decoder given above consider data word (1001)

a) Construct table as on p. 22 and p. 24.

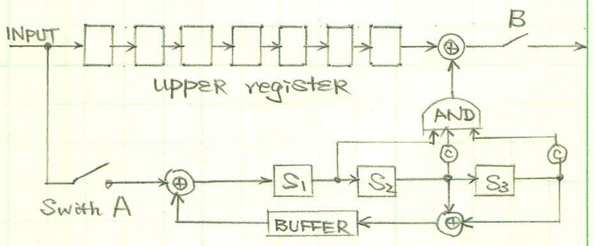
b) Add an error in the third of the codeword and construct a table as on p. 25

Solution:

ENCODER



DECODER



a) ENCODING TABLE data word (1 0 0 1)

SHIFT	INPUT	BUFFER	OUTPUT	S_1	S_2	S_3	$S_2 \oplus S_3$
0	0	0	0	0	0	0	0
1	1	0	1	1	0	0	0
2	0	0	0	0	1	0	1
3	0	1	0	1	0	1	1
4	1	1	1	0	1	0	1
5	Switch	1	1	0	0	1	1
6	to B	1	1	0	0	0	0
7		0	0	0	0	0	0
8	Next information word						

INFORMATION WORD: 1001   CODEWORD = 1001110

DECODING TABLE Received word (1001110)

	SHIFT	INPUT	BUFFER	S_1	S_2	S_3	$S_2 \oplus S_3$	← upper reg.	$S_1\bar{S}_2\bar{S}_3$	OUTPUT
A closed, B open	1	1	0	1	0	0	0			
	2	0	0	0	1	0	1			
	3	0	1	1	0	1	1			
	4	1	1	0	1	0	1			
	5	1	1	0	0	1	1			
	6	1	1	0	0	0	0			
	7	0	0	0	0	0	0			
B closed, A open	8		0	0	0	0	0	1	0	1
	9		0	0	0	0	0	0	0	0
	10		0	0	0	0	0	0	0	0
	11		0	0	0	0	0	1	0	1
	12		0	0	0	0	0	1	0	1
	13		0	0	0	0	0	1	0	1
	14		0	0	0	0	0	0	0	0

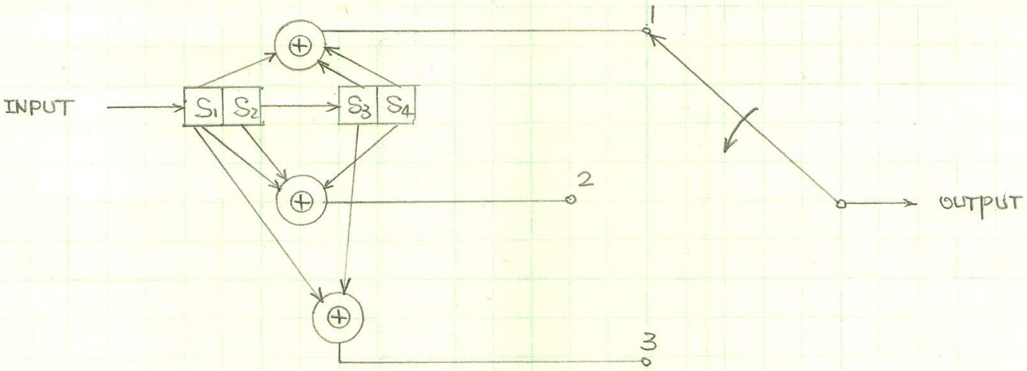
HOMEWORK 2 (CONT.)

b) Assuming Received word is (1011110)

↑ ERROR bit.

	SHIFT	INPUT	BUFFER	S_1	S_2	S_3	$S_2 \oplus S_3$	UPPER REG.	$S_1\bar{S}_2\bar{S}_3$	OUTPUT
A closed, B open	1	1	0	1	0	0	0			
	2	0	0	0	1	0	1			
	3	1	1	0	0	1	1			
	4	1	1	0	0	0	0			
	5	1	0	1	0	0	0			
	6	1	0	1	1	0	1			
	7	0	1	1	1	1	0			
B closed, A open	8		0	0	1	1	0	1	0	1
	9		0	0	0	1	1	0	0	0
	10		1	1	0	0	0	1	1	0
	11		0	0	1	0	1	1	0	1
	12		1	1	0	1	1	1	0	1
	13		1	1	1	0	1	1	0	1
	14		1	1	1	1	0	0	0	0

HOMEWORK 3-6: (P.23 LECTURE 9)



FOR PROBLEM 3 & 4: $k=1$, $n=3$, $L=4$ FOR PROBLEM 5 & 6: $k=2$, $n=3$, $L=2$

Convolutional Encoder

For the (1,3,4) and (2,3,2) convolutional code with encoder shown above

(Prob. 3 or 5) Find

a) Code rate

b) Tree diagram

c) Trellis diagram

d) State diagram

e) d_{free} and t

f) Transmitter sequence for information sequence 1110000 1

g) Recieved sequence for transmitter sequence found in (f) and transmission error sequence 00000100000 0

(Prob. 4 or 6) Illustrate Viterbi decoding of the received sequence found in (g). Use multiple diagram showing survivors and metrics as done in Figure D-5 of the text.

Solutions to problem 3 – 6 are on next pages.

PROBLEM 3 FOR (1, 3, ~~4~~ 3) convolutional code.

(5)

Not what I ask, but OK

a) $R_c = k/n = \frac{1}{3} = 0.33333$

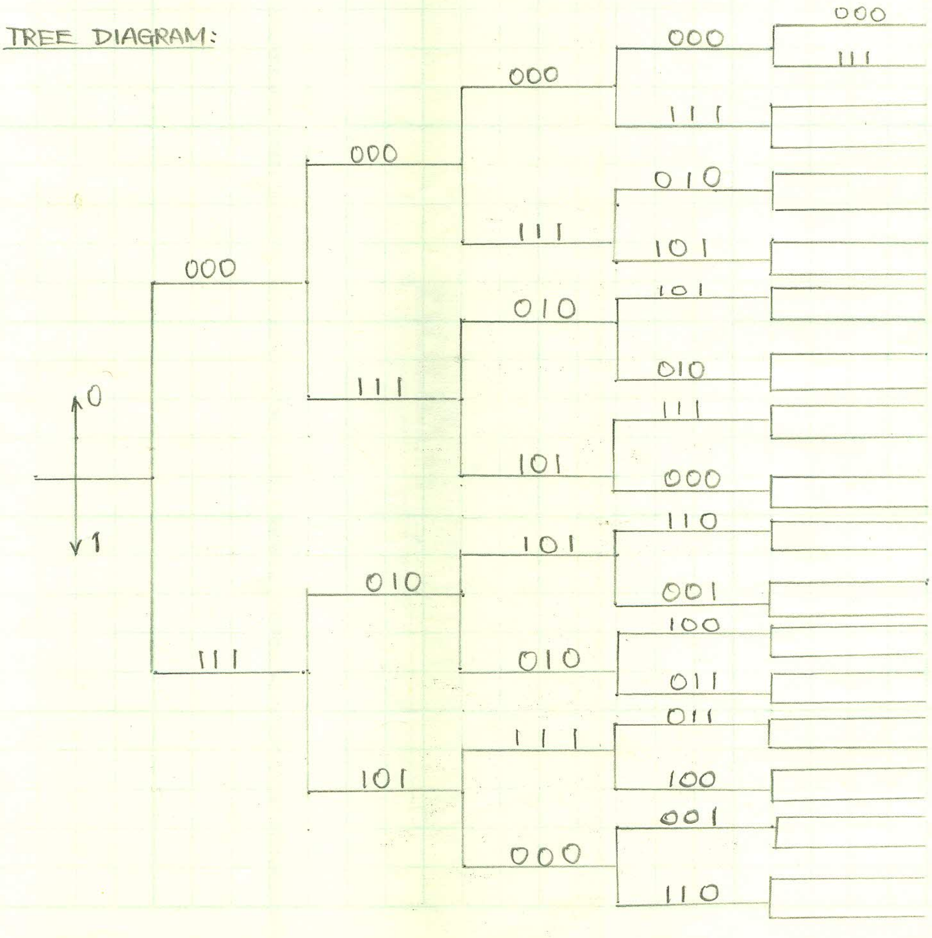
b) The output is related to the contents of the registers as follows:

S_1	S_2	S_3	S_4	1	2	3
0	0	0	0	0	0	0
0	0	0	1	1	1	0
0	0	1	0	1	0	1
0	0	1	1	0	1	1
0	1	0	0	0	1	0
0	1	0	1	1	0	0
0	1	1	0	1	1	1
0	1	1	1	0	0	1
1	0	0	0	1	1	1
1	0	0	1	0	0	1
1	0	1	0	0	1	0
1	0	1	1	1	0	0
1	1	0	0	1	0	1
1	1	0	1	0	1	1
1	1	1	0	0	0	0
1	1	1	1	1	1	0

← outputs

(note: This table is true for (2, 3, 2) convolutional code)

TREE DIAGRAM:

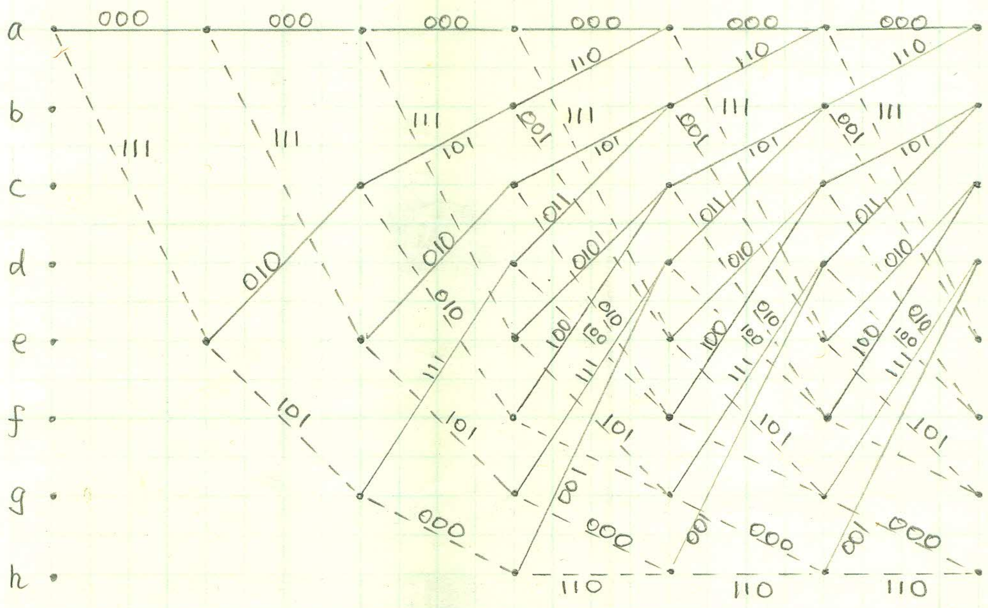


PROBLEM 3 (CONT.)

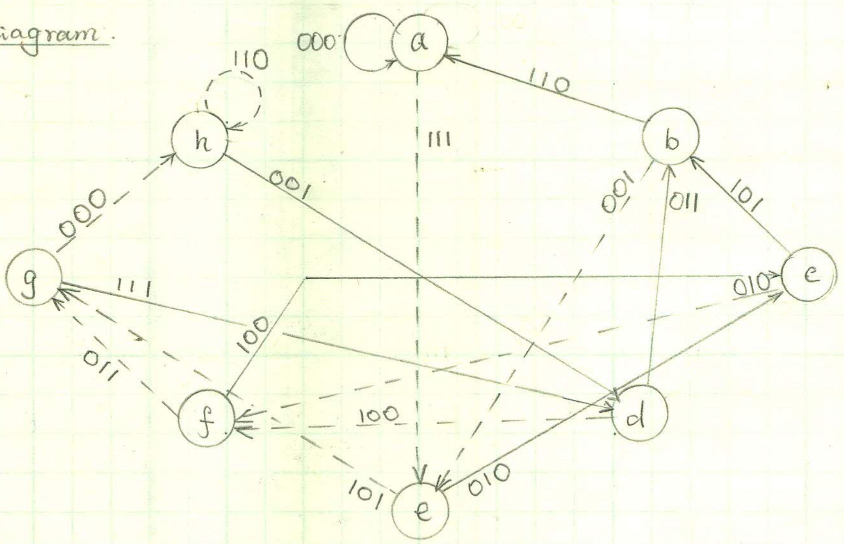
C) DEFINITION OF STATES

State	S_1	S_2	S_3
a	0	0	0
b	0	0	1
c	0	1	0
d	0	1	1
e	1	0	0
f	1	0	1
g	1	1	0
h	1	1	1

TRELLIS DIAGRAM



d) State diagram.



PROBLEM 3 (CONT.)

e) $d_{free} = W(111010101110) = 8$

$t \leqslant d_{free}/2 - 1 = 4 - 1 = 3$

f)

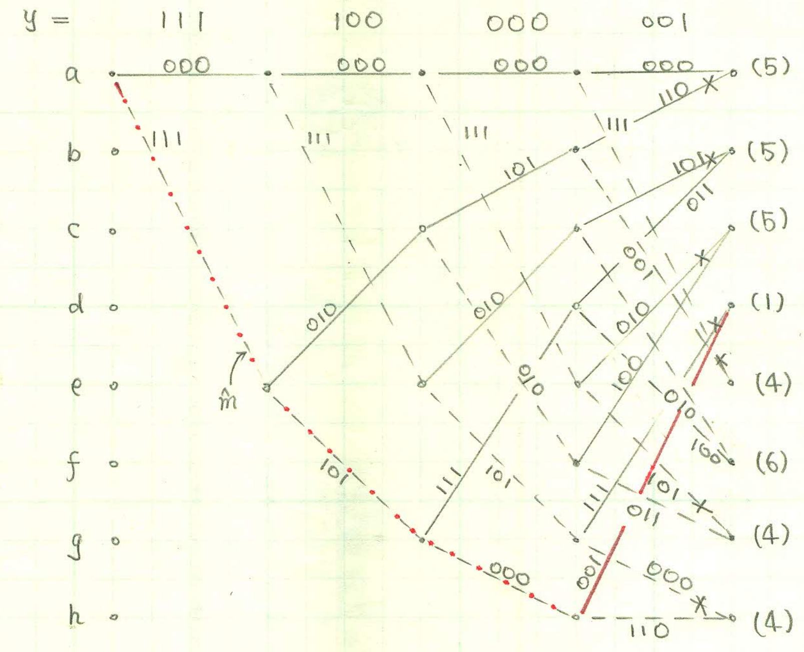
INFORMATION SEQUENCE	TRANSMITTER SEQUENCE
11100001	111101000001011110000111

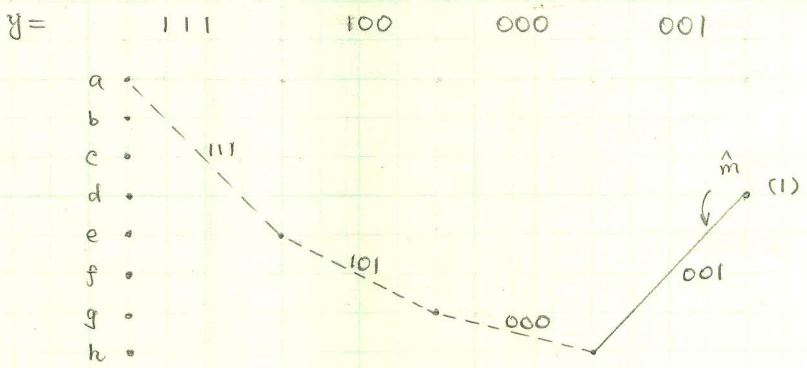
g) In order to match the problem for $k=1$, we assume that the information sequence is 1110 with codeword sequence 111101000001, which is found in part (f). Then, adding error sequence 000001000000, we have

$$y \triangleq \text{Received sequence} = 111100000001$$

PROBLEM 4.

(10)

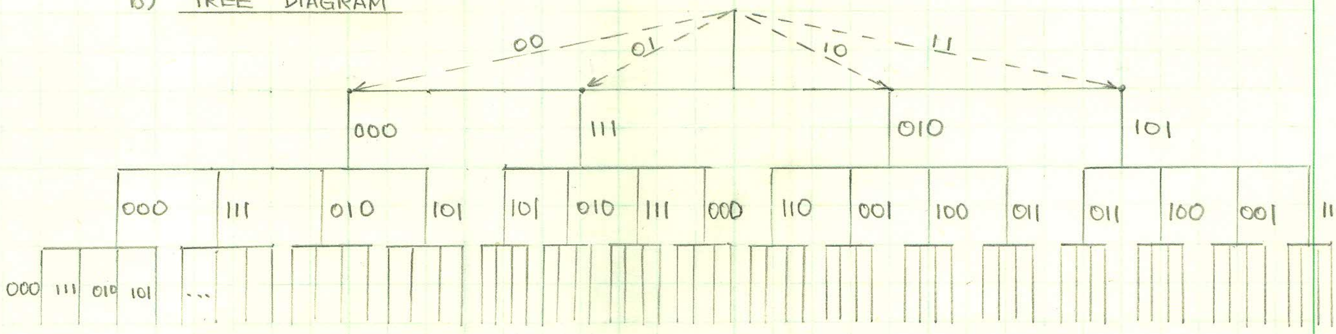




PROBLEM 5 FOR (2, 3, 2) convolutional code.

(10) a) $R_c = k/n = 2/3 = 0.66666$

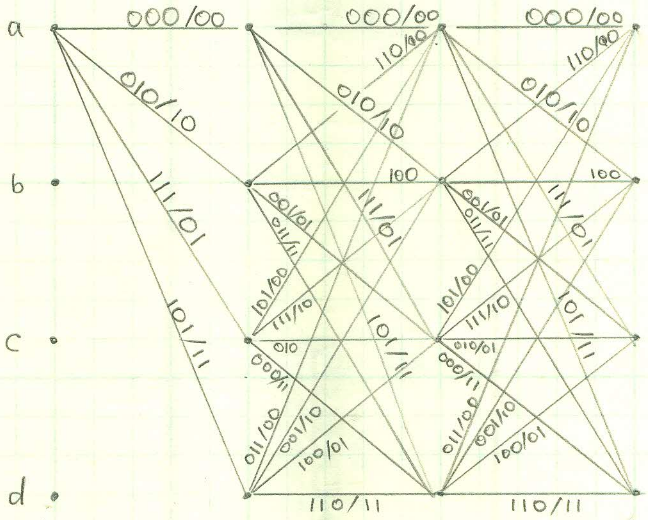
b) TREE DIAGRAM



C) DEFINITION OF STATES

State	S_1	S_2
a	0	0
b	0	1
c	1	0
d	1	1

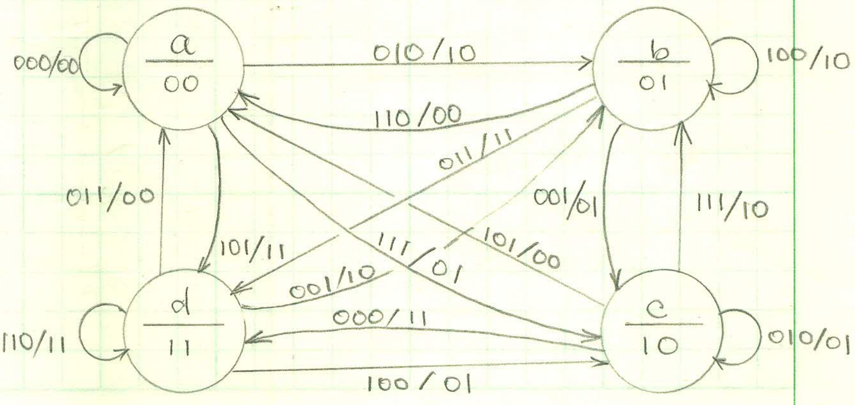
TRELLIS DIAGRAM



d) State diagram

XXX / XX

output input



PROBLEM 5 (CONT.)

e) $d_{free} = W(010110) = 3$

$t \leqslant (d_{free} - 1)/2 = 1$

f) INFORMATION SEQUENCE — TRANSMITTER SEQUENCE

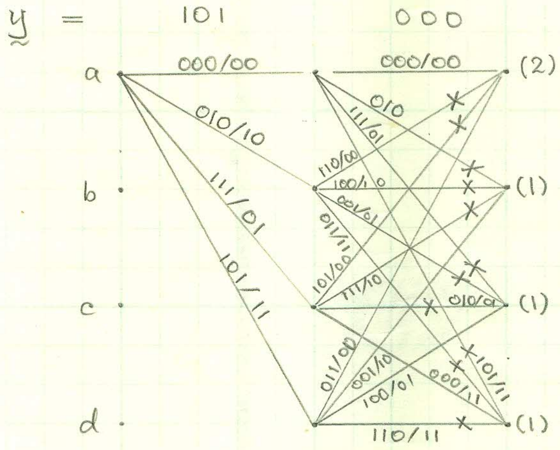
1 1 1 0 0 0 0 1 — 101001110111

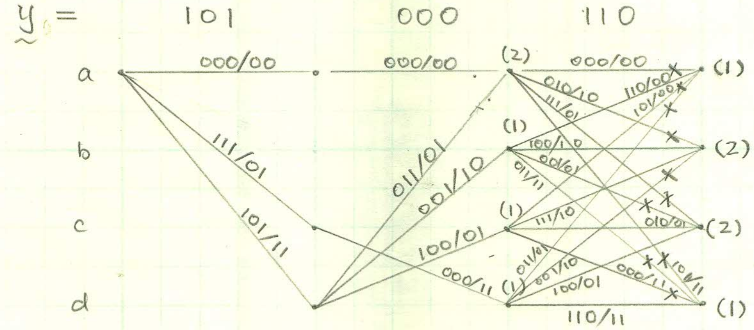
g) Error sequence 000 001000 000 is added, the

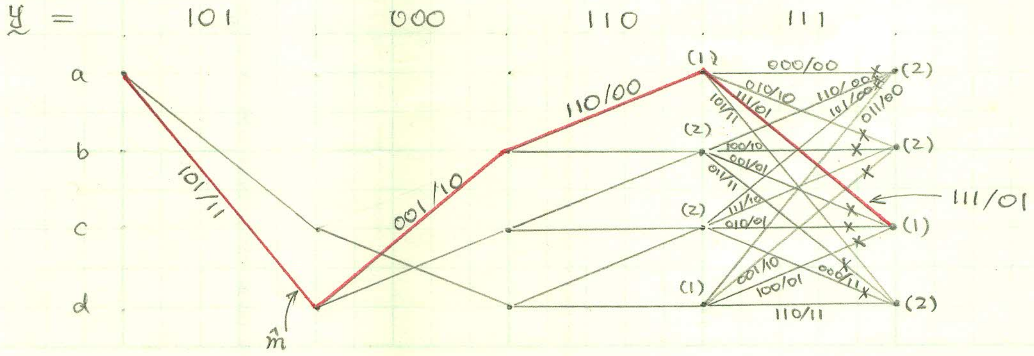
Received sequence = 101 000 110 111 = $\underset{\sim}{y}$

PROBLEM 6.

(10)







P16, LECTURE 10

48/51

1. Prove the following properties of Hilbert transforms:

(21)

(a) If $x(t) = x(-t)$, then $\hat{x}(t) = -\hat{x}(-t)$

According eq. (7) on page 5 in Lecture 10, we have

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{(t-\tau)}\, d\tau = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(t-\tau)}{\tau}\, d\tau$$

If $x(t) = x(-t)$, $\quad \hat{x}(-t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(-t-\tau)}{\tau}\, d\tau = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(t-(-\tau))}{-(-\tau)}\, d\tau$

(LET $s = -\tau$) $\quad = \frac{1}{\pi}\int_{\infty}^{-\infty} \frac{x(t-s)}{s}\, ds$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(t-s)}{s}\, ds = -\hat{x}(t)$$

$$\therefore \quad \hat{x}(t) = -\hat{x}(-t)$$ ✓

(b) If $x(t) = -x(-t)$, then $\hat{x}(t) = \hat{x}(-t)$

$$\hat{x}(-t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(-t-\tau)}{\tau}\, d\tau$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{-x(t+\tau)}{\tau}\, d\tau$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(t-(-\tau))}{-\tau}\, d\tau$$

(let $s = -\tau$) $\quad = +\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(t-s)}{s}\, ds = \hat{x}(t)$

$$\therefore \quad \hat{x}(t) = \hat{x}(-t)$$ ✓

(c) If $x(t) = \cos w_0 t$, then $\hat{x}(t) = \sin w_0 t$

From the definition of Hilbert transforms; we have

$$\hat{X}(f) = H(f)\cdot X(f) = -j\,\mathrm{sgn}(f)\tfrac{1}{2}\left[\delta\left(f - \tfrac{w_0}{2\pi}\right) + \delta\left(f + \tfrac{w_0}{2\pi}\right)\right]$$

$$= \frac{1}{2j}\,\delta\left(f - \frac{w_0}{2\pi}\right) - \frac{1}{2j}\,\delta\left(f + \frac{w_0}{2\pi}\right)$$

$$= \mathcal{F}\{\sin w_0 t\}$$

Thus,

$$\hat{x}(t) = \sin w_0 t$$ ✓

1. (d) If $x(t) = \sin\omega_0 t$, then $\hat{x}(t) = -\cos\omega_0 t$

$$x(t) = \sin\omega_0 t \quad \Rightarrow \quad X(f) = \frac{1}{2j}\delta\left(f - \frac{\omega_0}{2\pi}\right) - \frac{1}{2j}\delta\left(f + \frac{\omega_0}{2\pi}\right)$$

$$\hat{X}(f) = H(f)X(f) = -j\,sgn(f)\cdot\frac{1}{2j}\delta\left(f - \frac{\omega_0}{2\pi}\right) + j\,sgn(f)\cdot\frac{1}{2j}\delta\left(f + \frac{\omega_0}{2\pi}\right)$$

$$= -\frac{1}{2}\delta\left(f - \frac{\omega_0}{2\pi}\right) - \frac{1}{2}\delta\left(f + \frac{\omega_0}{2\pi}\right) = \mathcal{F}\{-\cos\omega_0 t\}$$

Thus, $\hat{x}(t) = -\cos\omega_0 t$

(e) $\hat{\hat{x}}(t) = -x(t)$

$$\hat{\hat{X}}(f) = H(f)\hat{X}(f) = H(f)H(f)\cdot X(f) = -j\,sgn(f)\cdot[-j\,sgn(f)]\cdot X(f)$$

$$= -\overset{1}{\cancel{[sgn(f)]^2}}\cdot X(f) = -X(f) = \mathcal{F}\{-x(t)\}$$

Thus, $\hat{\hat{x}}(t) = -x(t)$

(f) $\int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty}\hat{x}^2(t)\,dt$

If $x(t)$ is a real-valued signal, then from the definition of $\hat{x}(t)$, we know $\hat{x}(t)$ is a real-valued signal too. Apply the property of Fourier Xform:

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty}|X(f)|^2\,df$$

$$\int_{-\infty}^{\infty}\hat{x}^2(t)\,dt = \int_{-\infty}^{\infty}|\hat{X}(f)|^2\,df$$

Then

$$|\hat{X}(f)|^2 = |-j\,sgn(f)\,X(f)|^2 = |-j\,sgn(f)|^2\cdot|X(f)|^2 = |X(f)|^2$$

We see that

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty}\hat{x}^2(t)\,dt \qquad \cdots\cdots \quad (1)$$

If $x(t)$ is complex-valued signal, assuming $x(t) = r(t) + j\cdot i(t)$, then

$$X(f) = R(f) + j\,I(f)$$

$$\hat{X}(f) = \hat{R}(f) + j\,\hat{I}(f) \quad \text{and} \quad \hat{x}(t) = \hat{r}(t) + j\,\hat{i}(t)$$

$$\therefore \quad x^2(t) = r^2(t) - i^2(t) + 2j\,r(t)\,i(t)$$

$$\hat{x}^2(t) = \hat{r}^2(t) - \hat{i}^2(t) + 2j\,\hat{r}(t)\,\hat{i}(t)$$

(To be continued).

1 (f) (CONT.)

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty}[r^2(t) - i^2(t) + 2j\, r(t)i(t)]\,dt$$

$$= \int_{-\infty}^{\infty} r^2(t)\,dt - \int_{-\infty}^{\infty} i^2(t)dt + 2j \cdot \int_{-\infty}^{\infty} r(t)\,i(t)\,dt$$

From eg. (1) $$= \int_{-\infty}^{\infty} \hat{r}^2(t)dt - \int_{-\infty}^{\infty} \hat{i}^2(t)dt + 2j \cdot \int_{-\infty}^{\infty} r(t)\,i(t)\,dt \quad \cdots (2)$$

$$\int_{-\infty}^{\infty} \hat{x}^2(t)dt = \int_{-\infty}^{\infty} \hat{r}^2(t)dt - \int_{-\infty}^{\infty} \hat{i}^2(t)dt + 2j \cdot \int_{-\infty}^{\infty} \hat{r}(t)\hat{i}(t)dt \quad \cdots (3)$$

Now, if we let $r(t) = \delta(t)$ and $i(t) = \cos\omega_0 t$, then

$\hat{r}(t) = h(t) = \frac{1}{\pi t}$ and $\hat{i}(t) = \sin\omega_0 t$ (from (c))

$$\int_{-\infty}^{\infty} r(t)\,i(t)\,dt = \int_{-\infty}^{\infty} \cos\omega_0 t\ \delta(t)\,dt = \cos(\omega_0 \times 0) = 1$$

$$\int_{-\infty}^{\infty} \hat{r}(t)\,\hat{i}(t)\,dt = \int_{-\infty}^{\infty} \frac{\sin\omega_0 t}{\pi t}\,dt = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin\omega_0 t}{t}\,dt$$

$$= \begin{cases} 1 & \omega_0 > 0 \\ 0 & \omega_0 = 0 \\ -1 & \omega_0 < 0 \end{cases}$$

So, in this case, $\int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} \hat{x}^2(t)\,dt$ iff $\omega_0 > 0$

Dr. Birgenheier: Is $\int_{-\infty}^{\infty} r(t)i(t)dt = \int_{-\infty}^{\infty} \hat{r}(t)\hat{i}(t)dt$ a property of H. Xform ?

(g) $\int_{-\infty}^{\infty} x(t)\,\hat{x}(t)\,dt = 0$

Let $x(t) = \delta(t)$, then

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\,d\tau$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{t-\tau}\ \delta(\tau)d\tau = \frac{1}{\pi t}$$

−3

$$\int_{-\infty}^{\infty} x(t)\,\hat{x}(t) = \int_{-\infty}^{\infty} \frac{1}{\pi t}\ \delta(t)\,dt = \underline{\text{undefined}}$$

By Parseval $$\int_{-\infty}^{\infty} x(t)\,\hat{x}(t)\,dt = \int_{-\infty}^{\infty} X(f)\,\hat{X}^*(f)\,df$$

but $\hat{X}^*(f) = +j\,\mathrm{sgn}(f)\,X^*(f)$, so that

$$\int_{-\infty}^{\infty} x(t)\,\hat{x}(t)\,dt = j\int_{-\infty}^{\infty} \underbrace{\mathrm{sgn}(f)\,|X(f)|^2}_{\text{odd fcn.}}\,df = 0$$

2. Find the analytic signals and sketch the Fourier Xforms of the analytic signals corresponding to (c) and (d) above

(10)

For (c), we have analytic signals

$$Z_1(t) = \cos w_0 t + j \sin w_0 t$$

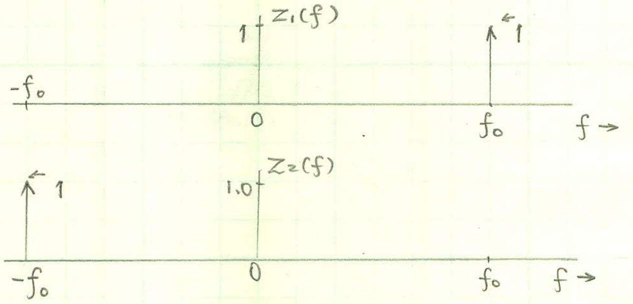
and $Z_2(t) = \cos w_0 t - j \sin w_0 t$

$$Z_1(f) = 2X(f) \qquad f > 0$$
$$= 0 \qquad f < 0$$

$$Z_2(f) = 0 \qquad f > 0$$
$$= 2X(f) \qquad f < 0$$

$$X(f) = \mathcal{F}\{\cos w_0 t\} = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0) \quad , \quad f_0 = \frac{w_0}{2\pi}$$

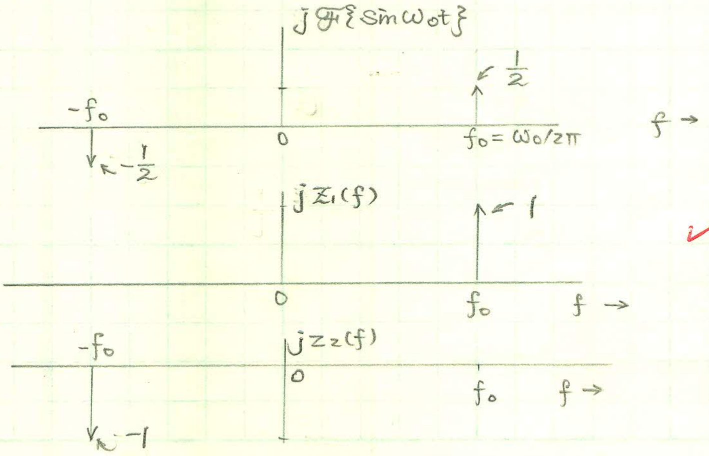
$\therefore$



For (d), $Z_1(t) = \sin w_0 t - j \cos w_0 t$ and $Z_2(t) = \sin w_0 t + j \cos w_0 t$

$$Z_1(f) = 2\mathcal{F}\{\sin w_0 t\} \quad f > 0 \;;\quad Z_1(f) = 0 \,,\; f < 0$$

$$Z_2(f) = 2\mathcal{F}\{\sin w_0 t\} \quad f < 0 \;;\quad Z_2(f) = 0 \,,\; f > 0$$



2-15. Given a message signal with the spectrum

(10)

$$M(f) = \Pi\left(\frac{f}{20}\right)$$

and a carrier signal

$$c(t) = \cos 200\pi t \qquad \omega_0 = 200\pi \Rightarrow f_0 = 100$$

Sketch the spectra of the following modulated signals:

(a) Double sideband.

(b) AM with modulation index 0.5.

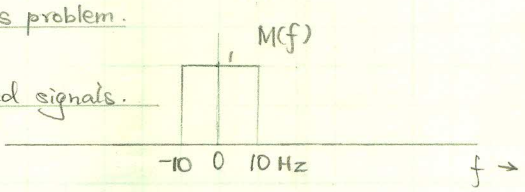
(c) Single sideband, upper sideband transmitted

(d) Single sideband, lower sideband transmitted.

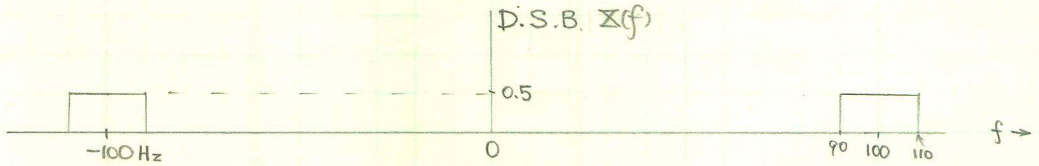
- - - - - - - - - - - - - - - - -

Solution: M(f) is real in this problem.

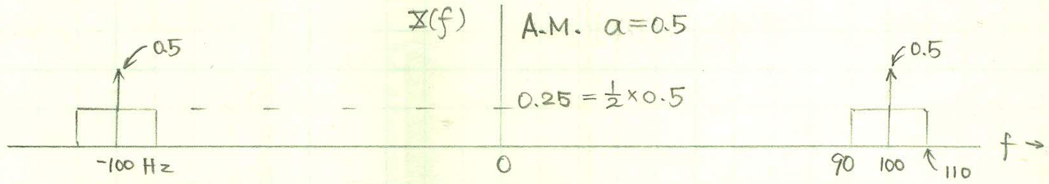
Let $X(f)$ be modulated signals.



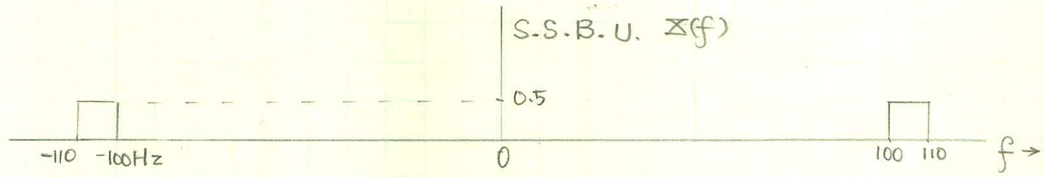
(a)



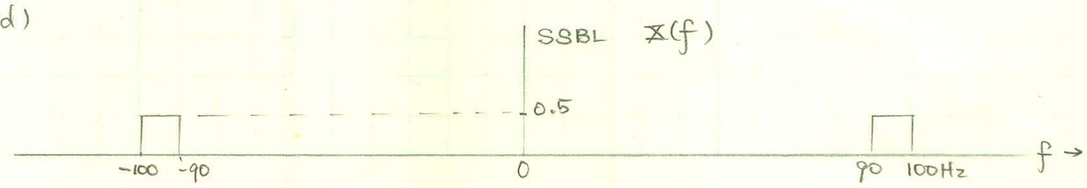
(b)



(c)



(d)



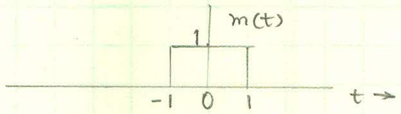
2-17. (a) Derive the Hilbert transform of the signal $m(t) = \Pi(t/2)$

(10)

(b) Use the result obtained in part (a) to obtain an expression for the analytic signal corresponding to $m(t) = \Pi(t/2)$

(c) Sketch the amplitude spectrum of the analytic signal above.

Solution: (a)



$$\hat{m}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{m(\tau)}{t-\tau}\,d\tau = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{t-\tau}\,d\tau$$

$$= \frac{1}{\pi}\int_{t+1}^{t-1} -\frac{1}{s}\,ds = \frac{1}{\pi}\int_{t-1}^{t+1} \frac{1}{s}\,ds$$

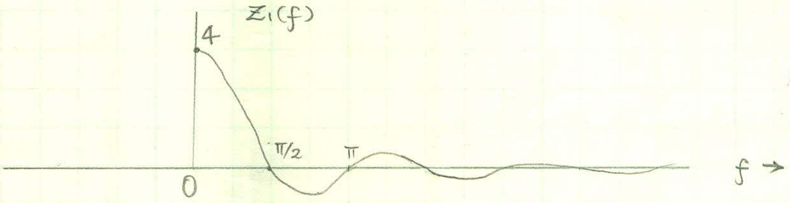
$$= \frac{1}{\pi}\left[\ln(t+1) - \ln(t-1)\right] = \frac{1}{\pi}\ln\left|\frac{t+1}{t-1}\right|$$

(b) $Z_1(t) = \Pi(t/2) + j\frac{1}{\pi}[\ln(t+1) - \ln(t-1)]$

$Z_2(t) = \Pi(t/2) - j\frac{1}{\pi}[\ln(t+1) - \ln(t-1)]$

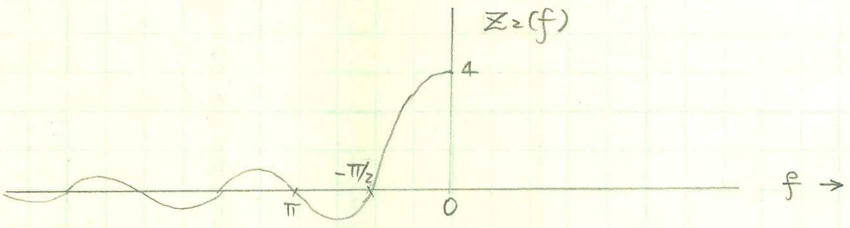
(c) $Z_1(f) = 2M(f) = 4\ \mathrm{sinc}(2f) \qquad f > 0$

$= 0 \qquad f < 0$



$Z_2(f) = 2M(f) = 4\ \mathrm{sin}(2f) \qquad f < 0$

$= 0 \qquad f > 0$



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Signal with abrupt 90° phase change.

$$x(t) = A[\cos(\omega_0 t)u(-t) - \sin(\omega_0 t)u(t)]$$

$$= \mathrm{Re}\{A[u(-t) + ju(t)]e^{j\omega_0 t}\}$$

For the bandpass filter considered in example 3 and the signal given above

(a) determined and sketch $y(t)$.

(b) the drop (in dB) of the envelope caused by phase change.

(c) comment on the advantage of this signal over that of example 3 if a hard limiter is employed after the bandpass filter

- -

Solution: From example 3, we have the bandpass filter

$$H(f) = \frac{1}{1 + j\frac{f-f_0}{B}} + \frac{1}{1 + j\cdot\frac{f+f_0}{B}} \quad \text{and}$$

$$H_e(f) = \frac{1}{1 + j\frac{f}{B}}, \quad H_e(0) = 1$$

(a) From the givens, we see that

$$\tilde{x}(t) = A[u(-t) + ju(t)]$$

$$\tilde{X}(f) = A\cdot\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]^* + jA\cdot\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]$$

$$= A\left[\frac{1}{2}\delta(f) + \frac{1}{2}j\delta(f) + \frac{1}{2\pi f} + j\frac{1}{2\pi f}\right]$$

$$\tilde{Y}(f) = H_e(f)\hat{X}(f) = \frac{A}{2}H_e(f)\delta(f) + \frac{A}{2}jH_e(f)\delta(f) + \frac{A}{2\pi f}H_e(f) + j\frac{A}{2\pi f}H_e(f)$$

$$= \frac{A}{2}H_e(0)\delta(f) + \frac{A}{2}j\cdot H_e(0)\delta(f) + \frac{A}{2\pi f(1+j\frac{f}{B})} - \frac{A}{j2\pi f(1+j\frac{f}{B})}$$

$$= A\cdot\left[\frac{1}{2}\delta(f) + j\frac{1}{2}\delta(f) + \frac{1}{2\pi f} - j\frac{1}{2\pi B + j2\pi f} + j\frac{1}{2\pi f} + \frac{1}{2\pi B + j2\pi f}\right]$$

$$= A\cdot\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]^* + jA\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] - j\frac{A}{2\pi B + j2\pi f} + \frac{A}{2\pi B + j2\pi f}$$

$$\therefore \quad \tilde{y}(t) = A[u(-t) + ju(t)] - jAe^{-2\pi Bt}u(t) + Ae^{-2\pi Bt}u(t)$$

$$y(t) = \mathrm{Re}\{\tilde{y}(t)e^{j\omega_0 t}\}$$

$$= \mathrm{Re}\{[Au(-t)+jAu(t)+jAe^{-2\pi Bt}u(t)+Ae^{-2\pi Bt}u(t)]\cdot[\cos\omega_0 t + j\sin\omega_0 t]\}$$

$$= A\cdot[\cos(\omega_0 t)u(-t) - \sin\omega_0 t\, u(t) + e^{-2\pi Bt}\cos\omega_0 t\, u(t) + e^{-2\pi Bt}\sin\omega_0 t\, u(t)]$$

$$= \begin{cases} A\cos\omega_0 t & ; \quad t<0 \\ A[(1-e^{-2\pi Bt})\sin\omega_0 t + e^{-2\pi Bt}\cos\omega_0 t] & ; \quad t>0 \end{cases}$$

$$\text{OR} \quad = \begin{cases} A\cos\omega_0 t & ; \quad t<0 \\ A\cdot\sqrt{2e^{-4\pi Bt}-2e^{-2\pi Bt}+1}\cdot\left[\dfrac{e^{-2\pi Bt}}{\sqrt{2e^{-4\pi Bt}-2e^{-2\pi Bt}+1}}\cos\omega_0 t + \dfrac{1-e^{-2\pi Bt}}{\sqrt{2e^{-4\pi Bt}-2e^{-2\pi Bt}+1}}\sin\omega_0 t\right] & t>0 \end{cases}$$

$$= \begin{cases} A\cos\omega_0 t & ; \quad t<0 \\ A\sqrt{2e^{-4\pi Bt}-2e^{-2\pi Bt}+1}\cdot\cos\left(\omega_0 t - \arccos\dfrac{e^{-2\pi Bt}}{\sqrt{2e^{-4\pi Bt}-2e^{-2\pi Bt}+1}}\right) ; & t>0 \end{cases}$$

Thus, for $t<0$, the envelope is A,

for $t>0$, the envelope is $A\sqrt{2e^{-4\pi Bt}-2e^{-2\pi Bt}+1} \triangleq E(t)$

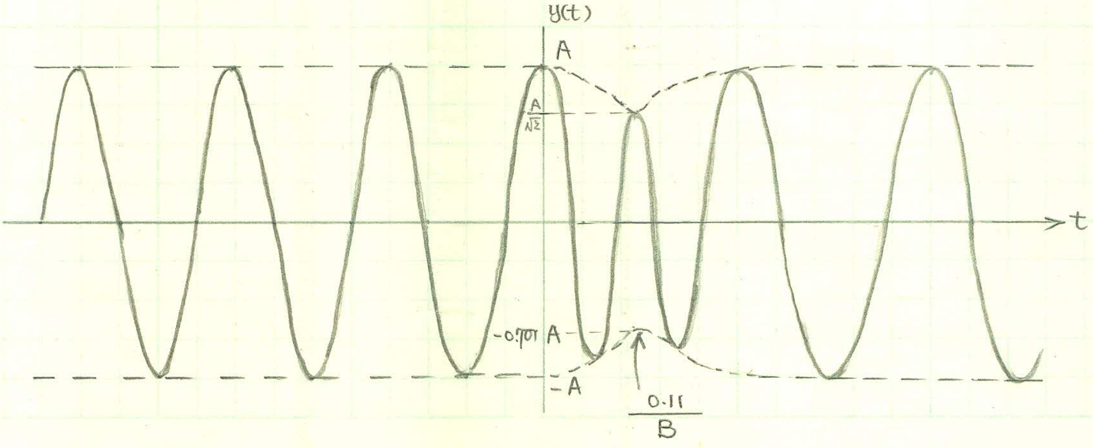
In order to find the droop of $E(t)$, we can find minimum value of $E^2(t)$ instead. And $E^2(t) = 2e^{-4\pi Bt}-2e^{-2\pi Bt}+1$

$$\frac{dE^2(t)}{dt} = 4\pi Be^{-2\pi Bt} - 8\pi Be^{-4\pi Bt} = 0$$

$$\therefore \quad e^{-2\pi Bt} = 0.5 \Rightarrow \quad t = \frac{0.11032}{B}$$

And

$$E^2_{min}(t) = 0.5 \quad \Rightarrow \quad E_{min}(t) = \sqrt{E^2_{min}(t)} = 0.707 \; (= -3\text{dB})$$



(b) From the calculations in part (a), we know that the droop by phase change is $-3dB$ at $t = 0.11032/B$

(c) If a hard limiter is employed after the band-pass filter, for this Signal, we can maintain the envelope at level A. because the droop caused by phase change is only 3dB. For the example 3, however, it is difficult or impossible to do so. because the magnitude of the envelope drops to zero when $t = 0.11/B$.

95.5/100

A-9 (a) Using the transformation of random variables technique discussed in Section A-4, show that the probability density function for the random process described in connection with (A-67) at any time t is given by

$$f_X(x,t) = \begin{cases} \dfrac{1}{\pi A\sqrt{1-(x/A)^2}} & |x| \leq A \\ 0 & \text{eleswhere} \end{cases}$$

(b) Given the value of X(t) at time $t = t_1$, obtain an expression for the conditional density function at time $t_2 > t_1$.

SOLUTION: (a) Refer to equation A-67 in text, we have

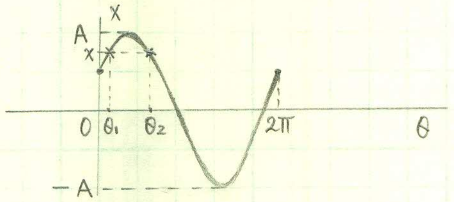
$$X(t) = A\cos(\omega_0 t + \theta) \qquad -\infty < t < \infty$$

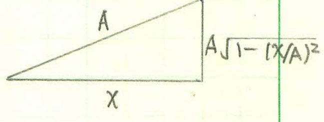
$$f_\theta(\theta) = \begin{cases} 1/2\pi & 0 \leq \theta \leq 2\pi \\ 0 & \text{eleswhere} \end{cases}$$

In this problem, we are only dealing with the transformation between θ and X. So, we can treat the time t as a 'constant'.

$$x = A\cos(\omega_0 t + \theta) \longrightarrow \frac{dx}{d\theta} = -A\sin(\omega_0 t + \theta)$$

And $\theta_1 = -\omega_0 t + \arccos(\frac{x}{A})$; $\theta_2 = -\omega_0 t + 2\pi - \arccos(\frac{x}{A})$





Using the results in notes (Lecture 3, page 2), we obtain

$$f_X(x,t) = \frac{1/2\pi}{|-A\sin(\arccos(\frac{x}{A}))|} + \frac{1/2\pi}{|-A\sin(2\pi-\arccos(\frac{x}{A}))|} \qquad |x| \leq A$$

$$= \frac{1}{\pi A\sqrt{1-(x/A)^2}} \qquad |x| \leq A$$

So,

$$f_X(x,t) = \begin{cases} \dfrac{1}{\pi A\sqrt{1-(x/A)^2}} & |x| \leq A \\ 0 & \text{eleswhere} \end{cases}$$

Showed!

<u>A-9</u> (b) Given $X(t)$ value at $t=t_1$, assuming that $x_1 = X(t_1)$, then

$$x_1 = A\cos(\omega_0 t_1 + \theta)$$

Let $t_2 = t_1 + \tau$, we have

$$X(t_2) = A\cos(\omega_0 t_2 + \theta)$$

$$= A\cos[\omega_0(t_1+\tau) + \theta]$$

$$= A\cos[\omega_0\tau + (\omega_0 t_1 + \theta)]$$

$$= A\cos\omega_0\tau\ \cos(\omega_0 t_1+\theta) - A\sin\omega_0\tau\ \sin(\omega_0 t_1 + \theta)$$

And

$$\sin(\omega_0 t_1+\theta) = \pm\sqrt{1-\cos^2(\omega_0 t_1+\theta)}$$

$$= \pm\sqrt{1-(x_1/A)^2}$$

So,

$$x_2 = X(t_2) = \cos\omega_0\tau \cdot x_1 \mp \sin\omega_0\tau\sqrt{A^2-x_1^2}$$

From this equation, we see that if the value of $X(t)$ at $t=t_1$ is given. then the value of $X(t)$ at $t=t_2$ can only have two defferent values $\cos\omega_0\tau\cdot x_1 - \sin\omega_0\tau\sqrt{A^2-x_1^2}$ and $\cos\omega_0\tau\cdot x_1 + \sin\omega_0\tau\sqrt{A^2-x_1^2}$, thus

$$f_{X_2|X_1}(x_2 | X(t_1)=x_1) = \frac{1}{2}\delta(x_2 - \cos\omega_0\tau\cdot x_1 + \sin\omega_0\tau\sqrt{A^2-x_1^2})$$
$$+ \frac{1}{2}\delta(x_2 - \cos\omega_0\tau\cdot x_1 - \sin\omega_0\tau\sqrt{A^2-x_1^2})$$

- -

A-10 A random process is defined as in (A-67) but with Θ having the density function

$$f_\Theta(\theta) = \begin{cases} \frac{2}{\pi} & 0 \le \theta \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the mean and variance of this random process.

(b) Obtain its autocorrelation function.

(c) Is this process stationary? Cyclostationary?

Solution: (a)

$$E[X(t)] = \int_0^{\frac{\pi}{2}} A\cos(\omega_0 t + \theta)\cdot\frac{2}{\pi}\cdot d\theta$$

$$= + \frac{2A}{\pi}\sin(\omega_0 t + \theta)\Big|_0^{\frac{\pi}{2}}$$

-1

$$= + \frac{2A}{\pi}\sin\left(\frac{\pi}{2} + \omega_0 t\right) + \frac{2A}{\pi}\sin\omega_0 t$$

$$\overset{-}{=} \frac{2A}{\pi}\cdot[\sin\omega_0 t - \cos\omega_0 t] = \underline{\frac{2\sqrt{2}\cdot A}{\pi}\cdot\sin\left(\omega_0 t + \frac{\pi}{4}\right)}$$ ←

$$E[X^2(t)] = \int_0^{\pi/2} A^2\cos^2(\omega_0 t + \theta)\cdot\frac{2}{\pi}\,d\theta$$

$$= \frac{A^2}{\pi}\int_0^{\pi/2}[1 + \cos(2\omega_0 t + 2\theta)]\,d\theta$$

$$= \frac{A^2}{\pi}\cdot\frac{\pi}{2} + \frac{A^2}{2\pi}\cdot\int_0^{\pi}\cos(2\omega_0 t + \theta_1)\,d\theta_1 \qquad (\theta_1 = 2\theta)$$

$$= \frac{A^2}{2} \overset{-}{\oplus} \frac{A^2}{\pi}\sin(2\omega_0 t)$$

$$\therefore\ \sigma_X^2(t) = \frac{A^2}{2} + \frac{A^2}{\pi}\sin(2\omega_0 t) - \frac{8A^2}{\pi^2}\sin^2\left(\omega_0 t - \frac{\pi}{4}\right)$$

$$= \frac{A^2}{2} \overset{-}{\oplus} \frac{A^2}{\pi}\sin(2\omega_0 t) - \frac{4A^2}{\pi^2} + \frac{4A^2}{\pi^2}\sin(2\omega_0 t)$$

$$= \underline{0.09472\cdot A^2 + 0.72359\cdot A^2\sin(2\omega_0 t)}$$ X ←

(b)

$$R_X(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= \frac{2}{\pi}A^2\cdot\int_0^{\frac{\pi}{2}}\cos(\omega_0 t + \theta)\cos[\omega_0(t+\tau) + \theta]\,d\theta$$

$$= \frac{2}{\pi}A^2\cdot\int_0^{\frac{\pi}{2}}\cos(\omega_0 t + \theta)[\cos\omega_0\tau\cos(\omega_0 t + \theta) - \sin\omega_0\tau\sin(\omega_0 t + \theta)]\,d\theta$$

$$= \frac{2}{\pi}A^2\int_0^{\frac{\pi}{2}}\cos\omega_0\tau\cdot\cos^2(\omega_0 t + \theta)\,d\theta - \frac{1}{\pi}A^2\sin\omega_0\tau\cdot\int_0^{\frac{\pi}{2}}\sin(2\omega_0 t + 2\theta)\,d\theta$$

$$= A^2\cos\omega_0\tau\left[\frac{1}{2} + \frac{1}{\pi}\sin(2\omega_0 t)\right] + \frac{A^2}{\pi}\sin\omega_0\tau\cos(2\omega_0 t)$$

$$= \underline{\frac{A^2}{2}\cos\omega_0\tau \overset{-}{\oplus} \frac{A^2}{\pi}\sin(2\omega_0 t + \omega_0\tau)}$$ X ←

-1

A-10 (c) Since $E[X(t)]$ and $R_X(t, t+\tau)$ are functions of time, $X(t)$ is not WSS. ✓

But

$$E[X(t+\frac{2\pi}{\omega_0})] = \frac{2\sqrt{2}A}{\pi} \sin[\omega_0(t+\frac{2\pi}{\omega_0}) - \frac{\pi}{4}]$$

$$= \frac{2\sqrt{2}A}{\pi} \sin[2\pi + \omega_0 t - \frac{\pi}{4}]$$

$$= \frac{2\sqrt{2}A}{\pi} \sin(\omega_0 t - \frac{\pi}{4}) = E[X(t)]$$

$$R_X(t+\frac{2\pi}{\omega_0}, t+\frac{2\pi}{\omega_0}+\tau)$$

$$= \frac{A^2}{2}\cos\omega_0\tau + \frac{A^2}{\pi}\sin(2\omega_0 t + 4\pi + \omega_0\tau)$$

$$= \frac{A^2}{2}\cos\omega_0\tau + \frac{A^2}{\pi}\sin(2\omega_0 t + \omega_0\tau)$$

$$= R_X(t, t+\tau)$$

Thus, $X(t)$ is cyclostationary. ✓ ←

A-11 Which of the following functions are suitable for autocorrelation functions? If a function is not suitable, tell why it isn't.

(a) $A\exp(-\alpha|\tau|)$, $\alpha > 0$ and $A > 0$ ANS: SUITABLE ✓

(b) $A\sin\omega_0\tau$ ANS: NOT SUITABLE BECAUSE IT DOESN'T SATISFY $|R(\tau)| \le R(0)$ ✓

(c) $A\exp(-\beta t)u(t)$, $\beta > 0$ ANS: NOT SUITABLE BECAUSE IT DOESN'T MEET $R(-\tau) = R^*(\tau)$ ✓

X (d) $A\Pi(\tau/\tau_0)$ Fourier transform not everywhere ~~for~~ non-neg. ANS: SUITABLE IFF $A > 0$

(e) $A\,\mathrm{sinc}(2W\tau)$ ANS: SUITABLE IFF $A > 0$ ✓

(f) $A\exp[-(\tau/2\tau_0)^2]$ ANS: SUITABLE IFF $A>0$ ✓

(g) $A\tau^2\exp(-\alpha|\tau|)$ ANS: NOT SUITABLE BECAUSE $R(0) = 0 \le |R(\tau)|$. ✓

$-1\frac{1}{2}$

A-12 Show that the time average mean and variance of the process defined by (A-67) with Θ uniform in $(0, 2\pi)$ are equal to the statistical average mean and variance.

SHOW: A-67 : $X(t) = A\cos(\omega_0 t + \Theta)$ $\quad -\infty < t < \infty$

$$f_\Theta(\theta) = \begin{cases} 1/2\pi & 0 < \theta < 2\pi \\ 0 & \text{eleswhere} \end{cases}$$

From the results in Example A-9 on page 670 in text, we have

$$E[X(t)] = 0 \quad \text{and} \quad \sigma_X^2(t) = \frac{A^2}{2} = E[X^2(t)]$$

Now, we compute the time average mean and variance,

$$\langle X(t) \rangle = \frac{1}{T}\int_0^T X(t)\,dt \qquad , X(t) \text{ is peoriodical.}$$

$$= \frac{\omega_0}{2\pi}\cdot\int_0^{\frac{2\pi}{\omega_0}} A\cos(\omega_0 t + \theta)\,dt$$

$$= -\frac{\omega_0 A}{2\pi\omega_0}\sin(\omega_0 t + \theta)\Big|_0^{\frac{2\pi}{\omega_0}} = 0$$

$$\langle X^2(t) \rangle = \frac{1}{T}\int_0^T X^2(t)\,dt$$

$$= \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} A^2\cos^2(\omega_0 t + \theta)\,dt$$

$$= \frac{\omega_0 A^2}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} \frac{1}{2}[1 + \cos(2\omega_0 t + 2\theta)]\,dt$$

$$= \frac{\omega_0 A^2}{4\pi}\cdot\frac{2\pi}{\omega_0} + \frac{\omega_0 A^2}{2\pi}\cdot\frac{1}{2\omega_0}\cdot\frac{1}{2}\int_0^{4\pi}\cos(x + 2\theta)\,dx$$

$$= \frac{A^2}{2}$$

Thus, we have shown that

$$E[X(t)] = \langle X(t) \rangle = 0$$

$$E[X^2(t)] = \langle X^2(t) \rangle = \sigma_X(t) = \frac{A^2}{2}$$

- -

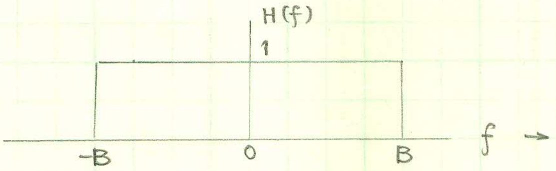
A-14. Obtain the autocorrelation functions and power spectral densities of the outputs of the following systems and input autocorrelation funtions or power spectral densities.

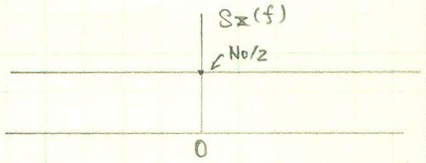
(a) $H(f) = \Pi(f/2B)$, $R_X(\tau) = (N_0/2)\,\delta(\tau)$, B and No positive constants.

(b) $h(t) = A\exp(-\alpha t)u(t)$, $S_X(f) = B/[1+(2\pi\beta f)^2]$, A, B, β and α positive constants.

Solution: (a) $R_X(\tau) = (N_0/2)\,\delta(\tau)$

$$S_X(f) = \frac{N_0}{2}$$





$$\therefore\ S_Y(f) = S_X(f)\,|H(f)|^2$$

$$= \frac{N_0}{2}\,\Pi(f/2B)$$

$$R_Y(\tau) = \mathcal{F}^{-1}\{S_Y(f)\} = N_0 B\,\mathrm{sinc}\,(2B\tau)$$

(b) $h(t) = A e^{-\alpha t} u(t)$

$$H(f) = \frac{A}{\alpha + j2\pi f} \rightarrow |H(f)|^2 = \frac{A^2}{\alpha^2 + (2\pi f)^2}$$

$$\therefore\ S_Y(f) = S_X(f)\cdot|H(f)|^2$$

$$= \frac{B}{1+(2\pi\beta f)^2}\cdot\frac{A^2}{\alpha^2+(2\pi f)^2}$$

$$R_Y(\tau) = \mathcal{F}^{-1}\{S_Y(f)\}$$

$$= \mathcal{F}^{-1}\left\{\frac{B}{1+(2\pi\beta f)^2}\cdot\frac{A^2}{\alpha^2+(2\pi f)^2}\right\}$$

$$= \mathcal{F}^{-1}\left\{\frac{-B\cdot\beta A^2}{2(1-\alpha^2\beta^2)}\cdot\frac{2\beta^{-1}}{(\beta^{-1})^2+(2\pi f)^2} + \frac{A^2}{2\alpha(1-\alpha^2\beta^2)}\cdot\frac{2\alpha}{\alpha^2+(2\pi f)^2}\right\}$$

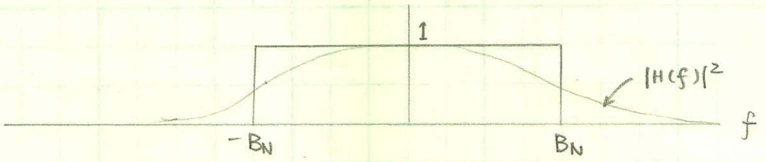
$$= \frac{-B\cdot\beta A^2}{2(1-\alpha^2\beta^2)}\cdot e^{-|\tau|/\beta} + \frac{A^2}{2\alpha(1-\alpha^2\beta^2)}\cdot e^{-\alpha|\tau|}$$

A-15 (a) Derive the general results for the ratio of equivalent noise to 3-dB bandwidth of an nth order Butterworth filter given in Table A-1

(b) Numerically verify the results given for Butterworth filter of order 1 though 6.

Solution : (a) From EE 448 and 449, we know the magnitude of nth Butterworth filter is given as (3dB-Bandwidth = 1 rad/sec = 1/2π Hz)

$$|H(f)|^2 = \frac{1}{1+(2\pi f)^{2n}}$$



$$B_n = \int_0^\infty |H(f)|^2 \, df$$

$$= \int_0^\infty \frac{1}{1+(2\pi f)^{2n}} \, df$$

$$= \frac{1}{2\pi} \int_0^\infty \frac{d\omega}{1+\omega^{2n}} .$$

From the table in Mathematics Handbook, we have (CRC 5th Ed. P437 F614)

$$B_n = \frac{1}{2\pi} \cdot \frac{\pi/2n}{\sin(\pi/2n)}$$

Thus, $$B_n/B_3 = \frac{\pi/2n}{\sin(\pi/2n)}$$

(b)

ORDER	B_N/B_3
1	1.57080
2	1.11072
3	1.04720
4	1.02617
5	1.01664
6	1.01152

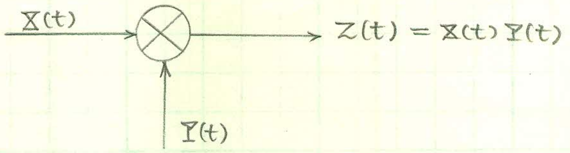
A-16 The two statistically independent inputs to a multiplier have autocorrelation functions

$$R_X(\tau) = A \cos \omega_0 \tau$$

$$R_Y(\tau) = B \operatorname{sinc}(2W\tau)$$

Obtain and plot the power spectrum of the multiplier output

Solution:



$$R_Z(\tau) = E[Z(t)\,Z(t+\tau)]$$

$$= E[X(t)\,Y(t)\;X(t+\tau)\,Y(t+\tau)]$$

$$= E[X(t)X(t+\tau)\cdot Y(t)Y(t+\tau)]$$

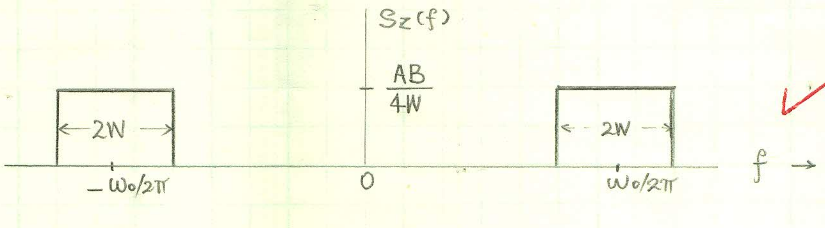
$$= E[X(t)X(t+\tau)]\cdot E[Y(t)Y(t+\tau)] \qquad (X, Y \text{ s. indep.})$$

$$= R_X(\tau)R_Y(\tau) = AB \operatorname{sinc}(2W\tau)\cos\omega_0\tau$$

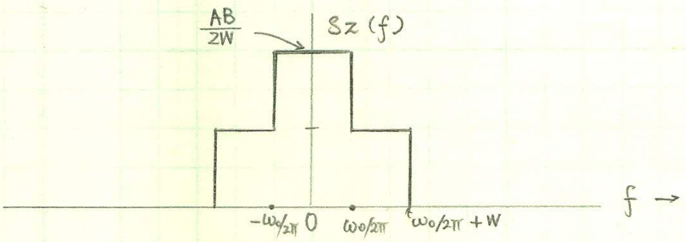
From table 2-4 and 2-5,

$$S_Z(f) = \frac{AB}{2W}\cdot\frac{1}{2}\left[\Pi\left(\frac{f-\omega_0/2\pi}{2W}\right) + \Pi\left(\frac{f+\omega_0/2\pi}{2W}\right)\right]$$

If $\omega_0/2\pi > W$, then



If $\omega_0/2\pi < W$, then



A-18 The input to a lowpass filter with impluse response

$$h(t) = e^{-10t}\,u(t)$$

is white, Gaussian noise with single-sided spectral density 1 W/Hz. Obtain the following:

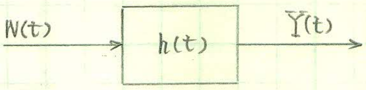
(a) The mean of the output

(b) The autocovariance function of the output

(c) The probability density function of the output at a single time instant, t_1.

(d) The joint probability of the output at time instants t_1 and $t_1 + 0.1S$.

Solution:



(a) $S_W(f) = \frac{1}{2} \qquad -\infty < f < \infty$

$$H(f) = \mathcal{F}\{h(t)\} = \frac{1}{10 + j2\pi f}$$

$$|H(f)|^2 = \frac{1}{100 + (2\pi f)^2}$$

$$\therefore\ S_Y(f) = S_W(f)\cdot|H(f)|^2 = \frac{1}{40}\cdot\frac{20}{100+(2\pi f)^2}$$

$$R_Y(\tau) = \frac{1}{40}e^{-10|\tau|}$$

Since $R_Y(\tau)$ has no periodical component. from property (6) of $R_Y(\tau)$

$$m_Y = E[Y(t)] = \lim_{|\tau|\to\infty} R_Y(\tau) = 0$$

(b) $C_Y(\tau) = R_Y(\tau) - |E[Y(t)]|^2 = R_Y(\tau) = \frac{1}{40}e^{-10|\tau|}$

(c) W(t) is white and Gaussian, so the output Y(t) is Gaussian too.

$$E[Y(t_1)] = 0$$

$$C_Y(t_1, t_1) = C_Y(0) = \frac{1}{40}$$

$$\therefore\ f_Y(y:t_1) = \frac{\sqrt{40}}{\sqrt{2\pi}}e^{-\frac{1}{2}\cdot 40\cdot y^2} = \sqrt{\frac{20}{\pi}}\cdot e^{-20y^2}$$

−1

(d) $Y_1 \triangleq Y(t_1)$, $Y_2 \triangleq Y(t_1+0.1S)$, $C_{\underline{Y}} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{pmatrix} = \begin{pmatrix} C_Y(0) & C_Y(0.1) \\ C_Y(0.1) & C_Y(0)\end{pmatrix} = \begin{pmatrix} 1/40 & 0.3679/40 \\ 0.3679/40 & 1/40\end{pmatrix}$

$\rho_{12} = \frac{R_Y(.1)}{\sigma_Y^2}$

$$f_Y(y_1, y_2) = \boxed{\frac{6.80152}{2\pi}}\, e^{-\frac{1}{2}(y, y_2)\begin{pmatrix} 1/40 & 0.3679/40 \\ 0.3679/40 & 1/40\end{pmatrix}^{-1}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}}$$

$$= 6.846\ \mathrm{exp}[-(23.13y_1^2 + 23.13y_2^2 - 29.43y_1y_2)]$$

A-19 Refering to Example A-13, obtain the p.d.f. of $Z(t)$ at an arbitary time t if $N_1 = N_2 = N_0$ and $f_1 = f_2 = f_0$. Assume that both inputs are stationary Gaussian inputs with zero means.

Solution: If $N_1 = N_2 = N_0$ and $f_1 = f_2 = f_0$, then

$$W_1(t) = W_2(t) \qquad PSD = N_0/2$$

$$H_1(f) = H_2(f) = \frac{1}{1 + j(f/f_0)}$$

From the results in Example A-13, we have

$$R_X(\tau) = R_Y(\tau) = N_0\left(\frac{\pi f_0}{2}\right)e^{-2\pi f_0|\tau|}$$

$$\therefore \quad E[X(t)] = E[Y(t)] = \lim_{|\tau|\to\infty} R_X(\tau) = 0$$

$$\sigma_X^2 = \sigma_Y^2 = R_X(0) = N_0\left(\frac{\pi f_0}{2}\right)$$

Using the fact that a linear transformation of a Gaussian process results in a Gaussian process, we can write

$$f_X(x,t) = \frac{1}{\sqrt{\pi^2 N_0 f_0}} \cdot e^{-\frac{x^2}{\pi N_0 f_0}}$$

$$f_Y(y,t) = \frac{1}{\sqrt{\pi^2 N_0 f_0}}\, e^{-\frac{y^2}{\pi N_0 f_0}}$$

Then, use the results in Example A-8.

$$f_Z(z,t) = \int_{-\infty}^{\infty} f_X(w,t)\, f_Y\!\left(\frac{z}{w}, t\right)\frac{dw}{|w|}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi^2 N_0 f_0}}\, e^{-\frac{w^2}{\pi N_0 f_0}} \cdot \frac{1}{\sqrt{\pi^2 N_0 f_0}}\, e^{-\frac{z^2}{w^2 \pi N_0 f_0}} \cdot \frac{dw}{|w|}$$

$$= \frac{1}{\pi^2 N_0 f_0} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{\pi N_0 f_0}\left(w^2 + \frac{z^2}{w^2}\right)} \frac{dw}{|w|} \quad \longleftarrow$$

$$\text{OR} \quad = \frac{1}{\pi^2 N_0 f_0} \int_0^{\infty} e^{-\frac{1}{\pi N_0 f_0}\left(w_1 + \frac{z^2}{w_1}\right)} \cdot \frac{dw_1}{w_1} \qquad (w_1 = w^2)$$

✓

$$\text{or} \quad \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma}\int_0^{\infty} \frac{\cos(z\nu)}{\sqrt{1+\nu^2\sigma^4}}\, d\nu = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma^3}\, K_0\!\left(\frac{|z|}{\sigma^2}\right)$$

A-21 (b) If $\alpha_{|m|} = a^{|m|}$ and $p(t) = A\exp(-\beta t)u(t)$, where a, A, β are positive constants, sum (A-105) to obtain a closed-form result for the autocorrelation function. Assume a is less than 1

Solution: [METHOD 1]: A-105 gives that

$$R_X(t+\tau, t) = \frac{1}{T}\sum_{m=-\infty}^{\infty} \alpha_{|m|}\, r(\tau + mT)$$

where $\quad r(\tau) = \int_{-\infty}^{\infty} p(t+\tau)p(t)\,dt$

If $\quad p(t) = Ae^{-\beta t}u(t)$

$$r(\tau) = \int_{-\infty}^{\infty} Ae^{-\beta(t+\tau)}u(t+\tau)\cdot Ae^{-\beta t}u(t)\,dt$$

$$= A^2\int_0^{\infty} e^{-\beta\tau}\cdot e^{-2\beta t}\,u(t+\tau)\,dt$$

For $\tau < 0$,

$$r(\tau) = A^2e^{-\beta\tau}\cdot\int_{-\tau}^{\infty} e^{-2\beta t}\,dt$$

$$= A^2e^{-\beta\tau}\cdot\left[-\frac{1}{2\beta}e^{-2\beta t}\right]\Big|_{-\tau}^{\infty}$$

$$= \frac{A^2}{2\beta}e^{\beta\tau}$$

For $\tau \geq 0$

$$r(\tau) = A^2e^{-\beta\tau}\cdot\int_0^{\infty} e^{-2\beta t}\,dt = \frac{A^2}{2\beta}e^{-\beta\tau}$$

Thus, $\quad r(\tau) = \frac{A^2}{2\beta}e^{-\beta|\tau|}$

$$r(\tau+mT) = \frac{A^2}{2\beta}e^{-\beta|\tau+mT|}$$

$$\therefore\quad R_X(t+\tau,t) = \frac{1}{T}\sum_{m=-\infty} a^{|m|}e^{-\beta|\tau+mT|}$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty} e^{|m|\ln a - \beta|\tau+mT|}$$

$$= \frac{1}{T}\sum_{m=-\infty}^{-1} e^{-m\ln a - \beta|\tau+mT|} + \frac{1}{T}\sum_{m=0}^{\infty} e^{m\cdot\ln a - \beta|\tau+mT|}$$

To be continued on next page.

A-21 (b) CONT.

For $\tau \leqslant 0$, there exists an integer n such that

$$\tau + (n-1)T < 0 \quad \rightarrow \quad n < -\frac{\tau}{T} + 1$$

$$\tau + nT \geqslant 0 \quad \rightarrow \quad n \geqslant -\frac{\tau}{T} \quad \geqslant 0$$

$$\therefore \quad n = \left[-\frac{\tau}{T}\right], \quad n \geqslant 0 \qquad \underline{[y] \triangleq \text{the smallest of integers which are greater than } -\frac{\tau}{T}.}$$

then

$$R_X(t+\tau,\tau) = \frac{1}{T}\Big[\sum_{m=-\infty}^{-1} e^{-m\ln a+\beta(\tau+mT)} + \sum_{m=0}^{n-1} e^{m\ln a+\beta(\tau+mT)} + \sum_{m=n}^{\infty} e^{m\ln a-\beta(\tau+mT)}\Big]$$

$$= \frac{1}{T}\Big[e^{\beta\tau}\cdot\sum_{m=1}^{\infty} e^{(\ln a-\beta T)m} + e^{\beta\tau}\cdot\sum_{m=0}^{n-1} e^{(\ln a+\beta T)m} + e^{-\beta\tau}\sum_{m=n}^{\infty} e^{(\ln a-\beta T)m}\Big]$$

$$= \frac{1}{T}\Big[\frac{ae^{-\beta T}}{1-ae^{-\beta T}}e^{\beta\tau} + \frac{1-a^n e^{\beta Tn}}{1-ae^{\beta T}}e^{\beta\tau} + \frac{a^n e^{-\beta Tn}}{1-ae^{-\beta T}}e^{-\beta\tau}\Big]$$

For $\tau > 0$, there exists an integer k also such that

$$\tau + kT < 0 \quad \rightarrow \quad k < -\frac{\tau}{T} \quad < 0$$

$$\tau + (k+1)T > 0 \quad \rightarrow \quad k > -\tau/T - 1$$

$$\therefore \quad k = \mathrm{INT}\left(-\frac{\tau}{T}\right) = -\left[\frac{\tau}{T}\right] \quad < 0$$

then

$$R_X(t+\tau,\tau) = \frac{1}{T}\Big[\sum_{m=-\infty}^{k} e^{-m\ln a+\beta(\tau+mT)} + \sum_{m=k+1}^{0} e^{-m\ln a-\beta(\tau+mT)} + \sum_{m=1}^{\infty} e^{m\ln a-\beta(\tau+mT)}\Big]$$

$$= \frac{1}{T}\Big[e^{\beta\tau}\cdot\sum_{m=-k}^{\infty} e^{m(\ln a-\beta T)} + e^{-\beta\tau}\sum_{m=0}^{-k-1} e^{m(\ln a+\beta T)} + e^{-\beta\tau}\sum_{m=1}^{\infty} e^{m(\ln a-\beta T)}\Big]$$

$$= \frac{1}{T}\Big[\frac{e^{-(\ln a-\beta T)k}}{1-e^{\ln a-\beta T}}e^{\beta\tau} + \frac{1-e^{-(\ln a+\beta T)k}}{1-e^{\ln a+\beta T}}e^{-\beta\tau} + \frac{e^{\ln a-\beta T}}{1-e^{\ln a-\beta T}}e^{-\beta\tau}\Big]$$

$$= \frac{1}{T}\Big[\frac{ae^{-\beta T}\cdot e^{-\beta\tau}}{1-ae^{-\beta T}} + \frac{1-a^{-k}e^{-\beta Tk}}{1-ae^{\beta T}}e^{-\beta\tau} + \frac{a^{-k}e^{\beta Tk}}{1-ae^{-\beta T}}e^{\beta\tau}\Big]$$

Finally, we have the closed-form for $R_X(t+\tau,t)$

I think this is OK!?

$$R_X(t+\tau,t) = \frac{1}{T}\cdot\left(\frac{ae^{-\beta T}}{1-ae^{-\beta T}} + \frac{1-a^{[|\tau/T|]}e^{\beta T[|\tau/T|]}}{1-ae^{\beta T}}\right)e^{-\beta|\tau|} \quad \swarrow$$

$$+ \frac{a^{[|\tau/T|]}e^{-\beta T[|\tau/T|]}}{T(1-ae^{-\beta T})}e^{\beta|\tau|} \quad \longleftarrow$$

where $[|\tau/T|] \triangleq$ the smallest of integers which are greater than $|\tau/T|$.

PROBLEM FROB NOTE (P20, LECTURE 15 & 16)

10/10

For $n_c(t)$ and $n_s(t)$ zero-mean, statistically independent. Gaussian random processes with equal varians $\sigma_{n_c}^2 = \sigma_{n_s}^2 = \sigma^2/2$, Find the pdf's of $R(t)$ and $\Phi(t)$.

Solution: From the givens, we have the p.d.f for $n_c(t)$ and $n_s(t)$

$$f_{n_c}(n_c,t) = \frac{1}{\sqrt{2\pi}\,\sigma_{n_c}} e^{-n_c^2/2\sigma_{n_c}^2} = \frac{1}{\sqrt{\pi}\,\sigma} e^{-n_c^2/\sigma^2}$$

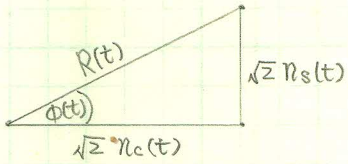
$$f_{n_s}(n_s,t) = \frac{1}{\sqrt{\pi}\,\sigma} \cdot e^{-n_s^2/\sigma^2} \quad ; \quad f_{n_s n_c}(n_s,t_1;n_c,t_2) = \frac{1}{\pi\sigma^2} e^{-(n_s^2+n_c^2)/\sigma^2}$$

From notes,

$$R(t) = \sqrt{2[n_c^2(t) + n_s^2(t)]} \qquad \geqslant 0$$

$$\Phi(t) = \tan^{-1}\left[\frac{n_s(t)}{n_c(t)}\right] \qquad (\text{Assume } 0 \leqslant \Phi(t) \leqslant 2\pi)$$

Although, $n_c(t)$, $n_s(t)$, $R(t)$ and $\Phi(t)$ are random processes, we can apply the result in transformation of random variables, because all the pdf's of them are independent of t.



$$\begin{pmatrix} n_s(t) \\ n_c(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} R(t) \sin\Phi(t) \\ \frac{1}{\sqrt{2}} R(t) \cos\Phi(t) \end{pmatrix}$$

We see that if we assume the value of $\Phi(t) = \tan^{-1}\left[\frac{n_s(t)}{n_c(t)}\right]$ goes from 0 to 2π, the transformation will be one-to-one.

$$J\left(\frac{n_s\ n_c}{R\ \Phi}\right) = \begin{vmatrix} \frac{\partial n_s(t)}{\partial R(t)} & \frac{\partial n_c(t)}{\partial R(t)} \\ \frac{\partial n_s(t)}{\partial \Phi(t)} & \frac{\partial n_c(t)}{\partial \Phi(t)} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{\sqrt{2}} \sin\Phi(t) & \frac{1}{\sqrt{2}} \cos\Phi(t) \\ \frac{1}{\sqrt{2}} R(t)\cos\Phi(t) & -\frac{1}{\sqrt{2}} R(t)\sin\Phi(t) \end{vmatrix} = -\frac{1}{2}R(t)$$

$$\left| J\left(\frac{n_s\, n_c}{R\ \Phi}\right) \right| = \frac{1}{2} R(t)$$

$$\therefore \quad f_{R\Phi}(r, t_1 ; \Phi, t_2) = f_{n_s n_c}\left(n_s = \frac{1}{\sqrt{2}} r \sin\Phi, t_1 ; n_c = \frac{1}{\sqrt{2}} r \cos\Phi, t_2\right) \cdot \left| J\left(\frac{n_s\, n_c}{r\ \Phi}\right) \right|$$

$$= \frac{1}{\pi \sigma^2} e^{-(\frac{1}{2} r^2 \sin^2\Phi + \frac{1}{2} r^2 \cos^2\Phi)/\sigma^2)} \cdot \frac{1}{2} r$$

$$= \begin{cases} \frac{1}{2\pi\sigma^2} r \cdot e^{-r^2/2\sigma^2} & r \geq 0 \ , \ 0 \leq \Phi \leq 2\pi \\ 0 & \text{eleswhere} \end{cases}$$

$$f_R(r, t) = \int_0^{2\pi} f_{R\Phi}(r, t_1 ; \phi, t_2)\, d\phi$$

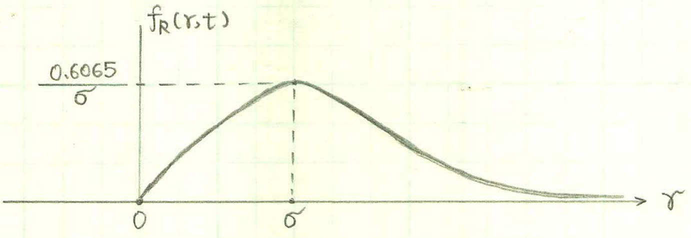
$$= \int_0^{2\pi} \frac{1}{2\pi\sigma^2} r\, e^{-r^2/2\sigma^2} d\Phi = \underbrace{\frac{1}{\sigma^2} r\, e^{-r^2/2\sigma^2} \quad r \geq 0}_{\text{Rayleigh p.d.f.}} \quad ✓ \quad \leftarrow$$

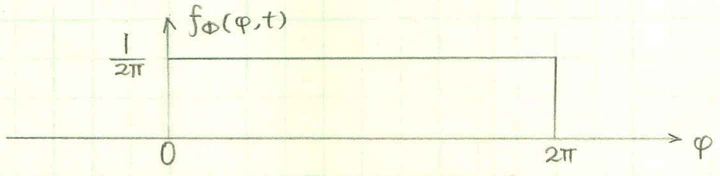
$$f_\Phi(\varphi, t) = \int_0^{\infty} \frac{1}{2\pi\sigma^2} r \cdot e^{-r^2/2\sigma^2}\, dr$$

$$= \frac{1}{2\pi} \int_0^{\infty} e^{-r^2/2\sigma^2}\, d\left(\frac{r^2}{2\sigma^2}\right)$$

$$= \frac{1}{2\pi} e^{-r^2/2\sigma^2} \Big|_0^{\infty} = \frac{1}{2\pi} \qquad 0 \leq \varphi \leq 2\pi \quad \leftarrow$$

$$\underbrace{= 0 \qquad \text{eleswhere}}_{\text{Uniform over } [0, 2\pi]}$$





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EE 425 Homework Problem

Due: Wed, March 30

Consider the random variable

$$R = \sqrt{(A + N_c)^2 + N_s^2}$$

where A is a known constant and N_c and N_s are zero-mean, statistically independent Gaussian random variable with variance $E\{N_c^2\} = E\{N_s^2\} = \sigma^2$.

Show that R is a Rician random variable with pdf

$$f_R(r) = \left(\frac{r}{\sigma^2}\right) exp\left[-\frac{r^2 + A^2}{2\sigma^2}\right] I_0\left(\frac{rA}{\sigma^2}\right) u(r)$$

where $I_0(x) \triangleq \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\, d\theta$

is the modified Bessel function of first kind and zero order.

(See problem A-4 in text)

Solution: From the results in Homework assignment eight, we have

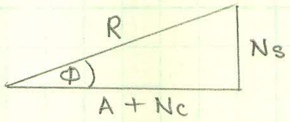
$$f_{N_c}(n_c) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{n_c^2}{2\sigma^2}}$$

$$f_{N_s}(n_s) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{n_s^2}{2\sigma^2}}$$

$$f_{N_cN_s}(n_c, n_s) = \frac{1}{2\pi\sigma^2} e^{-\frac{(n_c^2+n_s^2)}{2\sigma^2}}$$

$$R = \sqrt{(A+N_c)^2 + N_s^2} \qquad R \geq 0$$

And Let $\Phi = \tan^{-1}\left(\frac{N_s}{A+N_c}\right)$ $\qquad 0 \leq \Phi \leq 2\pi$



$$N_s = R\sin\Phi \qquad \text{OR} \qquad n_s = r\sin\varphi$$

$$N_c = R\cos\Phi - A \qquad\qquad n_c = r\cos\varphi - A$$

$$J\left(\frac{N_c N_s}{R\ \Phi}\right) = \begin{vmatrix} \frac{\partial n_c}{\partial r} & \frac{\partial n_s}{\partial r} \\ \frac{\partial n_c}{\partial \varphi} & \frac{\partial n_s}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{vmatrix} = r$$

$$f_{R\Phi}(r, \varphi) = f_{N_cN_s}(r\cos\varphi - A, r\sin\varphi) \cdot r$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{r^2\cos^2\varphi - 2Ar\cos\varphi + A^2 + r^2\sin^2\varphi}{2\sigma^2}}$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{(r^2+A^2) - 2Ar\cos\varphi}{2\sigma^2}} \qquad r \geq 0, \quad 0 \leq \varphi \leq 2\pi$$

$$\therefore \quad f_R(r) = \int_0^{2\pi} f_{R\Phi}(r, \varphi)\, d\varphi \qquad r \geq 0$$

$$= \int_0^{2\pi} \frac{r}{2\pi\sigma^2} \cdot e^{-\frac{r^2+A^2}{2\sigma^2}} \cdot e^{2Ar\cos\varphi/2\sigma^2}\, d\varphi \qquad r \geq 0$$

$$= \frac{r}{\sigma^2} e^{-\frac{r^2+A^2}{2\sigma^2}} \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{rA}{\sigma^2}\cos\varphi}\, d\varphi \qquad r \geq 0$$

$$= \frac{r}{\sigma^2} e^{-\frac{r^2+A^2}{2\sigma^2}} \cdot I_0\left(\frac{rA}{\sigma^2}\right) u(r)$$

And from problem A-4, we know R is Rician random variable. Showed

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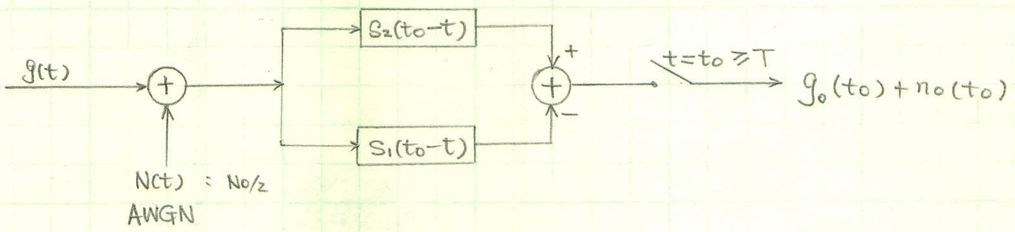
HOMEWORK FROM NOTES P22, LECTURE 17

1. (a) Find the optimum (matched) filter implulse response, $h_0(t)$, for signal $g(t) = S_2(t) - S_1(t)$. Note that $h_0(t)$ can be implemented as two filter in parallel feeding a sum (difference) point.

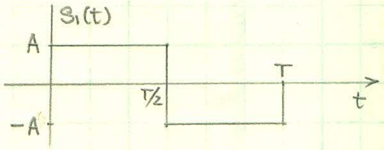
Solution: From notes, we have the matched filter for $g(t)$

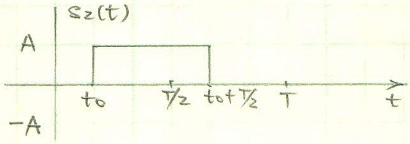
$$h_0(t) = g(t_0 - t) = S_2(t_0 - t) - S_1(t_0 - t)$$

And we see that both $S_2(t_0-t)$ and $S_1(t_0-t)$ are the matched filters corresponding to $S_2(t)$ and $S_1(t)$. the following filter is equivalent to $h_0(t) = g(t_0 - t)$

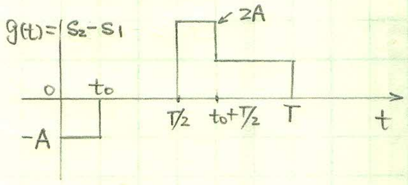


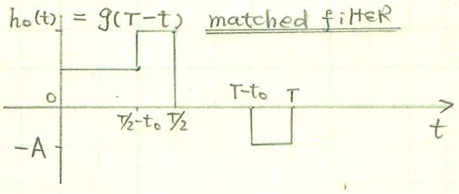
(b) Find $\xi^2 = g_0^2(T)/\sigma_N^2$ and plot as a function of t_0





Solution:



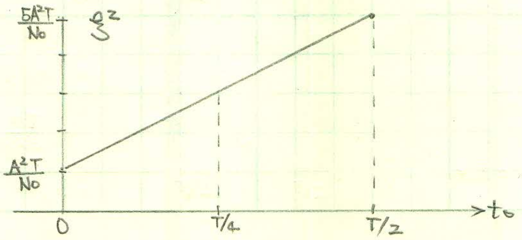


← (a)

From note, we see that (for matched filter)

$$\xi^2 = g_0^2(T)/\sigma_N^2 = \xi^2_{max} = \frac{2}{N_0} E_g = \frac{2}{N_0}\left[A^2 \cdot t_0 + 4A^2 t_0 + A^2 \cdot \left(\frac{T}{2} - t_0\right)\right]$$

$$= \frac{A^2}{N_0} T \cdot \left(1 + 8\frac{t_0}{T}\right)$$



(C) The correlation coefficient between $s_1(t)$ and $s_2(t)$ is defined as

$$\rho_{12} = \frac{\int_0^T s_1(t)\, s_2(t)\, dt}{\sqrt{\int_0^T s_1^2(t)\, dt \cdot \int_0^T s_2^2(t) dt}}$$

Find and plot ξ^2 as a function of ρ_{12}. (Generalize this result for any $s_1(t)$ and $s_2(t)$)

Solution: Here, I think ξ^2 is defined as ξ^2_{max} (matched filter). Otherwise, for different filter, we have different ξ^2. (we may pick any $h(t)$).

$$\xi^2_{max} = \frac{2}{N_0} E_g$$

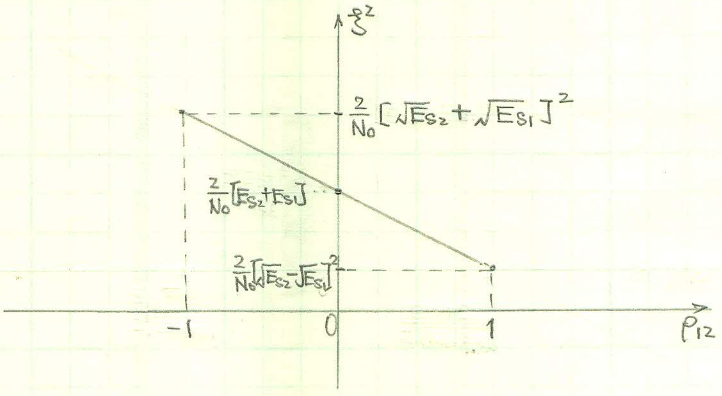
$$E_g = \int_0^T [s_2(t) - s_1(t)]^2\, dt$$

$$= \int_0^T [\, s_2^2(t) - 2 s_1(t)\, s_2(t) + s_1^2(t)]\, dt$$

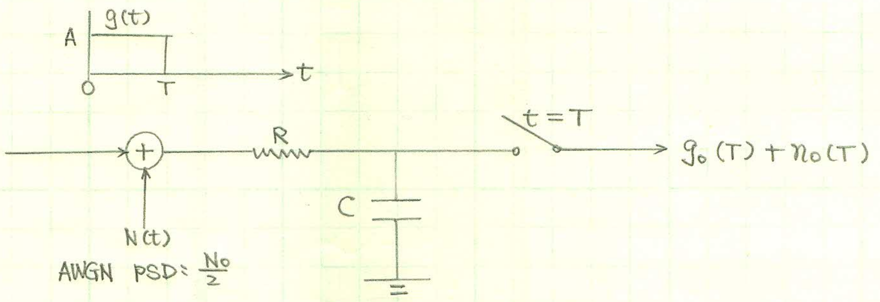
$$= \int_0^T s_2^2(t) dt + \int_0^T s_1(t) dt - 2\int_0^T s_1(t) s_2(t)\, dt$$

$$= E_{s_2} + E_{s_1} - 2\sqrt{E_{s_1} E_{s_2}} \cdot \rho_{12}$$

$$\therefore \quad \xi^2 = \frac{2}{N_0} \cdot [\, E_{s_2} + E_{s_1} - 2\sqrt{E_{s_1} E_{s_2}} \cdot \rho_{12}]$$



2. As an approximation to the integrate-and-dump detector for a rectagular pulse, the integrator may be replaced with a low-pass RC filter.



The transfer function of the filter is

$$H(f) = \frac{1}{1 + j(f/f_c)}$$

where $f_c = 1/2\pi RC$ is the 3-dB cut-off frequency of the filter.

(a) Find $\xi^2 = g_o^2(T) / E\{n_o^2(T)\}$

(Assume filter capacitor discharged at $t=0$) i.e.

$$n_o(T) = \int_0^T N(\lambda) h(T-\lambda) d\lambda$$

<u>Solution:</u> The RC Low-pass filter is not matched to $g(t)$. So, we can not use the results from matched filter.

$$H(f) = \frac{1}{1+j(f/f_c)} = \frac{2\pi f_c}{2\pi f_c + j \cdot 2\pi f}$$

From table,

$$h(t) = 2\pi f_c \cdot e^{-2\pi f_c t} u(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

$$n_o(T) = \int_0^T N(\lambda) h(T-\lambda) d\lambda$$

$$E[n_o^2(T)] = \int_0^T \int_0^T E\{N(\lambda_1) N(\lambda_2)\} h(T-\lambda_1) h(T-\lambda_2) d\lambda_1 d\lambda_2$$

$$= \frac{N_o}{2} \int_0^T h^2(T-\lambda) d\lambda = \frac{N_o}{2} \int_0^T h^2(\lambda) d\lambda$$

$$= \frac{N_o}{2} \cdot (2\pi f_c)^2 \cdot \int_0^T e^{-4\pi f_c \lambda} d\lambda$$

$$= \frac{N_o \pi f_c}{2} \cdot (1 - e^{-4\pi f_c T})$$

(To be continued)

$$g_o(T) = \int_0^T g(\lambda)h(T-\lambda)\,d\lambda$$

$$= A\cdot\int_0^T h(T-\lambda)\,d\lambda$$

$$= A\cdot\int_0^T h(\lambda)\,d\lambda$$

$$= -Ae^{-2\pi f_c t}\Big|_0^T = A(1-e^{-2\pi f_c T})$$

$$g_o^2(T) = A^2(1-2e^{-2\pi f_c T}+e^{-4\pi f_c T})$$

$$\therefore\quad \xi^2 = \frac{g_o^2(T)}{E\{n_o^2(T)\}} = \frac{2A^2}{N_o\pi f_c}\cdot\frac{1-2e^{-2\pi f_c T}+e^{-4\pi f_c T}}{1-e^{-4\pi f_c T}}$$

$$= \frac{2A^2T}{N_o\pi}\cdot\frac{1-2e^{-2\pi f_c T}+e^{-4\pi f_c T}}{f_cT\cdot(1-e^{-4\pi f_c T})}$$

(2) Plot ξ^2 (in dB) as a function of f_cT and show that ξ^2 is maximum when $f_cT=0$. Show that the maximum value obtained for ξ^2 is equal to ξ_{max} of the optimum integrate-and-dump filter.

Solution:

$$\frac{d\xi^2}{df_cT} = \frac{2A^2T}{N_o\pi}\cdot\frac{4\pi(e^{-2\pi f_cT}-e^{-4\pi f_cT})\cdot f_cT(1-e^{-4\pi f_cT})}{(f_cT)^2(1-e^{-4\pi f_cT})^2}$$

$$-\frac{2A^2T}{N_o\pi}\frac{(4\pi f_cT-1)e^{-4\pi f_cT}\cdot(1-e^{-2\pi f_cT})^2+(1-e^{-2\pi f_cT})^2}{(f_cT)^2(1-e^{-4\pi f_cT})^2}$$

$$= \frac{2A^2T}{N_o\pi}\cdot\frac{4\pi f_cT(e^{-2\pi f_cT}-e^{-4\pi f_cT}-e^{-6\pi f_cT}+e^{-8\pi f_cT}-e^{-4\pi f_cT}+2e^{-6\pi f_cT}-e^{-8\pi f_cT})-(1-e^{-2\pi f_cT})^2(1-e^{-4\pi f_cT})}{(f_cT)^2(1-e^{-4\pi f_cT})^2}$$

$$= \frac{2A^2T}{N_o\pi}\cdot\frac{4\pi f_cT(e^{-2\pi f_cT}+e^{-6\pi f_cT}-2e^{-4\pi f_cT})-(1-e^{-2\pi f_cT})^2(1-e^{-4\pi f_cT})}{(f_cT)^2(1-e^{-4\pi f_cT})^2} < 0\ , \quad \text{for } f_cT>0$$

So, ξ^2 is monotonous-decreased when f_cT increased. And

Using the fact that

$$\lim_{x\to0}\frac{f(x)}{g(x)} = \lim_{x\to0}\frac{f'(x)}{g'(x)} = \lim_{x\to0}\frac{f''(x)}{g''(x)}$$

$$\xi^2\Big|_{f_cT=0} = \lim_{f_cT\to0}\frac{2A^2T}{N_o\pi}\frac{(1-e^{-2\pi f_cT})^2}{f_cT(1-e^{-4\pi f_cT})}$$

$$= \lim_{f_cT\to0}\frac{2A^2T(e^{-2\pi f_cT}-e^{-4\pi f_cT})}{N_o\pi\cdot[(4\pi f_cT-1)e^{-4\pi f_cT}+1]}$$

$$= \lim_{f_cT\to0}\frac{4\pi\,2A^2T\cdot(4\pi e^{-4\pi f_cT}-2\pi e^{-2\pi f_cT})}{N_o\pi\cdot[4\pi e^{-4\pi f_cT}-(4\pi f_cT-1)\cdot4\pi e^{-4\pi f_cT}]}$$

$$= \frac{4\pi\cdot2A^2T\cdot2\pi}{N_o\pi\cdot(4\pi+4\pi)} = \frac{2A^2T}{N_o}$$

$\xi^2 \Big|_{f_cT=0} = \frac{2A^2T}{N_0}$ is the maximum value of ξ^2.

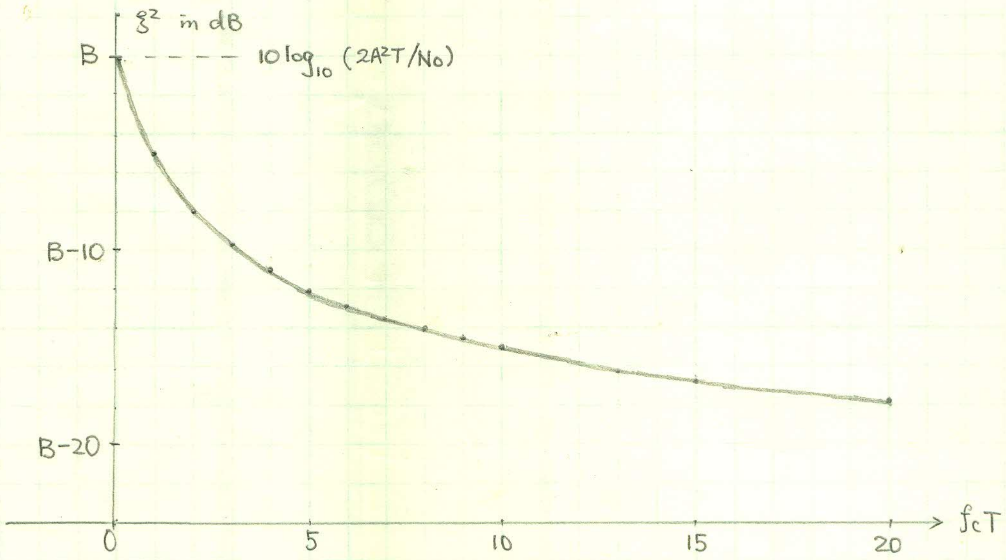
$\therefore \quad \xi^2_{max} = \frac{2A^2T}{N_0} =$ max. ξ^2 of the optimum integrate-and-dump filter (P20)

Recall that,

$$\xi^2 = \frac{2A^2T}{N_0} \frac{(1-e^{-2\pi f_cT})^2}{\pi f_cT(1-e^{-4\pi f_cT})}$$

$$\xi^2 \text{ in dB} = \underbrace{10\log_{10}\left(\frac{2A^2T}{N_0}\right)}_{B} + 20\log_{10}(1-e^{-2\pi f_cT}) - 10\log_{10}[\pi f_cT(1-e^{-4\pi f_cT})]$$

f_cT	ξ^2 in dB
0.01	B-0.00143
0.05	B-0.03552
0.10	B-0.13968
0.15	B-0.30579
0.20	B-0.52408
0.25	B-0.78322
0.30	B-1.07182
0.50	B-2.33678
1.00	B-4.98772
2.00	B-7.98183
3.00	B-9.74271
4.00	B-10.99210
5.00	B-11.96120
10.00	B-14.9715
20.00	B-17.8180



(c) $f_cT = 0$ can not be achieved in practice. Suppose we use $f_cT = 0.1$, What is the loss (in dB) of ξ^2 compared to optimum? For R = 1K ohm and $T = 10^{-5}$ second, what is the value of C? For A = 5 volts, what is the value of $g_o(T)$?

Solution: From part (b), we see that

$$\text{Loss of } \xi^2 \text{ (in dB)} = 0.13968 \text{ dB}$$

$$f_cT = \frac{T}{2\pi RC} = 0.1$$

$$C = \frac{T}{2\pi R \times 0.1} = \frac{10^{-5}}{2 \times 3.141593 \times 1000 \times 0.1} = 1.59155 \times 10^{-8} \text{ F}$$

From part (a) on page 4, we have

$$g_o(T) = A(1 - e^{-2\pi f_cT})$$

$$= 5(1 - e^{-2\pi \times 0.1})$$

$$= 2.33256 \text{ volts}$$

Homework no: 11 (Lecture 18) Digital Communication Systems 47/50 Ben Chen 01

1. For threshold $\frac{A_c}{2}$ and pdf's given by (35) and (37), show that the probability of error is approximately

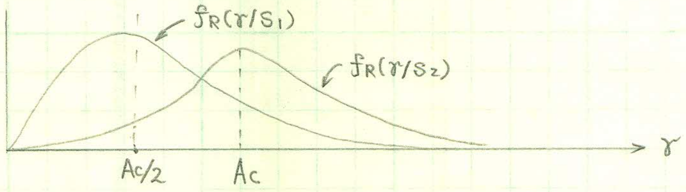
10

$$P_\varepsilon \approx \frac{1}{2} e^{-\frac{E_{b,ave}}{2N_0}} + \frac{1}{2} Q\left(\sqrt{\frac{E_{b,ave}}{N_0}}\right)$$

SHOW: Eq.(35) and (37)

$$f_R(r/S_1) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} u(r) \quad \cdots\cdots \quad (35)$$

$$f_R(r/S_2) = \sqrt{\frac{r}{2\pi A_c \sigma^2}} \cdot e^{-\frac{(r-A_c)^2}{2\sigma^2}} \cdot u(r) \quad \cdots \quad (37)$$



Using the hints on page 22, Lecture 18, we can approximate (37) with a Gaussian P.d.f.

$$f_R(r/S_2) \simeq \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(r-A_c)^2}{2\sigma^2}} \cdot u(r) \quad \cdots\cdots \quad (a)$$

$$\therefore \quad P_E(\varepsilon|S_2) \cong \int_0^{A_c/2} f_R(r|S_2)\, dr$$

$$\approx \int_{-\infty}^{A_c/2} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(r-A_c)^2}{2\sigma^2}} dr$$

Let $x = r - A_c$

$$= \int_{-\infty}^{-A_c/2} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} dx$$

$$= Q\left(\frac{A_c}{2\sigma}\right) = Q\left(\sqrt{\frac{A_c^2}{4\sigma^2}}\right)$$

Again, use the hint, we have $\frac{A_c^2}{2\sigma^2} = \frac{2E_{b,ave}}{N_0}$

$$P_E(\varepsilon|S_2) \cong Q\left(\sqrt{\frac{E_{b,ave}}{N_0}}\right)$$

$$P_E(\varepsilon|S_1) = \int_{A_c/2}^{\infty} f_R(r|S_1)\, dr$$

$$= \int_{A_c/2}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = \int_{A_c/2}^{\infty} e^{-\frac{r^2}{2\sigma^2}} d\left(\frac{r^2}{2\sigma^2}\right)$$

$$= -e^{-r^2/2\sigma^2}\Big|_{A_c/2}^{\infty} = e^{-\frac{A_c^2}{8\sigma^2}} = e^{-\frac{E_{b,ave}}{2N_0}}$$

Thus, we have shown that

$$P_\varepsilon = \frac{1}{2} P_E(\varepsilon|S_1) + \frac{1}{2} P_E(\varepsilon|S_2)$$

$$\approx \frac{1}{2} e^{-\frac{E_{b,ave}}{2N_0}} + \frac{1}{2} Q\left(\sqrt{\frac{E_{b,ave}}{N_0}}\right)$$

Q.E.D.

2. Use the expressions derived in these notes to make a table of SNR (E_b/N_0) for BPSK, coherent and non-coherent ASK, coherent and non-coherent FSK for $P_\varepsilon = 10^{-3}, 10^{-5}, 10^{-7}$

10

Solution: For BPSK, $P_\varepsilon = Q(\sqrt{\frac{2E_b}{N_0}}) = Q(\sqrt{2 \cdot SNR})$

For ASK — coherent: $P_\varepsilon = Q(\sqrt{\frac{E_{b,ave}}{N_0}}) = Q(\sqrt{SNR})$

nocoherent: $P_\varepsilon \cong \frac{1}{2} e^{-\frac{E_{b,ave}}{2N_0}} + \frac{1}{2} Q(\sqrt{\frac{E_{b,ave}}{N_0}}) = \frac{1}{2} e^{-\frac{1}{2}SNR} + \frac{1}{2} Q(\sqrt{SNR})$

For FSK — coherent: $P_\varepsilon = Q(\sqrt{\frac{E_b}{N_0}}) = Q(\sqrt{SNR})$

nocoherent: $P_\varepsilon = \frac{1}{2} e^{-\frac{E_b}{2N_0}} = \frac{1}{2} e^{-\frac{SNR}{2}}$

Then Define $x = Q^{-1}(P_\varepsilon)$, if $P_\varepsilon = Q(x)$

For BPSK: $SNR = [Q^{-1}(P_\varepsilon)]^2/2$

For ASK coherent: $SNR = [Q^{-1}(P_\varepsilon)]^2$

nocoherent: $SNR = [Q^{-1}(2P_\varepsilon - e^{-\frac{1}{2}SNR})]^2$ * pick and check

For FSK coherent: $SNR = [Q^{-1}(P_\varepsilon)]^2$

noncoherent $SNR = -2\ln(2P_\varepsilon)$

The Table (By using the Q-function from Math. Handbook, Xiamen Univ.)

P_ε	SNR (in dB)				
	BPSK	ASK coherent	ASK nocoherent	FSK coherent	FSK noncoherent
10^{-3}	6.79	9.80	10.94	9.80	10.94
10^{-5}	9.59	12.60	13.35	12.60	13.35
10^{-7}	11.31	14.32	14.89	14.32	14.89

Note: For small P_ε which requires high SNR, then

$$\frac{1}{2} e^{-\frac{1}{2}SNR} \gg \frac{1}{2} Q(\sqrt{SNR})$$

Thus, for FSK noncoherent.

$$P_\varepsilon \approx \frac{1}{2} e^{-\frac{1}{2}SNR}$$

3. Drive an expression for P_E for coherent FSK if the frequency separation of the two transmitted signals is chosen to give a minimum correlation coefficient. What is Δf in terms of T_b for this case? How much improvement (reduction) in E_b/N_o over the orthogoral - signal case is obtained?

10

Sltn: From notes, we have

$$S_1(t) = A_c \cos(2\pi f_o t)$$

$$S_2(t) = A_c \cos[2\pi(f_o + \Delta f)t]$$

$$S_1(t)S_2(t) = \frac{A_c^2}{2}\cos(2\pi\Delta f t) + \frac{A_c^2}{2}\cos[(4\pi f_o + 2\pi\Delta f)t]$$

$$\int_0^{T_b} S_1(t)S_2(t)\,dt = \int_0^{T_b} \frac{A_c^2}{2}\cos(2\pi\Delta f t)\,dt + 0$$

$$= \frac{A_c^2}{2}\cdot\frac{1}{2\pi\Delta f}\sin(2\pi\Delta f t)\Big|_0^{T_b}$$

$$= \frac{A_c^2}{2}\cdot\frac{1}{2\pi\Delta f}\sin(2\pi\Delta f T_b)$$

$$= \frac{A_c^2 T_b}{2}\cdot \mathrm{sinc}(2\pi\Delta f T_b)$$

When $2\pi\Delta f T_b = \frac{3}{2}\pi$, OR $\Delta f = \frac{3}{4T_b}$, $\mathrm{sinc}(2\pi\Delta f T_b)$ is the minimum with value $-\frac{2}{3\pi} = -0.21221$ ✓ close

$$\text{minimum} \int_0^{T_b} S_1(t)S_2(t)\,dt = -\frac{A_c^2 T_b}{3\pi} = -\frac{2}{3\pi}E_b$$

At $\Delta f = \frac{3}{4T_b}$

And $P_E = Q\left(\sqrt{\frac{2E_b + 4E_b/3\pi}{2N_o}}\right)$

$$= Q\left(\sqrt{1.21221\cdot\frac{E_b}{N_o}}\right)$$

$$\frac{E_b}{N_o} = [Q^{-1}(P_E)]^2/1.21221$$

$$\frac{E_b}{N_o} \text{ in dB} = 20\log_{10} Q^{-1}(P_E) - 10\log_{10} 1.21221$$

$$= \text{SNR (orthogoral)} - 0.83577 \text{ dB}$$

Thus, 0.83577 dB improvement is obtained for this case. ✓

close

4. Binary FSK transmissions are permitted in some high-frequency amateur radio bands using either $\Delta f = 170$ Hz ("narrowband") or $\Delta f = 850$ Hz ("wideband") with $\Delta f\, T = 3.75$, 18.75 respectively. Suppose the RMS (noise-free) signal strength is 10 μV across 50 ohms and that the transmissions are received in the presence of additive white Gaussian noise with two-sided power spectral density $\frac{N_o}{2} = 10^{-15}$ Watts/Hz

10

(a) Compute the theoretical minimum probability of error using coherent demodulation.

Solution: (a) $\Delta f \cdot T = 3.75$, $T_1 = 3.75/\Delta f = 0.022059$ sec. $\Delta f T = 18.75$, $T_2 = 0.022059$

RMS $V = 10\,\mu V$, then $A_c = 10\sqrt{2}\ \mu V$, $E_b = \frac{A_c^2}{2} T_b / 50 = 4.4118 \times 10^{-14}$ watts ; $\frac{E_b}{N_o} = 22.059$

Using the result in Problem 3 , we have

$$\int_0^T s_1(t) s_2(t)\, dt = E_b \cdot \operatorname{sinc}(2\cancel{\pi} \Delta f T)$$

For $\Delta f \cdot T = 3.75$, $\int_0^T s_1(t) s_2(t)\, dt = E_b \times (-0.0424413)$

For $\Delta f \cdot T = 18.75$, $\int_0^T s_1(t) s_2(t)\, dt = E_b \times (-0.008488264)$

$$P_{e,\,3.75} = Q\left(\sqrt{\frac{E_b + 0.0424413\, E_b}{N_o}}\right) = Q(\sqrt{1.0424413 \times 22.059}) = 8.13454 \times 10^{-7}$$

$$P_{e,\,18.75} = Q(\sqrt{1.008488264 \times 22.059}) = 1.1985 \times 10^{-6}$$

(b) Repeat (a) using non-coherent demodulation.

In this case , we only have

$$P_e = \frac{1}{2} e^{-\frac{E_b}{2N_o}} = 8.1081 \times 10^{-6}$$

(c) What are the relative advantages and disadvantages for each choice of Δf ?

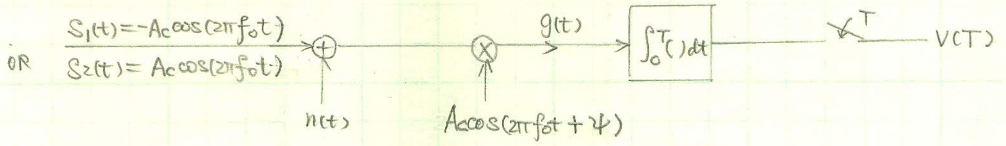
For coherent demodulation , If SNR remains the same , "narrowband" has lower P_e than "wideband". However , for non-coherent demodulation, there is no difference in P_e. More bandwidth for W.B.

5. Suppose a phase error is present in the correlation detector for a BPSK receiver so that

12/15 the reference signal is $A_c \cos(2\pi f_o t + \psi)$ where ψ is the phase error.

(a) Derive an expression for P_E

Solution:



If $S_2(t)$ is present,

$$g_2(t) = S_2(t) \cdot A_c \cos(2\pi f_o t + \psi) + \text{noise}$$

$$= A_c^2 \cos(2\pi f_o t)\cos(2\pi f_o t + \psi) + \text{noise}$$

$$= \frac{1}{2}A_c^2 \cos\psi + \underbrace{\frac{1}{2}A_c^2 \cos(4\pi f_o t + \psi)}_{\int_0^T (\,)dt = 0} + \text{noise}$$

If $S_1(t)$ is present,

$$g_1(t) = \frac{-1}{2}A_c^2 \cos\psi - \underbrace{\frac{1}{2}A_c^2 \cos(4\pi f_o t + \psi)}_{\int_0^T (\,)dt = 0} + \text{noise}$$

Now,

$$E_b = \frac{1}{4}A_c^4 \cos^2\psi \cdot T$$

$$N_o' = \frac{1}{2}A_c^2 \cdot N_o$$

$$P_E = Q\left(\sqrt{2E_b/N_o'}\right) = Q\left(\sqrt{\frac{2^{-1}A_c^4\cos^2\psi \cdot T}{2^{-1}A_c^2 \cdot N_o}}\right) = \underline{Q\left(\sqrt{\frac{A_c^2 T\cos^2\psi}{N_o}}\right)}$$ ✓

(b) Estimate the phase error ψ that would increas P_E from 10^{-5} to 10^{-4}

$P_E = 10^{-5}$, $\frac{A_c^2 T}{N_o} = 4.2647$. no error.

$P_E = 10^{-4}$, $\frac{A_c^2 T}{N_o}\cos^2\psi = 3.7187$

$\therefore$ $\cos^2\psi = 0.87197$ and $\underline{-20.96° = \psi = \cancel{20.96°}}$ 29.2° -3

(c) Suppose ψ is random and we are able to determine the probability density function, $f_\psi(\psi)$ for ψ.

Assume that ψ remains constant over a bit interval and set-up integral in terms of $f_\psi(\psi)$ for P_E

$$P_E = \int_{-\infty}^{\infty} Q\left(\sqrt{A_c^2 T\cos^2\psi/N_o}\right) f_\psi(\psi)\, d\psi = E\{P_{E,\psi}\}$$

Lecture 19, Page 22

89/100

1. Consider the digital amplitude modulated signal given as

$$s(t) = z(t)\cos(2\pi f_0 t)$$

where $$z(t) = \sum_{k=-\infty}^{\infty} A_K\, p(t - kT - \Delta)$$

Δ is a random variable uniformly distributed between 0 and T

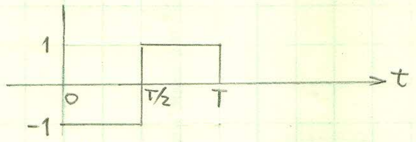
$A_K = A_c$ for bineary "1"

$= -A_c$ for binary "0"

A_K and A_{K+m} are statistically independent

$Pr\{\text{"1" transmitted}\} = Pr\{\text{"0" transmitted}\} = 0.5$

and p(t) is the given pulse



a) Determine and sketch the power spectral density of s(t).

b) Compare to ASK and BSK in terms of spectral shape and bandwidth efficiency. (-3dB BW)

c) Give a reason why this signaling format may be used.

Solution: (a)

$$P(f) = \int_{-\infty}^{\infty} p(t) e^{-j2\pi ft}\,dt = \int_{T/2}^{T} e^{-j2\pi ft}\,dt - \int_0^{T/2} e^{-j2\pi ft}\,dt$$

$$= -\frac{1}{j2\pi f} e^{-j2\pi ft}\Big|_{T/2}^{T} + \frac{1}{j2\pi f} e^{-j2\pi ft}\Big|_0^{T/2}$$

$$= \frac{1}{j2\pi f}\left[e^{-j\pi fT} - 1 + e^{-j\pi fT} - e^{-j2\pi fT}\right] = \frac{1}{j2\pi f}\left[2e^{-j\pi fT} - 1 - e^{-j2\pi fT}\right]$$

$$P(-f) = \frac{1}{-j2\pi f}\left[2e^{j\pi fT} - 1 - e^{j2\pi fT}\right]$$

$$|P(f)|^2 = \frac{1}{(2\pi f)^2}\cdot\left[6 - 8\cos(\pi fT) + 2\cos(2\pi fT)\right]$$

$$a = E\{A_K\} = \tfrac{1}{2}A_c - \tfrac{1}{2}A_c = 0$$

$$\sigma_A^2 = E\{A_K^2\} = \tfrac{1}{2}A_c^2 + \tfrac{1}{2}A_c^2 = A_c^2$$

$$\therefore\ S_z(f) = \frac{A_c^2}{T}\cdot\frac{1}{(2\pi f)^2}\cdot\left[6 + 2\cos(2\pi fT) - 8\cos(\pi fT)\right]$$

$$= \frac{A_c^2}{T}\cdot\frac{1}{(2\pi f)^2}\cdot\left[16\sin^2(\tfrac{\pi}{2}fT) - 4\sin^2(\pi fT)\right]$$

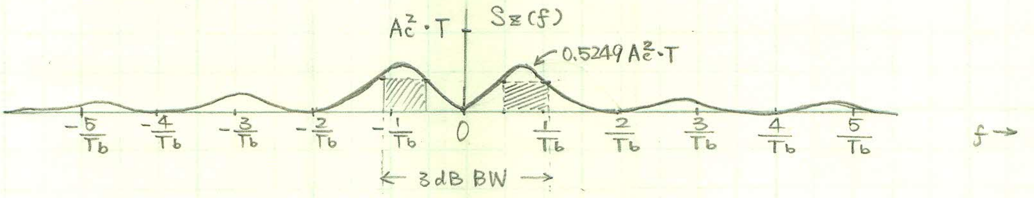
L19 1. (CONT.)

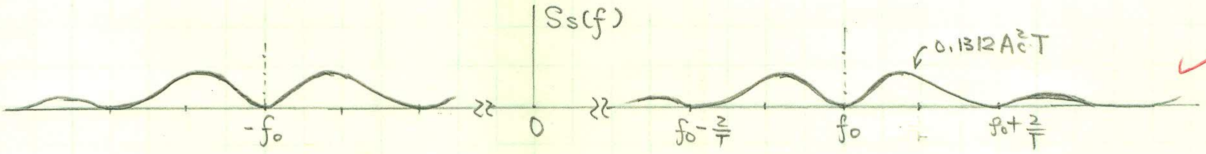
$$S_z(f) = A_c^2 \cdot T \cdot [\, \mathrm{sinc}^2(Tf/2) - \mathrm{sinc}^2(fT)]$$

$$S_s(f) = \frac{1}{4} S_z(f - f_0) + \frac{1}{4} S_z(f + f_0)$$

$$= \frac{1}{4} A_c^2 \cdot T \{ \mathrm{sinc}^2[T(f-f_0)/2] - \mathrm{sinc}^2[(f-f_0)T] \}$$

$$+ \frac{1}{4} A_c^2 T \cdot \{ \mathrm{sinc}^2[T(f+f_0)/2] - \mathrm{sinc}^2[(f-f_0)T] \}$$





(b) SHAPE: ASK : mainlobe from $f_0 - \frac{1}{T}$ to $f_0 + \frac{1}{T}$ with discrete spectral line at f_0.

BPSK : mainlobe from $f_0 - \frac{1}{T}$ to $f + \frac{1}{T}$ without discrete spectral line at f_0.

IN THIS CASE, $S_s(f)$ is sketched above.

Bandwidth : (-3 dB) ASK & BPSK = 0.88 R

$$\eta_{\text{ASK or BPSK}} = R/0.88R = 1.13636 \text{ bps/Hz}$$

For this problem, (using program in 41CX), -3 dB B.W. = 2.324 R

$$\eta = R/2.324R \doteq 0.4303 \text{ bps/Hz}$$

(c) Compared to ASK and BSK, this signal format has the worse bandwidth efficiency. However, most of power of s(t) is still distributed close to frequency f_0. We may this format if we don't have other choices.

-2

2. Repeat problem (1) for

$$S(t) = [A+z(t)]\cos(2\pi f_0 t)$$

where A is a given constant and $z(t)$ is as given in problem (1).

Solution: (a) $R_S(\tau) = E\{S(t)S(t+\tau)\}$

$$S(t)\,S(t+\tau) = [A+z(t)]\cos(2\pi f_0 t)\cdot[A+z(t+\tau)]\cos(2\pi f_0 t+2\pi f_0\tau)$$

$$= A^2\cos(2\pi f_0 t)\cos(2\pi f_0 t+2\pi f_0\tau) + Az(t)\cos(2\pi f_0 t)\cos(2\pi f_0 t+2\pi f_0\tau)$$

$$+ A\,z(t+\tau)\cos(2\pi f_0 t)\cos(2\pi f_0 t+2\pi f_0\tau) + z(t)z(t+\tau)\cos(2\pi f_0 t)\cos(2\pi f_0 t+2\pi f_0\tau)$$

$$\therefore\ R_S(\tau) = \frac{A^2}{2}\cos(2\pi f_0\tau) + \frac{1}{2}R_z(\tau)\cos(2\pi f_0\tau)$$

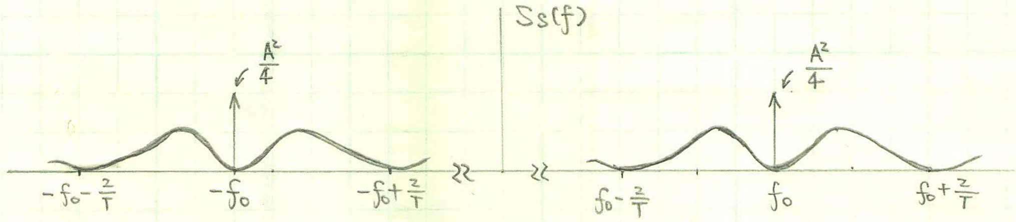
$$S_S(f) = \frac{A^2}{4}\cdot\delta(f-f_0) + \frac{A^2}{4}\cdot\delta(f+f_0) + \frac{1}{4}S_z(f-f_0) + \frac{1}{4}S_z(f+f_0)$$

From problem (1), we have

$$S_z(f) = A_c^2\cdot T\cdot[\operatorname{sinc}^2(Tf/2) - \operatorname{sinc}^2(fT)]$$

$$\therefore\ S_S(f) = \frac{A^2}{4}\delta(f-f_0) + \frac{A^2}{4}\delta(f+f_0) + \frac{A_c^2\cdot T}{4}\{\operatorname{sinc}^2[T(f-f_0)/2] - \operatorname{sinc}^2[(f-f_0)T]\}$$

$$+ \frac{A_c^2 T}{4}\cdot\{\operatorname{sinc}^2[T(f+f_0)/2] - \operatorname{sinc}^2[(f-f_0)T]\}$$



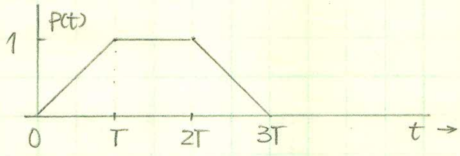
(b) SHAPE : Almost the same as those in part (b) of problem 1.

Bandwidth efficiency : $\eta_{\text{ASK OR BPSK}} = 1.13636$ bps/Hz , $\eta_{\text{this prob.}} = 0.4303$ bps/Hz.

(c) Compared to ASK and BSK, this signaling format has worse bandwidth efficiency. However, the most of power is still distributed near by frequency f_0. We may use this signaling format if we don't have other choices.

-2

3. Repeat problem (1) for the following trapezoidal pulse.



Solution:

$$P(f) = \int_{-\infty}^{\infty} p(t) e^{-j2\pi f t} dt = \int_0^T t/_T e^{-j2\pi f t} dt + \int_T^{2T} e^{-j2\pi f t} \cancel{dt} + \int_{2T}^{3T} (3 - \frac{t}{T}) e^{-j2\pi f t} dt$$

$$= \frac{-1}{T(2\pi f)^2} e^{-j2\pi f t} (-j2\pi f t - 1) \Big|_0^T - \frac{1}{j2\pi f} e^{-j2\pi f t} \Big|_T^{2T}$$

$$- \frac{3}{j2\pi f} e^{-j2\pi f t} \Big|_{2T}^{3T} + \frac{1}{T(2\pi f)^2} e^{-j2\pi f t} (-j2\pi f t - 1) \Big|_{2T}^{3T}$$

$$= - \frac{1}{T(2\pi f)^2} e^{-j2\pi f T} (-j\cancel{2\pi f} T - 1) - \frac{1}{T(2\pi f)^2}$$

$$\cancel{- \frac{1}{j2\pi f} e^{-j4\pi f T}} + \frac{1}{j2\pi f} e^{-j2\pi f T} - \frac{\cancel{3}}{j2\pi f} e^{-j6\pi f T} + \frac{\overset{3}{\cancel{2}}}{j2\pi f} e^{-j4\pi f T}$$

$$+ \frac{1}{T(2\pi f)^2} e^{-j6\pi f T} (-j6\cancel{\pi f} T - 1) - \frac{1}{T(2\pi f)^2} e^{j4\pi f T} (-j4\cancel{\pi f} T - 1)$$

$$= \frac{1}{T(2\pi f)^2} e^{-j2\pi f T} - \frac{1}{T(2\pi f)^2} - \frac{1}{T(2\pi f)^2} e^{-j6\pi f T} + \frac{1}{T(2\pi f)^2} e^{-j4\pi f T}$$

$$= \frac{1}{T(2\pi f)^2} [e^{-j2\pi f T} + e^{-j4\pi f T} - 1 - e^{-j6\pi f T}]$$

$$P(-f) = \frac{1}{T(2\pi f)^2} [e^{j2\pi f T} + e^{j4\pi f T} - 1 - e^{j6\pi f T}]$$

$$|P(f)|^2 = P(f) P(-f) = \frac{2}{T^2(2\pi f)^4} [2 + \cos(6\pi f T) - \cos(2\pi f T) - 2\cos(4\pi f T)]$$

$$= \frac{4}{T^2(2\pi f)^4} [\sin^2(\pi f T) + 2\sin^2(2\pi f T) - \sin^2(3\pi f T)]$$

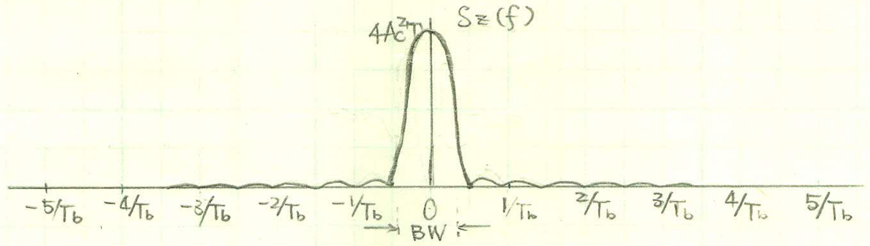
$$= \frac{T^2}{(2\pi f T)^2} [\mathrm{sinc}^2(fT) + 8\,\mathrm{sinc}^2(2fT) - 9\,\mathrm{sinc}^2(3fT)]$$

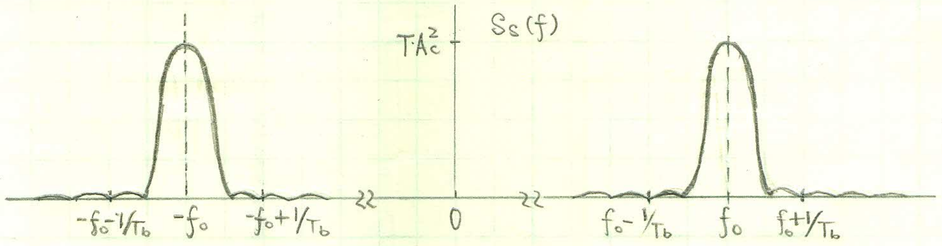
$a = 0$, $\sigma_A^2 = A_c^2$

$$\therefore \quad S_z(f) = A_c^2 \cdot T [\mathrm{sinc}^2(fT) + 8\,\mathrm{sinc}^2(2fT) - 9\,\mathrm{sinc}^2(3fT)] / (2\pi f T)^2$$

$$S_s(f) = \frac{1}{4} S_z(f - f_0) + \frac{1}{4} S_z(f + f_0)$$

3. (CONT.)





(b) Compared to ASK and BFSK, we have almost the same shapes of PDF for ASK, BFSK and this problem, but the bandwidth.

3dB Bandwith* for this case = $(0.2-(-0.2))R = 0.4R$ ✓

$\eta = R/0.4R = 2.5$ bps / Hz ✓

$\eta_{\text{ASK or BFSK}} = 1.13636$ bps / Hz ✓

(c) We can use this signaling format, because most of power of $s(t)$ is distributed very closed to carrier frequecy f_0.

-2

* 3dB Bandwidth for this problem is difined in Figure above.

Lecture 20:

1. Derive equation (8) , Page 17

Solution: $s(t)+n(t) = [A_c + n_c(t)]\cos(2\pi f_o t) - n_s(t)\sin(2\pi f_o t)$

$$s(t-T_b)+n(t-T_b)$$

$$= [A_c+n_c(t')]\cos(2\pi f_o t') - n_s(t')\sin(2\pi f_o t') \quad , \quad t' = t - T_b$$

$$\chi(t) \triangleq [s(t)+n(t)]\cdot[s(t-T_b)+n(t-T_b)]$$

$$= [A_c+n_c(t)]\cdot[A_c+n_c(t')]\cos(2\pi f_o t)\cos(2\pi f_o t')$$

$$- n_s(t)\,[A_c+n_c(t')]\sin(2\pi f_o t)\cos(2\pi f_o t')$$

$$- [A_c+n_c(t)]\,n_s(t')\cos(2\pi f_o t)\sin(2\pi f_o t')$$

$$+ n_s(t)\,n_s(t')\sin(2\pi f_o t)\sin(2\pi f_o t')$$

$$= [A_c+n_c(t)][A_c+n_c(t')]\cdot\tfrac{1}{2}\cos(2\pi f_o T_b) - n_s(t)[A_c+n_c(t')]\cdot\tfrac{1}{2}\sin(2\pi f_o T_b)$$

$$+ [A_c+n_c(t)]\,n_s(t')\cdot\tfrac{1}{2}\sin(2\pi f_o T_b) + \tfrac{1}{2}n_c(t)\,n_s(t')\cos(2\pi f_o T_b) + \underline{\text{high freq. terms}}$$

$S_o(T_b)+n_o(T_b) = \chi(t)$ pass through an ILPF. And since T_b is an integral multiple of $1/f_o$. So,

$$\cos(2\pi f_o T_b) = 1 \quad \text{and} \quad \sin(2\pi f_o T_b) = 0$$

$$\therefore \quad \underline{S_o(T_b)+n_o(T_b) = \tfrac{1}{2}[A_c+n_c(t)][A_c+n_c(t')] + \tfrac{1}{2}n_s(t)\,n_s(t')}$$

Q.E.D.

2. Prove equation (10), Page 9, That is

$$V_o = (\alpha^2 - \beta^2)/2$$

PROVE:

$$(\alpha^2 - \beta^2)/2 = \frac{1}{2}[\alpha_c^2 + \alpha_s^2 - \beta_c^2 - \beta_s^2]$$

$$= \frac{1}{2}[(\alpha_c + \beta_c)(\alpha_c - \beta_c) + (\alpha_s + \beta_s)(\alpha_s - \beta_s)]$$

$$= \frac{1}{2}\langle [A_c + n_c(t)]\cdot[A_c + n_c(t')] + n_s(t)\, n_s(t')\rangle$$ ✓

$$= \frac{1}{2}[A_c + n_c(t)]\cdot[A_c + n_c(t')] + \frac{1}{2} n_s(t)\, n_s(t') = \overline{V_o}$$ Q.E.D.

---

3. (a) Differentially encode the following binary messages.

i) 1 1 0 1 1 0 0 0 1 1 0

ii) 1 0 1 0 1 0 1 0 1 0 1

iii) 1 1 1 1 1 0 0 0 0 0 0

(b) Decode the received sequences for non-inverted and inverted phase.

Solution: (a) i) d_n : 1 1 0 1 1 0 0 0 1 1 0

C_n : 1 0 1 1 0 1 1 1 1 0 1 1 ← ✓

ii) d_n : 1 0 1 0 1 0 1 0 1 0 1

C_n : 1 0 0 1 1 0 0 1 1 0 0 1 ← ✓

iii) d_n : 1 1 1 1 1 0 0 0 0 0 0

C_n : 1 0 1 0 1 0 0 0 0 0 0 0 ← ✓

(b) i) non-inverted phase

$\hat{C}_n$: 1 0 1 1 0 1 1 1 1 0 1 1 ✓

$\hat{d}_n$: 1 1 0 1 1 0 0 0 1 1 0 ← correct

inverted-phase

$\hat{C}_n$: 0 1 0 0 1 0 0 0 0 1 0 0 ✓

$\hat{d}_n$: 1 1 0 1 1 0 0 0 1 1 0 ← correct again!

3 (b) ii) non-inverted phase

$\hat{c}_n$: 1 0 0 1 1 0 0 1 1 0 0 1 ✓

$\hat{d}_n$: 1 0 1 0 1 0 1 0 1 0 1 ← correct

inverted phase

$\hat{c}_n$: 0 1 1 0 0 1 1 0 0 1 1 0

$\hat{d}_n$: 1 0 1 0 1 0 1 0 1 0 1 ← ✓

iii) non-inverted phase

$\hat{c}_n$: 1 0 1 0 1 0 0 0 0 0 0 0 ✓

$\hat{d}_n$: 1 1 1 1 1 0 0 0 0 0 0 ←

inverted phase

$\hat{c}_n$: 0 1 0 1 0 1 1 1 1 1 1 1

$\hat{d}_n$: 1 1 1 1 1 0 0 0 0 0 0 ← ✓

---

4. Determine the SNR (E_b/N_0) in dB to acheive $P_\varepsilon = 10^{-5}$ for BPSK, DE-BPSK and DPSK.

Solution: For BPSK : $P(\varepsilon) = Q(\sqrt{2E_b/N_0})$

For DE-BPSK : $P(\varepsilon) = 2Q(\sqrt{2E_b/N_0})$

For DPSK : $P(\varepsilon) = \frac{1}{2} e^{-E_b/N_0}$

If $P_\varepsilon = 10^{-5}$,

For BPSK : SNR = 9.58751 dB ✓

For DE-BPSK : SNR = 9.89235 dB ✓

For DPSK : SNR = 10.34218 dB ✓

Lecture 21

1. (page 2) Show that the PSD of $x_c(t)$ is given as

$$S_{x_c}(f) = \frac{A_1^2}{4}[S_{m_1}(f+f_0) + S_{m_1}(f-f_0)] + \frac{A_2^2}{4}[S_{m_2}(f+f_0) + S_{m_2}(f-f_0)]$$

SHOW: $x_c(t) = A_1 m_1(t)\cos(2\pi f_0 t) + A_2 m_2(t)\sin(2\pi f_0 t)$

$$x_c(t)\,x_c(t+\tau)$$

$$= [A_1 m_1(t)\cos(2\pi f_0 t) + A_2 m_2(t)\sin(2\pi f_0 t)]\cdot[A_1 m_1(t+\tau)\cos(2\pi f_0 t + 2\pi f_0 \tau) + A_2 m_2(t+\tau)\sin(2\pi f_0 t + 2\pi f_0\tau)]$$

$$= A_1^2 m_1(t) m_1(t+\tau)\cos(2\pi f_0 t)\cos(2\pi f_0 t + 2\pi f_0\tau)$$
$$+ A_1 A_2 m_2(t) m_1(t+\tau)\sin(2\pi f_0 t)\cos(2\pi f_0 t + 2\pi f_0 \tau)$$
$$+ A_1 A_2 m_1(t) m_2(t+\tau)\cos(2\pi f_0 t)\sin(2\pi f_0 t + 2\pi f_0\tau)$$
$$+ A_2^2 m_2(t) m_2(t+\tau)\sin(2\pi f_0 t)\sin(2\pi f_0 t + 2\pi f_0\tau)$$

$$= \frac{A_1^2}{2} m_1(t) m_1(t+\tau)\cos(2\pi f_0\tau) + \frac{A_1^2}{2} m_1(t) m_1(t+\tau)\cos(4\pi f_0 t + 2\pi f_0\tau)$$
$$+ A_1 A_2 m_2(t) m_1(t+\tau)\cdot\underline{\text{sinewavies}} + A_1 A_2 m_1(t) m_2(t+\tau)\cdot\underline{\text{sinewavies}}$$
$$+ \frac{A_2^2}{2} m_2(t) m_2(t+\tau)\cos(2\pi f_0\tau) - \frac{A_1^2}{2} m_2(t) m_2(t+\tau)\cos(4\pi f_0 t + 2\pi f_0\tau)$$

$$E\{x_c(t)x_c(t+\tau)\} = \frac{A_1^2}{2} R_{m_1}(\tau)\cos(2\pi f_0\tau) + \frac{A_2^2}{2} R_{m_2}(\tau)\cos(2\pi f_0\tau)$$
$$+ \frac{A_1^2}{2} R_{m_1}(\tau)\cos(4\pi f_0 t + 2\pi f_0\tau) - \frac{A_2^2}{2} R_{m_2}(\tau)\cos(4\pi f_0 t + 2\pi f_0 \tau)$$

$$R_{x_c}(\tau) = \langle E\{x_c(t)x_c(t+\tau)\}\rangle$$
$$= \frac{A_1^2}{2} R_{m_1}(\tau)\cos(2\pi f_0\tau) + \frac{A_2^2}{2} R_{m_2}(\tau)\cos(2\pi f_0\tau) + 0$$

−3

what happened to cross term?

$$S_{x_c}(f) = \mathcal{F}\{R_{x_c}(\tau)\}$$
$$= \mathcal{F}\{\frac{A_1^2}{2} R_{m_1}(\tau)\cos(2\pi f_0\tau)\} + \mathcal{F}\{\frac{A_2^2}{2} R_{m_2}(\tau)\cos(2\pi f_0\tau)\}$$
$$= \frac{A_1^2}{4}[S_{m_1}(f+f_0) + S_{m_1}(f-f_0)] + \frac{A_2^2}{4}[S_{m_2}(f+f_0) + S_{m_2}(f-f_0)]$$

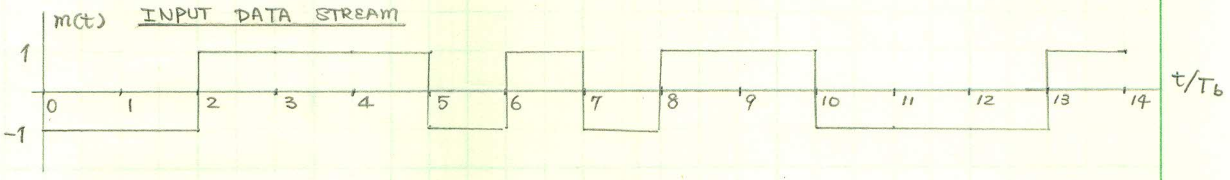
Q.E.D.

2. For QPSK and OQPSK draw waveforms for $m(t)$, $m_1(t)$ and $m_2(t)$ as on pages 4 and 15, and draw the phases as on page 18 for the following data sequence :

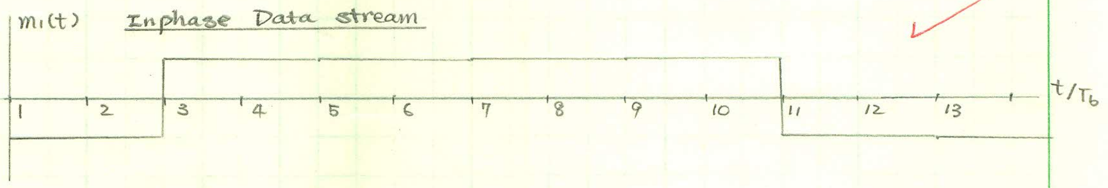
0 0 1 1 1 0 1 0 1 1 0 0 0 1

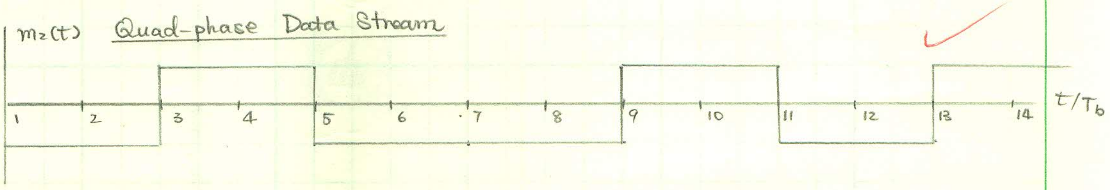
(represent "0" by −1 and "1" by 1)

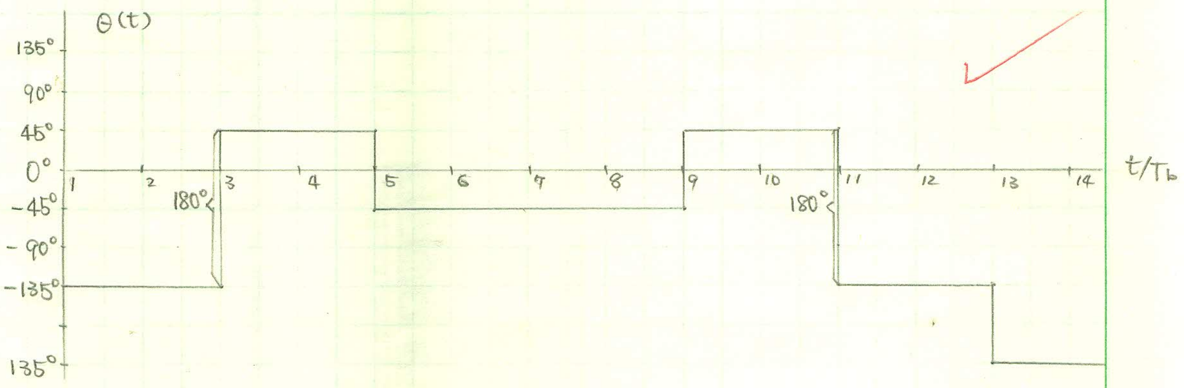
Solution:



For QPSK:

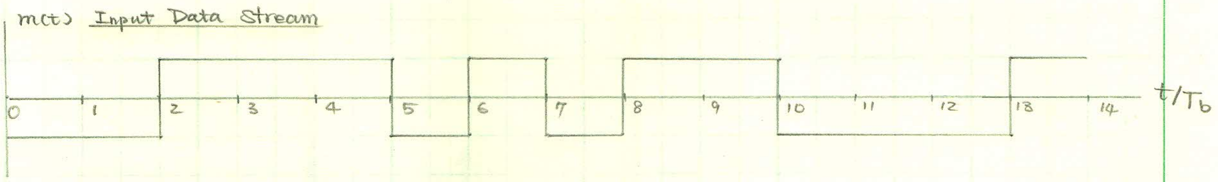


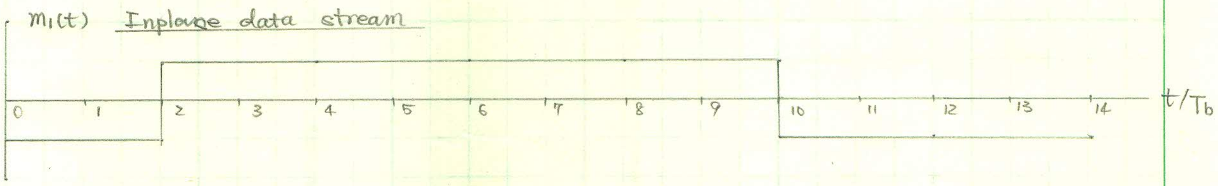


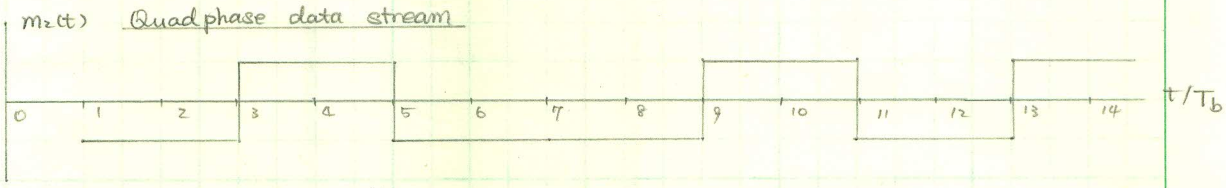


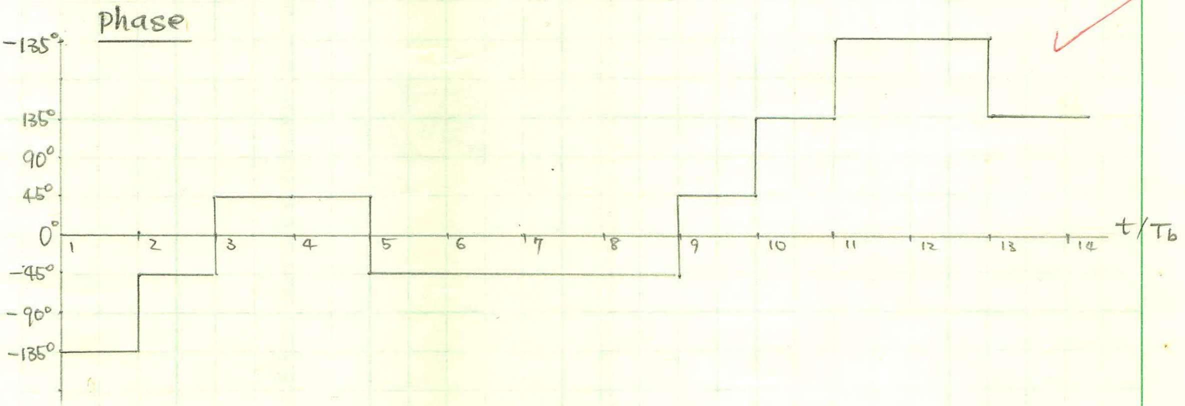
To be continued on next page!

2. (CONT.) For OQPSK:









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3. Suppose BPSK and QPSK transmit at the same symbol rate and the same energy per symbol. Compare their (a) bit rate, (b) bandwidth, (c) $P_s(\varepsilon)$, (d) $P_b(\varepsilon)$

Solution:

(a) $R_{b,BPSK} = R_{s,BPSK}$; $R_{b,QPSK} = 2R_{s,QPSK}$

If $R_{s,BPSK} = R_{s,QPSK}$ then $\underline{R_{b,QPSK} = 2R_{b,BPSK}}$

(b) And $T_{b,QPSK} = \frac{1}{2} T_{b,BPSK}$

$\underline{\text{QPSK Null to null Bandwidth}} = R_{b,QPSK} = 2R_{b,BPSK} = \underline{\text{BPSK Bandwidth}}$

(c) $\underline{P_{s,BPSK}(\varepsilon)} = P_{b,BPSK}(\varepsilon) = P_{b,QPSK}(\varepsilon) = \underline{\frac{1}{2} P_{s,QPSK}(\varepsilon)}$ −2

(d) $P_{b,BPSK}(\varepsilon) = P_{b,QPSK}(\varepsilon)$

Q.E.D.

Homework no: 13 Lecture 22 Digital Communication Sys: $\frac{30}{30}$ Ben Chen 01

1. Show that the phase $\varphi(t) = b_K \cdot \frac{\pi}{2T} t + \varphi_K$ is a continuous function of t.

SHOW: (1) For $kT \leq t < (k+1)T$, we see that b_K and φ_K are constants. And

$$\varphi'(t) = b_K \cdot \frac{\pi}{2T}$$

This shows that $\varphi(t)$ is continuous for $kT \leq t < (k+1)T$.

(2) Let $t_0 = (k+1)T$. And define

$$\varphi(t_0 -) = \lim_{t \to t_0 \text{(Left)}} \varphi(t) = b_K \cdot \frac{\pi}{2T}(k+1)T + \varphi_K$$

$$= \frac{\pi}{2}(k+1) \cdot b_K + \varphi_K$$

$$\varphi(t_0) = b_{K+1} \cdot \frac{\pi}{2T}(k+1)T + \varphi_{K+1}$$

$$= \frac{\pi}{2}(k+1) \cdot b_{K+1} + \varphi_{K+1}$$

$$\therefore \quad \varphi(t_0) - \varphi(t_0 -) = \frac{\pi}{2}(k+1) \cdot (b_{K+1} - b_K) + (\varphi_{K+1} - \varphi_K) \quad \cdots\cdots \quad (1)$$

From notes, on page 6, we have

$$b_K = a_1(kT_b) \cdot a_2(kT_b), \quad \varphi_K = \frac{\pi}{2}[1 - a_1(kT_b)] \quad ; \quad T_b = T$$

And $a_1(kT_b)$, $a_2(kT_b)$ can not change at the same time. So,

(i) if $b_{K+1} = b_K$, then $a_1[(k+1)T] = a_1(kT)$, $a_2[(k+1)T] = a_2(kT)$

Thus $\varphi_{K+1} = \varphi_K$ since $a_1(t)$ does not change.

$$\varphi(t_0) - \varphi(t_0 -) = \frac{\pi}{2}(k+1) \times 0 + 0 = 0$$

That is shown that $\varphi(t)$ is continuous at t_0 when $b_{K+1} = b_K$.

(ii) if $b_{K+1} \neq b_K$, OR $b_{K+1} = -b_K$, the we have two cases:

a. $a_2(t)$ changes. And $k+1$ must be even $\triangleq 2m$, $\varphi_{K+1} = \varphi_K$

$$\varphi(t_0) - \varphi(t_0 -) = \frac{\pi}{2} \cdot 2m \cdot 2 \cdot b_{K+1} + 0 = 2m b_{K+1} \pi$$

$$= 0 \quad (\text{if we only consider } 0 \leq \varphi(t) < 2\pi)$$

b. $a_1(t)$ changes. $k+1$ must be odd $\triangleq 2m+1$, $\varphi_{K+1} = \varphi_K \pm \pi$

$$\varphi(t_0) - \varphi(t_0 -) = 2m b_{K+1} \pi + \pi \pm \pi = 0 \quad (0 \leq \varphi(t) < 2\pi)$$

Thus, we have showed the statement.

Q.E.D.

2. Repeat the example presented on pp 20-23 for the sequence

a_K : -1 1 1 -1 1 1 -1 -1 1 -1 1 1 1 -1

Present the following : (a) sequence b_K,

(b) excess phase trellis (with trajectory for above sequence highlighted)

(c) Sketches of $\sin[\varphi(t)]$, $\cos[\varphi(t)]$, $\sin[\frac{\pi t}{2T_b}]$, $\cos[\frac{\pi t}{2T_b}]$, $\sin[\varphi(t)]\sin[\frac{\pi t}{2T_b}]$, $\cos[\varphi(t)]\cos[\frac{\pi t}{2T_b}]$

(d) Sequences $\hat{a}_1(k)$ and $\hat{a}_2(k)$

(e) sequences $\hat{a}_1(k) \oplus \hat{a}_1(k-2)$, $\hat{a}_2(k) \oplus \hat{a}_2(k-2)$

(f) $\hat{a}_k$

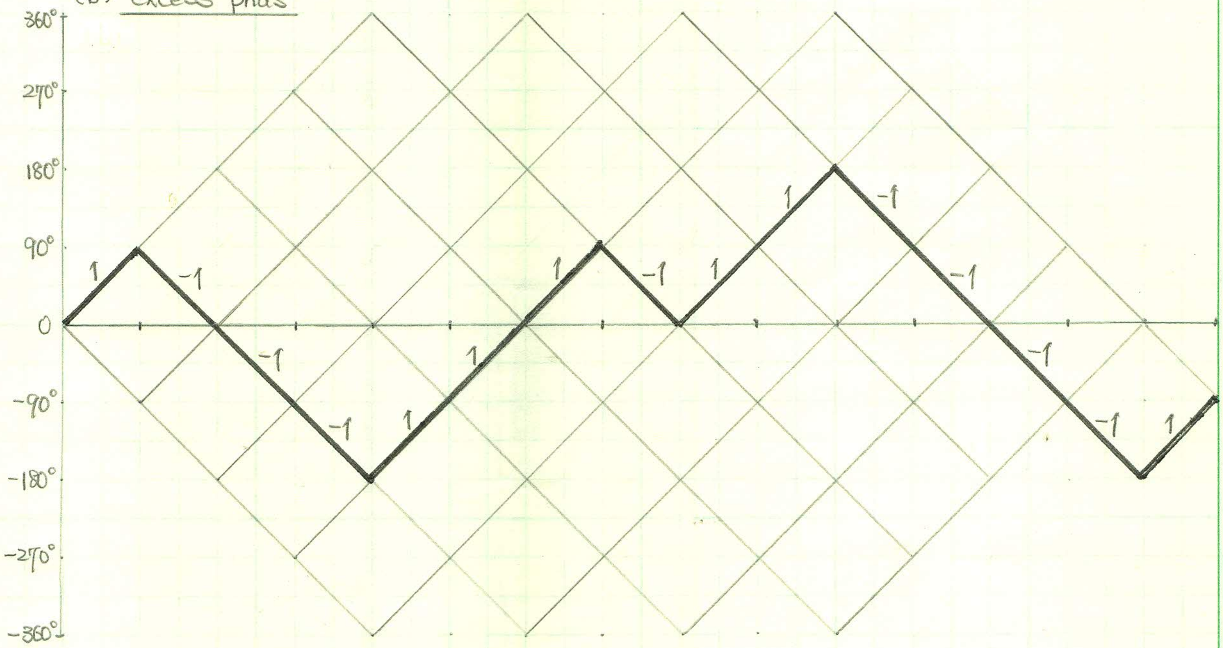
(g) Compare to a_K

(h) label the block diagram on p23 with the appropriate signal notation.

Solution : (a) a_K : -1 1 1 -1 1 1 -1 -1 1 -1 1 1 1 -1

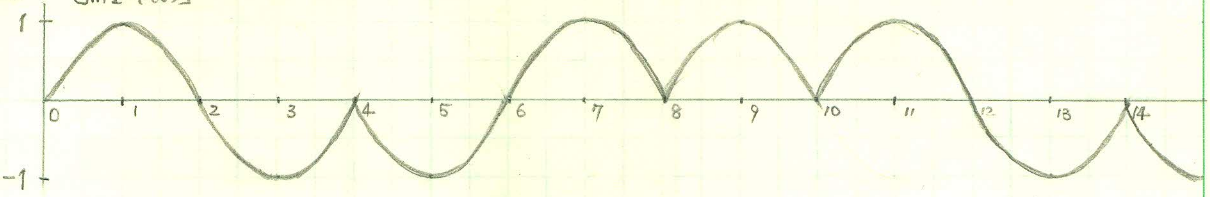
b_K : 1 -1 -1 -1 1 1 1 -1 1 1 -1 -1 -1 -1 1

(b) excess phas

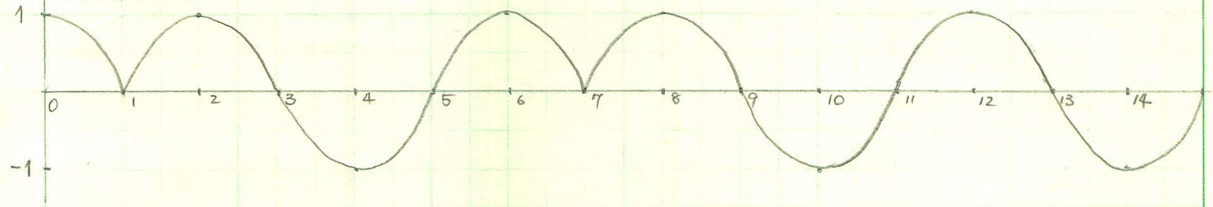


(c)

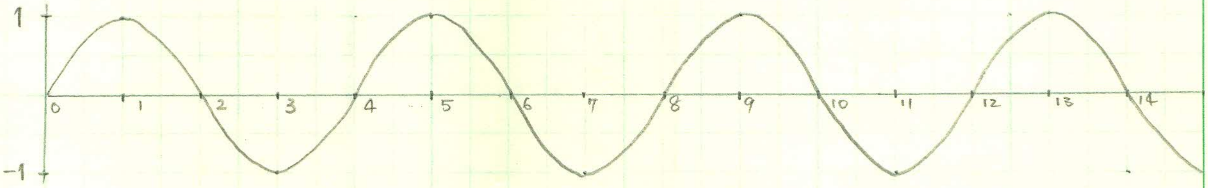
$\sin[\varphi(t)]$



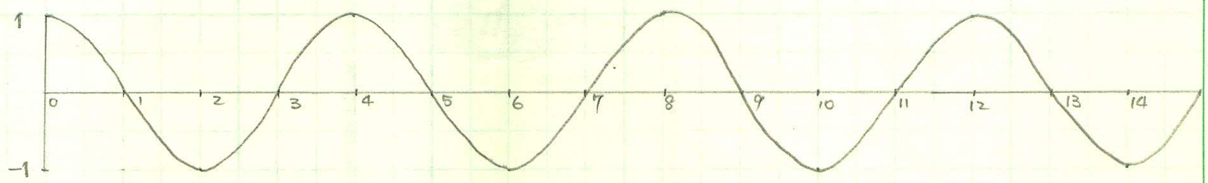
$\cos[\varphi(t)]$



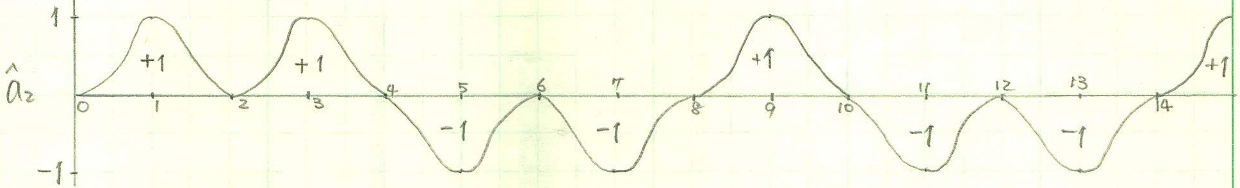
$\sin\left[\frac{\pi t}{2T_b}\right]$



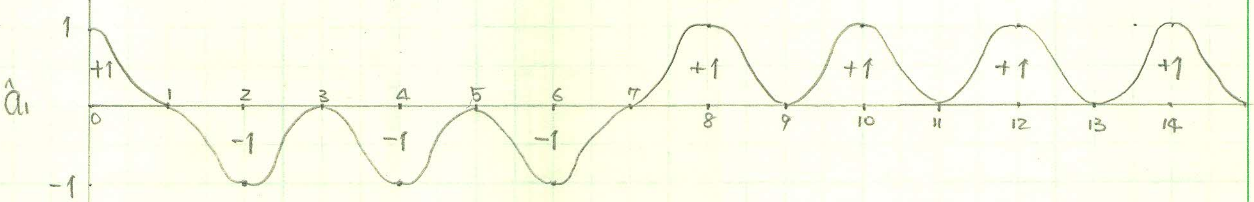
$\cos\left[\frac{\pi t}{2T_b}\right]$



$\sin[\varphi(t)]\sin\left[\frac{\pi t}{2T_b}\right]$



$\cos[\varphi(t)]\cos\left[\frac{\pi t}{2T_b}\right]$



(d) k: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

$\hat{a}_1(k)$: 1 -1 -1 -1 1 1 1 1

$\hat{a}_2(k)$: 1 1 -1 -1 1 -1 -1 1

(e) $\hat{a}_1(k) \oplus \hat{a}_1(k-2)$: -1 1 1 -1 1 1 1

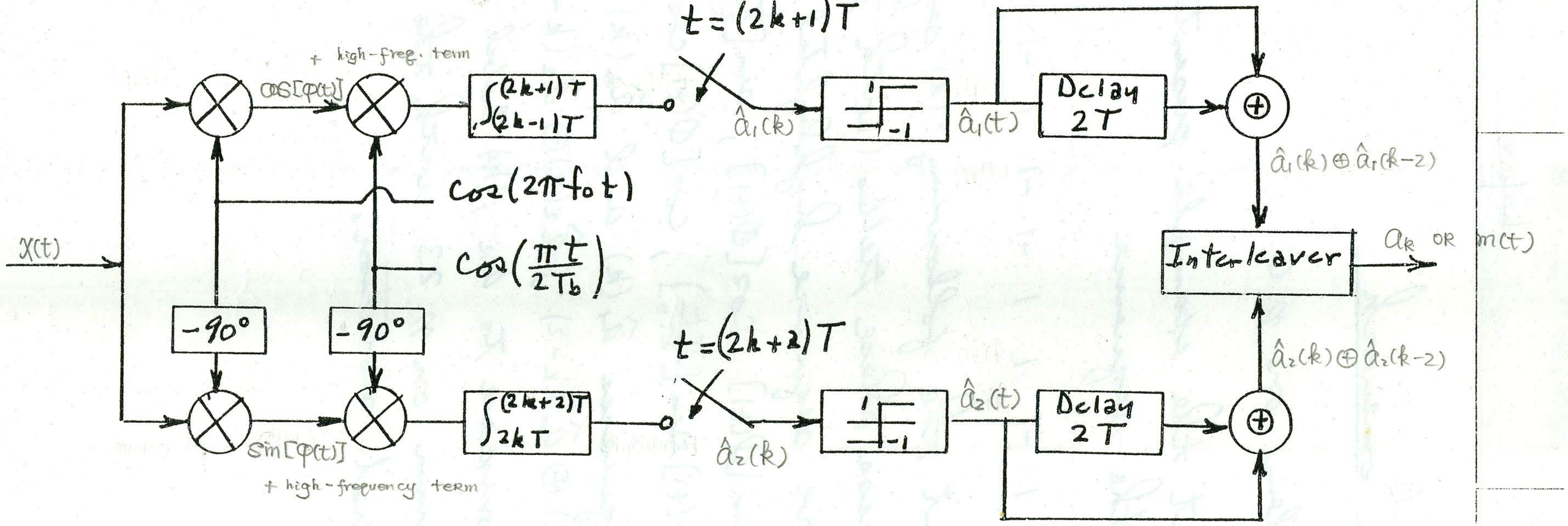
$\hat{a}_2(k) \oplus \hat{a}_2(k-2)$: 1 -1 1 -1 -1 1 -1

(f) $\hat{a}_K$: -1 1 1 -1 1 1 -1 -1 1 -1 1 1 1 -1

(g) compare a_K : -1 1 1 -1 1 1 -1 -1 1 -1 1 1 1 -1

(h) Done on next page.

MSK Coherent Differential Demodulator/Decoder



$x(t) = \cos[\varphi(t)]\cos(2\pi f_0 t) + \sin[\varphi(t)]\sin(2\pi f_0 t)$

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