Communication Systems

EE 423

Instructor

Dr. R. A. Birgenheier

Fall 1986

Homework and Tests

Benmei Chen

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Find examples for each operations in table 3.2

(1) Linearity (superposition) $a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$

Let $f_1(t) = e^{-at} u(t)$, $f_2(t) = t e^{-at} u(t)$

$\therefore F_1(\omega) = 1/(a+j\omega)$ ✓ $F_2(\omega) = 1/(a+j\omega)^2$ ✓

$$\mathcal{F}\{a_1 f_1(t) + a_2 f_2(t)\}$$

$$= \int_{-\infty}^{\infty} [a_1 f_1(t) + a_2 f_2(t)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} (a_1 + a_2 t) e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_0^{\infty} (a_1 + a_2 t) e^{-(a+j\omega)t} dt$$

$$= -\frac{1}{a+j\omega} \int_0^{\infty} (a_1 + a_2 t) \, d\, e^{-(a+j\omega)t}$$

$$= -1/(a+j\omega) \cdot \left[(a_1 + a_2 t) e^{-(a+j\omega)t} \Big|_0^{\infty} - a_2 \int_0^{\infty} e^{-(a+j\omega)t} dt \right]$$

$$= -1/(a+j\omega) \cdot [-a_1 - a_2 \cdot /(a+j\omega)]$$

$$= \frac{a_2 + a_1(a+j\omega)}{(a+j\omega)^2}$$ ✓

$$a_1 F_1(\omega) + a_2 F(\omega) = a_1/(a+j\omega) + a_2/(a+j\omega)^2$$

$$= \frac{a_2 + a_1(a+j\omega)}{(a+j\omega)^2}$$ ✓

$$\therefore \mathcal{F}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(\omega) + a_2 F_2(\omega)$$ ✓

(2) Complex conjugate $f^*(t) \leftrightarrow F^*(-\omega)$

Let $f(t) = \cos\omega_0 t + j\sin\omega_0 t$ ✓ $\therefore f^*(t) = \cos\omega_0 t - j\sin\omega_0 t$ ✓

$$\mathcal{F}\{f(t)\} = F(\omega) = 2\pi\,\delta(\omega - \omega_0)$$ ✓

$$\mathcal{F}\{f^*(t)\} = 2\pi\,\delta(\omega + \omega_0) = F(-\omega) = F^*(-\omega)$$ ✓

*(Because $F(\omega) = F^*(\omega)$ $\delta(-\omega) = \delta(\omega)$)

(3) Scaling $\quad f(\alpha t) \leftrightarrow \frac{1}{|\alpha|} F(\frac{\omega}{\alpha})$

Let $\quad f(t) = e^{-at} u(t) \qquad \alpha > 0$

$$\therefore f(\alpha t) = e^{-a\alpha t} u(\alpha t) = e^{-a\alpha t} u(t)$$

$$\mathcal{F}\{f(\alpha t)\} = \mathcal{F}\{e^{-a\alpha t} u(t)\} = 1/(a\alpha + j\omega)$$

$$\frac{1}{|\alpha|} F(\frac{\omega}{\alpha}) = \frac{1}{\alpha} \cdot [1/(a + j\frac{\omega}{\alpha})] = 1/(a\alpha + j\omega)$$

$$\therefore \mathcal{F}\{f(\alpha t)\} = \frac{1}{|\alpha|} F(\frac{\omega}{\alpha})$$

(4) Delay $\quad f(t - t_0) \leftrightarrow e^{-j\omega t_0} F(\omega)$

Let $\quad f(t) = e^{-at} u(t) \xrightarrow{\mathcal{F}} F(\omega) = 1/(a + j\omega)$

$$\mathcal{F}\{f(t - t_0)\} = \int_{-\infty}^{\infty} e^{-a(t-t_0)} u(t - t_0) e^{-j\omega t} dt$$

$$= e^{-j\omega t_0} \int_{t_0}^{\infty} e^{-a(t-t_0)} e^{-j\omega(t-t_0)} d(t - t_0)$$

$$= e^{-j\omega t_0} \cdot \int_0^{\infty} e^{-au} e^{-j\omega u} du$$

$$= e^{-j\omega t_0} \cdot 1/(a + j\omega) = e^{-j\omega t_0} \cdot F(\omega)$$

(5) Frequency traslation $\quad e^{j\omega_0 t} f(t) \leftrightarrow F(\omega - \omega_0)$

Let $\quad f(t) = e^{j\omega_0 t} \qquad \therefore F(\omega) = 2\pi\,\delta(\omega - \omega_0)$

$$\mathcal{F}\{e^{j\omega_0 t} f(t)\} = \mathcal{F}\{e^{j2\omega_0 t}\} = 2\pi\,\delta(\omega - 2\omega_0) = F(\omega - \omega_0)$$

(6) Amplitude modulation $\quad f(t)\cos\omega_0 t \leftrightarrow \frac{1}{2} F(\omega + \omega_0) + \frac{1}{2} F(\omega - \omega_0)$

Let $\quad f(t) = 2\sin\omega_0 t \xrightarrow{\mathcal{F}} F(\omega) = -j2\pi [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$

$$\mathcal{F}\{f(t)\cos\omega_0 t\} = \mathcal{F}\{2\sin\omega_0 t \cos\omega_0 t\} = \mathcal{F}\{\sin 2\omega_0 t\}$$

$$= -j\pi [\delta(\omega - 2\omega_0) - \delta(\omega + 2\omega_0)]$$

And $\quad \frac{1}{2} F(\omega + \omega_0) + \frac{1}{2} F(\omega - \omega_0) = -j\pi [\delta(\omega) - \delta(\omega + 2\omega_0)] - j\pi [\delta(\omega - 2\omega_0) - \delta(\omega)]$

$$= -j\pi [\delta(\omega - 2\omega_0) - \delta(\omega + 2\omega_0)]$$

$$\therefore \mathcal{F}\{f(t)\cos\omega_0 t\} = \frac{1}{2} F(\omega + \omega_0) + \frac{1}{2} F(\omega - \omega_0)$$

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(7) Time convolution $\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,d\tau \leftrightarrow F_1(\omega) F_2(\omega)$

Let $f_1(t) = e^{-a|t|}$ and $f_2(t) = 1$

$\therefore F_1(\omega) = 2a/(a^2+\omega^2)$, $F_2(\omega) = 2\pi\delta(\omega)$

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,d\tau = \int_{-\infty}^{\infty} e^{-a|\tau|}\cdot 1\cdot d\tau = \int_{-\infty}^{0} e^{a\tau}\,d\tau + \int_{0}^{\infty} e^{-a\tau}\,d\tau$$

$$= \frac{1}{a} e^{a\tau}\Big|_{-\infty}^{0} - \frac{1}{a} e^{-a\tau}\Big|_{0}^{\infty} = \frac{2}{a}$$

$$\therefore \mathcal{F}\{f(t)\} = \mathcal{F}\left\{\frac{2}{a}\right\} = \frac{2}{a}\cdot 2\pi\delta(\omega) = \frac{4\pi}{a}\delta(\omega)$$

$$F_1(\omega)\cdot F_2(\omega) = 2a/(a^2+\omega^2)\cdot 2\pi\delta(\omega) = \frac{4\pi}{a}\delta(\omega)$$

* (Because $\delta(\omega) = \begin{cases} \text{undefined}, & \omega = 0 \\ 0 & \text{elsewhere} \end{cases}$) ✓

(8) Frequency convolution $f_1(t) f_2(t) \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(u) F_2(\omega_2 - u)\,du$

Let $f_1(t) = \delta(t)$, $f_2(t) = 1$

$\therefore F_1(\omega) = 1$ AND $F_2(\omega) = 2\pi\delta(\omega)$

$$\mathcal{F}\{f_1(t) f_2(t)\} = \mathcal{F}\{\delta(t)\} = 1$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(u) F_2(\omega - u)\,du = \int_{-\infty}^{\infty} 1\cdot\delta(\omega-u)\,du = 1$$

$$\therefore \mathcal{F}\{f_1(t) f_2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(u) F_2(\omega - u)\,du$$

(9) Duality : time-frequency $F(t) \leftrightarrow 2\pi f(-\omega)$

Let $f(t) = sgn(t) \leftrightarrow F(\omega) = 2/(j\omega)$

$F(t) = 2/jt$

$$\mathcal{F}\{2/jt\} = \mathcal{F}\{-2\pi\cdot j/(\pi t)\} = -2\pi\,\mathcal{F}\{j/(\pi t)\} = -2\pi\,sgn(\omega)$$

$$= 2\pi\,sgn(-\omega) = 2\pi f(-\omega)$$ ✓

(10) Symmetry : even – odd $f_e(t) \leftrightarrow F_e(\omega)$ [real]

$f_o(t) \leftrightarrow F_o(\omega)$ [Imaginary]

Let $f_e(t) = \cos\omega_0 t$

$F(\omega) = \mathfrak{F}\{\cos\omega_0 t\} = \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

$F(-\omega) = \pi[\delta(-\omega-\omega_0)+\delta(-\omega+\omega_0)] = \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] = F(\omega)$

(Because $\delta(\omega)$ is even) ✓

∴ $F(\omega)$ is even and real.

Let $f_o(t) = \sin\omega_0 t$ ✓

$F(\omega) = \mathfrak{F}\{\sin\omega_0 t\} = -j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

$F(-\omega) = -j\pi[\delta(-\omega-\omega_0)-\delta(-\omega+\omega_0)]$

$= j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)] = -F(\omega)$ ✓

∴ $F(\omega)$ is odd and imaginary.

(11) Time differentiation $\frac{d}{dt}f(t) \leftrightarrow j\omega F(\omega)$

Let $f(t) = \sin\omega_0 t$ ∴ $\frac{d}{dt}f(t) = \omega_0\cos\omega_0 t$

$F(\omega) = -j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$ ✓

$\mathfrak{F}\{\frac{d}{dt}f(t)\} = \mathfrak{F}\{\omega_0\cos\omega_0 t\}$

$= \omega_0\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

$j\omega F(\omega) = -j\omega\cdot j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

$= \omega\pi\delta(\omega-\omega_0) - \omega\pi\delta(\omega+\omega_0)$

$= \omega_0\pi\delta(\omega-\omega_0) + \omega_0\pi\delta(\omega+\omega_0)$

$= \omega_0\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

So, $\mathfrak{F}\{\frac{d}{dt}f(t)\} = j\omega F(\omega)$ ✓

(12) Time integration $\int_{-\infty}^{t} f(\tau)\,d\tau \leftrightarrow \frac{1}{j\omega}F(\omega) + \pi F(0)\,\delta(\omega)$

WHERE $F(0) = \int_{-\infty}^{\infty} f(t)\,dt$

Let $f(t) = e^{-at}u(t)$

$F(\omega) = 1/(a+j\omega)$

$F(0) = \frac{1}{a}$

AND $\int_{-\infty}^{t} f(\tau)\,d\tau = \int_{-\infty}^{t} e^{-a\tau}u(\tau)\,d\tau = \int_{0}^{t} e^{-a\tau}\,d\tau$

$= -\frac{1}{a}e^{-a\tau}\Big|_0^t$

$= -\frac{1}{a}e^{-at} + \frac{1}{a} \qquad (t>0)$

$= -\frac{1}{a}e^{-at}u(t) + \frac{1}{a}u(t)$

(If $t \le 0$, $\int_{-\infty}^{t} f(\tau)\,d\tau = \int_{-\infty}^{t} e^{-a\tau}u(\tau)\,d\tau = 0$)

$\mathcal{F}\{\int_{-\infty}^{t} f(\tau)\,d\tau\} = \mathcal{F}\{-\frac{1}{a}e^{-at}u(t) + \frac{1}{a}u(t)\}$

$= -\frac{1}{a}\cdot 1/(a+j\omega) + \frac{1}{a}\cdot(\pi\delta(\omega) + 1/j\omega)$

$= \frac{1}{-a(a+j\omega)} + \frac{1}{a}\cdot\pi\,\delta(\omega) + 1/(ja\omega)$

$= \frac{1}{j\omega}\left[\frac{1}{a} - \frac{1}{a(-j\frac{a}{\omega}+1)}\right] + \pi\cdot\frac{1}{a}\delta(\omega)$

$= \frac{1}{j\omega}\cdot 1/(a+j\omega) + \pi\cdot\frac{1}{a}\delta(\omega)$

$= \frac{1}{j\omega}F(\omega) + \pi\cdot F(0)\,\delta(\omega)$ ✓

Benmei Chen

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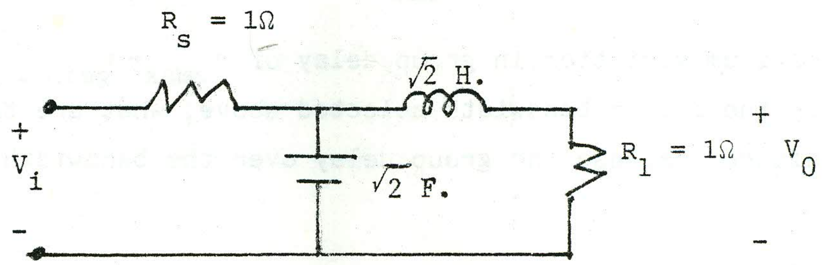
E-E 423 Communication Systems
Homework #2 - Butterworth Lowpass Filter
September 12, 1986

We will analyze a 2nd order Butterworth LPF for which the normalized network is given below.



Normalized 2nd order Butterworth LPF

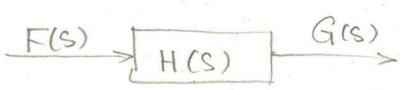
a) Determine the voltage transfer function, $H(s) = \dfrac{V_0(s)}{V_i(s)}$, of the normalized filter and draw the pole constellation.

b) Determine and plot the (i) magnitude response, (ii) phase response and (iii) group delay. Plot magnitude in dB, phase in degrees and group delay in seconds. Use a logarithm frequency scale of 3 decades from 0.1 rps to 100 rps. (3 cycle semi-log paper is available at the bookstore.)

c) Determine and plot the (i) unit impulse response function and (ii) unit step response function. Verify that $h(t) = \dfrac{dg(t)}{dt}$ where g(t) is the unit step response and h(t) is the unit impulse response.

d) Find the 10% to 90% rise-time, t_r. How is t_r related to the 3dB bandwidth?

e) Frequency scale to a cut-off frequency of $\omega_c = 2\pi \times 10^4$ rps and impedance scale to $R_s = R_1 = 1K\Omega$. Draw a new circuit schematic with the scaled component values. Give new scales to the plots drawn for part b.



$u(t) \longrightarrow \frac{1}{s}$

$G(s) \longrightarrow \frac{1}{s} H(s)$

f) We wish to use a 2nd order Butterworth LPF to pass a band-limited signal with bandwidth

$$B_{sig} = 1\text{KHz}$$

(i) Determine the minimum bandwidth of the filter if we allow a maximum variation in magnitude response of

$$20\log_{10}\frac{|H(\omega)|_{max}}{|H(\omega)|_{min}} \leq 1\text{dB}$$

and a maximum variation in group delay of $\tau_{gmax} - \tau_{gmin} \leq 50\mu\text{sec}$

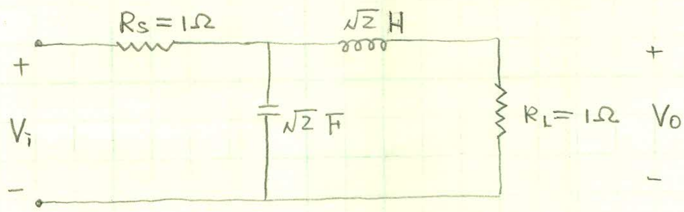
(ii) For the filter bandwidth selected above, what are the variations in magnitude response and group delay over the bandwidth of signal?

$I =$

$i = 10\cos(1000t + 30°) \Rightarrow 10\angle 30°$

$\Rightarrow$

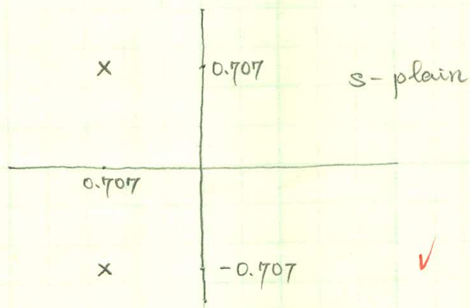
Butterworth Lowpass Filter



a) The voltage transfer function

$$H(S) = V_o(S)/V_i(s) = 1/(2S^2 + 2\sqrt{2}S + 2) \quad ✓$$

$$= 1/2(S + 0.707 - j0.707)(S + 0.707 + j0.707)$$



✓

a − 10
b − 9
c − 10
d − 10
e − 10
f − 10

b) $H(\omega) = H(S)|_{s=j\omega} = 1/(-2\omega^2 + 2 + j2\sqrt{2}\omega)$

$$|H(\omega)| = 1/\sqrt{4\omega^4 - 8\omega^2 + 4 + 8\omega^2} = 1/(2\sqrt{\omega^4+1})$$

$$\angle H(\omega) = -\tan^{-1} \cdot 2\sqrt{2}\omega/(2 - 2\omega^2) = -\tan^{-1}\sqrt{2}\omega/(1-\omega^2)$$

$$\tau_g = -\frac{d}{d\omega}\angle H(\omega) = 1\Big/\left(1 + \frac{2\omega^2}{(1-\omega^2)^2}\right) \cdot \frac{\sqrt{2}(1-\omega^2) + \sqrt{2}\omega \cdot 2\omega}{(1-\omega^2)^2}$$

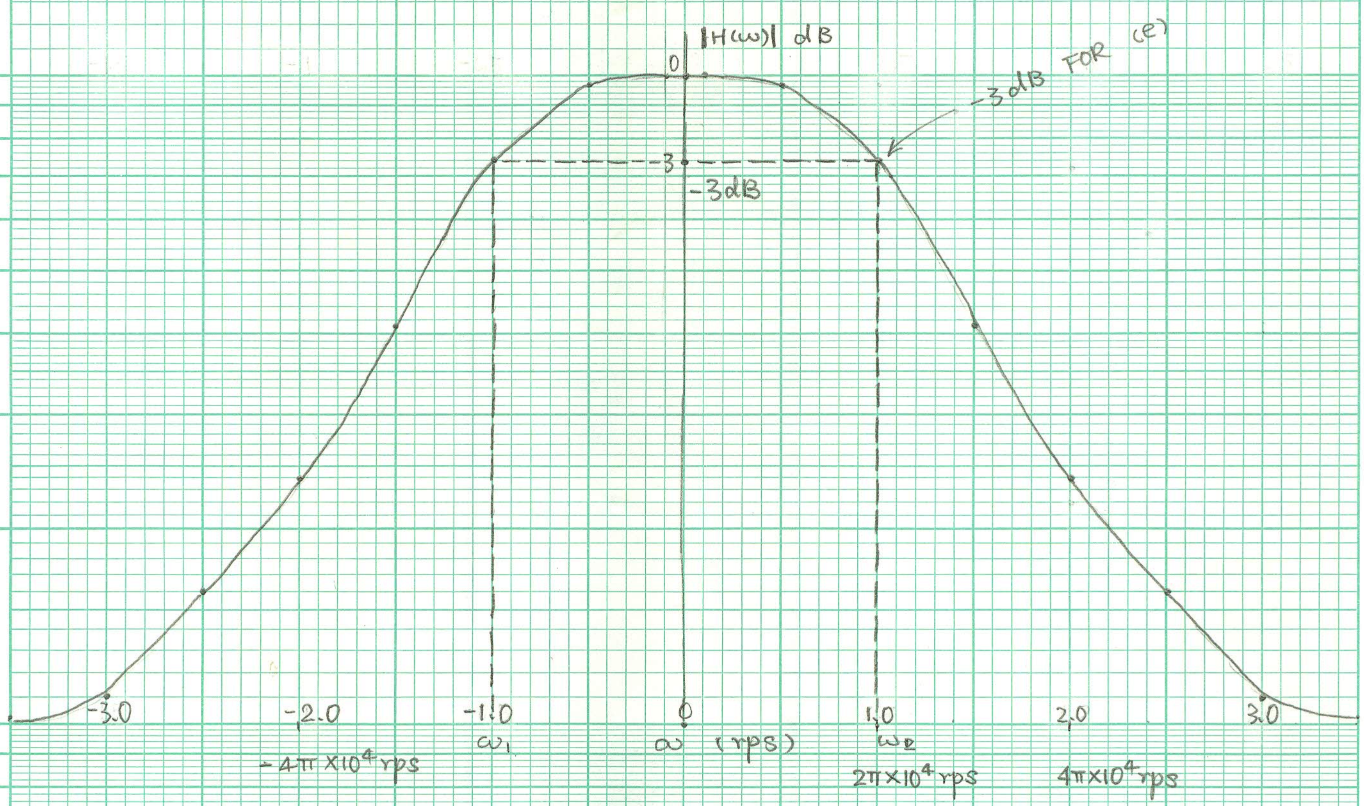
$$= \sqrt{2}(1+\omega^2)/(1+\omega^4) \quad ✓$$

ω (rps)	−10	−1	−0.5	−0.1	−0.05	0	0.05	0.1	0.5	0.71	1	1.2	5
|H(ω)|	0.005	0.354	0.485	0.500	0.500	0.500	0.500	0.500	0.485	0.446	0.354	0.285	0.020
∠H(ω) DEG.	—	90°	43.3°	8.1°	4.1°	0	−4.1°	−8.1°	−43.3°	−63.7°	−90°	—	—
τ_g sec.	0.01	1.41	1.66	1.43	1.42	1.41	1.42	1.43	1.66	1.70	1.41	1.12	0.06

$$|H(\omega)|\,dB \triangleq 20\log_{10}|H(\omega)|/|H(0)| = 20\log_{10}|H(\omega)| + 6$$

The figures are on page 2 and 3

8 a−10 b−9


|H(ω)| dB
0
-3dB FOR (e)
-3
-3dB
-3.0
-2.0
-1.0
0
1.0
2.0
3.0
ω₁
ω (rps)
ω₂
-4π x10⁴ rps
2π x10⁴ rps
4π x10⁴ rps

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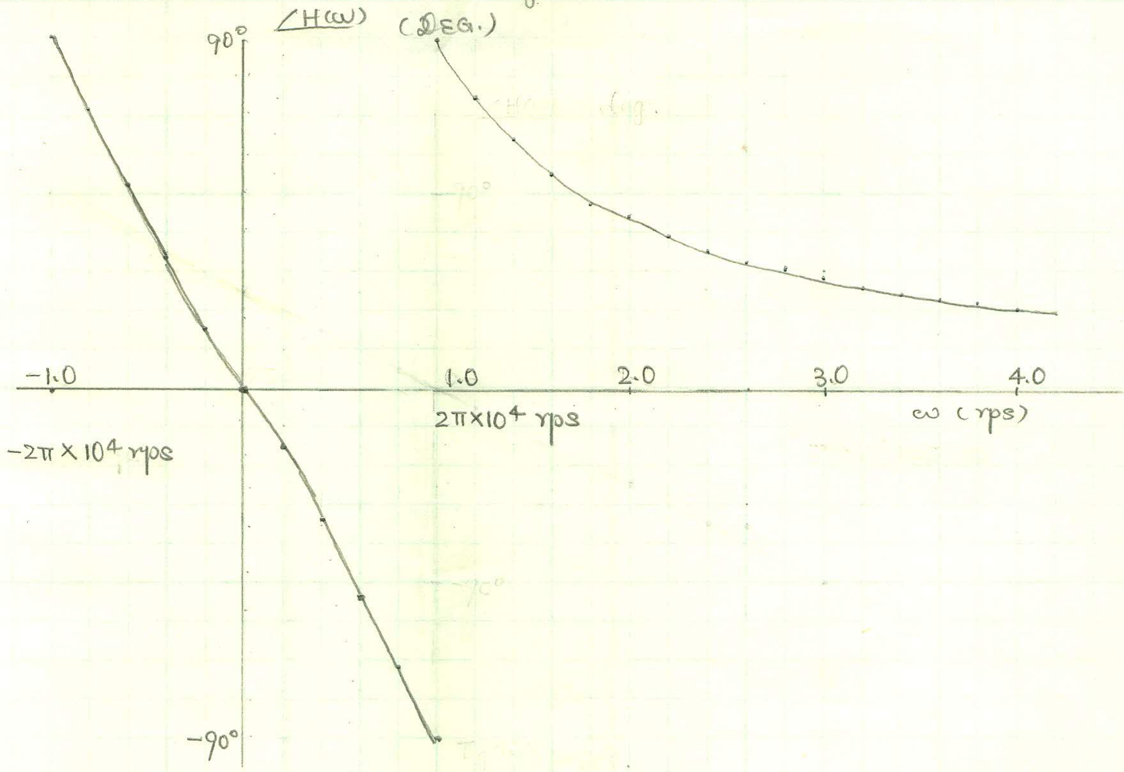
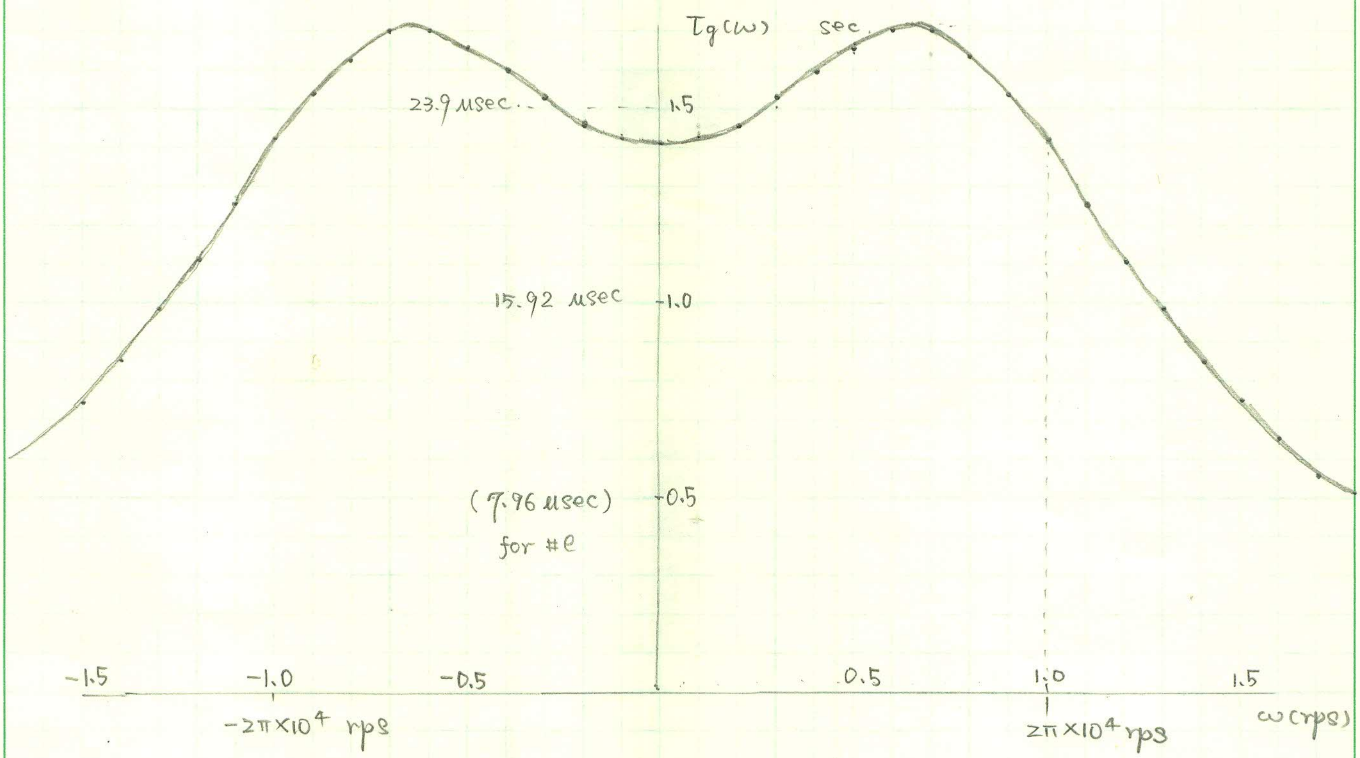

$\angle H(\omega)$ (DEG.)
90°
-90°
-1.0
1.0
2.0
3.0
4.0
$-2\pi \times 10^4$ rps
$2\pi \times 10^4$ rps
ω (rps)


$T_g(\omega)$ sec
23.9 μsec
1.5
15.92 μsec
1.0
(7.96 μsec)
for #e
0.5
-1.5
-1.0
-0.5
0.5
1.0
1.5
$-2\pi \times 10^4$ rps
$2\pi \times 10^4$ rps
ω (rps)

c) unit impulse response function

$$H(\omega) = 1/(-2\omega^2 + 2 + j2\sqrt{2}\,\omega)$$

$$= 1/\{2[0.707 + j(\omega - 0.707)]\cdot[0.707 + j(\omega + 0.707)]\}$$

$$= \frac{\sqrt{2}}{4}\cdot\frac{1}{j}\left[\,1/[0.707 + j(\omega - 0.707)] - 1/[0.707 + j(\omega + 0.707)]\right]$$

$$h(t) = \frac{\sqrt{2}}{4j}\cdot\left[e^{-0.707t}u(t)\cdot e^{+j0.707t} - e^{-0.707t}u(t)\cdot e^{-j0.707t}\right]$$

$$= \frac{\sqrt{2}}{4j}e^{-0.707t}u(t)\left[e^{+j0.707t} - e^{-j0.707t}\right]$$

$$= \frac{\sqrt{2}}{2}e^{-0.707t}\cdot\sin(0.707t)\cdot u(t)$$ ✓

$$G(s) = \mathcal{L}\{u(t)\}\cdot H(s)$$

$$= \frac{1}{s}\cdot 1/(2s^2 + 2\sqrt{2}s + 2)$$

$$= \frac{1}{2}\left[\frac{1}{s} - s/(s^2 + \sqrt{2}s + 1) - \sqrt{2}/(s^2 + \sqrt{2}s + 1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - (s + \tfrac{\sqrt{2}}{2})/(s^2 + \sqrt{2}s + 1) - \tfrac{\sqrt{2}}{2}/(s^2 + \sqrt{2} + 1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s + 0.707 - j0.707} - \frac{1}{2}\frac{1}{s + 0.707 + j0.707} + \frac{1}{2j}\cdot\frac{1}{s + 0.707 + j0.707} - \frac{1}{2j}\cdot\frac{1}{s + 0.707 - j0.707}\right]$$

$$\xrightarrow{\mathcal{L}^{-1}} g(t) = \frac{1}{2}\left[u(t) - \frac{1}{2}e^{(-0.707 + j0.707)t}\cdot u(t) - \frac{1}{2}e^{(-0.707 - j0.707)t}\cdot u(t) + \frac{1}{2j}\left(e^{-(0.707 + j0.707)t}\cdot u(t) - e^{-(0.707 - j0.707)t}\cdot u(t)\right)\right]$$

$$= \frac{1}{2}u(t)\left[1 - \frac{1}{2}e^{-0.707t}\cdot\left[[e^{j0.707t} + e^{-j0.707t}] + \frac{1}{j}[e^{j0.707t} - e^{-j0.707t}]\right]\right]$$

$$= \frac{1}{2}u(t)\left[1 - e^{-0.707t}\cdot[\cos 0.707t + \sin 0.707t]\right]$$ ✓

$$\frac{d}{dt}g(t) = \frac{1}{2}\delta(t)\left[1 - e^{-0.707t}\{\cos 0.707t + \sin 0.707t\}\right] + \frac{1}{2}u(t)\left[\sqrt{2}\,e^{-0.707t}\sin 0.707t\right]$$

$$= \frac{\sqrt{2}}{2}e^{-0.707t}\cdot\sin 0.707t\cdot u(t) = h(t)$$ ✓

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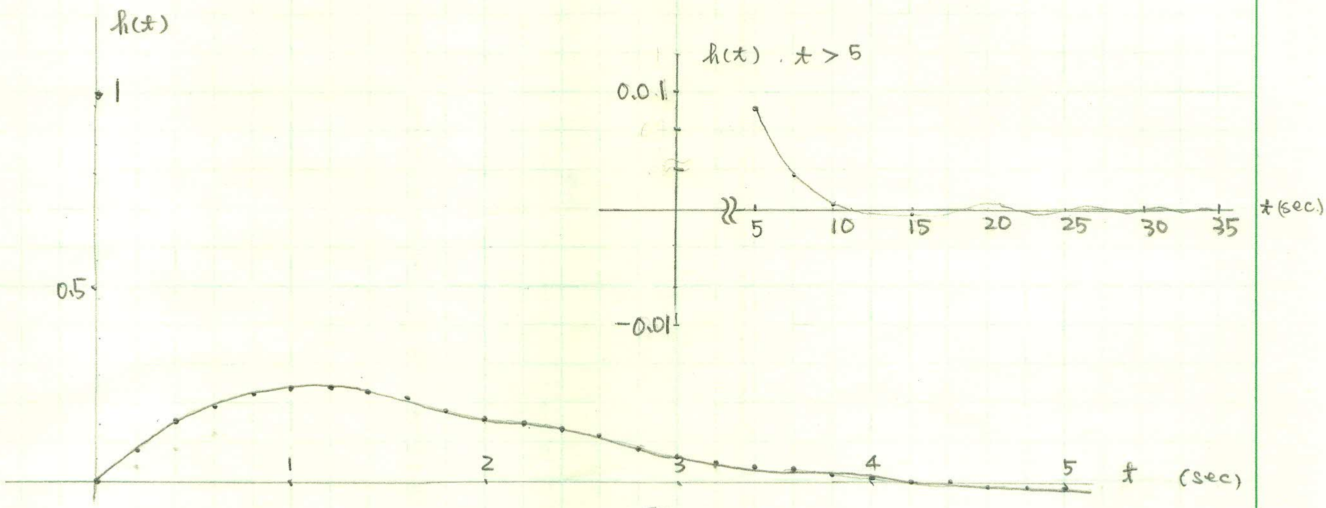

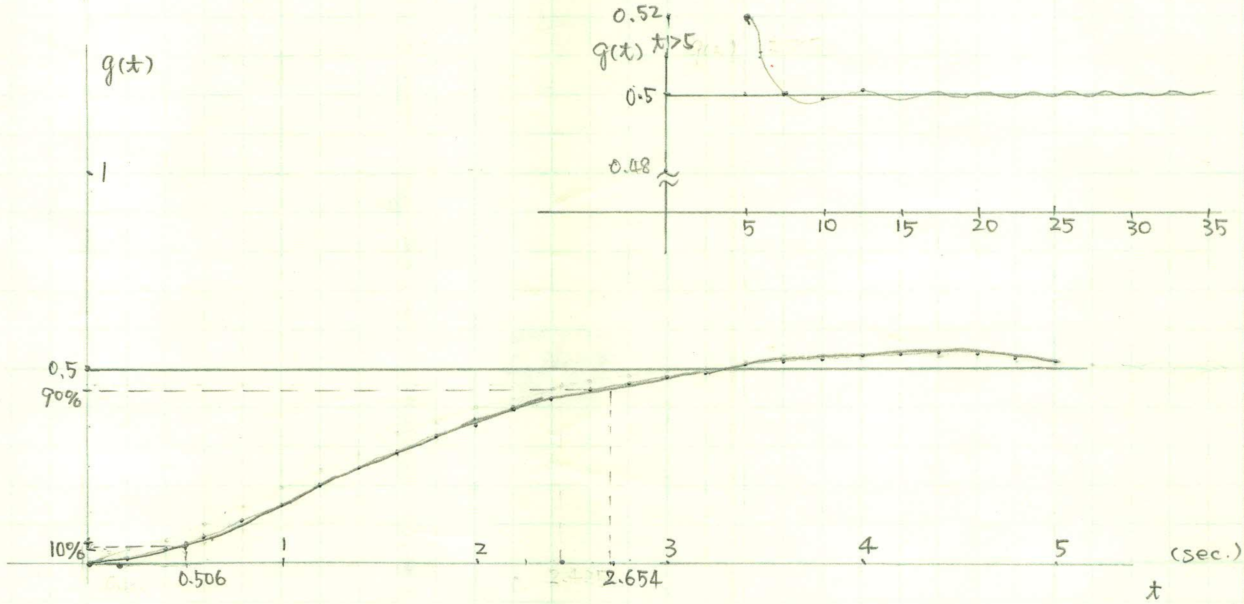

d) $t_r = 2.654 - 0.506 = 2.139$

$t_r \cdot B = 2.139 \cdot W/2\pi = 2.139 \cdot 1/2\pi = 0.34$

$t_r = 0.34/B$

✓

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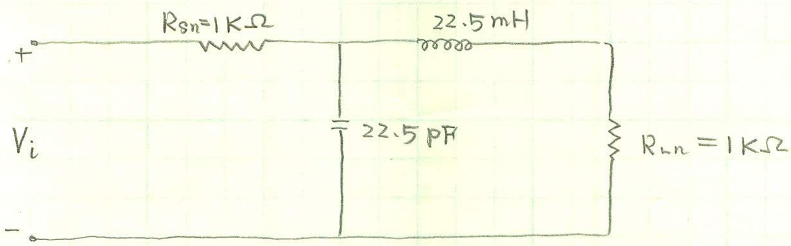
e) $\omega_c = 2\pi \times 10^4$ rps , $K_f = 2\pi \times 10^4$

$R_s = R_1 = 1K\Omega$, $K_m = 1000$

$\therefore R_{sn} = R_{Ln} = 1K\Omega$

$C_n = C/(K_m \cdot K_f) = 22.5$ pF

$L_n = K_m L / K_f = \sqrt{2} \times 1000 / 2\pi \times 10^4 = 22.5$ mH



✓

f) $B_{sig} = 1$ KHz $W_{sig} = 2\pi \cdot B = 2000\pi$ rps

(i) *Assume the minimum bandwidth is ω_c*

$K_f = \omega_c$

$|H(\omega)| = 1 / (2\sqrt{1 + (\frac{\omega}{\omega_c})^4}$

$\tau_g = \sqrt{2}\,[1 + (\frac{\omega}{\omega_c})^2] / [1 + (\frac{\omega}{\omega_c})^4] \cdot \frac{1}{\omega_c}$

$|H(\omega)|_{max} = |H(0)| = \frac{1}{2}$

$|H(\omega)|_{min} = |H(2000\pi)| = 1/(2\sqrt{1 + (\frac{\omega}{\omega_c})^4})$

Because

$$20 \cdot \log_{10} \frac{|H(\omega)|_{max}}{|H(\omega)|_{min}} \leq 1\,dB$$

$|H(\omega)|_{max} / |H(\omega)|_{min} \leq 1.122$

$|H(\omega)|_{min} = 1/(2\sqrt{1 + (\frac{\omega}{\omega_c})^4}) \geq |H(\omega)|_{max} / 1.122 = 0.5/1.122 = 0.4456$

$\therefore \omega_c \geq 2803\pi$ rps $= 8808$ rps

$\tau_{g\,min} = \tau_g(0) = \frac{\sqrt{2}}{\omega_c}$

$\tau_{g\,max} = \tau_g(0.644\,\omega_c) = 1.707/\omega_c$

$\tau_{g\,max} - \tau_{g\,min} = \frac{1}{\omega_c}(1.707 - 1.414) \leq 50\,\mu sec = 5\times10^{-5}\ sec$

$\omega_c \geq 5858$ rps

So, the minimum bandwidth of the filter is 8808 rps.

ii) The variation in magnitude response over the bandwidth of signal is 1 dB

The variation in group delay over the bandwidth of signal,

$\tau_{g\,max} - \tau_{g\,min} = 1.707/\omega_c - \sqrt{2}/\omega_c$

$= 33.26\ \mu sec$ ✓ Very good!

(Because $0.644\,\omega_c \doteq 5672 < 2000\pi$)

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EE 423 - COMMUNICATION SYSTEMS

Benmei Chen

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Homework #3
Transversal Filter and Sampling

Dr. R. A. Birgenheier
September 19, 1986

DUE SEPTEMBER 24, 1986

1. Prob. 3.10.1 in Stremler (simplify answer as much as possible).

2.

8.1. In the system shown in Figure P8.1, two time functions $x_1(t)$ and $x_2(t)$ are multiplied and the product $w(t)$ is sampled by a periodic impulse train. $x_1(t)$ is bandlimited to ω_1 and $x_2(t)$ is bandlimited to ω_2, that is,

$$X_1(\omega) = 0, \qquad |\omega| > \omega_1$$

$$X_2(\omega) = 0, \qquad |\omega| > \omega_2$$

Determine the *maximum* sampling interval T such that $w(t)$ is recoverable from $w_p(t)$ through the use of an ideal lowpass filter.

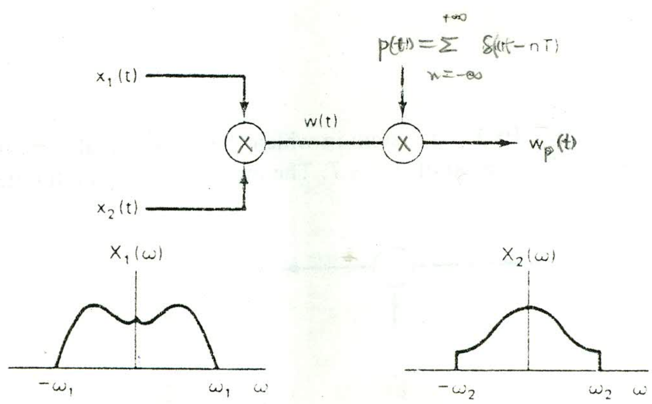


Figure P8.1

3. **8.2.** Shown in Figure P8.2 is a system in which the sampling signal is an impulse train with alternating sign. The Fourier transform of the input signal is as indicated in Figure P8.2.

(a) For $\Delta < \pi/2\omega_M$, sketch the Fourier transform of $x_p(t)$ and $y(t)$.
(b) For $\Delta < \pi/2\omega_M$, determine a system which will recover $x(t)$ from $x_p(t)$.
(c) For $\Delta < \pi/2\omega_M$, determine a system which will recover $x(t)$ from $y(t)$.
(d) What is the *maximum* value of Δ in relation to ω_M for which $x(t)$ can be recovered from either $x_p(t)$ or $y(t)$.

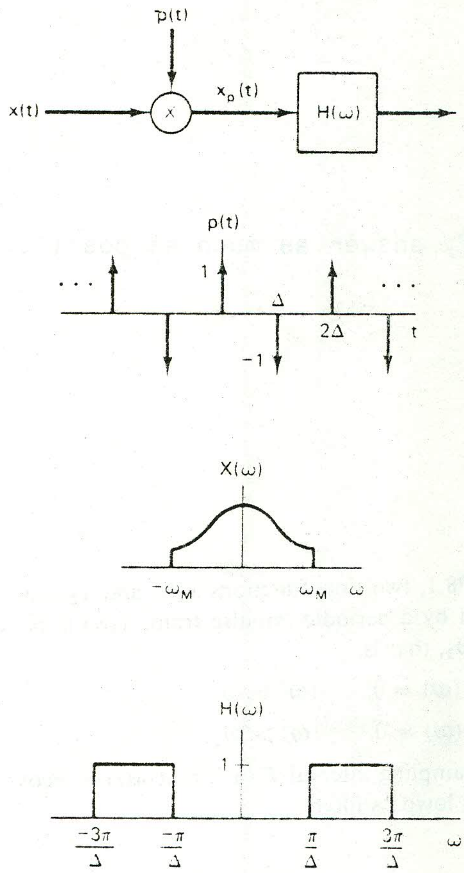


Figure P8.2

4. **8.3.** Shown in Figure P8.3 is a system in which the input signal is multiplied by a periodic square wave. The period of $s(t)$ is T. The input signal is bandlimited with $|X(\omega)| = 0$, $|\omega| \geq \omega_M$.

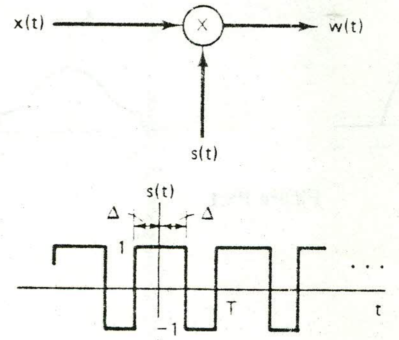


Figure P8.3

(a) For $\Delta = T/3$ determine in terms of ω_M the *maximum* value of T for which $x(t)$ can be recovered from $w(t)$. With this maximum value, determine a system to recover $x(t)$ from $w(t)$.
(b) For $\Delta = T/4$, determine in terms of ω_M the *maximum* value of T for which $x(t)$ can be recovered from $w(t)$. With this maximum value, determine a system to recover $x(t)$ from $w(t)$.

5. **8.6.** The sampling theorem as we have derived it states that a signal $x(t)$ must be sampled at a rate greater than its bandwidth (or equivalently a rate greater than twice its highest frequency). This implies that if $x(t)$ has a spectrum as indicated in Figure P8.6-1, then $x(t)$ must be sampled at a rate greater than $2\omega_2$. Since the signal has most of its energy concentrated in a narrow band, it would seem reasonable to expect that a sampling rate lower than twice the highest frequency could be used. A signal whose energy is concentrated in a frequency band is often referred to as a *bandpass signal*. There are a variety of techniques for sampling such signals, and these techniques are generally referred to as *bandpass sampling*.

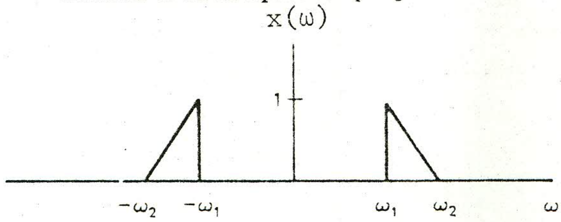


Figure P8.6-1

To examine the possibility of sampling a bandpass signal at a rate less than the total bandwidth, consider the system shown in Figure P8.6-2. Assuming that $\omega_1 > (\omega_2 - \omega_1)$, find the maximum value of T and the values of the constants A, ω_a, and ω_b such that $x_r(t) = x(t)$.

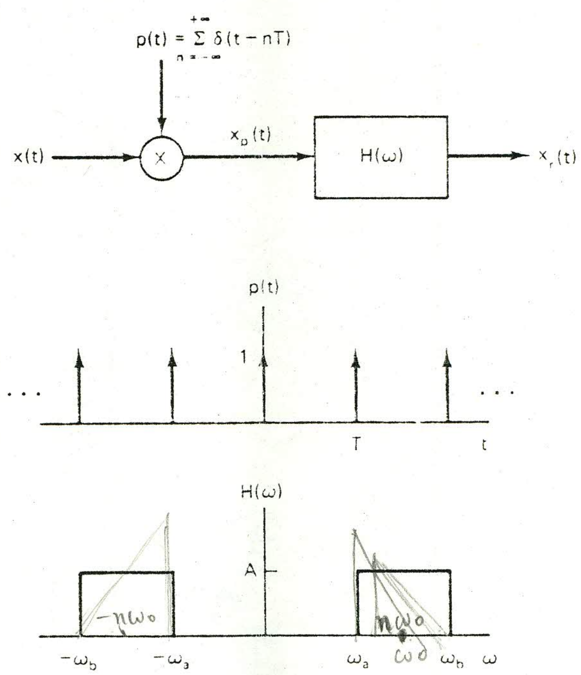
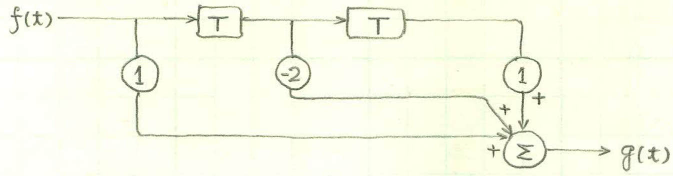


Figure P8.6-2

6. Derive and plot Fig. 3.22 in Stremler (carefully label the scales).

1. [3.10.1 (IN STREMLER)]



SOLUTION: $g(t) = f(t) - 2f(t-T) + f(t-2T)$

$$G(\omega) = F(\omega) - 2\cdot e^{-j\omega T}\cdot F(\omega) + e^{-j\omega\cdot 2T}\cdot F(\omega)$$

$$= (1 - 2e^{-j\omega T} + e^{-j\omega\cdot 2T})\cdot F(\omega)$$

$$H(\omega) = G(\omega)/F(\omega) = 1 - 2e^{-j\omega T} + e^{-j\omega\cdot 2T}$$

$$|H(\omega)| = |1 - 2e^{-j\omega T} + e^{-j\omega\cdot 2T}|$$

$$= |1 - 2\cos\omega T + j2\sin\omega T + \cos 2\omega T - j\sin 2\omega T|$$

$$= |(1 - 2\cos\omega T + \cos 2\omega T) + j2\sin\omega T(1-\cos\omega T)|$$

$$= |2\cos\omega T(\cos\omega T - 1) + j2\sin\omega T(1-\cos\omega T)|$$

$$= 2(1-\cos\omega T)$$

$\cos WT = 1 - 2\sin^2(\frac{WT}{2})$

$$= 2[1-(1-2\sin^2(\tfrac{WT}{2}))] = \boxed{4\sin^2\frac{WT}{2}}$$

-1

2. SOLUTION: $X_1(\omega) = 0 \quad |\omega| > \omega_1$

$$X_2(\omega) = 0 \quad |\omega| > \omega_2$$

$$W(t) = X_1(t)\cdot X_2(t)$$

$$W(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(u)\cdot X_2(\omega-u)\,du$$

$$= \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} X_1(u)\, X_2(\omega-u)\,du$$

If Let $|\omega'| > \omega_1 + \omega_2$,

$$W(\omega') = \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} X_1(u)\, X_2(\omega-u)\,du = 0$$

$$\therefore \quad W(\omega) = 0 \qquad |\omega| > \omega_1 + \omega_2$$

$$\omega_w = \omega_1 + \omega_2, \qquad B_w = \frac{\omega_w}{2\pi} = (\omega_1+\omega_2)/2\pi$$

According to the sampling theorem, $T \le \frac{1}{2B_w} = \frac{\pi}{\omega_1+\omega_2}$

$$T_{max} = \pi/(\omega_1+\omega_2)$$

✓

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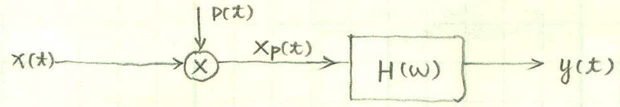
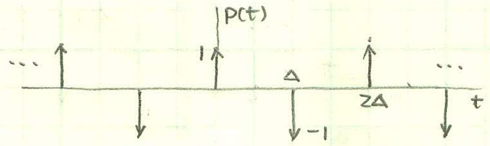
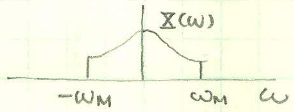
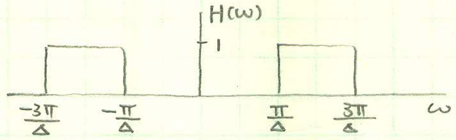
3.









(a) $\Delta < \pi/2\omega_M$, $\quad p(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t - n\Delta)$

$$X_p(t) = x(t)\,p(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} (-1)^n \delta(t - n\Delta)$$

$$\sum_{n=-\infty}^{\infty} (-1)^n \delta(t - n\Delta) = \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_0 t}$$

where $\omega_0 = 2\pi/2\Delta = \pi/\Delta$ and

$$P_n = \frac{1}{2\Delta} \int_{\frac{\Delta}{2}}^{\frac{5}{2}\Delta} \left(-\delta(t-\Delta) + \delta(t-2\Delta)\right) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{2\Delta}\left[- e^{-jn\cdot\frac{\pi}{\Delta}\cdot\Delta} + e^{-jn\cdot\frac{\pi}{\Delta}\cdot 2\Delta}\right]$$

$$= \frac{1}{2\Delta}\left[-\cos(n\pi) + j\sin(n\pi) + \cos(2n\pi) - j\sin(2n\pi)\right]$$

$$= \frac{1}{2\Delta}\left[1 - \cos(n\pi)\right]$$

$$\therefore\quad X_p(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_0 t}$$

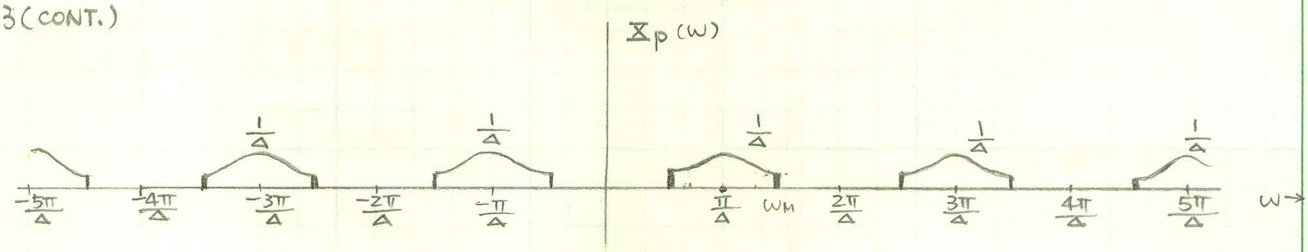
$$= \sum_{n=-\infty}^{\infty} P_n \cdot x(t) e^{jn\omega_0 t}$$

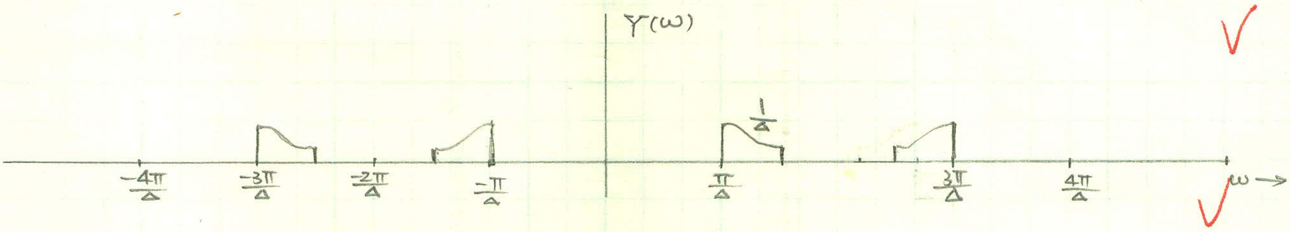
$$X_p(\omega) = \sum_{n=-\infty}^{\infty} P_n \cdot X(\omega - n\omega_0)$$

$$= \sum_{n=-\infty}^{\infty} P_n \cdot X(\omega - n\pi/\Delta)$$

$$Y(\omega) = X_p(\omega) \cdot H(\omega)$$

3 (CONT.)





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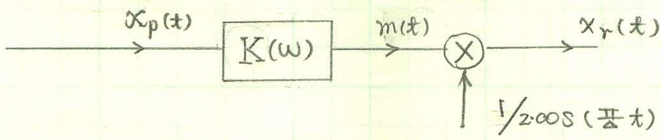
(b) THE FOLLOWING SYSTEM will RECOVER $x(t)$ FROM $x_p(t)$



$$K(\omega) = \begin{cases} \Delta & |\omega \pm \frac{\pi}{\Delta}| \leq \omega_M \\ 0 & \text{elsewhere} \end{cases}$$

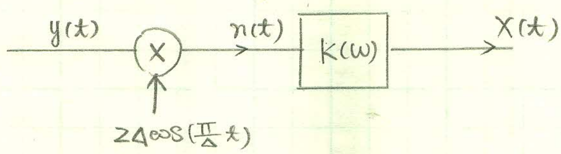
$$m(t) = \mathcal{F}^{-1}\{X_p(\omega)\cdot K(\omega)\} = \mathcal{F}^{-1}\{X(\omega-\tfrac{\pi}{\Delta}) + X(\omega+\tfrac{\pi}{\Delta})\}$$

$$= 2\cos(\tfrac{\pi}{\Delta}t)\cdot x(t)$$

~~$X_p(\omega)$ is periodic~~ ?

$$x_r(t) = m(t)\cdot 1/[2\cos(\tfrac{\pi}{\Delta}t)] = x(t)$$

(c)



$$N(\omega) = 2\Delta\left[\tfrac{1}{2}Y(\omega-\tfrac{\pi}{\Delta}) + \tfrac{1}{2}Y(\omega+\tfrac{\pi}{\Delta})\right]$$

$$= \Delta Y(\omega-\tfrac{\pi}{\Delta}) + \Delta Y(\omega+\tfrac{\pi}{\Delta}) = \begin{cases} X(\omega) & |\omega| \leq \omega_M \\ X'(\omega) & |\omega| > \omega_M \end{cases}$$

$$K(\omega) = \begin{cases} 1 & |\omega| \leq \omega_M \\ 0 & |\omega| > \omega_M \end{cases}$$

(d) $\Delta_{max} = \frac{\pi}{\omega_M}$ (corrected in red: Δ_{min})

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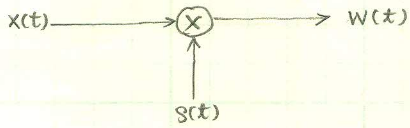
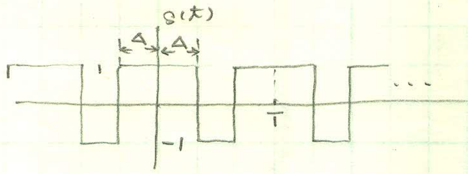
4.





(a)

$$S(t) = \sum_{n=-\infty}^{\infty} P_n \cdot e^{-jn\omega_0 t}$$

$$P_n = \frac{1}{T}\int_0^T s(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^{\Delta} e^{-jn\omega_0 t} dt - \frac{1}{T}\int_{\Delta}^{T-\Delta} e^{-jn\omega_0 t} dt + \frac{1}{T}\int_{T-\Delta}^{T} e^{-jn\omega_0 t} dt \qquad (\Delta = \tfrac{T}{3})$$

$$= \begin{cases} 1/3 & n = 0 \\ \left(\sin\frac{2n\pi}{3} - \sin\frac{4n\pi}{3}\right)/n\pi & n \neq 0 \end{cases}$$

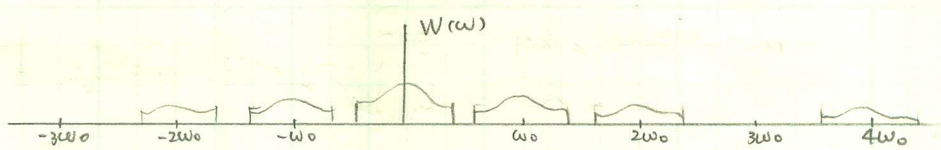
$$= \begin{cases} 1/3 & n = 0 \\ \sqrt{3}/n\pi & n = 3k+1, \quad k = 0, 1, 2, \dots \\ 0 & n = 3k, \quad k = 0, 1, 2, \dots \\ -\sqrt{3}/n\pi & n = 3k-1, \quad k = 0, 1, 2, \dots \end{cases}$$

$$W(t) = x(t) \cdot s(t)$$

$$= x(t) \cdot \sum_{n=-\infty}^{\infty} P_n \cdot e^{jn\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T}$$

$$= \sum_{n=-\infty}^{\infty} P_n \cdot x(t) e^{jn\omega_0 t}$$

$$W(\omega) = \sum_{n=-\infty}^{\infty} P_n \, X(\omega - n\omega_0) = \frac{1}{3} X(\omega) + \sum_{n\neq 0} P_n X(\omega - n\omega_0)$$

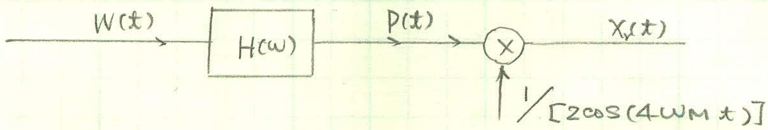


If we let $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\omega_M} = \frac{\pi}{\omega_M}$, It is easy to recover x(t) from w(t) by using a low-pass filter $H(\omega) = \begin{cases} 3 & |\omega| \le \omega_M \\ 0 & |\omega| > \omega_M \end{cases}$.

4(a) (CONT.)

Let $T = \frac{2\pi}{\omega_M}$, we also can use the system below to recover $x(t)$ from $w(t)$:

$$\omega_0 = 2\pi / T = \omega_M$$



$$H(\omega) = \begin{cases} \frac{4\pi}{\sqrt{3}} = 7.255 & |\omega \pm 4\omega_M| < \omega_M \\ 0 & \text{elsewhere} \end{cases}$$

$$p(\omega) = X(\omega - 4\omega_c) + X(\omega + 4\omega_c)$$

$$p(t) = 2\cos(4\omega_M t)\, x(t)$$

$$x_r(t) = p(t) \cdot 1/[2\cos(4\omega_M t)] = x(t)$$

So, the maximum T is $\frac{2\pi}{\omega_M}$.

4. (b) $T_{max} = \pi/\omega_M$ (same as part (a))

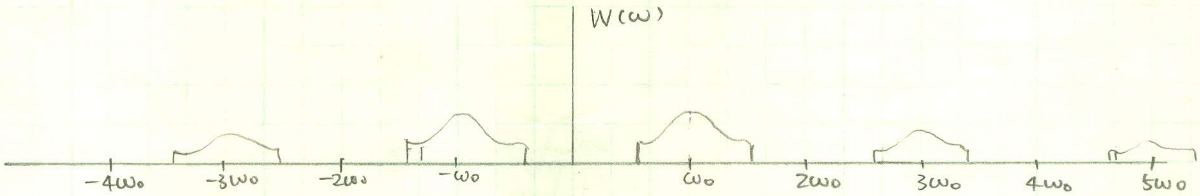
$$S(t) = \sum_{n=-\infty}^{\infty} P_n \cdot e^{-jn\omega_0 t}, \quad \Delta = T/4, \quad \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2\Delta}$$

$$P_n = \frac{1}{4\Delta}\left[\int_0^{\Delta} e^{-jn\omega_0 t}dt - \int_{\Delta}^{3\Delta} e^{-jn\omega_0 t}dt + \int_{3\Delta}^{4\Delta} e^{-jn\omega_0 t}dt\right]$$

$$= \begin{cases} 0 & n = 0 \\ \left(\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right)/n\pi & n \neq 0 \end{cases}$$

$$= \begin{cases} 2/n\pi & n = 4k+1 \\ 0 & n = 2k \\ -2/n\pi & n = 4k-1 \end{cases}$$

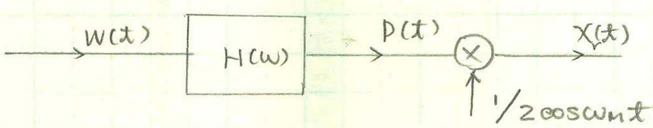
$$W(\omega) = \sum_{n=-\infty}^{\infty} P_n \cdot X(\omega - n\omega_0) = \frac{2}{\pi}\left(X(\omega - \tfrac{\pi}{2\Delta}) + X(\omega - \tfrac{\pi}{2\Delta})\right) + \sum_{n\neq 1,-1} P_n X(\omega - n\omega_0)$$



The minimum $\omega_0 = \omega_M$

SO, the maximum $T = \frac{2\pi}{\omega_0} = 2\pi/\omega_M$

AND the following system will recover $x(t)$ from $w(t)$:



$$H(\omega) = \begin{cases} \frac{\pi}{2} = 1.571 & |\omega| \leq 2\omega_M \\ 0 & |\omega| > \omega_M \end{cases}$$

$$P(\omega) = X(\omega + \omega_M) + X(\omega - \omega_M)$$

$$p(t) = 2\cos\omega_M t \cdot x(t)$$

$$x_r(t) = p(t) \cdot 1/2 \cos\omega_M t = x(t)$$

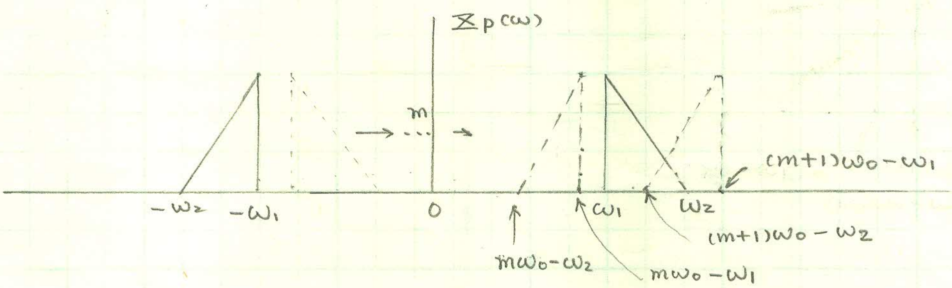
5. $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$

$$p(t) = \sum_{n=-\infty}^{\infty} P_n \cdot e^{jn\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T}$$

$$P_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} dt = 1/T$$

$$X_p(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)\, e^{jn\omega_0 t}$$

$$X_p(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)$$



IF $X_p(t)$ is recoverable,

$$\begin{cases} 2mB \leq 2\omega_1 \\ \omega_1 \geq m\omega_0 - \omega_1 \quad \text{AND} \quad (m+1)\omega_0 - \omega_2 \geq \omega_2 \end{cases}$$

$$\Rightarrow \begin{cases} m \leq \omega_1/B \\ 2\omega_1 \geq m\omega_0 \quad \text{AND} \quad (m+1)\omega_0 \geq 2\omega_2 \end{cases}$$

Let $m = \left[\frac{\omega_1}{B}\right] \leq \omega_1/B$ $\qquad$ * $\left[\frac{\omega_1}{B}\right] = \text{INT}\left(\frac{\omega_1}{B}\right)$

$$\omega_0 \geq 2\omega_2/(m+1) = 2\omega_2 \Big/ \left(\left[\frac{\omega_1}{B}\right] + 1\right)$$

SO; $\underline{\omega_{0,min} = 2\omega_2 \Big/ \left(\left[\frac{\omega_1}{B}\right] + 1\right)}$

$$T_{max} = 2\pi/\omega_{0,min} = \pi\left(\left[\frac{\omega_1}{B}\right]+1\right)/\omega_2 = \pi\left(\left[\frac{\omega_1}{\omega_2 - \omega_1}\right]+1\right)/\omega_2$$

$A = T$

$\omega_a = \omega_1$

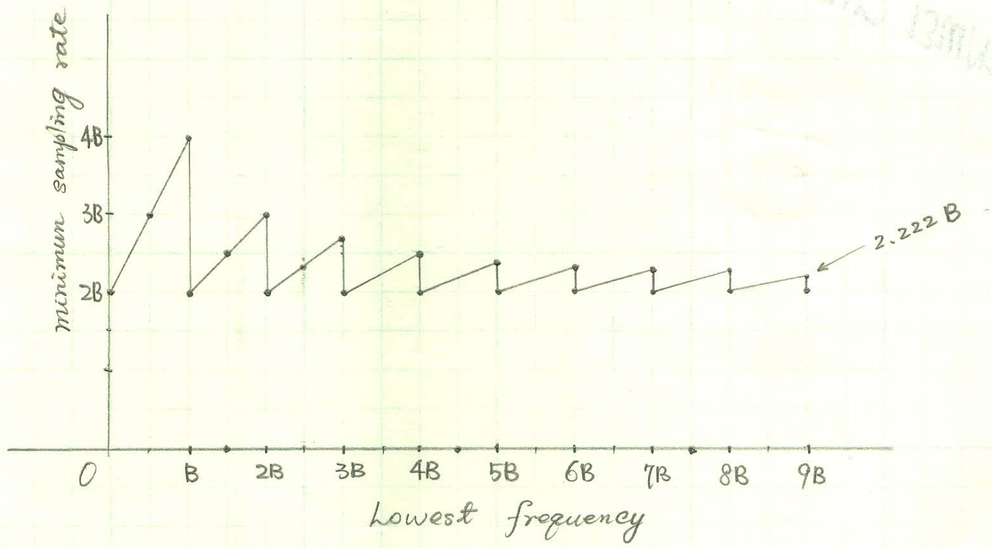
$\omega_b = \omega_2$

2

6. From the result in Problem #5, we got

$$\omega_{o,min} = 2W_2 / \left(\left[\frac{\omega_1}{B}\right] + 1 \right)$$

$$= 2(\omega_1 + B) / \left(\left[\frac{\omega_1}{B}\right] + 1 \right)$$



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Drill Problem 3.17.1

$$\tilde{F}(\omega) = \int_{0-}^{NT-} f(t) e^{-j\omega t}\, dt$$

$$f(t) \longrightarrow f(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t/T - k)$$

$$\therefore \tilde{F}(\omega) = \int_{0-}^{NT-} f(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t/T - k) \cdot e^{-j\omega t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} \left\{ \int_{0-}^{NT-} f(t)\, \delta(t/T - k) e^{-j\omega t}\, dt \right.$$

$$= \sum_{k=-\infty}^{\infty} \left[\int_{0-}^{NT-} f(t) e^{-j\omega t} \cdot T \cdot \delta(t - kT)\, dt \right]$$

$$= \sum_{k=0}^{N-1} \left[T \cdot f(kT) \cdot e^{-j\omega \cdot kT} \right]$$

$$\tilde{F}(n\Omega) = \sum_{k=0}^{N-1} f(kT) e^{-jn\Omega \cdot kT} \cdot T$$

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D17.1 - 10
D17.2 - 10
D17.3 - 9
P17.1 - 10
P17.2 - 10
P17.3 - 10

Drill Problem 3.17.2 (a) $\{1, 0, 1, 0\}$

$$F_D[n] = \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N} nk} \quad , n = 0, 1, 2, 3$$

$$F_D[0] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 0\cdot 0} + 0 \cdot e^{-j\frac{\pi}{2}\cdot 0\cdot 1} + 1 \cdot e^{-j\frac{\pi}{2}\cdot 0\cdot 2} + 0 \cdot e^{-j\frac{\pi}{2}\cdot 0\cdot 3} = 2$$

$$F_D[1] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 1\cdot 0} + 1 \cdot e^{-j\frac{\pi}{2}\cdot 1\cdot 2} = 0$$

$$F_D[2] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 2\cdot 0} + 1 \cdot e^{-j\frac{\pi}{2}\cdot 2\cdot 2} = 2$$

$$F_D[3] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 2\cdot 0} + 1 \cdot e^{-j\frac{\pi}{2}\cdot 3\cdot 2} = 0$$

$$F_D[n] = \{2, 0, 2, 0\} \quad n = 0, 1, 2, 3$$

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(b) $\{1, 1, 0, 0\}$

$$F_D[0] = 1 + 1 + 0 + 0 = 2$$

$$F_D[1] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 1\cdot 0} + 1 \cdot e^{-j\frac{\pi}{2}\cdot 1\cdot 1} = 1 - j1$$

$$F_D[2] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 2\cdot 0} + 1 \cdot e^{-j\frac{\pi}{2}\cdot 2\cdot 1} = 0$$

$$F_D[3] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 3\cdot 0} + 1 \cdot e^{-j\frac{\pi}{2}\cdot 3\cdot 1} = 1 + j1$$

$$F_D[n] = \{2, 1-j1, 0, 1+j1\} \quad n = 0, 1, 2, 3.$$

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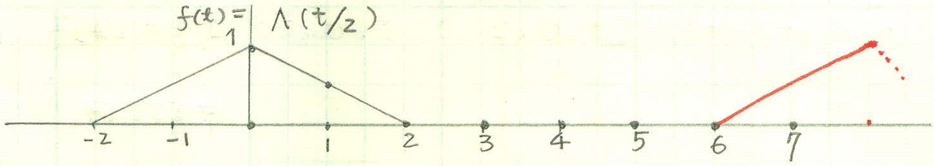
Drill Problem 3.17.3. Compute the Fourier transform of $\Lambda(t/2)$ using the DFT and eight samples, each taken at one second intervals.

Solution:



$f[k] = \{1, 0.5, 0, 0, 0, 0, 0, \tfrac{1}{2}\}$

$F[n] = \sum_{k=0}^{7} f[k]\, e^{-j\frac{\pi}{4}\cdot n\cdot k}$

$F[0] = 1\cdot e^{-j\frac{\pi}{4}\cdot 0\cdot 0} + 0.5\cdot e^{-j\frac{\pi}{4}\cdot 0\cdot 1} = 1.5$

$F[1] = 1\cdot e^{-j\frac{\pi}{4}\cdot 1\cdot 0} + 0.5\, e^{-j\frac{\pi}{4}\cdot 1\cdot 1} = 1 + 1/2\sqrt{2} - j\,0.5/\sqrt{2}$

$F[2] = 1\cdot e^{-j\frac{\pi}{4}\cdot 2\cdot 0} + 0.5\, e^{-j\frac{\pi}{4}\cdot 2\cdot 1} = 1 - j\,1$

$F[3] = 1\cdot e^{-j\frac{\pi}{4}\cdot 3\cdot 0} + 0.5\, e^{-j\frac{\pi}{4}\cdot 3\cdot 1} = 1 - 1/2\sqrt{2} - j\,1/2\sqrt{2}$

$F[4] = 1\cdot e^{-j\frac{\pi}{4}\cdot 4\cdot 0} + 0.5\, e^{-j\frac{\pi}{4}\cdot 4\cdot 1} = 0$

$F[5] = 1\cdot e^{-j\frac{\pi}{4}\cdot 5\cdot 0} + 0.5\, e^{-j\frac{\pi}{4}\cdot 5\cdot 1} = 1 - 0.5/\sqrt{2} + j\,0.5/\sqrt{2}$

$F[6] = 1\cdot e^{-j\frac{\pi}{4}\cdot 6\cdot 0} + 0.5\, e^{-j\frac{\pi}{4}\cdot 6\cdot 1} = 1 + j\,1$

$F[7] = 1\cdot e^{-j\frac{\pi}{4}\cdot 7\cdot 0} + 0.5\, e^{-j\frac{\pi}{4}\cdot 7\cdot 1} = 1 + 0.5/\sqrt{2} + j\,0.5/\sqrt{2}$

$F[n] = \{1.5,\ 1+1/\sqrt{2} - j\,1/\sqrt{2},\ 1-j,\ 1-1/\sqrt{2} - j\,1/\sqrt{2},\ 0,\ 1-1/\sqrt{2}+j\,1/\sqrt{2},\ 1+j,\ 1+1/\sqrt{2}+j\,1/\sqrt{2}\}$ $\quad n = 0, 1, 2, \dots, 7$.

* WHY IS MY ANSWER DIFFERENT FROM THE ANSWER IN THE TEXT BOOK ?

3-17-1. Compute the DFT of the following sequences:

(a) $\{1, 0, 0, 0\}$

$$F[n] = \sum_{k=0}^{3} f[k] \cdot e^{-j\frac{2\pi}{N}\cdot n\cdot k}$$

$$F[0] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 0\cdot 0} + 0 + 0 + 0 = 1$$

$$F[1] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 1\cdot 0} + 0 + 0 + 0 = 1$$

$$F[2] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 2\cdot 0} + 0 + 0 + 0 = 1$$

$$F[3] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 2\cdot 0} + 0 + 0 + 0 = 1$$

$F[n] = \{1, 1, 1, 1\}$ $n = 0, 1, 2, 3$ ✓

(b) $\{0, 1, 0, -1\}$

$$F[n] = \sum_{k=0}^{3} f[k] \cdot e^{-j\frac{2\pi}{4}\cdot n\cdot k}$$

$$F[0] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 0\cdot 1} - 1 \cdot e^{-j\frac{\pi}{2}\cdot 0\cdot 3} = 0$$

$$F[1] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 1\cdot 1} - 1 \cdot e^{-j\frac{\pi}{2}\cdot 1\cdot 3} = -j2$$

$$F[2] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 2\cdot 1} - 1 \cdot e^{-j\frac{\pi}{2}\cdot 2\cdot 3} = 0$$

$$F[3] = 1 \cdot e^{-j\frac{\pi}{2}\cdot 3\cdot 1} - 1 \cdot e^{-j\frac{\pi}{2}\cdot 3\cdot 3} = j2$$

$F[n] = \{0, -j2, 0, j2\}$ $n = 0, 1, 2, 3$ ✓

(c) $\exp(-k)$, $N = 4$

$f[k] = \{1, 0.3679, 0.1353, 0.0498\}$ $k = 0, 1, 2, 3$

$F[0] = 1.5530$

$$F[1] = e^{-j\frac{\pi}{2}\cdot 0} + 0.3679 e^{-j\frac{\pi}{2}\cdot 1} + 0.1353 e^{-j\frac{\pi}{2}\cdot 2} + 0.0498 e^{-j\frac{\pi}{2}\cdot 3} = 0.8647 - j0.3181$$

$$F[2] = e^{-j\frac{\pi}{2}\cdot 0} + 0.3679 e^{-j\frac{\pi}{2}\cdot 2} + 0.1353 e^{-j\frac{\pi}{2}\cdot 4} + 0.0498 e^{-j\frac{\pi}{2}\cdot 6} = 0.7176$$

$$F[3] = e^{-j\frac{\pi}{2}\cdot 0} + 0.3679 e^{-j\frac{\pi}{2}\cdot 3} + 0.1353 e^{-j\frac{\pi}{2}\cdot 6} + 0.0498 e^{-j\frac{\pi}{2}\cdot 9} = 0.8647 + j0.3181$$

$F[n] = \{1.5530, 0.8649 - j0.3181, 0.7176, 0.8647 + j0.3181\}$ ✓

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THE FOLLOWING IS THE BASIC PROGRAM FOR THE PROBLEMS 3.17.2 & 3.17.3 :

```
1010 DIM F(100),G1(100),G2(100)
1020 PRINT:PRINT:INPUT"        N=";N
1030 PRINT:PRINT:INPUT"       NT=";NT
1040 PRINT:PRINT:INPUT"       IS IT0 NECESSARY TO DEFINE f(0+)?   (Y/N)";DE$
1050 IF DE$ <> "Y" THEN DE=0 : GOTO 1080
1060 PRINT:PRINT:INPUT"      f(0+)=";F(0)
1070 DE=1
1080 FOR I=DE TO N-1
1090 GOSUB 2000
1100 NEXT I
1110 FOR I=0 TO N-1
1120 COSG=0:SING=0
1130 FOR K=0 TO N-1
1140 COSG=F(K)*COS(2*3.14159*I*K/N)+COSG
1150 SING=-F(K)*SIN(2*3.14159*I*K/N)+SING
1160 NEXT K
1170 G1(I)=COSG
1180 G2(I)=SING
1190 NEXT I
1200 CLS
1210 FOR I=0 TO N-1
1211 IF ABS(G1(I)) < .00001 THEN G1(I)=0
1212 IF ABS(G2(I)) < .00001 THEN G2(I)=0
1220 PRINT :PRINT"  F(";I;")=";G1(I);"+j*";G2(I)
1222 PRINT"  F[";I;"]=";G1(I);"+j*";G2(I)
1230 NEXT I
1240 END
2000 F(I)=F(NT*I/N)    *    F(NT*I/N) ≙ f(NT*I/N)
2010 RETURN
```

3.17.2 (a) f(t)= exp (-t)u(t) over (0,2) using the DFT and 16 sample points.

Assign a value 0.5 to the point at t=0

F(0)= 6.858655 +j* 0
F(1)= .5777969 +j*-1.970937
F(2)= .112509 +j*-1.016593
F(3)= .0190036 +j*-.6389363
F(4)=-1.390703E-02 +j*-.4289788
F(5)=-2.870311E-02 +j*-.2872521
F(6)=-3.607388E-02 +j*-.178264
F(7)=-3.961872E-02 +j*-8.565106E-02
F(8)=-4.068194E-02 +j* 0
F(9)=-3.961775E-02 +j* 8.564395E-02
F(10)=-3.607082E-02 +j* .1782571
F(11)=-2.869859E-02 +j* .287246
F(12)=-1.389933E-02 +j* .4289698
F(13)= 1.901226E-02 +j* .6389269
F(14)= .1125206 +j* 1.016577
F(15)= .5778076 +j* 1.970901

(b) Compare your result with that of magnitude of the continuous Fourier transform evaluated at the discrete frequency points

POINTS	w	|F(w)|	|Fd[n]*T|
0	0	1	.8573319
1	3.14159	.3033147	.2567355
2	6.28318	.1571768	.12785
3	9.424771	.1055111	7.990235E-02
4	12.56636	7.932676E-02	5.365052E-02
5	15.70795	6.353341E-02	3.608532E-02
6	18.84954	.0529772	2.273467E-02
7	21.99113	4.542594E-02	1.179628E-02
8	25.13272	3.975731E-02	5.085243E-03
9	28.27431	.0353457	1.179543E-02
10	31.4159	.0318149	2.273375E-02
11	34.55749	2.892518E-02	3.608451E-02
12	37.69908	2.651652E-02	5.364936E-02
13	40.84067	2.447806E-02	7.990121E-02
14	43.98226	2.273056E-02	.1278482
15	47.12385	.0212159	.2567316

✓

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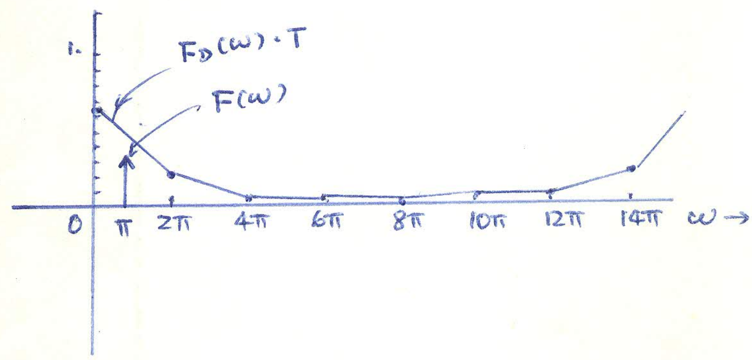
3.17.3 Compute the magnitude of the Fourier transform of the following functions using the DFT and eight samples over (0,1) :

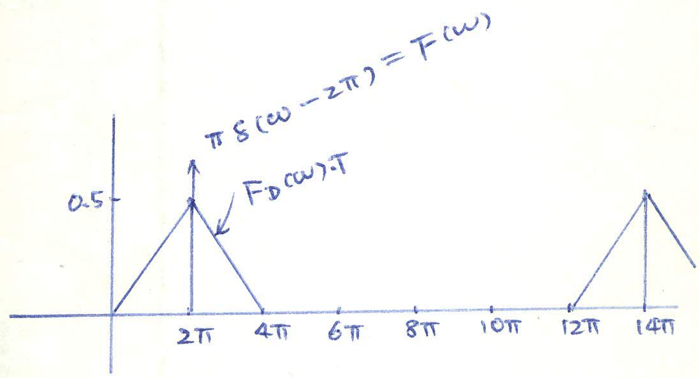
(a) sin (3.14159265 * t)

F[0]=	.6284179	$= F_D(0)$
F[1]=	.2206712	$= F_D(2\pi)$
F[2]=	5.177686E-02	$= F_D(4\pi)$
F[3]=	2.932935E-02	$= F_D(6\pi)$
F[4]=	2.486417E-02	$= F_D(8\pi)$
F[5]=	2.932921E-02	$= F_D(10\pi)$
F[6]=	5.177648E-02	$= F_D(12\pi)$
F[7]=	.2206675	$= F_D(14\pi)$



(b) sin (2*3.14159265 t)

F[0]=	0	$= F_D(0)$
F[1]=	.5000004	$= F_D(2\pi)$
F[2]=	0	$= F_D(4\pi)$
F[3]=	0	$= F_D(6\pi)$
F[4]=	0	$= F_D(8\pi)$
F[5]=	0	$= F_D(10\pi)$
F[6]=	2.339482E-06	$= F_D(12\pi)$
F[7]=	.4999989	$= F_D(14\pi)$



The significant difference between these two spectral densites is that the second one is closed to the continuous Fourier transform.

✓

50/50

COMMUNICATION SYSTEMS

HOMEWORK #5

BENMEI CHEN BENMEI CHEN

OCTORBER 8, 1986

E-E 423 COMMUNICATION SYSTEMS
Homework #5 - The DFT and Windows
Dr. R. A. Birgenheier
October 3, 1986

Due October 13, 1986

Three commonly used window functions are the rectangle, Hamming and Hanning windows. If we choose 64 sample points and 192 augmenting zeros in computing a 256 point DFT, then the three window functions are given as follows:

i) Rectangle

$$w[k] = 1 \quad , \quad |k| \leq 32$$
$$= 0 \quad , \quad |k| > 32$$

ii) Hamming

$$w[k] = 0.54 + 0.46\cos(\pi k/32) \quad ; \quad |k| \leq 32$$
$$= 0 \quad ; \quad |k| > 32$$

iii) Hanning

$$w[k] = 0.50 + 0.50\cos(\pi k/32) \quad ; \quad |k| \leq 32$$
$$= 0 \quad ; \quad |k| > 32$$

Observe that the three functions can be written as

$$w[k] = \alpha + (1-\alpha)\cos(\pi k/32) \quad ; \quad |k| \leq 32$$
$$= 0 \quad ; \quad |k| > 32$$

where

α = 1 for rectangle
= 0.54 for Hamming
= 0.5 for Hanning

These windows are referred to as "cos-on-pedestal" or "raised-cosine" functions with parameter α.

a) Use a 256 point DFT to compute the spectrum, $W_D[n]$, of the three windows. Plot $W_D[n]$ in dB for $0<n\leq$ folding frequency (N/2)

b) Compare the three magnitude spectra on the basis of (1) peak-to-null widths of the mainlobes; (2) levels of the first sidelobes relative to the level of the mainlobe and (3) the rate of decrease of the sidelobes; i.e., how many dB do the sidelobes decrease each time n is doubled? (dB per octave?)

c) Without further computation, answer the following questions regarding the spectra of the window functions.

1) Is $|W[n]|$ even or odd? Why?

2) Is W[n] real, imaginary or complex? Why?

3) What is the phase of W[n]? Why?

d) List the relative advantages and disadvantages of the three window functions considered above.

256

8

$\log_{10}$

256

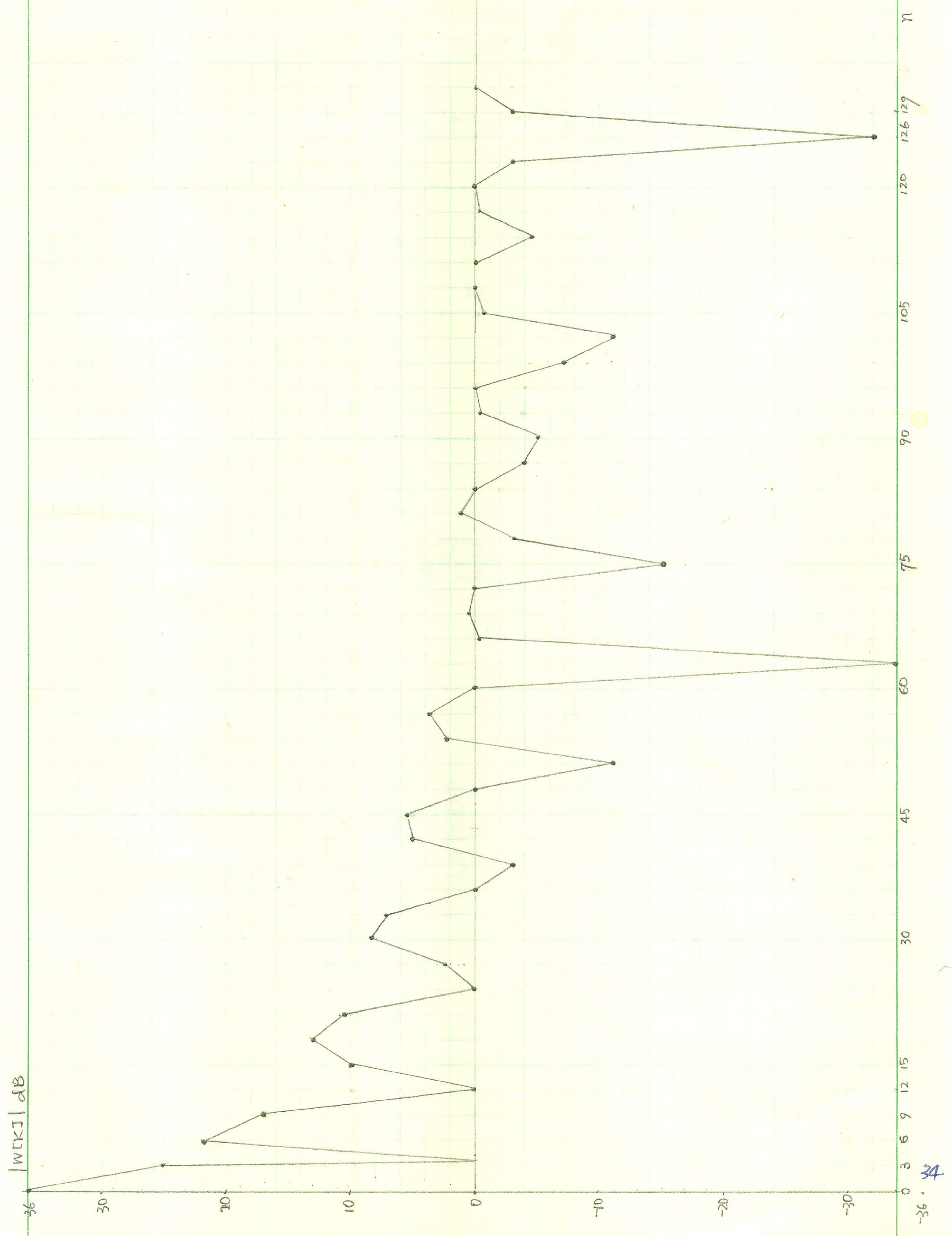
$$\log_{10} x = \frac{\log_e x}{\log_e 10}$$

THE FOLLOWING IS THE BASIC PROGRAM FOR HOMEWORK #5:

```
1000 CLS:PRINT"              YOU CAN COMPUTE THE DFT OF "
1010 PRINT:PRINT"                  1)  Rectangle   Window "
1020 PRINT:PRINT"                  2)  Hamming     Window "
1030 PRINT:PRINT"                  3)  Hanning     Window "
1040 PRINT:INPUT"            WHICH ONE ( 1,2 OR 3 )";WO
1050 IF WO=1 THEN AA=1
1060 IF WO=2 THEN AA=.54
1070 IF WO=3 THEN AA=.5
1080 DIM F(300),G1(300),G2(300), BBB(300)
1090 PRINT:PRINT:INPUT"            N=";N
1100 PRINT:PRINT:INPUT"           NT=";NT
1110 PRINT:PRINT:INPUT"           IS IT NECESSARY TO DEFINE f(0+)?   (Y/N)";DE$
1120 IF DE$ <> "Y" THEN DE=0 : GOTO 1150
1130 PRINT:PRINT:INPUT"         f(0+)=";F(0)
1140 DE=1
1150 FOR I=DE TO N-1
1160 GOSUB 1480
1170 NEXT I
1180 FOR I=0 TO N-1
1190 COSG=0:SING=0
1200 FOR K=0 TO 32
1210 COSG=F(K)*COS(2*3.14159*I*K/N)+COSG
1220 SING=-F(K)*SIN(2*3.14159*I*K/N)+SING
1230 NEXT K
1240 FOR K=224 TO 255
1250 COSG=F(K)*COS(2*3.14159*I*K/N)+COSG
1260 SING=-F(K)*SIN(2*3.14159*I*K/N)+SING
1270 NEXT K
1280 G1(I)=COSG
1290 G2(I)=SING
1300 BBB(I)=SQR(G1(I)*G1(I)+G2(I)*G2(I))
1310 BBB(I)=20*LOG(BBB(I))/LOG(10)
1320 NEXT I
1330 CLS
1340 FOR I=0 TO N-1
1350 IF (I/45)=INT(I/45) THEN FOR J=1 TO 21:LPRINT:NEXT J
1360 IF ABS(G1(I)) < .00001 THEN G1(I)=0
1370 IF ABS(G2(I)) < .00001 THEN G2(I)=0
1380 PRINT :PRINT TAB(15);"F(";I;")=";TAB(25);G1(I);TAB(42);"+j*";TAB(47);G2(I)
1390 LPRINT TAB(15);"F(";I;")=";TAB(25);G1(I);TAB(42);"+j*";TAB(47);G2(I)
1400 NEXT I
1410 PRINT :INPUT "         GO ON COMPUTING THE MAGNITUDE "; KK
1420 FOR I=0 TO N-1
1430 IF (I/45)=INT(I/45) THEN FOR J=1 TO 21:LPRINT:NEXT J
1440 PRINT :PRINT TAB(15);"I F(";I;") I  =";TAB(30);BBB(I);TAB(48);"dB"
1450 LPRINT TAB(15);"I F(";I;") I  =";TAB(30);BBB(I);TAB(48);"dB"
1460 NEXT I
1470 END
1480 IF I<=32 THEN F(I) = AA + ( 1- AA )*COS( 3.14159*I/32):GOTO 1510
1490 IF (256-I)<=32 THEN F(I) = AA+(1-AA)*COS(3.14159*(256-I)/32) : GOTO 1510
1500 F(I)=0
1510 RETURN
```

Homework #5 Communication Systems Benmei Chen

THE MAGNITUDE OF THE DFT OF RECTANGLE WINDOW: FOR $0 < n \leq N/2$



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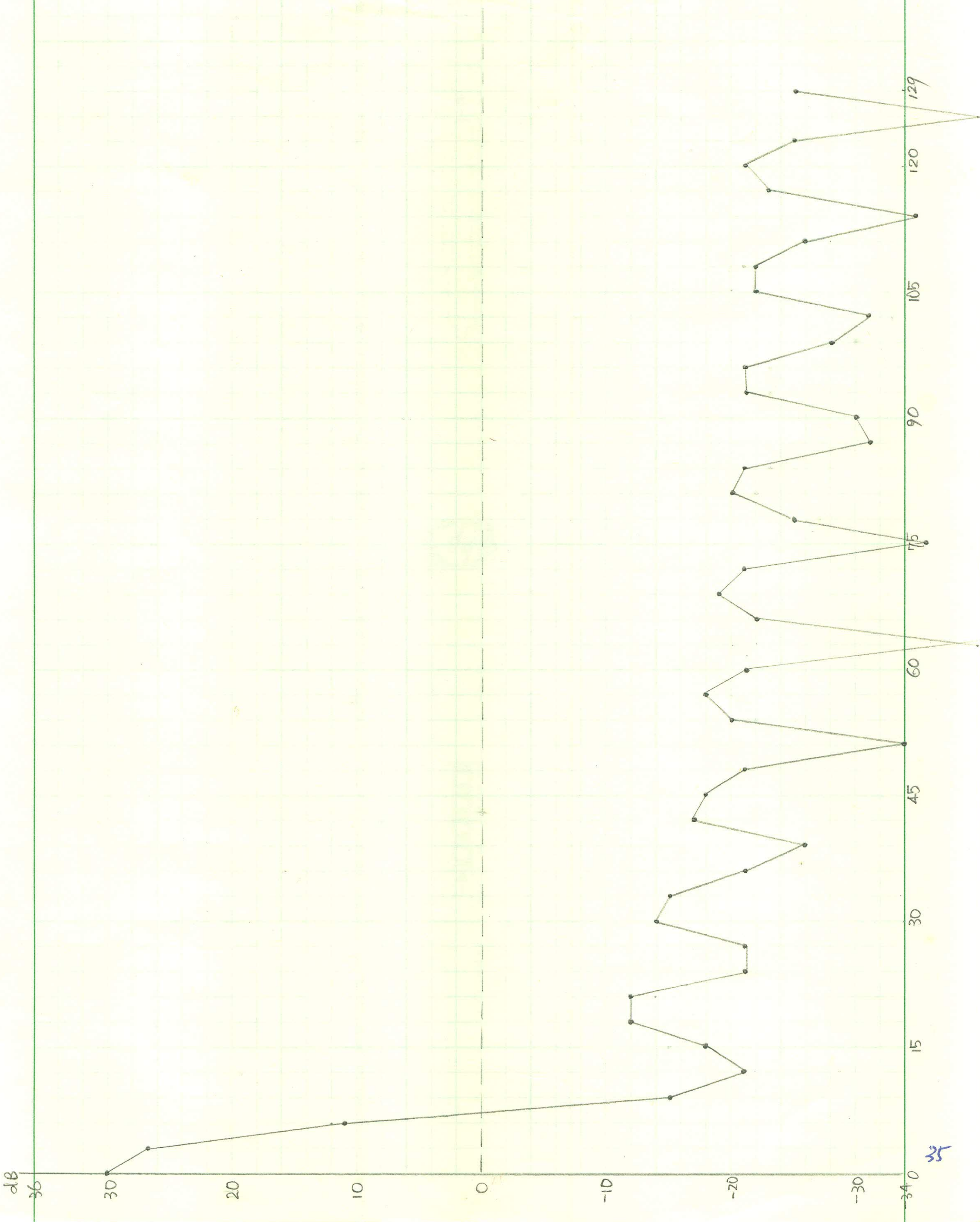
THE MAGNITUDE OF THE DFT OF HAMMING WINDOW:



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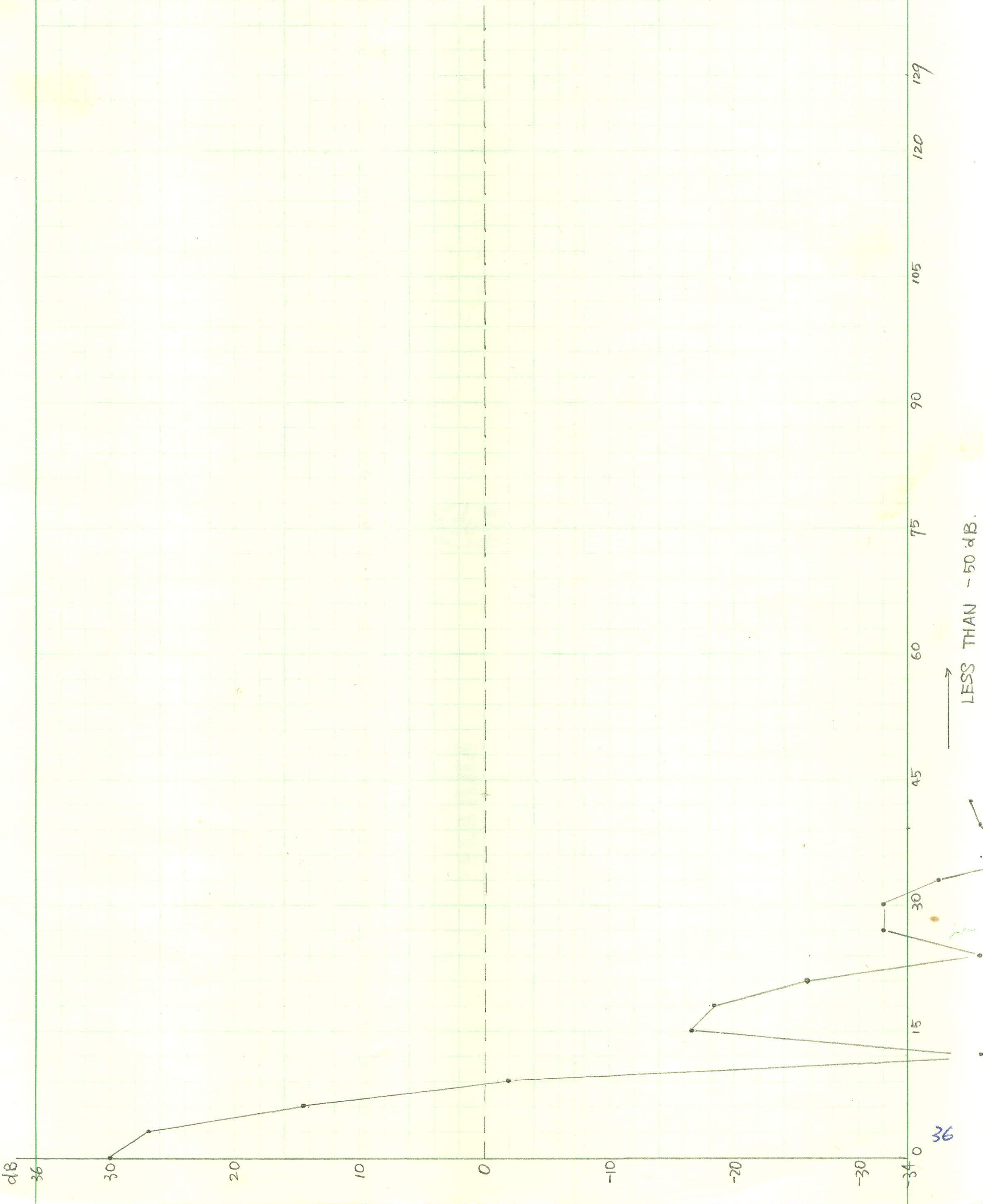
The magnitude of the DFT of Hanning window :



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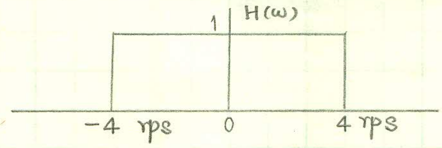
HOMEWORK FOR CHAPTER FOUR

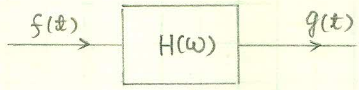
BENMEI CHEN

NOV. 10, 1986

54/60

4.1.1





$G(\omega) = F(\omega)\cdot H(\omega)$ $\qquad E_{out} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2\, d\omega$

a) $f(t) = \delta(t)$; $F(\omega) = 1$

$$E_{out} = \frac{1}{2\pi}\int_{-4}^{4} 1^2\, d\omega = \frac{8}{2\pi} = \frac{4}{\pi} \text{ joules}$$ ✓

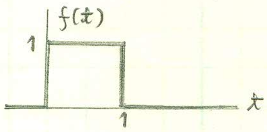
b) $f(t) = \exp(-4t)\,u(t)$; $F(\omega) = 1/(4+j\omega)$

$$E_{out} = \frac{1}{2\pi}\int_{-4}^{4} 1/(16+\omega^2)\, d\omega = \frac{1}{8\pi}\int_{-4}^{4} 1/(1+(\tfrac{\omega}{4})^2)\, d(\tfrac{\omega}{4})$$

$$= \frac{1}{8\pi}\int_{-1}^{1} 1/(1+u^2)\, du = \frac{1}{8\pi}\arctan u \Big|_{-1}^{1} = \frac{1}{8\pi}(\tfrac{\pi}{4}+\tfrac{\pi}{4}) = \frac{1}{16} \text{ joules}$$ ✓

10

4.1.2



$$F(\omega) = e^{-j\omega_0\cdot\frac{1}{2}}\cdot Sa(\omega/2)$$

a) $f_c = 0.1$ Hz , $\omega_c = 2\pi\times 0.1 = 0.2\pi$

$$E_{out} = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} |F(\omega)|^2\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} Sa^2(\omega/2)\, d\omega = \frac{1}{\pi}\int_0^{\omega_c} Sa^2(\omega/2)\, d\omega$$

$$= \frac{1}{\pi}\int_0^{\omega_c} Sa^2(\omega/2)\, d\omega = \frac{1}{\pi}\int_0^{\omega_c} \frac{2^2\sin^2(\omega/2)}{\omega^2}\, d\omega$$

$$= \frac{4}{\pi}\int_0^{\omega_c} \sin^2(\omega/2)/\omega^2\, d\omega$$

$$= \frac{2^2}{\pi}\left[\frac{1}{2}\int_0^{\omega_c}\frac{\sin\omega}{\omega}\, d\omega - \frac{\sin^2\omega/2}{\omega}\Big|_0^{\omega_c}\right]$$

$$= \frac{2^2}{\pi}\left[\frac{1}{2}\cdot Si(\omega_c) - \sin^2 \omega_c/2\right]$$

$\omega_c = 0.2\pi$ rps

$$E_{out} = \frac{2^2}{\pi}\left[\frac{1}{2}Si(0.628) - \sin^2 0.314\right] = \frac{2}{\pi}\left[Si(0.628) - 0.1908\right] = 0.2697 \text{ joules}$$ ok.

b) $f_c = 10$ Hz , $\omega_c = 2\pi f_c = 20\pi = 62.832$ rps

$$E_{out} = \frac{2^2}{\pi}\left[\frac{1}{2}\cdot Si(62.832) - \sin^2 31.416\right]$$

$$= \frac{2^2}{\pi}\left[\frac{1}{2}Si(62.832)\right] = \frac{2}{\pi}Si(62.832) \doteq 1.0186 \text{ joules}$$

ok

* $Si(0.628) = 0.6144$ $\qquad Si(62.832) \doteq 1.6$

42

10

4.1.3 THE TWO-SIDE EXPONENTIAL VOLTAGE $f(t)=10e^{-|t|}$ is developed across a 50-Ω resistor. a) Calculate the total energy dissipated in the resistor.

b) What fraction of this energy is in the freq. range 0 to 1 rps?

Solution: a) $f(t)=10e^{-|t|}$

$$F(\omega)=20/(1+\omega^2)$$

$$E_R=\frac{1}{50}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega$$

$$=\frac{1}{100\pi}\int_{-\infty}^{\infty}400/(1+\omega^2)^2\,d\omega$$

$$=\frac{400}{100\pi}\left[\frac{\omega}{2(1+\omega^2)}+\frac{1}{2}\tan^{-1}(\omega)\right]\Big|_{-\infty}^{\infty}$$

$$=\frac{400}{200}=2 \text{ joules}$$

4.1.3 wasn't assigned

b) $$E_R'=\frac{400}{100\pi}\left[\frac{\omega}{2(1+\omega^2)}+\frac{1}{2}\tan^{-1}(\omega)\right]\Big|_0^1$$

$$=\frac{400}{100\pi}\left[\frac{1}{4}+\frac{1}{8}\pi\right]=0.8183 \text{ joules}$$

-5

$$E_R'/E_R\times 100\%=40.92\%$$

4.2.2. A given voltage signal is $f(t)=4\cos 20\pi t+2\cos 30\pi t$ aross 1 Ω.

a) Determine the power spectral density of $f(t)$

b) Sketch $S_f(\omega)$

c) Calculate the mean (average) power, both in the time domain and in the frequency domain, that is dissipated by $f(t)$ aross the one-ohm resistor.

Solution: a) $f(t)=4\cos 20\pi t+2\cos 30\pi t$

$$=2(e^{j20\pi t}+e^{-j20\pi t})+(e^{j30\pi t}+e^{-j30\pi t})$$

$$=2\cdot e^{j2\cdot(10\pi)t}+2e^{-j2\cdot(10\pi)t}+e^{j3\cdot(10\pi)t}+e^{-j3\cdot(10\pi)t}$$

$\omega_0=10\pi$, $F_2=2$, $F_{-2}=2$, $F_3=1$, $F_{-1}=1$

According to the Equation (4.20)

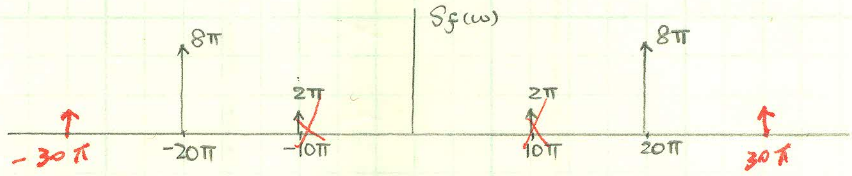
4.2.2 (CONT.) $S_f(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0)$

$$= 2\pi \left[4\delta(\omega - 2\omega_0) + 4\delta(\omega + 2\omega_0) + \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right]$$

$$= 2\pi \left[4\delta(\omega - 20\pi) + 4\delta(\omega + 20\pi) + \delta(\omega - 30\pi) + \delta(\omega + 30\pi) \right]$$

✓

b) $S_f(\omega) = 2\pi [4\delta(\omega - 20\pi) + 4\delta(\omega + 20\pi) + \delta(\omega - 10\pi) + \delta(\omega + 10\pi)]$



c) mean-power, $T = 2\pi/\omega_0 = \frac{1}{5}$ sec.

Time domain: $P = 5 \int_{-0.1}^{0.1} |f(t)|^2 dt$

$$= 5 \int_{-0.1}^{0.1} [4\cos 20\pi t + 2\cos 30\pi t]^2 dt$$

$$= 5 \cdot \int_{-0.1}^{0.1} [16\cos^2 20\pi t + 4\cos^2 30\pi t + 16\cos 20\pi t \cos 30\pi t] dt$$

$$= 5 \cdot \left[16\left(\frac{t}{2} + \frac{\sin 40\pi t}{80\pi}\right) + 4\left(\frac{t}{2} + \frac{\sin 60\pi t}{120\pi}\right) + 16\left(\frac{\sin 10\pi t}{20\pi} + \frac{\sin 50\pi t}{100\pi}\right)\right] \Big|_{-0.1}^{0.1}$$

$$= 10 \ (W)$$

Frequency domain: $P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_f(\omega) d\omega$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi [4\delta(\omega - 20\pi) + 4\delta(\omega + 20\pi) + \delta(\omega - 10\pi) + \delta(\omega + 10\pi)] d\omega$$

$$= 4 + 4 + 1 + 1 = 10 \ W$$ ✓

4.2.3. Solution: The Fourier siries of symmetric square.

$F_n = Sa(n\pi/2)$ FOR $n \neq 0$; $F_n = 0$ FOR $n = 0$

$\therefore \ S_f(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0)$, $\omega_0 = \frac{2\pi}{T}$

$$|F_n|^2 = \left| \frac{\sin(n\pi/2)}{n\pi/2} \right|^2 = \frac{4\sin^2(n\pi/2)}{(n\pi)^2} = \begin{cases} 0 & \text{FOR } n = 2k \\ \frac{4}{\pi^2} \cdot \frac{1}{n^2} & \text{FOR } n = 2k+1 \end{cases}$$

$$\therefore \ S_f(\omega) = \frac{8}{\pi} \Big[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) + \frac{1}{3^2}\delta(\omega + 3\omega_0) + \frac{1}{3^2}\delta(\omega - 3\omega_0)$$

$$+ \cdots + \frac{1}{(2k+1)^2}\delta(\omega + (2k+1)\omega_0) + \frac{1}{(2k+1)^2}\delta(\omega - (2k+1)\omega_0) + \cdots \Big]$$

$$= \frac{8}{\pi} \sum_{\substack{n=-\infty \\ n = \text{odd}}}^{\infty} \left[\frac{1}{n^2} \delta(\omega - n\omega_0) \right]$$

4.2.3 (CONT.) a) $P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_f(\omega)\, d\omega$

$$= \frac{1}{2\pi} \cdot \frac{8}{\pi} \cdot \int_{-\infty}^{\infty} \left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0) + \frac{1}{9}\delta(\omega-3\omega_0) + \frac{1}{9}\delta(\omega+3\omega_0) + \cdots \right] d\omega$$

$$= \frac{4}{\pi^2} \left[2 + \frac{2}{3^2} + \frac{2}{5^2} + \cdots + \frac{2}{(2k+1)^2} + \cdots \right]$$

$$= \frac{8}{\pi^2} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2k+1)^2} + \cdots \right]$$

$$= \frac{8}{\pi^2} \cdot \frac{\pi^2}{8} = 1 \quad W$$

✓

$$S_{out}(\omega) = S_f(\omega)\, |H(\omega)|^2$$

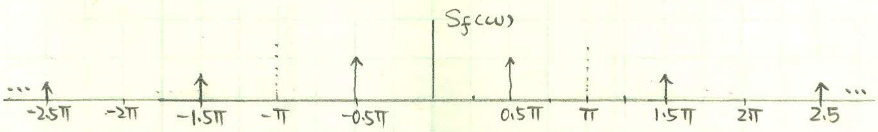
$$= \frac{8}{\pi} \cdot \sum_{\substack{n=-\infty \\ n=odd}}^{\infty} \left[\frac{1}{n^2} \delta(\omega - n\omega_0) \right] \cdot |H(\omega)|^2$$

$$H(\omega) = \begin{cases} (1+\cos\omega)/2 & |\omega| < \pi \\ 0 & \text{eleswhere} \end{cases}$$

$$|H(\omega)|^2 = \begin{cases} \cos^4 \frac{\omega}{2} & |\omega| < \pi \\ 0 & \text{eleswhere} \end{cases}$$

$$S_{out}(\omega) = \begin{cases} \frac{8}{\pi} \sum\limits_{\substack{n=-\infty \\ n=odd}}^{\infty} \frac{1}{n^2} \delta(\omega - n\omega_0) \cdot \cos^4 \frac{\omega}{2} , & |\omega| < \pi \\ 0 , & \text{eleswhere} \end{cases}$$

b) $T = 4$ sec. $\qquad \omega_0 = \frac{2\pi}{T} = 0.5\pi$



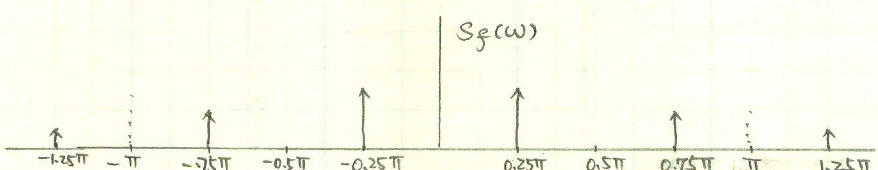
$$S_{out}(\omega) = \frac{8}{\pi} \cos^4(-0.25\pi)\, \delta(\omega + 0.5\pi) + \frac{8}{\pi} \cos^4(0.25\pi)\, \delta(\omega - 0.5\pi)$$

$$P_{out} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{out}(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{8}{\pi} \cos^4(-0.25\pi) + \frac{8}{\pi} \cos^4(0.25\pi) \right] = 0.2026 \quad W$$

$P_{out} / P_{in} \times 100\% = 20.26\ \%$ ✓

c) $T = 8$ sec. $\qquad \omega_0 = \frac{2\pi}{T} = 0.25\pi$



4.2.3 (CONT.) $S_{out}(\omega) = \frac{8}{\pi}[\cos^4(\frac{\pi}{8})\delta(\omega-0.25\pi) + \cos^4(\frac{\pi}{8})\delta(\omega+0.25\pi)$

$+ \frac{1}{9}\cos^4(\frac{3\pi}{8})\delta(\omega-0.75\pi) + \frac{1}{9}\cos^4(\frac{3\pi}{8})\delta(\omega+0.75\pi)]$

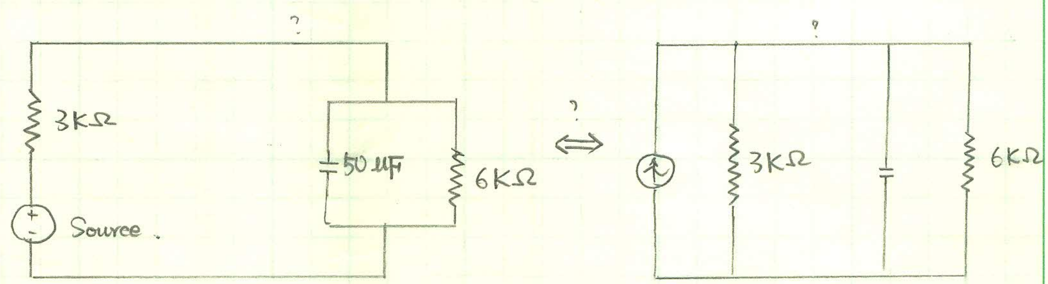
$$P_{out} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{out}(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\cdot\frac{8}{\pi}\left[\cos^4(\tfrac{\pi}{8}) + \cos^4(\tfrac{\pi}{8}) + \tfrac{1}{9}\cos^4(\tfrac{3\pi}{8}) + \tfrac{1}{9}\cos^4(\tfrac{3\pi}{8})\right]$$

$$= 0.5925 \text{ W}$$

$P_{out}/P_{in} \times 100\% = 59.25\%$ ✓

4.2.4.



According to the circuit theorem, the mean power aross the capacitor is zero. ∴

$$\begin{cases} P_{soure} = P_{3K\Omega} + P_{6K\Omega} \\ P_{3K\Omega} = 2P_{6K\Omega} \end{cases}$$

$$\therefore P_{6K\Omega} = \frac{1}{3}P_{soure} = \frac{1}{6\pi}\int_{-\infty}^{\infty} S_{soure}(\omega)\,d\omega$$

a) $S_{source}(\omega) = 10^{-2}$ W/Hz

$$P_{6K\Omega} = \frac{1}{3}P_{source} = \frac{1}{6\pi}\int_{-\infty}^{\infty} 2\pi\times10^{-2}\,d\omega = \infty$$ ← Physically Unrealizable

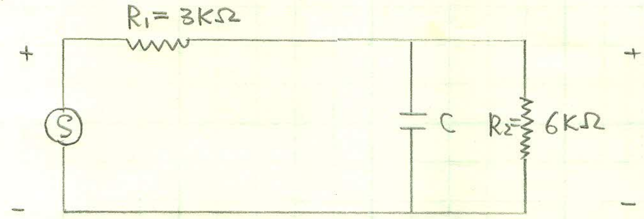
b) $S_{source}(\omega) = \delta(\omega+10) + \delta(\omega-10)$ W/Hz

$$P_{6K\Omega} = \frac{1}{6\pi}\int_{-\infty}^{\infty}[\delta(\omega+10)+\delta(\omega-10)]\,d\omega$$

$$= \frac{2}{6\pi} = 0.106 \text{ W}.$$

Go on next page for more.

4.2.4.



check carefully

$$H(\omega) = \frac{jR_2}{R_1R_2C\omega - j(R_1+R_2)}$$

$$|H(\omega)|^2 = \frac{R_2^2}{R_1^2 \cdot R_2^2 C^2 \omega^2 + (R_1+R_2)^2}$$

$$= \frac{36\times10^6}{9\times10^6 \times 36\times10^6 \times 50^2\times10^{-12}\omega^2 + 9^2\times10^6}$$

$$= \frac{36}{0.81\,\omega^2 + 81} = \frac{4}{9}\cdot\frac{1}{1+(0.1\omega)^2}$$

$$S_g(\omega) = S_f(\omega)\cdot|H(\omega)|^2 = \frac{4}{9}\cdot S_f(\omega)\cdot\frac{1}{1+(0.1\omega)^2}$$

a) $S_f(\omega) = S_{source}(\omega) = 10^{-2}$ W/Hz

$$P_{6\Omega} = \frac{1}{2\pi R_2}\int_{-\infty}^{\infty} S_g(\omega)\,d\omega = \frac{4\times10^{-2}}{9\times0.1}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R_2}\cdot\frac{1}{1+(0.1\omega)^2}\,d(0.1\omega) = 3.70\ \mu W$$

b) $S_f(\omega) = \delta(\omega+10) + \delta(\omega-10)$

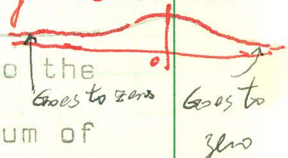
$$P_{6\Omega} = \frac{1}{2\pi R_2}\cdot\frac{4}{9}\int_{-\infty}^{\infty}\frac{1}{1+(0.1\omega)^2}\left[\delta(\omega+10)+\delta(\omega-10)\right]d\omega$$

$$= \frac{1}{2\pi\times6\times10^3}\cdot\frac{4}{9}\cdot\left[\frac{1}{2}+\frac{1}{2}\right] = 11.79\ \mu W$$

Dear Our ~~Great~~ Grader:

I have a question in problem 4.2.4. According to the results above, The mean power aross the 6-kohm resistor is finity and even is very small. But the input mean power is infinity (we can find the input mean power in the following way, with Ssource(w)=10^{-2}W/Hz)

$$P\ in = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ssource(w)\cdot dw = \frac{10^{-2}}{2\pi}\cdot\int_{-\infty}^{\infty} d\omega = \infty$$ ← physically Unrealizable



Now, we regard this problem as a circuit one. According to the circuit theorem, The mean-power of source equels to the sum of the mean-powers which absorbed by the elements in the circuit. The capacitor doesn't consume any average-power. So, where is the left power gone?

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THks for the Question

50/50

Excellent!

E-E 423 COMMUNICATION SYSTEMS

Homework #7 Signal-to-Noise Ratio and Autocorrelation

Dr. R., A. Birgenheier October 17, 1986

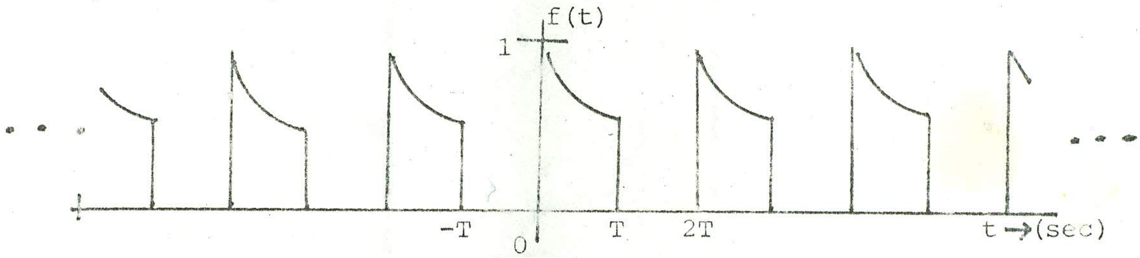
Due October 22, 1986 Bonmei Chen Oct 21. 1986

1. (4.3.2 in Stremler)

2. Consider a periodic waveform, f(t), for which the pulse

$$f_T(t) = e^{-at}[u(t) - u(t-T)]$$

is repeated every 2T seconds



(a) Find (and sketch for a=1, T=1) the autocorrelation function, $R_f(\tau)$. (Observe that $R_f(\tau)$ is periodic)

(b) Find the Fourier transform of $R_f(\tau)$ to obtain the power-spectral-density, $S_f(\omega)$. (Hint: First find the Fourier series of $R_f(\tau)$ and then obtain the Fourier transform of the Fourier series.)

(c) Find the Fourier series of f(t) to obtain the power-spectral density from the relationship

$$S_f(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_o)$$

(Check the results of (b) and (c); if they do not agree, find your error.)

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(d) Plot $S_f(\omega)$ for $|\omega| < 5\omega_o$ with a=T=1.

(e) Find the inverse Fourier transform of the expression for $S_f(\omega)$ given in (c). This provides an expression for $R_f(\tau)$ in terms of the Fourier coefficients of the periodic function.

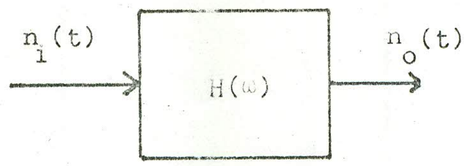
(f) Repeat part (a) using the expression for $R_f(\tau)$ derived in part (e).

3. A white noise source with two-sided power-spectral density

$$S_{ni}(\omega) = \eta \text{ Watts/Hz}$$

is connected to a low-pass RC filter with transfer function

$$H(\omega) = \frac{1}{1 + j\omega RC}$$



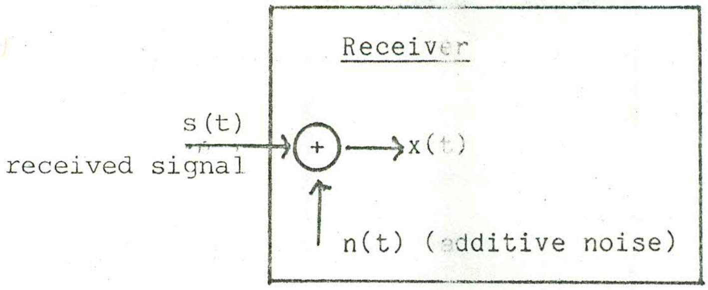
(a) Determine the autocorrelation functions of the noise at the input and output of the filter in terms of R, C and η.

(b) Determine the power-spectral density, $S_{no}(\omega)$, of the output noise.

(c) Determine the average power (mean-squared value) of the output noise.

4. (4.5.3 in Stemler)

A sinusoidal signal is transmitted. On reception, the signal plus additive noise is present.



The composite signal at the receiver is

$$x(t) = s(t) + n(t)$$

with autocorrelation function

$$R_x(\tau) = R_s(\tau) + R_n(\tau)$$

where

$$R_s(\tau) = a \cos \omega_o \tau \qquad \text{(signal)}$$

and

$$R_n(\tau) = b \exp(-c|\tau|) \qquad \text{(noise)}$$

(a) Directly from $R_s(\tau)$ and $R_n(\tau)$, determine the mean-square signal-to-noise ratio (SNR) of $x(t)$.

(b) Determine (and sketch on the same graph) the power-spectral densities of the signal and noise, $S_s(\omega)$ and $S_n(\omega)$.

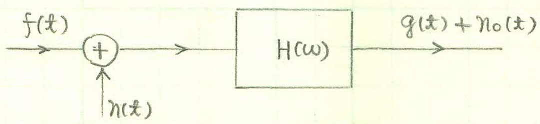
(c) Integrate the spectra found in (b) to obtain the SNR of $x(t)$, and, thereby, verify the results of part (a).

1. 4.3.2. $H(\omega) = 1/(1+j\omega RC)$

$S_n(\omega) = 0.1\ W/Hz$

$f(t) = 10\cos 30\pi t = 5\,(e^{j30\pi t} + e^{-j30\pi t})$

$S_f(\omega) = 50\pi\ (\delta(\omega - 30\pi) + \delta(\omega + 30\pi))$ $\quad S_f(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0)$



(a) $S_{n_o}(\omega) = S_n(\omega) \cdot |H(\omega)|^2$

$= 0.1 \cdot \dfrac{1}{1 + \omega^2 \cdot R^2 \cdot C^2}$

$\overline{n_o^2(t)} = P_{n_o} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} S_{n_o}(\omega)\,d\omega$

$= \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \dfrac{0.1}{1+\omega^2 \cdot R^2 C^2}\,d\omega$

$= \dfrac{0.1}{2\pi RC}\int_{-\infty}^{\infty} \dfrac{1}{1+(\omega RC)^2}\,d(\omega RC)$

$= 0.1/2\pi RC \cdot \arctan(\omega RC)\Big|_{-\infty}^{\infty}$

$= \dfrac{0.1}{2RC} = \dfrac{0.1 \times 30\pi}{2} = 4.7124$

$\sqrt{\overline{n_o^2(t)}} = 2.1708\ V$ ✓

(b) $S_g(\omega) = S_f(\omega) \cdot |H(\omega)|^2 = 50\pi\,[\delta(\omega - 30\pi) + \delta(\omega + 30\pi)] \cdot \dfrac{1}{1+\omega^2 R^2 C^2}$

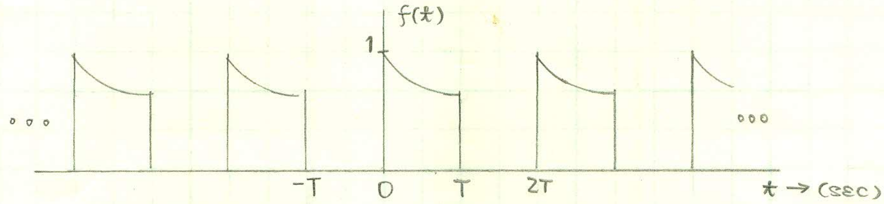
$P_{out,\,signal} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} S_g(\omega)\,d\omega$

$= \dfrac{50\pi}{2\pi}\left[\dfrac{1}{1+(30\pi RC)^2} + \dfrac{1}{1+(-30\pi RC)^2}\right] = 25\ W$

$P_{out,\,noise} = 4.7124$

$P_{out,\,signal} / P_{out,\,noise} = 25/4.7124 = 5.31$ (7.25 dB) ✓

2. $f_T(t) = e^{-at}\,[u(t) - u(t-T)]$



(a) $R_{f_T}(\tau) = \frac{1}{2T}\int_{-T}^{T} f_T(t)\, f_T(t+\tau)\, dt$

$= \frac{1}{2T}\int_{-T}^{0} f_T(t) f_T(t+\tau)\, dt + \frac{1}{2T}\int_{0}^{T} f_T(t) f_T(t+\tau)\, dt$

$= \frac{1}{2T}\int_{0}^{T} e^{-at}\cdot f_T(t+\tau)\, dt$

$= \frac{1}{2T}\int_{0}^{T} e^{-a(t+\tau)}\cdot f_T(t+\tau)\cdot e^{a\tau}\cdot d(t+\tau)$

$= \frac{e^{a\tau}}{2T}\int_{\tau}^{T+\tau} e^{-at'} f_T(t')\, dt'$ $\quad 0 \le \tau \le T$

$= e^{a\tau}/2T \cdot \int_{\tau}^{T} e^{-at'}\cdot e^{-at'}\, dt'$

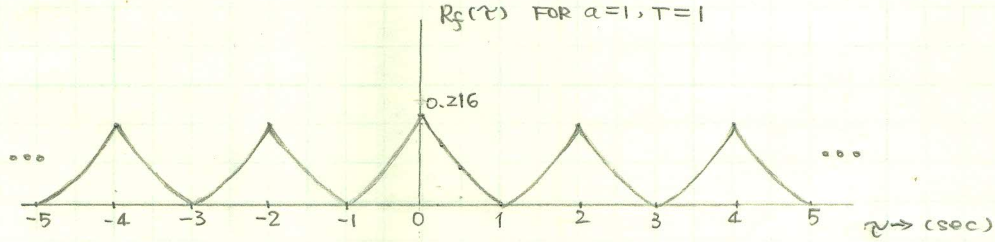
$= e^{a\tau}/2T \cdot \left(-\frac{1}{2a} e^{-2at'}\Big|_{\tau}^{T}\right)$

$= \frac{e^{a\tau}}{4aT}\cdot\left(e^{-2a\tau} - e^{-2aT}\right)$ $\quad 0 \le \tau \le T$

$R_{f_T}(\tau) = \frac{e^{a|\tau|}}{4aT}\left(e^{-2a|\tau|} - e^{-2aT}\right)$ $\quad -T \le \tau \le T$

FOR $a=1$, $T=1$

$R_{f_T}(\tau) = \frac{1}{4} e^{|\tau|}\left(e^{-2|\tau|} - e^{-2}\right)$ ✓



✓

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(b) $R_f(\tau) = \sum_{n=-\infty}^{\infty} R_n \cdot e^{jn\omega_0\tau}$, WHERE $\omega_0 = 2\pi/2T = \pi/T$

$$R_n = \frac{1}{2T}\int_{-T}^{T} R_{f_T}(\tau)e^{-jn\omega_0\tau}\,d\tau$$

$$= \frac{1}{2T}\int_{-T}^{0} \cdot \frac{e^{-a\tau}}{4aT}(e^{2a\tau} - e^{-2aT})e^{-jn\omega_0\tau}d\tau + \frac{1}{2T}\int_0^T \frac{e^{a\tau}}{4aT}(e^{-2a\tau} - e^{-2aT})e^{-jn\omega_0\tau}d\tau$$

$$= \frac{e^{-aT}}{8aT^2}\cdot\left[\int_{-T}^{0}(e^{a(\tau+T)} - e^{-a(\tau+T)})e^{-jn\omega_0\tau}d\tau + \int_0^T(e^{-a(\tau-T)} - e^{a(\tau-T)})e^{-jn\omega_0\tau}d\tau\right]$$

$$= \frac{e^{-aT}}{8aT^2}\cdot\left[e^{aT}\cdot\frac{e^{(a-jn\omega_0)\tau}}{a-jn\omega_0}\Big|_{-T}^{0} + e^{-aT}\frac{e^{-(a+jn\omega_0)\tau}}{a+jn\omega_0}\Big|_{-T}^{0} - e^{aT}\frac{e^{-(a+j\omega_0 n)\tau}}{a+jn\omega_0}\Big|_0^T - e^{-aT}\frac{e^{(a-jn\omega_0)\tau}}{a-jn\omega_0}\Big|_0^T\right]$$

$$= \frac{e^{-aT}}{8aT^2}\cdot\left[\frac{e^{aT}}{a-jn\omega_0} - \frac{e^{jn\omega_0T}}{a-jn\omega_0} + \frac{e^{-aT}}{a+jn\omega_0} - \frac{e^{jn\omega_0T}}{a+jn\omega_0} - \frac{e^{-jn\omega_0T}}{a+jn\omega_0} + \frac{e^{aT}}{a+jn\omega_0} - \frac{e^{-jn\omega_0T}}{a-jn\omega_0} + \frac{e^{-aT}}{a-jn\omega_0}\right]$$

$$= \frac{e^{-aT}}{8aT^2}\cdot\left[\frac{2a(e^{aT}+e^{-aT})}{a^2+n^2\omega_0^2} - \frac{2ae^{jn\omega_0T}}{a^2+n^2\omega_0^2} - \frac{2a\cdot e^{-jn\omega_0T}}{a^2+n^2\omega_0^2}\right]$$

$$= \frac{e^{-aT}}{4T^2}\cdot\left[\frac{e^{aT}+e^{-aT}-2\cos(n\omega_0T)}{a^2+n^2\omega_0^2}\right] = \frac{1}{4T^2}\left[1-2\cos n\omega_0T\cdot e^{-aT} + e^{-2aT}\right]/(a^2+n^2\omega_0^2)$$

$$R_n = \frac{1}{4T^2}\cdot\left[1-2\cos n\omega_0T\cdot e^{-aT}+e^{-2aT}\right]/(a^2+n^2\omega_0^2)$$

$$R_f(\tau) = \sum_{n=-\infty}^{\infty} R_n\cdot e^{jn\omega_0\tau}$$

$$S_f(\omega) = \mathcal{F}\{R_f(\tau)\} = 2\pi\sum_{n=-\infty}^{\infty} R_n\cdot\delta(\omega-n\omega_0)$$

$$= 2\pi\sum_{n=-\infty}^{\infty}\frac{1-2e^{-aT}\cos n\omega_0T + e^{-2aT}}{4T^2\,(a^2+n^2\omega_0^2)}\cdot\delta(\omega-n\omega_0)$$ ✓

(c) $f(t) = \sum_{n=-\infty}^{\infty} F_n\cdot e^{jn\omega_0 t}$, WHERE, $\omega_0 = 2\pi/2T = \pi/T$

$$F_n = \frac{1}{2T}\int_{-T}^{T} f_T(t)\,e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{2T}\int_0^T e^{-at}\cdot e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{2T}\cdot\left(-\frac{1}{a+jn\omega_0}e^{-(a+jn\omega_0)t}\Big|_0^T\right)$$

$$= \frac{1}{2T}\left(\frac{1-e^{-(a+jn\omega_0)T}}{a+jn\omega_0}\right) = \frac{1}{2T}\left(\frac{1-e^{-aT}\cdot\cos(n\omega_0T)+je^{-aT}\sin n\omega_0T}{a+jn\omega_0}\right)$$

$$|F_n|^2 = \frac{1}{4T^2}\left(\frac{1-2e^{-aT}\cos(n\omega_0T)+e^{-2aT}}{a^2+n^2\omega_0^2}\right)$$

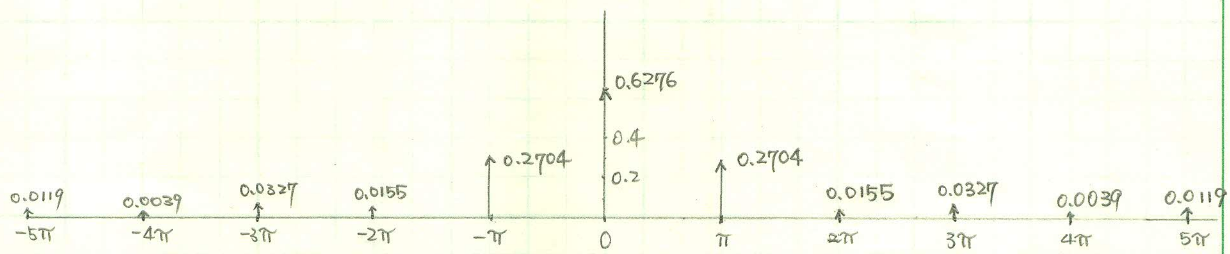
$$\therefore S_f(\omega) = 2\pi\cdot\sum_{n=-\infty}^{\infty}|F_n|^2\cdot\delta(\omega-n\omega_0)$$

$$= 2\pi\cdot\sum_{n=-\infty}^{\infty}\frac{1-2e^{-aT}\cos n\omega_0T+e^{-2aT}}{4T^2\,(a^2+n^2\omega_0^2)}\cdot\delta(\omega-n\omega_0)$$ ✓

THE ANSWERS FOR (b) AND (c) ARE JUST THE SAME. Verified.

(d) FOR $a = T = 1$. $\omega_0 = \pi$

$$S_f(\omega) = 2\pi \cdot \sum_{n=-\infty}^{\infty} \frac{1+e^{-2} - 2e^{-1}\cos n\pi}{4(1+\pi^2 n^2)} \delta(\omega - n\omega_0)$$



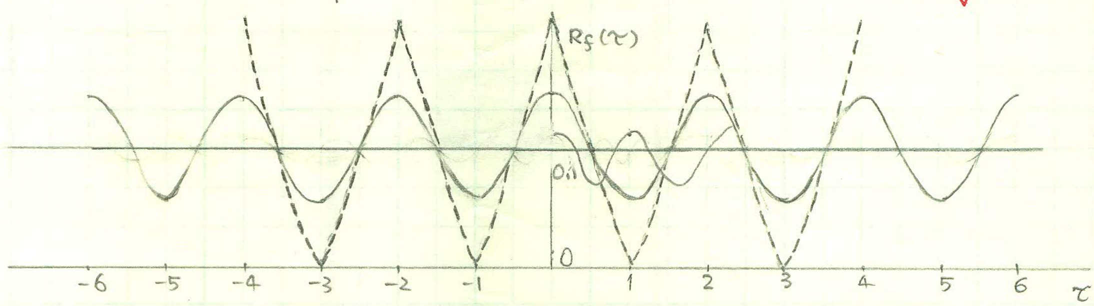
(e) Because $S_f(\omega)$ given in (c) is just the same as it in (b), so

$$R_f(\tau) = \sum_{n=-\infty}^{\infty} \frac{1-2e^{-aT}\cos n\omega_0 T + e^{-2aT}}{4T^2(a^2+n^2\omega_0^2)} \cdot e^{jn\omega_0\tau}, \quad \text{WHERE } \omega_0 = \pi/T$$

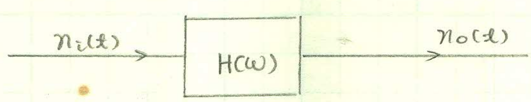
(f) FOR $a=1$ and $T=1$

$$R_f(\tau) = \sum_{n=-\infty}^{\infty} \frac{1+e^{-2}-2e^{-1}\cos n\pi}{4(1+n^2\pi^2)} \cdot e^{jn\pi\tau}$$

$$= \frac{1+e^{-2}-2e^{-1}}{4} + \sum_{n=1}^{\infty} \frac{1+e^{-2}-2e^{-1}\cos n\pi}{2(1+n^2\pi^2)} \cos n\pi\cdot\tau$$



3. $S_{ni}(\omega) = \eta$ Watts/Hz, $H(\omega) = \frac{1}{1+j\omega RC}$



a) $R_{ni}(\tau) = \mathcal{F}^{-1}\{S_{ni}(\omega)\} = \mathcal{F}^{-1}\{\eta\} = \eta\,\delta(\tau)$

$R_{no}(\tau) = \mathcal{F}^{-1}\{S_{no}(\omega)\}$

$$S_{no}(\omega) = S_{in}(\omega)\cdot|H(\omega)|^2 = \eta/(1+\omega^2 R^2C^2) = \frac{\eta}{2RC}\cdot\left(\frac{\frac{2}{RC}}{(\frac{1}{RC})^2+\omega^2}\right)$$

SO,

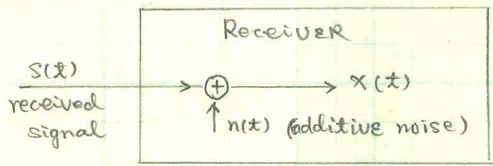
$$R_{no}(\tau) = \frac{\eta}{2RC} e^{-\frac{1}{RC}|\tau|}$$

(b) $S_{no}(\omega) = \eta / (1+\omega^2 \cdot R^2C^2)$ ✓

(c) $P = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta/(1+\omega^2R^2C^2)\, d\omega$

$= \frac{\eta}{\pi}\int_0^{\infty} 1/(1+\omega^2R^2C^2)\, d\omega = \frac{\eta}{\pi RC}\int_0^{\infty} 1/(1+(\omega RC)^2)\, d(\omega RC) = \frac{\eta}{2RC}$ ✓

4. (4.5.3)



$x(t) = s(t) + n(t)$

$R_x(\tau) = R_s(\tau) + R_n(\tau)$

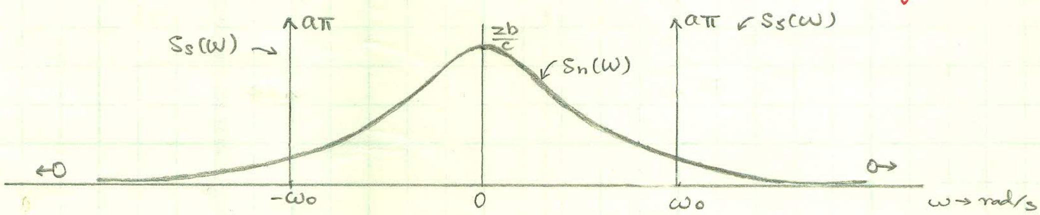
$R_s(\tau) = a\cos\omega_0\tau$, $R_n(\tau) = b\exp(-c|\tau|)$

(a) $\overline{s^2(t)} = R_s(0) = a$, $\overline{n^2(t)} = R_n(0) = b$.

SNR = a/b ✓

(b) $S_s(\omega) = \mathcal{F}\cdot\{R_s(\tau)\} = a\cdot\mathcal{F}\{\cos\omega_0\tau\} = a\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

$S_n(\omega) = \mathcal{F}\{R_n(\tau)\} = b\cdot\mathcal{F}\{\exp -c|\tau|\} = 2bc/(c^2+\omega^2)$ ✓



(c) $P_s = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_s(\omega)\, d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} a\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\, d\omega = \frac{1}{2\pi}[a\pi + a\pi] = a$

$P_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)\, d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2bc/(c^2+\omega^2)\, d\omega$

$= \frac{1}{\pi}\int_0^{\infty} 2b\cdot 1/(1+(\frac{\omega}{c})^2)\cdot d(\frac{\omega}{c}) = b$

SNR = a/b CHECKED!!

✓

55

80/80

HOMEWORK #8 FOR EE 423

COMMUNICATION SYSTEMS

BENMEI CHEN

OCTOBER , 29 , 1986

PROBLEMS : 3.24 , 4.6.1 , 4.6.4 , 4.7.1

4.7.3 , 4.7.4 , 4.7.6 , 4.7.7

3.2.4 $f(t) = (1/\sigma\sqrt{2\pi})e^{-t^2/2\sigma^2}$. FIND ITS FOURIER TRANSFORM.

$$F(\omega) = \mathcal{F}\{f(t)\} = \mathcal{F}\{(1/\sigma\sqrt{2\pi})e^{-t^2/2\sigma^2}\}$$

$$= (1/\sigma\cdot\sqrt{2\pi})\cdot\mathcal{F}\{e^{-t^2/2\sigma^2}\}$$

$$= (1/\sigma\sqrt{2\pi})\int_{-\infty}^{\infty} e^{-t^2/2\sigma^2}\cdot e^{-j\omega t}\,dt$$

$$= (1/\sigma\cdot\sqrt{2\pi})\int_{-\infty}^{\infty} e^{-(t^2/2\sigma^2 + j\omega t)}\,dt$$

$$= (1/\sigma\cdot\sqrt{2\pi})\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t^2 + j\omega\cdot 2\sigma^2 t - \omega^2\sigma^4 + \omega^2\sigma^4)}\,dt$$

$$= (1/\sigma\cdot\sqrt{2\pi})\cdot\int_{-\infty}^{\infty} e^{-1/2\sigma^2 (t+j\omega\sigma^2)^2 - \frac{1}{2}\cdot\omega^2\sigma^2}\,dt$$

$$= (1/\sigma\cdot\sqrt{2\pi})\cdot e^{-\frac{1}{2}\omega^2\sigma^2}\cdot\int_{-\infty}^{\infty} e^{-1/2\sigma^2(t+j\omega\sigma^2)^2}\,dt$$

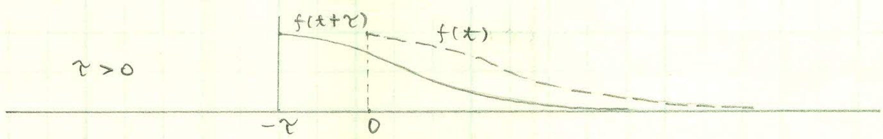
$$= (1/\sigma\cdot\sqrt{2\pi})\,e^{-\frac{1}{2}\omega^2\sigma^2}\cdot\int_{-\infty}^{\infty} e^{-[(t+j\omega\sigma^2)/\sqrt{2}\sigma]^2}\,d(t+j\omega\sigma^2)/\sqrt{2}\sigma\cdot\sqrt{2}\sigma$$

$$= 1/\sqrt{\pi}\cdot e^{-\frac{1}{2}\omega^2\sigma^2}\cdot\int_{-\infty}^{\infty} e^{-u^2}\,du = 1/\sqrt{\pi}\cdot e^{-\frac{1}{2}\omega^2\sigma^2}\cdot\sqrt{\pi}$$

$$= e^{-\omega^2\sigma^2/2}$$ ✓ (Agree at the result in table 3.1)

4.6.1 $f(t) = [\exp(-at)]\,u(t)$, FIND THE ENERGY SPECTRAL DENSITY $E(\omega)$

(a) FINDING ITS AUTOCORRELATION FUNCTION FIRST.



$$r_f(\tau) = \int_{-\infty}^{\infty} f^*(t)\cdot f(t+\tau)\,dt$$

$$= \int_0^{\infty} e^{-at}\cdot e^{-a(t+\tau)}\,dt$$

$$= e^{-a\tau}\cdot\int_0^{\infty} e^{-2at}\,dt = e^{-a\tau}\cdot\left(-\frac{1}{2a}e^{-2at}\Big|_0^{\infty}\right)$$

$$= \frac{1}{2a}e^{-a\tau} \qquad \tau > 0$$

$$r_f(\tau) = \frac{1}{2a}e^{-a|\tau|}$$

$$E(\omega) = \mathcal{F}^{-1}\{r_f(\tau)\} = \frac{1}{2a}\mathcal{F}^{-1}\{e^{-a|\tau|}\} = 1/(a^2+\omega^2)$$ ✓

(b) FINDING FOURIER TRANSFORM FIRST.

$$F(\omega) = \mathcal{F}^{-1}\{\exp(-at)\,u(t)\} = 1/(a+j\omega)$$

$$|F(\omega)|^2 = 1/(a^2+\omega^2)$$ ✓

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4.6.4. $f(t) = (1/\sigma\sqrt{2\pi})\, e^{-t^2/2\sigma^2}$

IT IS EASY TO CHECK $f(t)$ HAS FINITY ENEGY.

IN PROBLEM 3.2.4., WE FIND

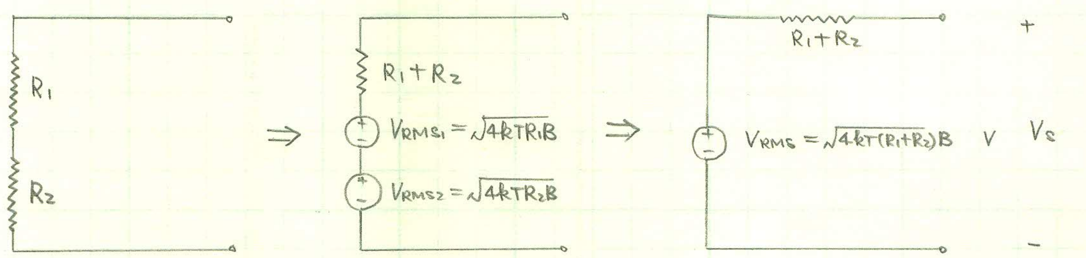
$$F(\omega) = e^{-\omega^2\sigma^2/2} \quad ; \quad |F(\omega)|^2 = e^{-\omega^2\sigma^2}$$

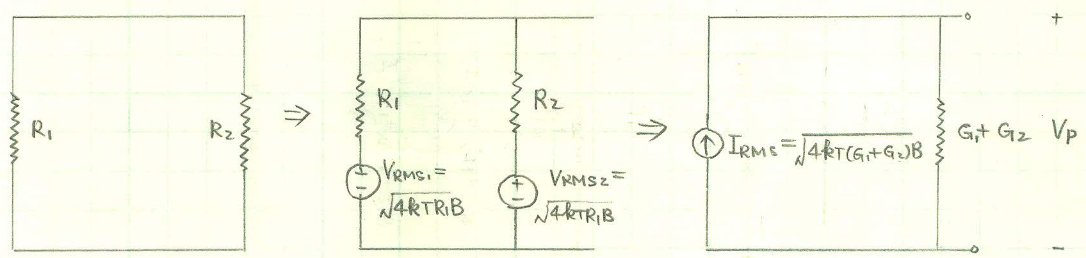
$$r_f(\tau) = \mathcal{F}^{-1}\{|F(\omega)|^2\} = \mathcal{F}^{-1}\{e^{-\omega^2\sigma^2}\} = 1/\sigma\sqrt{2\pi}\ \mathcal{F}^{-1}\{\sigma\sqrt{2\pi}\cdot e^{-\sigma^2(\sqrt{2}\omega)^2/2}\}$$

$$= 1/\sigma\sqrt{2\pi}\cdot\frac{1}{\sqrt{2}}\cdot e^{-t^2/4\sigma^2} = \underline{\underline{1/2\sigma\sqrt{\pi}\cdot e^{-t^2/4\sigma^2}}}$$

✓

4.7.1.





$$V_s = V_{RMS} = \sqrt{4kT(R_1+R_2)B}$$

$$V_p = \frac{I_{RMS}}{G_1+G_2} = I_{RMS}\cdot\frac{R_1\cdot R_2}{R_1+R_2} = \sqrt{4kTR_1R_2B/(R_1+R_2)}$$

$$V_s = 2V_p \qquad \therefore \quad \sqrt{4kT(R_1+R_2)B} = 2\sqrt{4kTR_1R_2B/(R_1+R_2)}$$

$$\sqrt{R_1+R_2} = 2\sqrt{\frac{R_1R_2}{R_1+R_2}}$$

$$(R_1+R_2)^2 = 4\cdot R_1R_2$$

$$R_1^2+2R_1R_2+R_2^2 = 4R_1R_2$$

$$R_1^2-2R_1R_2+R_2^2 = 0$$

$$(R_1-R_2)^2 = 0$$

✓

$$R_1 = R_2$$

4.7.3. $V_{in,RMS} = \sqrt{4KTRB}$

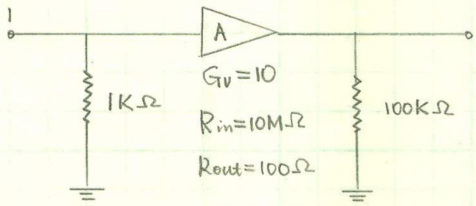
$= \sqrt{4 \times 1.38 \times 10^{-23} \times 295 \times 100 \times 10^3 \times 100 \times 10^3}$

$= 1.276 \times 10^{-5}$ V rms

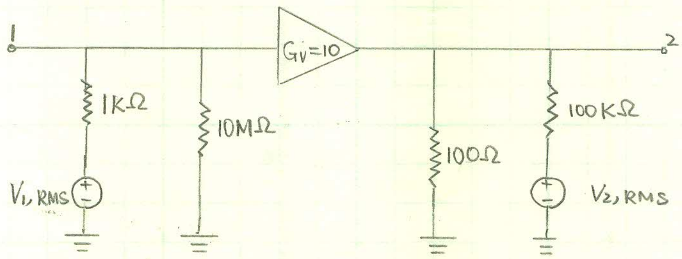
$V_{out,RMS} = 2$ V rms

$G_V = V_{out,RMS} / V_{in,RMS} = 156729 \doteq 104$ dB ✓

4.7.4.



$B_N = 1MHz$, $T = 300K$



$V_{1,RMS} = \sqrt{4KTR_1B_N} = \sqrt{4 \times 1.38 \times 10^{-23} \times 300 \times 10^3 \times 10^6} = 4.07 \times 10^{-6}$ V rms

$V_{2,RMS} = \sqrt{4KTR_2B_N} = \sqrt{4 \times 1.38 \times 10^{-23} \times 300 \times 100 \times 10^3 \times 10^6} = 4.07 \times 10^{-5}$ V rms

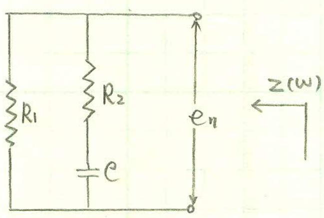
(a) $V_1 = \frac{V_{1,RMS}}{1K\Omega + 10M\Omega} \times 10M\Omega \doteq V_{1,RMS} = 4.07 \times 10^{-6}$ V rms ✓

(b) $V_2' = \frac{V_{2,RMS}}{100\Omega + 100K\Omega} \times 100 = 4.065 \times 10^{-8}$ V rms

$V_{out} = 10 \cdot V_1 = 4.07 \times 10^{-6} \times 10 = 4.07 \times 10^{-5}$ V rms

$V_2 = \sqrt{V_2'^2 + V_{out}^2} \cong 4.07 \times 10^{-5}$ V rms ✓

4.7.6



SOLUTION: $Z(\omega) = \frac{R_1 \cdot (R_2 + \frac{1}{j\omega C})}{R_1 + R_2 + \frac{1}{j\omega C}}$

$= \frac{R_1 + j\omega \cdot C \cdot R_1 R_2}{1 + j\omega \cdot C(R_1 + R_2)}$

$= \frac{(R_1 + j\omega C R_1 R_2)(1 - j\omega C(R_1 + R_2))}{1 + \omega^2 C^2 \cdot (R_1 + R_2)^2}$

$R(\omega) = (R_1 + \omega^2 C^2 R_1 R_2 (R_1 + R_2)) / (1 + \omega^2 C^2 \cdot (R_1 + R_2)^2)$

$S_{en}(\omega) = 2kT\,R(\omega) = 2kT \cdot R_1 \cdot [1 + \omega^2 C^2 R_2 (R_1 + R_2)] / [1 + \omega^2 C^2 \cdot (R_1 + R_2)^2]$ V^2/Hz

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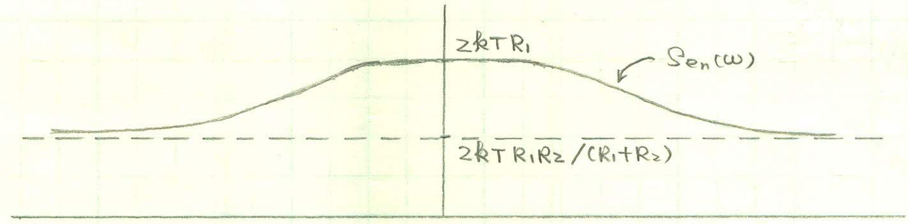
4.7.6 (CONT.)

$S_{en}(\omega) = 2kTR_1[1+\omega^2C^2R_2(R_1+R_2)]/[1+\omega^2C^2\cdot(R_1+R_2)^2]$ V^2/Hz

$R_1 \to 0$; $S_{en}(\omega) \to 0$, $e_n \to 0$: SHORT CIRCUIT.

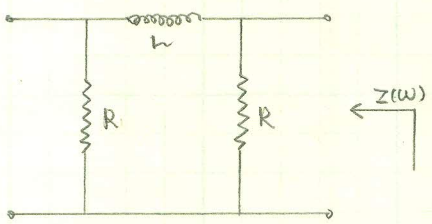
$R_2 \to 0$; $S_{en}(\omega) \to 2kTR_1/(1+\omega^2c^2R_1^2)$

$R_1 \to 0$ and $R_2 \to 0$; $S_{en}(\omega) \to 0$; PURE Noiseless circuit.



V

4.7.7



(a) $Z(\omega) = R(R+j\omega L)/(2R+j\omega L) = R(R+j\omega L)(2R-j\omega L)/(4R^2+\omega^2L^2)$

$R(\omega) = R(2R^2+\omega^2L^2)/(4R^2+\omega^2L^2)$

$S_v(\omega) = 2kTR(2R^2+\omega^2L^2)/(4R^2+\omega^2L^2)$ V^2/Hz

(b) $H(\omega) = \dfrac{R}{R+j\omega L}$

$|H(\omega)|^2 = R^2/(R^2+\omega^2L^2) = 1/(1+(\frac{\omega L}{R})^2)$

$|H(0)|^2 = 1$

$B_N = \frac{1}{2\pi}\int_0^\infty 1/[1+(\frac{\omega L}{R})^2]\,d\omega = \frac{R}{2\pi L}\int_0^\infty 1/[1+(\frac{\omega L}{R})^2]\,d(\frac{\omega L}{R})$

$= \frac{R}{2\pi L}\cdot\frac{\pi}{2} = \frac{R}{4L}$ IS THIS REASONABLE? TED.

Yes

$\overline{v^2(t)} = \frac{1}{2\pi}\int_{-2\pi B_N}^{2\pi B_N} S_v(\omega)\,d\omega$

$= \frac{1}{\pi}\cdot kT\cdot R\int_{-2\pi B_N}^{2\pi B_N}(2R^2+\omega^2L^2)/(4R^2+\omega^2L^2)\,d\omega$

$= \frac{2}{\pi}\cdot kT\cdot R\left[2R^2\cdot\frac{1}{2RL}\tan^{-1}(\frac{L}{2R}\omega) + \frac{\omega L^2}{L^2} - \frac{2R}{L}\tan^{-1}(\frac{L}{2R}\omega)\right]\Big|_0^{2\pi B_N}$

$= \frac{2}{\pi}kTR[2\pi B_N - R/L\cdot\tan^{-1}(L/2R\cdot 2\pi\cdot B_N)]$

$= 4kTRB_N - \frac{2kTR^2}{\pi L}\cdot\tan^{-1}(\pi LB_N/R)$

60

10

20/20

4.7.8. $|H(\omega)| = 1/\sqrt{1+\omega^6}$

$|H(\omega)|^2 = 1/(1+\omega^6)$; $|H(0)|^2 = 1$

$$\int_0^\infty |H(\omega)|^2 d\omega = \int_0^\infty \frac{d\omega}{1+\omega^6} = \frac{\pi}{3}$$

$$B_N = \frac{1}{2\pi}\cdot\int_0^\infty |H(\omega)|^2 d\omega \,/\, |H(0)|^2 = \frac{1}{2\pi}\cdot\frac{\pi}{3} = \frac{1}{6} = 0.1667 \text{ Hz}$$

-3dB Bandwidth $\omega = 1$ rad/s , so $B = 0.15915$

Noise Bandwidth is about 4.73 % greater than the -3dB Bandwidth.

THE FIRST-ORDER IS 57% GREATER.

THE SECOND-ORDER IS 11 % GREATER.

THE THIRD-ORDER IS 4.73 % GREATER.

4.7.9 $|H(\omega)| = \exp(-a\omega^2)$

$|H(\omega)|^2 = \exp(-2a\omega^2)$; $|H(0)| = 1$

$$\int_0^\infty |H(\omega)|^2 d\omega = \int_0^\infty e^{-2a\omega^2} d\omega = \frac{1}{\sqrt{2a}}\int_0^\infty e^{-(\sqrt{2a}\,\omega)^2}\cdot d(\sqrt{2a}\,\omega) = \frac{\sqrt{\pi}}{2\sqrt{2a}}$$

$$\therefore\ B_N = \frac{1}{2\pi}\cdot\frac{\sqrt{\pi}}{2\sqrt{2a}} = \frac{1}{4\sqrt{2a\pi}} = 0.09974/\sqrt{a} \quad \text{Hz}$$

$-3\text{dB} = 20\log(\exp(-a\omega^2))$

$\exp(-a\omega^2) = 0.70795$

$-a\omega^2 = -0.34539$

$\omega = \dfrac{0.58771}{\sqrt{a}}$

$B = \omega/2\pi = 0.09353/\sqrt{a}$ Hz

$B_N/B \times 100\% = 106.64\ \%$

SO, THE NOISE BANDWIDTH IS 6.64 % GREATER THAN THE -3dB BANDWIDTH.

60/60

Excellent!

COMMUNICATION SYSTEMS

HOMEWORK #10

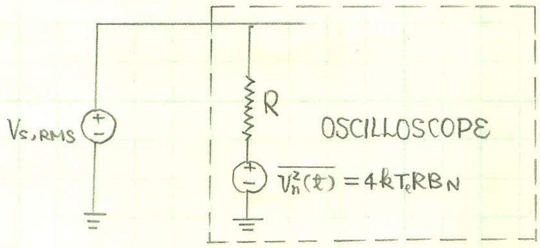
NOV.5,1986

BENMEI CHEN

CHAPTER 4 PROBLEM 4.7.11 4.7.12 4.7.13

" 4.7.14 4.7.15 4.7.16

4.7.11.



$T_e = 600°K$

$B_N = 500\ MHz = 5 \times 10^8\ Hz$

$R \quad 50\Omega \longrightarrow 10^6\Omega$

$P_{signal} / P_{noise} \geq 10$

SOLUTION: $P_{signal} = V_{s,RMS}^2 / R$

$P_{noise} = 4kT_eB_N = 4 \times 1.38 \times 10^{-23} \times 600 \times 5 \times 10^8 = 1.656 \times 10^{-11}\ W$

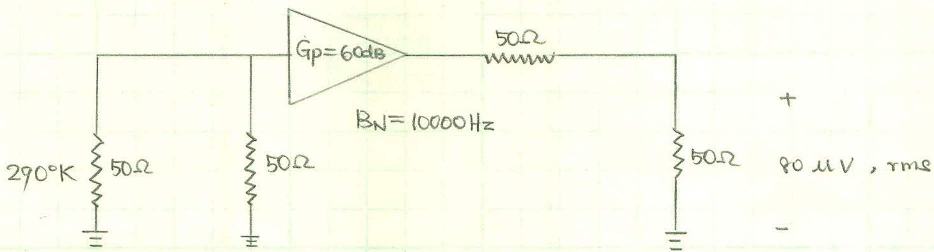
$P_{signal} / P_{noise} = V_{s,RMS}^2 / 1.656 \times 10^{-11} \cdot R \geq 10$

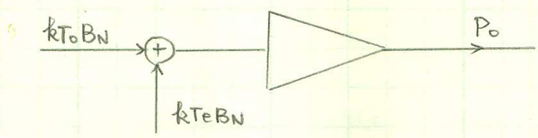
$V_{s,RMS} \geq \sqrt{1.656 \times 10^{-10} \cdot R}$

FOR $R = 50\ \Omega$, $V_{s,RMS} \geq 90.99\ \mu V$ $\quad V_{s,RMS,min} = 90.99\ \mu V$

FOR $R = 1\ M\Omega$, $V_{s,RMS} \geq 12.87\ mV$ $\quad V_{s,RMS,min} = 12.87\ mV$ ✓

4.7.12.





$G_p = 60dB = 10^6$

Gp is great enough FOR us TO ~~take regardless~~ neglect the noise-powe R generated by the output resistors. So,

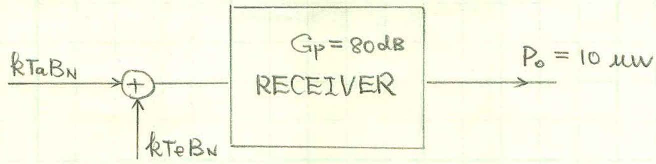
$P_o = (80\ \mu V)^2 / 50 = 1.28 \times 10^{-10}\ W$

$P_i = kT_oB_N + kT_eB_N = P_o / G_p$ ✓

$$T_e = \frac{P_o}{G_p \cdot k \cdot B_N} - T_o$$

$$= \frac{1.28 \times 10^{-10}}{10^6 \times 1.38 \times 10^{-23} \times 10^4} - 290 = 637.54\ °K$$

4.7.13.



$B_N = 10MHz = 10^7 Hz$, $G_p = 80dB = 10^8$

$F = 3dB = 1.995$; FIND T_a (Is this the same meaning as grader? Ted)

Y

SOLUTION: $T_e = (F-1)T_o = (1.995-1)\times 290 = 288.626$

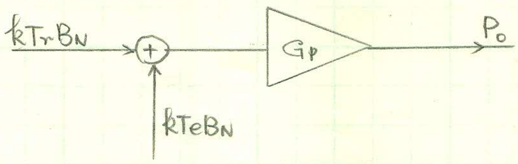
$P_i = kT_aB_N + kT_eB_N = kB_N(T_a+T_e) = P_o/G_p$

$$T_a = \frac{P_o}{G_pkB_N} - T_e$$

$$= \frac{10\times10^{-6}}{10^8\times1.23\times10^{-23}\times10^7} - 288.626 = 436\ °K$$

✓

4.7.14.



WHEN $T_r = T_o$, $P_{oo} = P$

$T_r = T_R$, $P_{oR} = 2P$

$P_{oo} = (kT_oB_N + kT_eB_N)G_p = kB_N(T_o+T_e)G_p = P$

$P_{oR} = (kT_RB_N + kT_eB_N)\cdot G_p = kB_N(T_R+T_e)G_p = 2P$

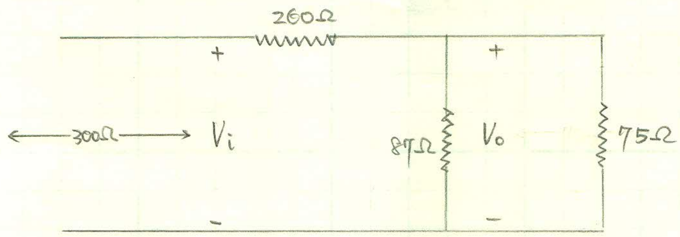
$$\therefore \frac{T_o+T_e}{T_R+T_e} = \frac{1}{2}$$

$T_e = T_R - 2T_o$

$$F = (1+\frac{T_e}{T_o}) = (1+\frac{T_R}{T_o}-2) = \frac{T_R}{T_o} - 1$$

✓

4.7.15



$$V_o = \frac{V_i}{300} \cdot \frac{75 \times 87}{75 + 87} = 0.134\ V_i$$

$$P_i = V_i^2 / 300 = 33.333 \times 10^{-4} \cdot V_i^2$$

$$P_o = V_o^2 / 75 = 0.018\, V_i^2 / 75 = 2.403 \times 10^{-4}\ V_i^2$$

$$\alpha = \frac{1}{G_P} = \frac{P_i}{P_o} = 13.872$$

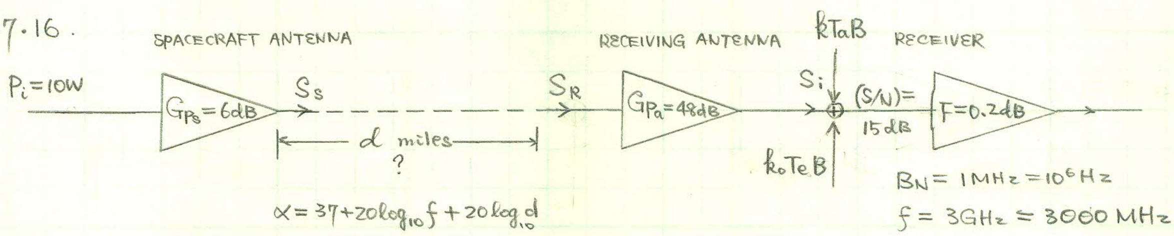
FROM THE RESULT OF EXAMPLE 4.7.6

$$T_e = (\alpha - 1) T_o = (13.872 - 1) \times 290 = 3732.75\ °K$$

$$F = \alpha = 13.872 = 11.42\ dB.$$

✓

4.7.16.



SOLUTION: $S/N = 15dB = 31.623$, $G_{Pa} = 48dB = 63095.7$; $G_{P_S} = 6dB = 3.981$; $F = 0.2 = 1.047$

$S_s = P_i \cdot G_{P_S} = 39.81\ W$, $T_e = (F-1)T_o = 13.63\ °K$

$N_i = kT_eB = 1.38 \times 10^{-23} \times 13.63 \times 10^6 = 1.881 \times 10^{-16}$ (FROM FIG. 4.16, FIND $T_a = 0$)

$S_i = (S/N) \cdot N_i = 1.881 \times 10^{-16} \times 31.623 = 5.948 \times 10^{-15}\ W$

$S_R = S_i / G_{Pa} = 5.948 \times 10^{-15} / 63095.7 = 9.427 \times 10^{-20}\ W$

$\alpha = S_S / S_R = 39.81 / 9.427 \times 10^{-20} = 4.223 \times 10^{20} = 206.256$

$\alpha = 37 + 20 \log_{10} f + 20 \log_{10} d$

$= 37 + 20 \log_{10}(3000) + 20 \log_{10} d = 106.542 + 20 \log_{10} d = 206.256$

$\underline{d = 96760.9\ miles}$

65

OK

50/50

D.P. 5.1.2: SHOW THAT $f_1(t)$ AND $f_2(t)$ ESSENTIALLY CONTROL BOTH THE AMPLITUDE AND THE AND THE PHASE OF $\phi(t)$ IN THE QUADRATURE MULTIPLEXING SYSTEM OF EXAPLE 5.1.1. IN OTHER WORDS, DERIVE AN EXPRESSION FOR $\phi(t)$ WHICH EXHIBITS THE ENVELOPE AND PHASE ANGLE IN TERMS OF $f_1(t)$ AND $f_2(t)$. ALSO FIND $\phi(t)$ FOR THE SPECIAL CASE WHERE $f_1(t) = \cos\theta$, $f_2(t) = \sin\theta$.

SOLUTION: FROM THE EXAMPLE 5.1.1, WE HAVE

$$\phi(t) = f_1(t)\cos\omega_c t + f_2(t)\sin\omega_c t$$

$$= \sqrt{f_1^2(t)+f_2^2(t)} \cdot \left[\frac{f_1(t)}{\sqrt{f_1^2(t)+f_2^2(t)}} \cos\omega_c t + \frac{f_2(t)}{\sqrt{f_1^2(t)+f_2^2(t)}} \sin\omega_c t \right]$$

LET $a = \frac{f_1(t)}{\sqrt{f_1^2(t)+f_2^2(t)}}$; $b = \frac{f_2(t)}{\sqrt{f_1^2(t)+f_2^2(t)}}$

$a^2 + b^2 = 1$, SO

LET $a = \cos[\theta(t)]$; $b = \sin(\theta(t))$

$$\theta(t) = \tan^{-1}\frac{b}{a} = \tan^{-1}[f_2(t)/f_1(t)]$$

$$\therefore \phi(t) = \sqrt{f_1^2(t)+f_2^2(t)} \cdot [\cos\theta(t)\cos\omega_c t + \sin\theta(t)\sin\omega_c t]$$

$$= \sqrt{f_1^2(t)+f_2^2(t)}\cos(\omega_c t - \theta(t))$$

$$= \sqrt{f_1^2(t)+f_2^2(t)}\cos\{\omega_c t + \tan^{-1}[-f_2(t)/f_1(t)]\}$$

IF $f_1(t) = \cos\theta$, $f_2(t) = \sin\theta$

$$\phi(t) = \cos\omega_c t \cdot \cos\theta + \sin\omega_c t \sin\theta$$

$$= \cos(\omega_c t - \theta)$$

Drill Problem 5.1.5

a) Looking at the Fig 5.12 (b) on Page 219. WE HAVE

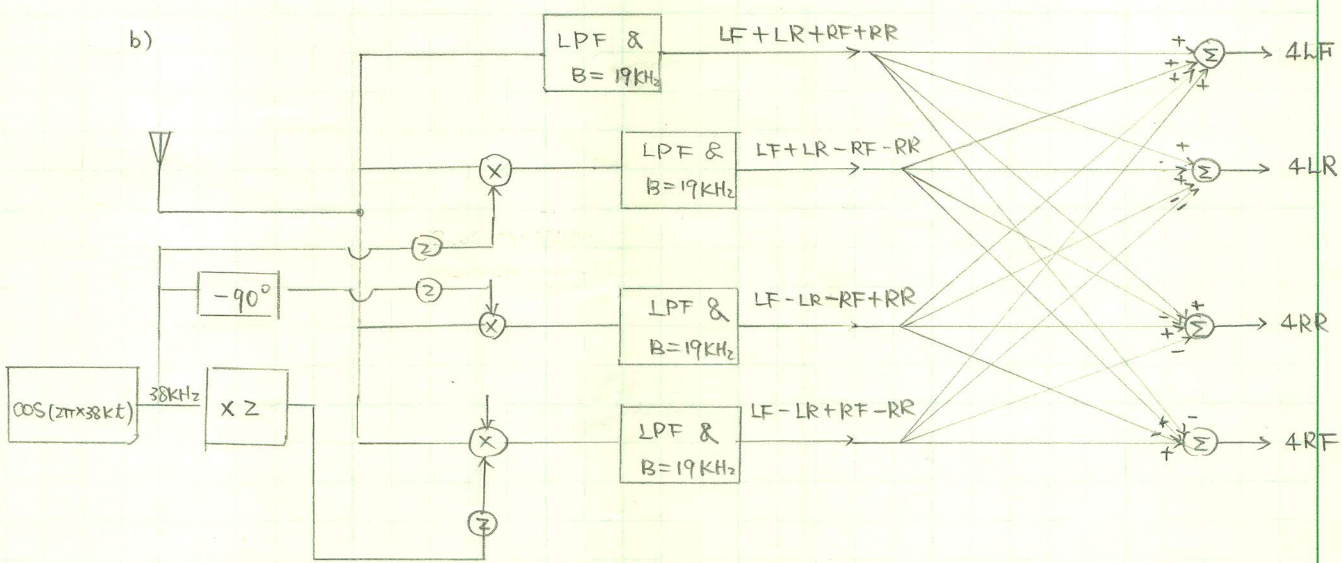
0-15 KHz = L+R : LF + LR + RF + RR

23-53 KHz = L-R : LF + LR - RF - RR

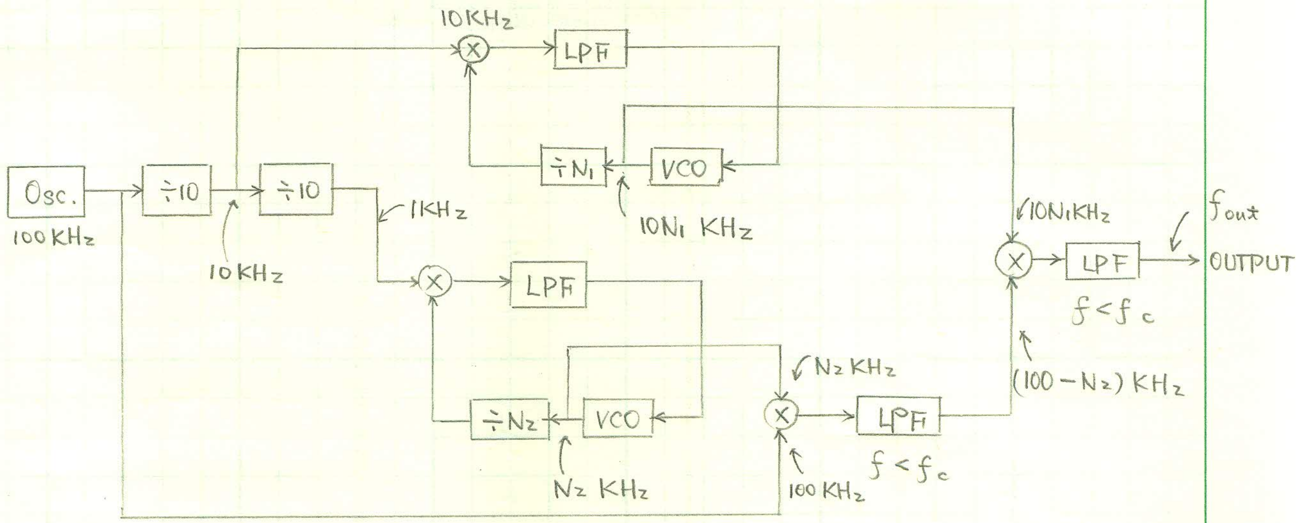
FROM GIVEN : LF - LR - RF + RR

REMAINED : LF - LR + RF - RR

b)



5.1.12



$$f_{out} = \begin{cases} 100 - N_2 - 10N_1 & \text{IF } 100 - N_2 \geq 10N_1 \\ 10N_1 - 100 + N_2 & \text{IF } 100 - N_2 \leq 10N_1 \quad \Rightarrow \quad N_1 = 10 \end{cases}$$

(Because $100 - N_2 + 10N_1 = 100 + (10N_1 - N_2) > 100$ KHz, $1 \leq N_1, N_2 \leq 10$)

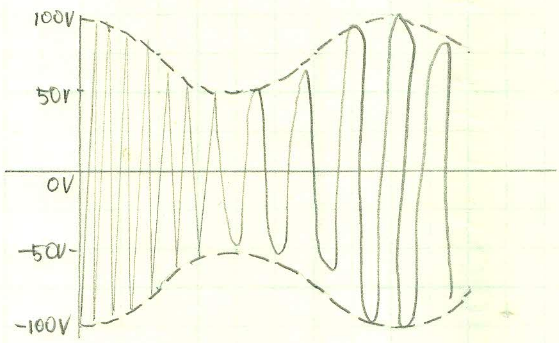
a) $N_1 = 4$, $N_2 = 2$

$f_{out} = 100 - 2 - 10 \times 4 = 58$ KHz

b) $f_{minimum} = 0$ KHz ($N_1 = 9$, $N_2 = 10$)

$f_{max} = 89$ KHz ($N_1 = 1$, $N_2 = 1$)

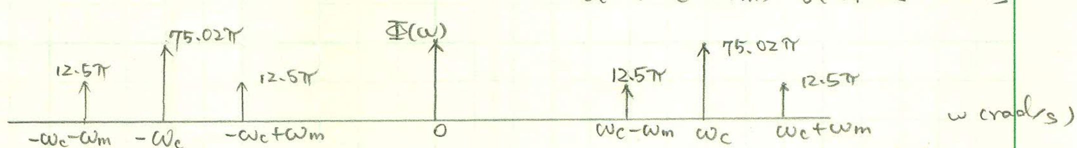
5.2.4.



a) $m = \frac{100 - 50}{100 + 50} = 0.3333 = 33.33\%$

b) $\phi(t) = A(1 + m\cos\omega_m t)\cos\omega_c t = 75.02(1 + 0.3333\cos\omega_m t)\cos\omega_c t$

$\Phi(\omega) = 75.02\pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] + 12.5\pi[\delta(\omega - \omega_c - \omega_m) + \delta(\omega - \omega_c + \omega_m)$
$+ \delta(\omega + \omega_c - \omega_m) + \delta(\omega + \omega_c + \omega_m)]$



c) $P_s / P_c = m^2/2 = 0.0556 = 5.56\%$

d) $mA = 25$, $A = 25/m = 25/10\% = 250$ V, $A_{add.} = 250 - 75 = 175$V.

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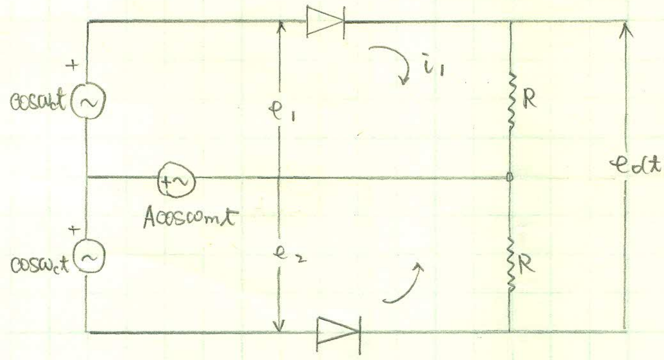
5.2.9.



$$e_1(t) = \cos\omega_c t + A\cos\omega_m t$$

$$e_2(t) = -\cos\omega_c t + A\cos\omega_m t$$

$$i_1(t) = a_1 e_1(t) + a_2 e_1^2(t) = a_1\cos\omega_c t + a_1 A\cos\omega_m t + a_2\cos^2\omega_c t + 2a_2 A\cos\omega_c t\cos\omega_m t + a_2 A^2\cos^2\omega_m t$$

$$i_2(t) = a_1 e_2(t) + a_2 e_2^2(t) = -a_1\cos\omega_c t + a_1 A\cos\omega_m t + a_2\cos^2\omega_c t - 2a_2 A\cos\omega_c t\cos\omega_m t + a_2 A^2\cos\omega_m t$$

$$e_o(t) = (i_1 - i_2)R = 2a_1 R\cos\omega_c t + 4a_2 AR\cos\omega_c t\cos\omega_m t$$

(a) $$m = \frac{4a_2 AR}{2a_1 R} = \frac{2a_2 A}{a_1} \leq 1$$

$$A \leq \frac{a_1}{2a_2} \qquad A_{max} = a_1/2a_2$$

(b) IF ONE OF THE DIODES IS OPEN-CIRCUITED

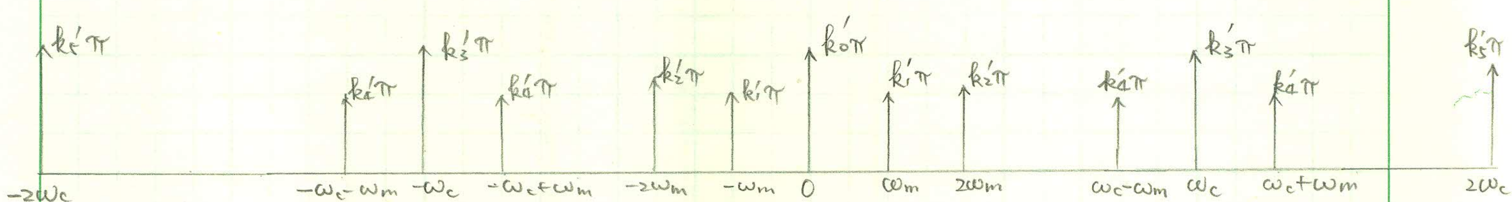
$$\therefore\ e_o(t) = k_1\cos\omega_c t + k_2\cos\omega_m t + k_3\cos^2\omega_c t + k_4\cos\omega_c t\cos\omega_m t + k_5\cos^2\omega_m t$$

$$= k_1\cos\omega_c t + k_2\cos\omega_m t + k_3/2 + k_3/2\cos 2\omega_c t + \frac{k_4}{2}\cos[(\omega_c+\omega_m)t] + \frac{k_4}{2}[(\omega_c-\omega_m)t]$$

$$+ \frac{k_5}{2} + \frac{k_5}{2}\cos 2\omega_m t$$

$$= k_0' + k_1'\cos\omega_m t + k_2'\cos 2\omega_m t + k_3'\cos\omega_c t + k_4'\cos(\omega_c+\omega_m)t + k_4'\cos(\omega_c-\omega_m)t$$

$$+ k_5'\cos 2\omega_c t$$



$$B_{max} = (2\omega_c - 2\omega_m)/2\pi = 2(f_c - f_m)$$

$$B_{min} = \omega_c + \omega_m - \omega_c + \omega_m = 2\pi\cdot\omega_2\cdot 2 = 2f_m$$

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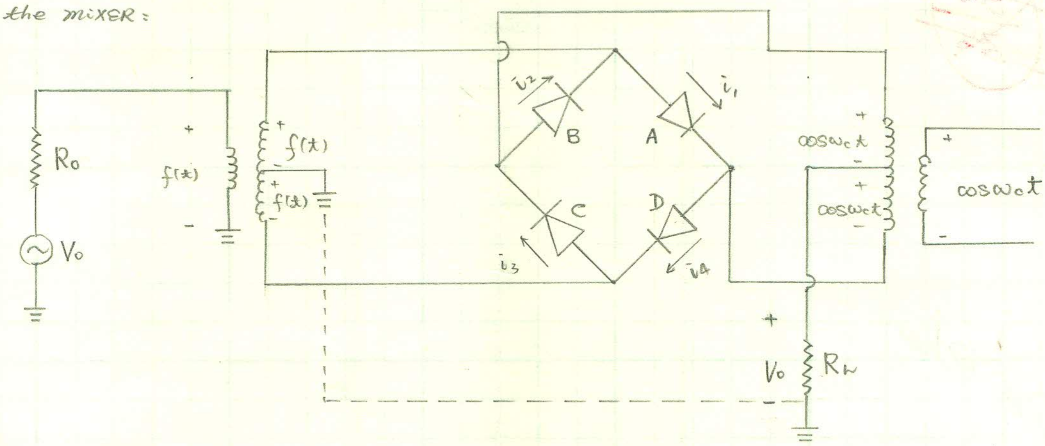
$$f = \frac{\omega}{2\pi} \sim T = \frac{2\pi}{\omega}$$

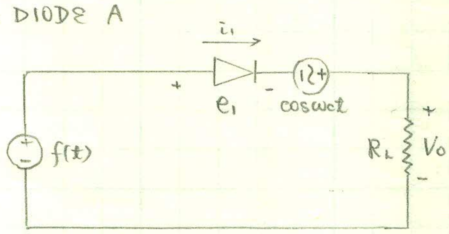
70/100

Aanlysis the mixer:

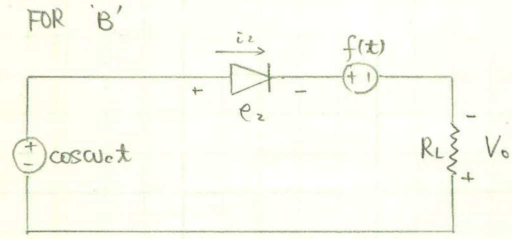


FOR DIODE A, B, C, D, WE HAVE THE EQUAVELENT CIRCUITS BELOW.

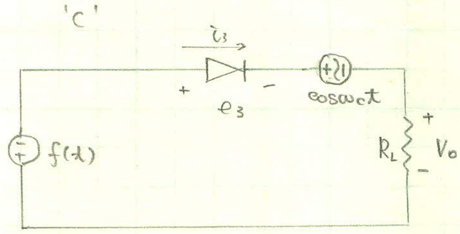
FOR DIODE A



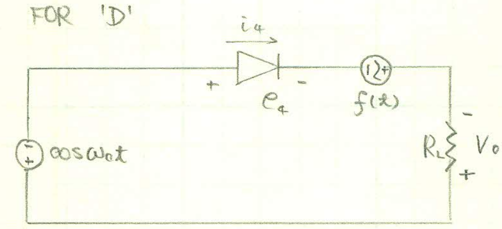
FOR 'B'



FOR 'C'



FOR 'D'



$$\begin{cases} i_1(t) = a_0 + a_1 e_1(t) + a_2 e_1^2(t) & e_1(t) = f(t) + \cos\omega_c t - V_o \\ i_2(t) = a_0 + a_1 e_2(t) + a_2 e_2^2(t) & e_2(t) = \cos\omega_c t - f(t) + V_o \\ i_3(t) = a_0 + a_1 e_3(t) + a_2 e_3^2(t) & e_3(t) = -f(t) - \cos\omega_c t - V_o \\ i_4(t) = a_0 + a_1 e_4(t) + a_2 e_4^2(t) & e_4(t) = f(t) - \cos\omega_c t + V_o \end{cases}$$

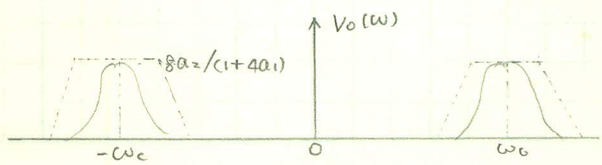
$$\begin{cases} i_1(t) = a_0 + a_1 f(t) + a_1\cos\omega_c t - a_1 V_o + a_2 f^2(t) + a_2\cos^2\omega_c t + a_2 V_o^2 + 2a_2 f(t)\cos\omega_c t - 2a_2 V_o f(t) - 2a_2 V_o\cos\omega_c t \\ i_2(t) = a_0 - a_1 f(t) + a_1\cos\omega_c t + a_1 V_o + a_2 f^2(t) + a_2\cos^2\omega_c t + a_2 V_o^2 - 2a_2 f(t)\cos\omega_c t - 2a_2 V_o f(t) + 2a_2 V_o\cos\omega_c t \\ i_3(t) = a_0 - a_1 f(t) - a_1\cos\omega_c t - a_1 V_o + a_2 f^2(t) + a_2\cos^2\omega_c t + a_2 V_o^2 + 2a_2 f(t)\cos\omega_c t + 2a_2 V_o f(t) + 2a_2 V_o\cos\omega_c t \\ i_4(t) = a_0 + a_1 f(t) - a_1\cos\omega_c t + a_1 V_o + a_2 f^2(t) + a_2\cos^2\omega_c t + a_2 V_o^2 - 2a_2 f(t)\cos\omega_c t + 2a_2 V_o f(t) - 2a_2 V_o\cos\omega_c t \end{cases}$$

$$V_o = (i_1 - i_2 + i_3 - i_4) R_L$$

$$= [2a_1 f(t) - 2a_1 V_o + 4a_2 f(t)\cos\omega_c t - 4a_2 V_o\cos\omega_c t - 2a_1 f(t) - 2a_1 V_o + 4a_2 f(t)\cos\omega_c t + 4a_2 V_o\cos\omega_c t]$$

$$= (-4a_1 V_o + 8a_2 f(t)\cos\omega_c t) R_L$$

$$\therefore V_o = \frac{8a_2}{1+4a_1} \cdot f(t)\cos\omega_c t \quad R_L$$



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Communication Systems (EE423)

Homework #11

Benmei Chen Nov. 18, 1986

Chapter 5: 5.1.2 ; 5.1.6 ; 5.1.7

5.1.2. The modulating signal $f(t) = 2\cos 100\pi t + \cos 400\pi t$ is applied as the input to a double-sideband supressed-carrier modulator operating at a carrier frequency of 1 KHz. Sketch the spectral density of f(t) and the resulting DSB-SC waveform identifying the upper and lower sidebands.

SOLUTION: $f(t) = 2\cos 100\pi t + \cos 400\pi t$

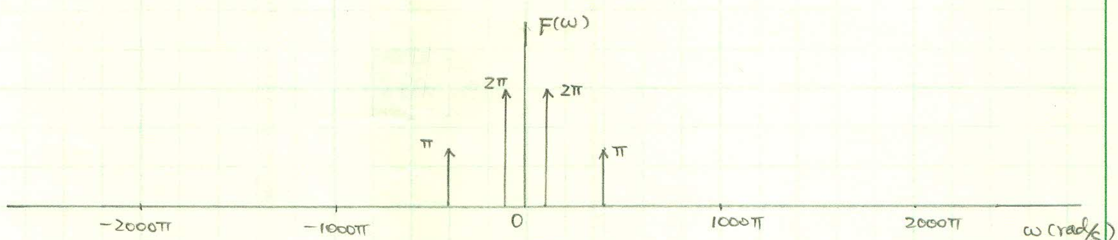
$$f = 1\text{ KHz} = 1000\text{ Hz} \Rightarrow \omega_c = 2\pi f = 2000\pi \text{ rad/s}$$

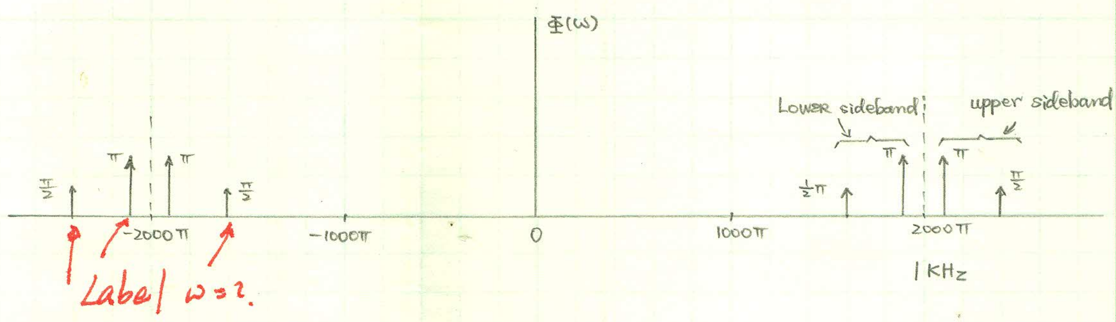
$$\phi(t) = f(t)\cos(2000\pi t)$$

$$= (2\cos 100\pi t + \cos 400\pi t)\cos 2000\pi t$$

$$= 2\cos 100\pi t \cos 2000\pi t + \cos 400\pi t \cos 2000\pi t$$

$$= \cos 1900\pi t + \cos 2100\pi t + \frac{1}{2}\cos 1600\pi t + \frac{1}{2}\cos 2400\pi t$$





5.1.6. When the input to given audio amplifier is $(4\cos 800\pi t + \cos 2000\pi t)$ mV, the measured frequency component at 1 KHz in the output is 1V and the frequency component at 600 Hz is 1 mV. Represent the amplifier output-input characteristic by $e_o = a_1 e_i + a_2 e_i^2$ and evaluate the numerical values of a_1, a_2 from the data given.

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The solution is given on next page.

SOLUTION TO 5.1.6: LET $e_i = 4\cos 800\pi t + \cos 2000\pi t$ mV $= (4\cos 800\pi t + \cos 2000\pi t)/10^3$ V

$$e_o = a_1 e_i + a_2 e_i^2 = \frac{a_1}{10^3}(4\cos 800\pi t + \cos 2000\pi t) + a_2(4\cos 800\pi t + \cos 2000\pi t)^2/10^6$$

$$= \frac{a_1}{10^3}(4\cos 800\pi t + \cos 2000\pi t) + a_2(16\cos^2 800\pi t + 8\cos 800\pi t \cos 2000\pi t + \cos^2 2000\pi t)/10^6$$

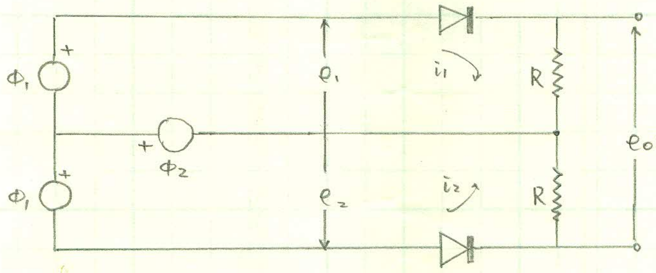
$$= \frac{a_1}{10^3}(4\cos 800\pi t + \cos 2000\pi t) + a_2(8\cos 1600\pi t + 8 + 4\cos 2800\pi t + 4\cos 1200\pi t + \tfrac{1}{2}\cos 4000\pi t + \tfrac{1}{2})/10^6$$

$$= \frac{a_1}{10^3}\cos 2000\pi t + \frac{4a_2}{10^6}\cos 1200\pi t + [4a_1\cos 800\pi t + 8a_2\cos 1600\pi t + 4a_2\cos 2800\pi t + \tfrac{1}{2}a_2\cos 4000\pi t + 8.5a_2]_*$$

$\therefore$ $a_1 = 1000$ (at the frequency of 1 KHz $\to 2000\pi$ rad/s)

$a_2 = 250$ (at the frequency of 600 Hz $\to 1200\pi$ rad/s) ✓

5.1.7. In the balanced modulator shown in Fig P-5.1.7, $\phi_1(t) = \cos(\omega_c t + \theta)$ and $\phi_2(t) = A\cos\omega_c t$. Show that the output voltage contains a term proportional to $\cos\theta$. Assume that the diode characteristics are piece-wise linear [that is, $i(t) = e(t)/r_d$ for $e(t) > 0$; $i(t) = 0$ for $e(t) < 0$] and that $A \gg 1$.



Solution: $e_1(t) = \phi_1(t) + \phi_2(t) = \cos(\omega_c t + \theta) + A\cos\omega_c t = (A + \cos\theta)\cos\omega_c t - \sin\theta \sin\omega_c t$

$e_2(t) = \phi_2(t) - \phi_1(t) = A\cos\omega_c t - \cos(\omega_c t + \theta) = (A - \cos\theta)\cos\omega_c t + \sin\theta\sin\omega_c t$

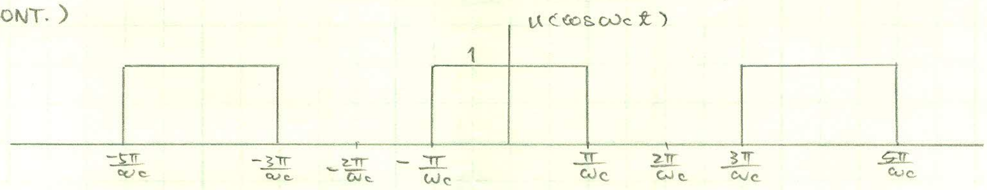
Because $A \gg 1$,

$$i_1(t) = e_1(t)/(r_d + R) \begin{cases} \frac{1}{R + r_d}\cos(\omega_c t + \theta) + \frac{A}{R + r_d}\cos\omega_c t & A\cos\omega_c t > 0 \\ 0 & A\cos\omega_c t < 0 \end{cases}$$

$$i_2(t) = e_2(t)/(r_d + R) = \begin{cases} \frac{A}{R + r_d}\cos\omega_c t - \frac{1}{R + r_d}\cos(\omega_c t + \theta) & A\cos\omega_c t > 0 \\ 0 & A\cos\omega_c t < 0 \end{cases}$$

$$e_o(t) = [i_1(t) - i_2(t)]R = \frac{2R}{R + r_d}\cos(\omega_c t + \theta)\, u(\cos\omega_c t)$$

5.17. (CONT.)



$$u(\cos\omega_c t) = \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_0 t}$$

$$\omega_0 = 2\pi/T = 2\pi/\left(\frac{2\pi}{\omega_c} + \frac{2\pi}{\omega_c}\right) = \omega_c/2$$

$$P_n = \frac{\omega_c}{4\pi}\int_{-\frac{\pi}{\omega_c}}^{\frac{\pi}{\omega_c}} 1 \cdot e^{-jn\omega_c/2\, t}\, dt$$

$$= \frac{1}{jn\cdot 2\pi}\int_{-\pi/\omega_c}^{\pi/\omega_c} e^{-jn\omega_c/2\, t}\, d(jn\omega_c/2\, t)$$

$$= \frac{1}{j2n\pi}\left(-e^{-jn\omega_c/2\, t}\Big|_{-\pi/\omega_c}^{\pi/\omega_c}\right)$$

$$= +\frac{1}{j2n\pi}\cdot\left(e^{+j\frac{n}{2}\pi} - e^{-j\frac{n\pi}{2}}\right)$$

$$= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$P_n = \begin{cases} 0 & n = 2k, \quad k = 1, 2, \cdots \; -1, -2, \cdots \\ \frac{1}{n\pi} & n = 4k+1 \\ -\frac{1}{n\pi} & n = 4k+3 \\ \frac{1}{2} & n = 0 \end{cases}$$

$$\therefore\ u(\cos\omega_c t) = \frac{1}{2} + \frac{2}{\pi}\cos\omega_c t - \frac{2}{3\pi}\cos 3\omega_c t + \cdots$$

$$e_o(t) = \frac{2R}{R+r_d}\cos(\omega_c t+\theta)\left(\frac{1}{2} + \frac{2}{\pi}\cos\omega_c t - \frac{2}{3\pi}\cos 3\omega_c t + \cdots\right)$$

$$= \frac{R}{R+r_d}\cos(\omega_c t+\theta) + \frac{4R}{\pi(R+r_d)}\cos(\omega_c t+\theta)\cos\omega_c t + \cdots$$

$$= \frac{2R}{\pi(R+r_d)}\left[\cos\theta + \cos(2\omega_c t+\theta)\right] + \cdots$$

$$= \frac{2R}{\pi(R+r_d)}\cos\theta + \frac{R}{R+r_d}\cos(\omega_c t+\theta) + \frac{2R}{\pi(R+r_d)}\cos(2\omega_c t+\theta) - \frac{2R}{3\pi(R+r_d)}\cos(\omega_c t+\theta)\cos 3\omega_c t + \cdots$$

USING A LPF @ $B = \frac{1}{2}\omega_c/2\pi = \omega_c/4\pi$, WE GET

$$e_o(t) = \frac{2R}{\pi(R+r_d)}\cos\theta$$

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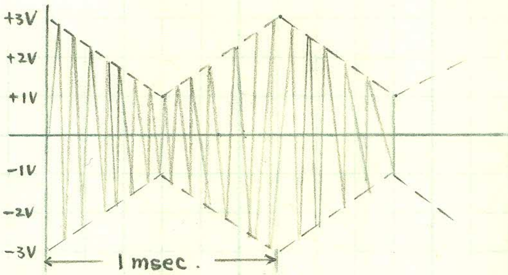
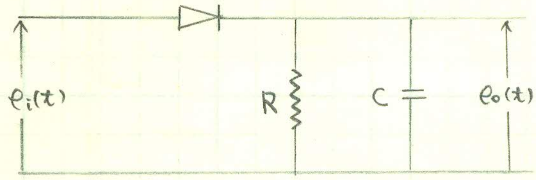
5.2.12





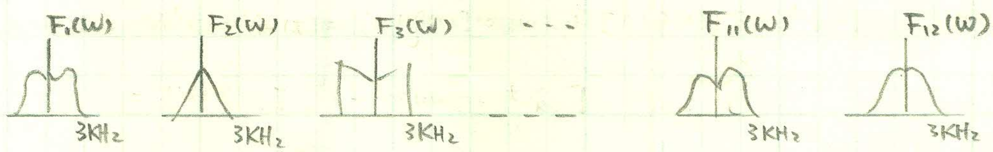
THE ENVELOPE DETECTOR

$$V_{RC}(t) = V_o(t)\, e^{-t/RC} \approx V_o(t)\left[1 + \left(-\frac{t}{RC}\right) + \cdots\right] = V_o(t)\left[1 - \frac{t}{RC}\right]$$

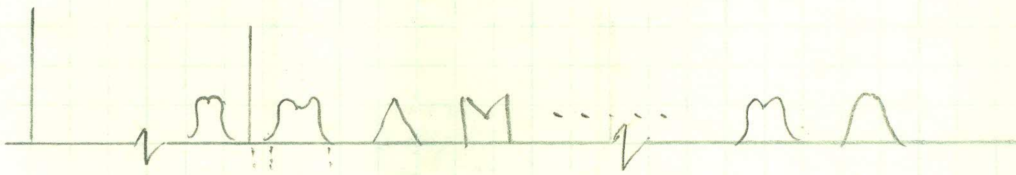
LET $t = 0$, $V_o(t) = 3V$ $\therefore -3/RC \geq -(3-1)/0.5 = -4$

$RC \leq 3/4 = 0.75$ ✓ 10

5.3.2

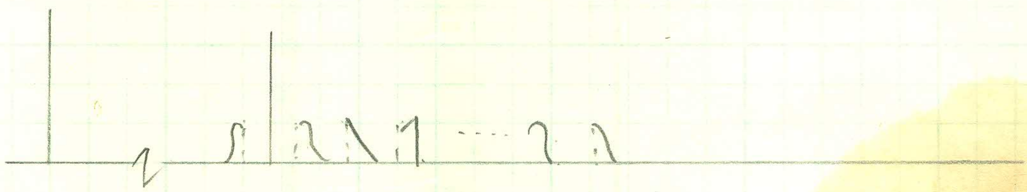


a)



$B = (6+1) \times 12 \times 2 = 168$ KHz ✓

b)



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$B = (3+1) \times 12 \times 2 = 96$ KHz ✓

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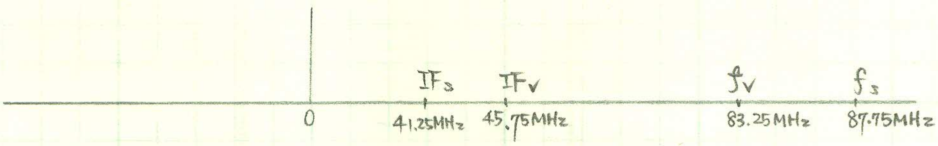
5.4.2



a) $f_0 = 45.75 + 83.25 = 41.25 + 87.75 = 129$ MHz

b) IF ONLY HAVE VIDEO CARRIER , WE HAVE TWO LOCAL OSCILLATOR FREQ. $f_{01} = 129$MHz , $f_{02} = 37.5$ MHz

IF ONLY HAVE SOUND CARRIER , WE HAVE TWO LOAL OSCILLATOR FREQ. $f_{01}' = 129$ MHz , $f_{02}' = 46.5$ MHz

SO, $f_0 = 129$ MHz is THE ONLY CHOICE.

c) $f_{IMAGE_1} = 129 + 83.25 = 212.25$ MHz

$f_{IMAGE_2} = 129 + 87.75 = 216.75$ MHz

ANSWER UHF CHANNEL 7.

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5.4.4.

5.4.6 SOLUTION: $f(t) = \cos\omega_m t$, $\hat{f}(t) = \cos(\omega_m t - 90° - \alpha) = \sin(\omega_m t - \alpha)$

UNDESIRED: $\phi_{SSB\mp}(t) = f(t)\cos\omega_c t \pm \hat{f}(t)\sin\omega_c t$

$$= \cos\omega_m t \cos\omega_c t \pm \sin(\omega_m t - \alpha)\sin\omega_c t$$

$$= \frac{1}{2}\cos(\omega_c + \omega_m)t + \frac{1}{2}\cos(\omega_c - \omega_m)t \pm \frac{1}{2}\cos(\omega_c - \omega_m + \alpha)t \mp \frac{1}{2}\cos(\omega_c + \omega_m - \alpha)$$

$$= \frac{1}{2}\cos(\omega_c + \omega_m)t + \frac{1}{2}\cos(\omega_c - \omega_m)t$$

$$\pm \frac{1}{2}\cos(\omega_c - \omega_m)t\cos\alpha \mp \frac{1}{2}\sin(\omega_c - \omega_m)t\sin\alpha$$

$$\mp \frac{1}{2}\cos(\omega_c + \omega_m)t\cos\alpha \pm \frac{1}{2}\sin(\omega_c + \omega_m)t\sin\alpha$$

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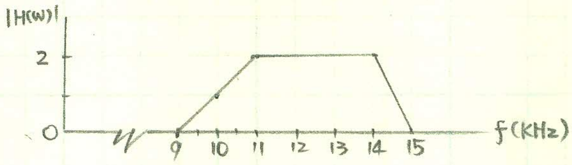
$$\frac{\text{Magnitude of desired output}}{\text{Magnitude of undesired output}} = \frac{\sqrt{(1-\cos\alpha)^2 + \sin^2\alpha}}{(1+\cos\alpha)^2 + \sin^2\alpha} = \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}} = \operatorname{tg}\alpha$$

#

5.5.1 DSB-LC , $f_c = 10\,KHz$, $m = 1$



a) $f(t) = \cos(1000\pi t)$ ✓

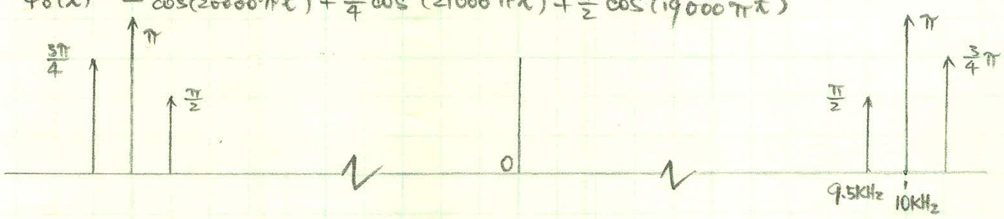
$$\phi(t) = [A + f(t)] \cos(20000\pi t)$$

$$= \cos(20000\pi t) + \cos(1000\pi t)\cos(20000\pi t)$$

$$\Phi(\omega) = \pi[\delta(\omega - 20000\pi) + \delta(\omega + 20000\pi)] + \pi/2[\delta(\omega + 21000\pi) + \delta(\omega + 19000\pi) + \delta(\omega - 19000\pi) + \delta(\omega - 21000\pi)]$$

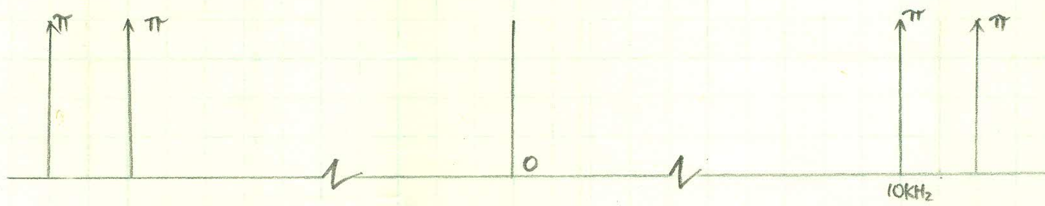
$$\Phi(\omega)\Big|_{H(\omega)} = \pi[\delta(\omega - 20000\pi) + \delta(\omega + 20000\pi)] + \frac{\pi}{2} \times \frac{3}{2}[\delta(\omega + 21000\pi) + \delta(\omega - 21000\pi)] + \frac{\pi}{2} \times \frac{1}{2}[\delta(\omega + 1900\pi) + \delta(\omega - 1900\pi)]$$

$$\phi_o(t) = \cos(20000\pi t) + \frac{3}{4}\cos(21000\pi t) + \frac{1}{2}\cos(19000\pi t)$$



b) $f(t) = \cos 2000\pi t$

$$\phi_o(t) = \cos 20000\pi t + \cos 21000\pi t$$



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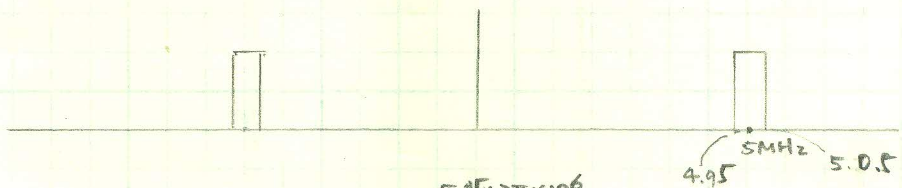
5.6.1

$$S_n(\omega) = \begin{cases} 10^{-7}\left(1 - \frac{|\omega|}{2\pi \times 10^7}\right) & |\omega| < 2\pi \times 10^7 \\ 0 & \text{elsewhere} \end{cases} \quad W/Hz$$

$n(t) = n_c(t)\cos\omega_0 t - n_s(t)\sin\omega_0 t$; $\omega_0 = 2\pi \times 5MHz = 10\pi \times 10^6$ rps



$$\overline{n_c^2(t)} = \overline{n_s^2(t)} = \overline{n^2(t)} = \frac{1}{\pi}\int_{4.95\times 2\pi\times 10^6}^{5.05\times 2\pi\times 10^6} 10^{-7}\left(1 - \frac{\omega}{2\pi\times 10^7}\right) d\omega$$

$$= 10^{-7}\cdot\frac{1}{\pi}\times 0.1\times 2\pi\times 10^6 - \frac{1}{2\pi}\times 10^{-14}\cdot\frac{1}{2\pi}\,\omega^2 \Big|_{4.95\times 2\pi\times 10^6}^{5.05\times 2\pi\times 10^6}$$

$= 0.01$ W ✓

$\sqrt{\overline{n_c^2(t)}} = \sqrt{\overline{n_s^2(t)}} = \sqrt{\overline{n^2(t)}} = 0.1$ V rms ✓

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5.7.1

DSB-SC

SSB-SC

a) $N_D = \frac{1}{\pi}\times 10^{-3}\times 10^{-6}\times 3\times 2\pi\times 10^3\times 2 = 12\ \mu W = 0.012$ mW

$N_S = \frac{1}{\pi}\times 10^{-3}\times 10^{-6}\times 3\times 2\pi\times 10^3 = 6\ \mu W = 0.006$ mW

FOR DSB-SC : $S/N = 1/0.012 = 83.33$

FOR SSB-SC : $S/N = 1/0.006 = 166.67$ ✓

b) FOR DSB-SC : $S_o/N_o = 2\,S/N = 166.67$

FOR SSB-SC : $S_o/N_o = S/N = 166.67$ ✓

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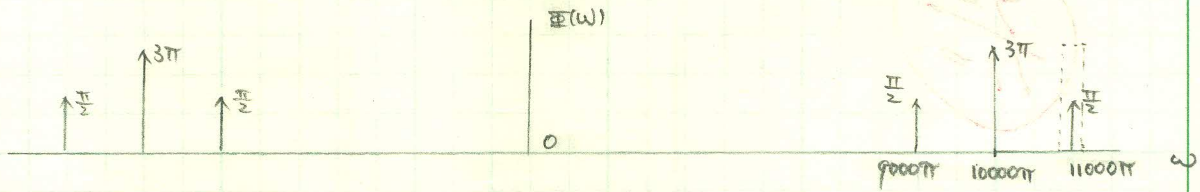
c) $f \approx 1MHz = 10^6 Hz$

$10^3 (1/|f|) \doteq 10^{-3}\ \mu W/Hz$

THE ANSWER : YES.

5.7.2.



a) $S = 9/2 + 1/2 + 1/2 = 5.5$

$N = \frac{1}{\pi}\int_0^{10000 \times 2\pi} 10^{-6}\, d\omega = 0.02\ W$ ✓

$S/N = 5.5/0.02 = 275 = 24.39\ dB$ ok.

b) $S_o = \frac{1}{4}$

$N_o = \frac{1}{\pi} \times 10^{-6} \times 100 = 0.000031831$

$S/N = 7853.98 = 38.95\ dB$ X. 625.

c) $S/N = \frac{1}{2} / 0.000159155 = 31415.92 = 44.97\ dB$ X 1250

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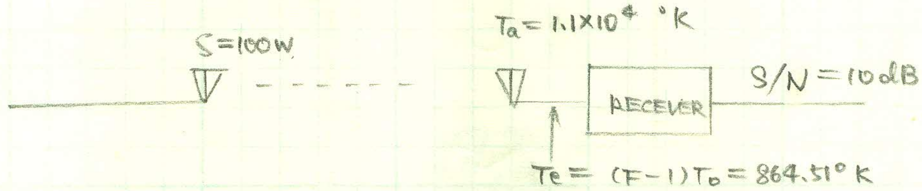
5.8.2



$N = k(T_a + T_e)B_N = 1.38 \times 10^{-23}(11000 + 864.51) \times 3 \times 10^3 = 4.91 \times 10^{-16} W$

$S = N \times (S/N) = 4.91 \times 10^{-16} \times 10 = 4.91 \times 10^{-15}\ W$

$\alpha = 37 + 20\log_{10} 10 + 20 \log d = 57 + 20 \log d = 100 / 4.91 \times 10^{-15} = 2.04 \times 10^{16} = 163.09$

$d = 201546.34$ miles.

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