TE 2401 Linear Algebra and Numerical Methods

Tutorial Set 1: Two Hours

1. (a) Show that the product $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$ is a symmetric matrix.

   (b) Show that any square matrix $\boldsymbol{A}$ can be written as the sum of a symmetric matrix $\boldsymbol{S}$ and a skew-symmetric matrix $\boldsymbol{T}$.

   (c) For the 2×2 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ given below, verify that $\boldsymbol{AB} \neq \boldsymbol{BA}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 \\ 6 & 4 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}.$$

   (d) Give an example to show that if the product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is the all-zero matrix, it does not imply that $\boldsymbol{A}$ or $\boldsymbol{B}$ has to be all-zero matrix.

2. (a) Show that the product of two upper triangular square matrices is a upper triangular matrix.

   (b) Can you make a similar statement about the product of two lower triangular square matrices?

   (c) It is given to us that the product of two symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is a symmetric matrix. Under what conditions does this property hold?

   (d) A matrix is called idempotent if $\boldsymbol{A}^2 = \boldsymbol{A}$. Give examples of 2×2 and 3×3 idempotent matrices.

3. Solve that the set of linear equations: $x - y = 3$ and $2x - 3y = k$. For which values of the constant k does it have no solution? many solutions? unique solutions?

4. Show that the matrix

$$\boldsymbol{A} = \begin{bmatrix} a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}$$

   can be reduced to the following echelon form

$$\boldsymbol{B} = \begin{bmatrix} a & b & c & d \\ 0 & b(b-a) & c(c-a) & d(d-a) \\ 0 & 0 & c(c-a)(c-b) & d(d-a)(d-b) \end{bmatrix}.$$

5. (a) Express $[0 \quad 0 \quad 0]$ as a non-trivial linear combination of $\boldsymbol{u} = [2 \quad 1 \quad 4]$, $\boldsymbol{v} = [1 \quad -1 \quad 3]$ and $\boldsymbol{w} = [3 \quad 2 \quad 5]$, if possible. Otherwise, give your comments.

   (b) Express $7+8x+9x^2$ as a linear combination of $\boldsymbol{p}_1 = 2+x+4x^2$, $\boldsymbol{p}_2 = 1-x+3x^2$ and $\boldsymbol{p}_3 = 3+2x+5x^2$.

6. Compute the inverse of the square matrix $\boldsymbol{A}$ given by

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 5 & 6 \end{bmatrix}.$$

Note that the inverse of a matrix $\boldsymbol{A}$ is defined as the matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{I}$, the identity matrix. It is represented by $\boldsymbol{A}^{-1}$. It can be computed by treating $\boldsymbol{AB} = \boldsymbol{I}$ as a set of linear equations to be solved for the elements of $\boldsymbol{B}$ where $\boldsymbol{A}$ is the coefficient matrix and successive n columns of $\boldsymbol{I}$ constitute the data vectors.

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Tutorial Set 2: Two Hours

1. (a) Determine whether the vectors $\boldsymbol{v}_1 = [\,3 \quad 1 \quad 4\,]$, $\boldsymbol{v}_2 = [\,2 \quad -3 \quad 5\,]$, $\boldsymbol{v}_3 = [\,5 \quad -2 \quad 9\,]$ and $\boldsymbol{v}_4 = [\,1 \quad 4 \quad -1\,]$ span $\mathbb{R}^3$.

   (b) Determine whether the following polynomials span P_2, i.e., the set of all polynomials of order less than or equal to 2:

$$\boldsymbol{p}_1 = 1 - x + 2x^2, \quad \boldsymbol{p}_2 = 3 + x, \quad \boldsymbol{p}_3 = 5 - x + 4x^2, \quad \boldsymbol{p}_4 = -2 - 2x + 2x^2.$$

2. Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be a basis for a vector space V. Show that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is also a basis where $\boldsymbol{u}_1 = \boldsymbol{v}_1$, $\boldsymbol{u}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_2$, $\boldsymbol{u}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3$.

3. (a) A vector space V consists of the vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ and all their linear combinations. Determine the dimension and the basis for V if

$$\boldsymbol{v}_1 = [\,1 \quad 2 \quad 3 \quad 4 \quad 5\,], \quad \boldsymbol{v}_2 = [\,1 \quad 3 \quad 5 \quad 6 \quad 7\,],$$

   and

$$\boldsymbol{v}_3 = [\,1 \quad 3 \quad 6 \quad 8 \quad 9\,], \quad \boldsymbol{v}_4 = [\,1 \quad 3 \quad 6 \quad 10 \quad 12\,].$$

   Is the basis unique? Justify your answer.

   (b) Express $\boldsymbol{v} = [\,5 \quad 4 \quad 3 \quad 2 \quad 1\,]$ as a linear combination of the basis vectors found in (a) if this vector belongs to the vector space V.

4. (a) If $\boldsymbol{A}$ is a $m \times p$ matrix, what is the largest possible value of its rank and the smallest possible value of its nullity.

   (b) Are there values of r and s for which the rank of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

   is one or two. If so, find those values.

5. (a) Show that the following set of linear equations has a unique solution and then find it:

$$\begin{array}{ccccccc} 4x_1 & - & x_2 & & & = & 1 \\ & & x_2 & + & 3x_3 & = & 0 \\ x_1 & & & - & 4x_3 & = & -2 \end{array}$$

(b) Show that the following homogeneous system has a non-trivial solution. Find the solution space and its dimensions:

$$\begin{array}{rcrcrcrcl} & & 9x_2 & + & x_3 & - & 5x_4 & = & 0 \\ x_1 & + & x_2 & & & - & 4x_4 & = & 0 \\ & & & & x_3 & + & 8x_4 & = & 0 \end{array}$$

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TE 2401 Linear Algebra and Numerical Methods

Tutorial Set 3: Two Hours

1. (a) Find the characteristic equation, eigenvalues and the corresponding eigenvectors for the following matrix,

$$\boldsymbol{A} = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

   (b) From the characteristic equation, find the determinant of $\boldsymbol{A}$. What is the rank of $\boldsymbol{A}$?

   (c) Now consider $\boldsymbol{A}^{\mathrm{T}}$. Find its eigenvalues and eigenvectors. Let the eigenvalues be λ_1, λ_2 and λ_3 and the corresponding eigenvectors of $\boldsymbol{A}$ be $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ and those of $\boldsymbol{A}^{\mathrm{T}}$ be $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$. Verify that for $i \neq j$, $\boldsymbol{x}_i$ and $\boldsymbol{y}_j$ are orthogonal.

2. (a) Consider the symmetric matrix $\boldsymbol{A}$, where

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

   Find its eigenvalues and normalized eigenvectors.

   (b) Show that these eigenvectors are orthogonal to each other.

   (c) Arrange these eigenvectors as the columns of a matrix $\boldsymbol{P}$. Find the inverse of $\boldsymbol{P}$. Verify that in this case $\boldsymbol{P}^{-1} = \boldsymbol{P}^{\mathrm{T}}$.

3. Show that the determinant of a Vandermonde matrix $\boldsymbol{V}$ given by

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix},$$

   is non-zero if a, b and c are distinct.

4. Given a matrix

$$\boldsymbol{A} = \begin{bmatrix} -2 & 2 & 1 \\ -1 & 1 & 1 \\ -2 & 2 & 1 \end{bmatrix},$$

   compute $\boldsymbol{A}^{314150}$, $\cos(\boldsymbol{A}\pi)$ and $e^{\mathbf{A}}$.

5. Show that $Q = 3x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_3 + 2kx_2x_3$ can be expressed as $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$. Find the symmetric matrix $\boldsymbol{A}$. Find all values of k for which the quadratic form Q is positive definite.

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TE 2401 Linear Algebra and Numerical Methods

Tutorial Set 4: Two Hours

1. Assume that the true value of π is 3.14159265 to eight decimal places.

   (a) Find the absolute and relative errors in approximating π by $\bar{\pi} = 3.1416$

   (b) Find the absolute and relative errors in approximating 100π by $100\bar{\pi} = 314.16$

2. Discuss the convergence of the following two iterative schemes for possible solution of $f(x) = e^x - 1 - 2x = 0$

   (a) $x_{n+1} = (e^{x_n} - 1)/2$

   (b) $x_{n+1} = \ln(1 + 2x_n)$

3. Solve the equation $e^x - 1 - 2x = 0$ by Newton's method. The solution is expected to be computed up to 7 decimal places. Compare the speed of convergence with that using 2(b), assuming the same starting point of $x_0 = 1.5$.

4. Consider the problem of computing $\sqrt{a}$ for a positive real number a.

   (a) Use Newton's method to design an iterative equation for this purpose.

   (b) Derive the absolute and relative iterative error formulas for the equation obtained in (a).

5. (a) Complete Newton's Divided Deference table for $\sin(x)$ with $x_j = 0,\ 0.1\pi,\ 0.2\pi,\ \cdots,\ 0.5\pi$.

x_j	$\sin(x_j)$	$f_0^{[1]}$	$f_0^{[2]}$	$f_0^{[3]}$	$f_0^{[4]}$	$f_0^{[5]}$
0	0					
0.1π	0.30902					
0.2π	0.58779					
0.3π	0.80902					
0.4π	0.95106					
0.5π	1					

(b) From this table, compute $\sin(\pi/4)$ using linear and quadratic interpolation from the intervals: $(0.2\pi, \cdots)$.

(c) The error of the n-th order polynomial $p_n(x)$ interpolating $\{x_i\}$ is given by

$$e_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\eta)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where η is a point in the smallest interval containing $\{x_i\}$. Also note that x_0 in the above formula is the actual starting point of the intervals for interpolation while in case (b), $x_0 = 0.2\pi$.

Use the above formula to compute error estimates for the linear and quadratic interpolations in (b) and compare these estimates with the actual errors.

6. (a) Find the least square quadratic fit to the following shifted data for the function:

$$f(x) = \frac{x+2}{4}e^{(x+2)/4} - 1$$

x_i	-3	-2	-1	0	1	2	3
$f(x_i)$	-1.195	-1	-0.679	-0.1756	0.5878	1.718	3.363

(b) Compute the error at the data points and describe how they are distributed?

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TE 2401 Linear Algebra and Numerical Methods

Tutorial Set 5: Two Hours

1. Compute $J = \int_0^1 f(x)dx$ with $f(x) = x^3$ using the following numerical approximations:

   (a) The Trapezoidal Rule:

   $$J_n = \frac{1}{2n}\left[f(0) + f(1) + 2\sum_{j=1}^{n-1} f\left(\frac{j}{n}\right)\right], \quad n = 1, 2, 4$$

   and compute $J - J_i$, $i = 1, 2, 4$.

   (b) The simpler (midpoint) Rectangular Rule

   $$I_n = \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{2j-1}{2n}\right), \quad n = 1, 2, 4$$

   and compute $J - I_i$, $i = 1, 2, 4$.

   (c) Simpson's $\frac{1}{3}$ Rule with $h = 0.5$,

   $$K_1 = \frac{h}{3}\big[f(0) + 4f(h) + f(1)\big]$$

   and with $h = 0.25$

   $$K_2 = \frac{h}{3}\big[f(0) + 4f(h) + 2f(2h) + 4f(3h) + f(1)\big]$$

   and compute $J - K_i$, $i = 1, 2$.

2. Evaluate the following integrals numerically as indicated.

   $$\mathrm{Si}(x) = \int_0^x \frac{\sin x^*}{x^*}dx^*, \quad \mathrm{C}(x) = \int_0^x \cos(x^{*2})dx^*$$

   These are non-elementary integrals. $\mathrm{Si}(x)$ is called the sine integral and $\mathrm{C}(x)$ the Fresnel integral.

   (a) $\mathrm{Si}(1)$ by the trapezoidal rule with $n = 5$ and $n = 10$.

   (b) $\mathrm{Si}(1)$ by the Simpson's $\frac{1}{3}$ rule with $h = 0.5$ and $h = 0.1$.

   (c) $\mathrm{C}(2)$ by the trapezoidal rule with $n = 10$.

   (d) $\mathrm{C}(2)$ by the Simpson's $\frac{1}{3}$ rule with $h = 0.2$.

3. Given the following initial condition problem

$$y'(x) = y(x), \quad y(0) = 1$$

(a) Show that the solution of Euler's method with the step-size h can be written as

$$y_n = c(h)^{x_n}, \quad n = 0, 1, 2, \cdots$$

where $c(h) = (1+h)^{1/h}$ (note that $\lim_{h\to 0} c(h) = e$) and $x_n = nh$.

(b) Show that

$$\max_{0\le x\le 1} |y(x_n) - y_n| = e - c(h) \approx \frac{h}{2}e$$

where $e = 2.718281828\cdots$, the natural number.

4. Two second order methods for solving numerically the initial condition problem:

$$\frac{dy}{dx} = f(x, y)$$

with $x_0 = x(0)$ and $y_0 = y(0)$ are:

Method I	Method II
$x_0 = x(0), \quad y_0 = y(0)$	$x_0 = x(0), \quad y_0 = y(0)$
$x_{n+1} = x_n + h$	$x_{n+1} = x_n + h$
$k_1 = h \cdot f(x_n, y_n)$	$k_1 = h \cdot f(x_n, y_n)$
$k_2 = h \cdot f(x_{n+1}, y_n + k_1)$	$k_3 = h \cdot f(x_{n+1} + h, y_n + 2k_1)$
$y_{n+1} = y_n + (k_1 + k_2)/2$	$y_{n+1} = y_n + (3k_1 + k_3)/4$

(a) Show that these methods give identical results for $f(x, y) = x + y$.

(b) For $f(x, y) = y + 1 + e^x$, take four steps with each method, and compare your results with analytical solution,

$$y = xe^x - 1$$

Use $h = 0.05$ and initial conditions: $x(0) = 0$ and $y(0) = -1$.

5. Find a numerical solution for a vibrating string which is characterized by a wave equation

$$u_{tt} = u_{xx}, \quad 0 \le x \le 1, \;\; t \ge 0$$

with an initial displacement

$$u(x,0) = \sin(\pi x)$$

and an initial velocity

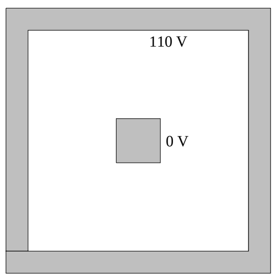
$$u_t(x,0) = 0$$

and boundary conditions

$$u(0,t) = u(1,t) = 0$$

Let $h = k = 0.2$ and compute the solution $u(x,t)$ for t up to 1 second. Note that the exact solution to this problem is given by $u(x,t) = \sin(\pi x)\cos(\pi t)$. Compare your results with the exact solution.

6. The cross-section of a coaxial cable has an inner 1-cm-square conductor at the center of an outer conductor which has a 5-cm-square inner boundary, as shown in the figure. The inner conductor is kept at zero volts while the outer conductor is kept at 110 volts.



To find an approximation of the potential between the two conductors, place a grid with horizontal mesh spacing $h = 1$ cm and vertical mesh spacing $k = 1$ cm on the region:

$$R = \{(x,y) \mid 0 \le x \le 5, \; 0 \le y \le 5\}$$

where we want to approximate the solution of Laplace's equation $u_{xx} + u_{yy} = 0$. Use the central difference approximations and the given boundary conditions to derive the linear system that needs to be solved. Solve it using any software package.

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TE 2401 Linear Algebra and Numerical Methods

Solutions to Tutorial Set 1

1. (a) Show that the product $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$ is a symmetric matrix.

   (b) Show that any square matrix $\boldsymbol{A}$ can be written as the sum of a symmetric matrix $\boldsymbol{S}$ and a skew-symmetric matrix $\boldsymbol{T}$.

   (c) For the 2×2 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ given below, verify that $\boldsymbol{A}\boldsymbol{B} \neq \boldsymbol{B}\boldsymbol{A}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 \\ 6 & 4 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}.$$

   (d) Give an example to show that if the product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is the all-zero matrix, it does not imply that $\boldsymbol{A}$ or $\boldsymbol{B}$ has to be all-zero matrix.

Solution: (a)

$$(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = (\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$$

Hence, $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$ is a symmetric matrix.

(b) Let

$$\boldsymbol{S} := \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}), \quad \boldsymbol{T} := \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}).$$

We have

$$\boldsymbol{S}^{\mathrm{T}} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \frac{1}{2}(\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A}) = \boldsymbol{S}.$$

Thus, $\boldsymbol{S}$ is symmetric. Similarly, we have

$$\boldsymbol{T}^{\mathrm{T}} = \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \frac{1}{2}(\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{A}) = -\boldsymbol{T},$$

which implies that $\boldsymbol{T}$ is skew-symmetric. Finally, it is simple to see that $\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{T}$.

(c)

$$\boldsymbol{A}\boldsymbol{B} = \begin{bmatrix} 1 & 5 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 20 & 9 \\ 42 & 2 \end{bmatrix}$$

$$\boldsymbol{B}\boldsymbol{A} = \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 21 \\ 15 & 23 \end{bmatrix}$$

Clearly, $\boldsymbol{A}\boldsymbol{B} \neq \boldsymbol{B}\boldsymbol{A}$.

(d)

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

2. (a) Show that the product of two upper triangular square matrices is a upper triangular matrix.

   (b) Can you make a similar statement about the product of two lower triangular square matrices?

   (c) It is given to us that the product of two symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is a symmetric matrix. Under what conditions does this property hold?

   (d) A matrix is called idempotent if $\boldsymbol{A}^2 = \boldsymbol{A}$. Give examples of 2×2 and 3×3 idempotent matrices.

Solution: (a) An upper triangular matrix $\boldsymbol{A}$ has elements $a_{ij} = 0$ for $j < i$. The product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, say $\boldsymbol{C}$, whose elements are given by

$$c_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}$$

For the case when both $\boldsymbol{A}$ and $\boldsymbol{B}$ are upper triangular, we have $a_{i\ell} = 0$ for all $\ell < i$ and $b_{\ell j} = 0$ for all $j < \ell$. Hence,

$$c_{ij} = \sum_{\ell=i}^{j} a_{i\ell} b_{\ell j}$$

For $j < i$, $c_{ij} = 0$, which implies that $\boldsymbol{C} = \boldsymbol{AB}$ is an upper triangular matrix.

(b) Yes.

(c) $\boldsymbol{AB} = (\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}} \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{BA}$. $\implies$ $\boldsymbol{A}$ and $\boldsymbol{B}$ commute.

(d) Let

$$\boldsymbol{A} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \implies \quad \boldsymbol{A}^2 = \begin{bmatrix} a^2 + bc & (a+d)b \\ (a+d)c & d^2 + bc \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Thus, $a + d = 1$. Let $a = d = \frac{1}{2}$, which implies that $bc = \frac{1}{4}$. Let us choose $b = 1$ and $c = \frac{1}{4}$. Here is a 2×2 idempotent matrix,

$$\boldsymbol{A} = \frac{1}{4} \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}.$$

Here are some more,

$$\boldsymbol{A} = \frac{1}{3} \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix}, \quad \boldsymbol{A} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Here are 3×3 idempotent matrices,

$$\boldsymbol{A} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Solve that the set of linear equations: $x - y = 3$ and $2x - 3y = k$. For which values of the constant k does it have no solution? many solutions? unique solutions?

Solution:

$$x - y = 3 \tag{0.1}$$

$$2x - 3y = k \tag{0.2}$$

Equation (0.2) $-2\times$ Equation (0.1) yields

$$-y = k - 6 \quad \Longrightarrow \quad y = 6 - k$$

$$\Downarrow$$

$$x = 9 - k$$

There is always a unique solution for every k.

4. Show that the matrix

$$\boldsymbol{A} = \begin{bmatrix} a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}$$

can be reduced to the following echelon form

$$\boldsymbol{B} = \begin{bmatrix} a & b & c & d \\ 0 & b(b-a) & c(c-a) & d(d-a) \\ 0 & 0 & c(c-a)(c-b) & d(d-a)(d-b) \end{bmatrix}.$$

Proof:

$$\begin{bmatrix} a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix} \quad \text{2nd row } - \, a\times \text{ 1st row } \ldots$$

$$\rightarrow \begin{bmatrix} a & b & c & d \\ 0 & b(b-a) & c(c-a) & d(d-a) \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix} \quad \text{3rd row } - \, a^2\times \text{ 1st row } \ldots$$

$$\rightarrow \begin{bmatrix} a & b & c & d \\ 0 & b(b-a) & c(c-a) & d(d-a) \\ 0 & b(b^2-a^2) & c(c^2-a^2) & d(d^2-a^2) \end{bmatrix} \quad \text{3rd row } - \, (b+a)\times \text{ 2nd row } \ldots$$

$$\rightarrow \begin{bmatrix} a & b & c & d \\ 0 & b(b-a) & c(c-a) & d(d-a) \\ 0 & 0 & c(c-a)(c-b) & d(d-a)(d-b) \end{bmatrix}$$

5. (a) Express $[\,0 \quad 0 \quad 0\,]$ as a non-trivial linear combination of $\boldsymbol{u} = [\,2 \quad 1 \quad 4\,]$, $\boldsymbol{v} = [\,1 \quad -1 \quad 3\,]$ and $\boldsymbol{w} = [\,3 \quad 2 \quad 5\,]$, if possible. Otherwise, give your comments.

(b) Express $7+8x+9x^2$ as a linear combination of $\boldsymbol{p}_1 = 2+x+4x^2$, $\boldsymbol{p}_2 = 1-x+3x^2$ and $\boldsymbol{p}_3 = 3+2x+5x^2$.

Solution: (a) Let a, b and c be scalars such that $a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = \mathbf{0} \Longrightarrow$

$$a\,[\,2 \quad 1 \quad 4\,] + b\,[\,1 \quad -1 \quad 3\,] + c\,[\,3 \quad 2 \quad 5\,] = [\,0 \quad 0 \quad 0\,]$$

$$[\,2a+b+3c \quad a-b+2c \quad 4a+3b+5c\,] = [\,0 \quad 0 \quad 0\,]$$

$$\Downarrow$$

$$\begin{array}{rcrcrcr} 2a & + & b & + & 3c & = & 0 \\ a & - & b & + & 2c & = & 0 \\ 4a & + & 3b & + & 5c & = & 0 \end{array}$$

The augmented matrix is then given by

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & -2/3 & 0 \end{bmatrix}$$

$\Longrightarrow a = b = c = 0$. Thus, $\mathbf{0}$ cannot be expressed by a non-trivial linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$. The given vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent.

(b) Let a, b and c be the scalars such that

$$a\boldsymbol{p}_1 + b\boldsymbol{p}_2 + c\boldsymbol{p}_3 = 7 + 8x + 9x^2 \quad \Longrightarrow$$

$$a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) = 7 + 8x + 9x^2$$

Equating coefficients of x^0, x and x^2, we obtain

$$\begin{array}{rcrcrcr} 2a & + & b & + & 3c & = & 7 \\ a & - & b & + & 2c & = & 8 \\ 4a & + & 3b & + & 5c & = & 9 \end{array}$$

$\Longrightarrow$

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 2 & 1 & 3 & 7 \\ 1 & -1 & 2 & 8 \\ 4 & 3 & 5 & 9 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 1 & -1/3 & -3 \\ 0 & 0 & -2/3 & -2 \end{bmatrix}$$

$\Longrightarrow a = 0$, $b = -2$ and $c = 3$.

6. Compute the inverse of the square matrix $\boldsymbol{A}$ given by

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 5 & 6 \end{bmatrix}.$$

Note that the inverse of a matrix $\boldsymbol{A}$ is defined as the matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{I}$, the identity matrix. It is represented by $\boldsymbol{A}^{-1}$. It can be computed by treating $\boldsymbol{AB} = \boldsymbol{I}$ as a set of linear equations to be solved for the elements of $\boldsymbol{B}$ where $\boldsymbol{A}$ is the coefficient matrix and successive n columns of $\boldsymbol{I}$ constitute the data vectors.

Solution: The inverse matrix of $\boldsymbol{A}$, say $\boldsymbol{B}$, if existent, is defined as $\boldsymbol{AB} = \boldsymbol{I}$. If we partition $\boldsymbol{B} = [\,\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \cdots \quad \boldsymbol{b}_n\,]$, i.e., $\boldsymbol{b}_i$ is the i-th column of $\boldsymbol{B}$, then $\boldsymbol{AB} = \boldsymbol{I}$ can be written as

$$\boldsymbol{AB} = \boldsymbol{A}\,[\,\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \cdots \quad \boldsymbol{b}_n\,] = \boldsymbol{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\boldsymbol{Ab}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{Ab}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \boldsymbol{Ab}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Writing the augmented matrix for these system together, we have

$$\tilde{\boldsymbol{A}} = [\,\boldsymbol{A} \quad | \quad \boldsymbol{I}\,]$$

In our case,

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 3 & 2 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 1 & | & 0 & 1 & 0 \\ 0 & 5 & 6 & | & 0 & 0 & 1 \end{bmatrix}$$

and the echelon form is given by

$$\begin{bmatrix} 1 & 2/3 & 1/3 & | & 1/3 & 0 & 0 \\ 0 & 4/3 & 2/3 & | & -1/3 & 1 & 0 \\ 0 & 0 & 7/10 & | & 1/4 & -3/4 & 1/5 \end{bmatrix}$$

The solution vectors are

$$\boldsymbol{b}_1 = \frac{1}{14}\begin{bmatrix} 7 \\ -6 \\ 5 \end{bmatrix}, \quad \boldsymbol{b}_2 = \frac{1}{14}\begin{bmatrix} -7 \\ 18 \\ -15 \end{bmatrix}, \quad \boldsymbol{b}_3 = \frac{1}{14}\begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix},$$

and the inverse of $\boldsymbol{A}$, i.e., $\boldsymbol{A}^{-1} = \boldsymbol{B}$, is given by

$$\boldsymbol{A}^{-1} = \frac{1}{14}\begin{bmatrix} 7 & -7 & 0 \\ -6 & 18 & -2 \\ 5 & -15 & 4 \end{bmatrix}.$$

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TE 2401 Linear Algebra and Numerical Methods

Solutions to Tutorial Set 2

1. (a) Determine whether the vectors $\boldsymbol{v}_1 = [\,3 \quad 1 \quad 4\,]$, $\boldsymbol{v}_2 = [\,2 \quad -3 \quad 5\,]$, $\boldsymbol{v}_3 = [\,5 \quad -2 \quad 9\,]$ and $\boldsymbol{v}_4 = [\,1 \quad 4 \quad -1\,]$ span $\mathbb{R}^3$.

   (b) Determine whether the following polynomials span P_2, i.e., the set of all polynomials of order less than or equal to 2:

$$\boldsymbol{p}_1 = 1 - x + 2x^2, \quad \boldsymbol{p}_2 = 3 + x, \quad \boldsymbol{p}_3 = 5 - x + 4x^2, \quad \boldsymbol{p}_4 = -2 - 2x + 2x^2.$$

Solution: Let $\boldsymbol{r} = (\,r_1 \quad r_2 \quad r_3\,)$ be any arbitrary vector in $\mathbb{R}^3$. If $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ span $\mathbb{R}^3$, then $\boldsymbol{r}$ can be expressed as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$, i.e., there exist scalars a, b, c and d such that

$$a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3 + d\boldsymbol{v}_4 = \boldsymbol{r} = (\,r_1 \quad r_2 \quad r_3\,).$$

$\Longrightarrow$

$$\begin{array}{rcrcrcrcl} 3a & + & 2b & + & 5c & + & d & = & r_1 \\ a & - & 3b & - & 2c & + & 4d & = & r_2 \\ 4a & + & 5b & + & 9c & - & d & = & r_3 \end{array}$$

The augmented matrix is the given by

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 3 & 2 & 5 & 1 & r_1 \\ 1 & -3 & -2 & 4 & r_2 \\ 4 & 5 & 9 & -1 & r_3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -3 & -2 & 4 & r_2 \\ 0 & 11 & 11 & -11 & r_1 - 3r_2 \\ 0 & 0 & 0 & 0 & r_3 + 7r_2/11 - 17r_1/11 \end{bmatrix}$$

Clearly, the system has an solution only if $11r_3 + 7r_2 - 17r_1 = 0$. Therefore, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ do not span $\mathbb{R}^3$.

(b) An arbitrary polynomial $\boldsymbol{p}$ in P_2 can be expressed as

$$\boldsymbol{p} = a_0 + a_1 x + a_2 x^2.$$

If $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$ and $\boldsymbol{p}_3$ span P_2, there exist scalars a, b, c and d such that

$$\boldsymbol{p}(x) = a\boldsymbol{p}_1 + b\boldsymbol{p}_2 + c\boldsymbol{p}_3 + d\boldsymbol{p}_4$$

$\Longrightarrow$

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 1 & 3 & 5 & -2 & a_0 \\ -1 & 1 & -1 & -2 & a_1 \\ 2 & 0 & 4 & 2 & a_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 3 & 5 & -2 & a_0 \\ 0 & 1 & 1 & -1 & (a_0 + a_1)/4 \\ 0 & 0 & 0 & 0 & -a_0 + 3a_1 + 2a_2 \end{bmatrix}$$

A solution exists only if $-a_0 + 3a_1 + 2a_2 = 0$. The given polynomials do not span P_2.

2. Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be a basis for a vector space V. Show that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is also a basis where $\boldsymbol{u}_1 = \boldsymbol{v}_1$, $\boldsymbol{u}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_2$, $\boldsymbol{u}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3$.

   Proof: $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are a basis of the vector space V if and only if

   (a) For any $\boldsymbol{v} \in V$, $\boldsymbol{v}$ can be expressed as a linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$; and

   (b) $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are linearly independent.

   Next, $\boldsymbol{u}_1 = \boldsymbol{v}_1$, $\boldsymbol{u}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_2$ and $\boldsymbol{u}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3$ imply that $\boldsymbol{v}_1 = \boldsymbol{u}_1$, $\boldsymbol{v}_2 = \boldsymbol{u}_2 - \boldsymbol{u}_1$ and $\boldsymbol{v}_3 = \boldsymbol{u}_3 - \boldsymbol{u}_2$. Since $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are a basis of V, thus for any $\boldsymbol{v}$ in V, there exist a_1, a_2 and a_3 such that

$$\begin{aligned}
\boldsymbol{v} &= a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 \\
&= a_1\boldsymbol{u}_1 + a_2(\boldsymbol{u}_2 - \boldsymbol{u}_1) + a_3(\boldsymbol{u}_3 - \boldsymbol{u}_2) \\
&= (a_1 - a_3)\boldsymbol{u}_1 + (a_2 - a_3)\boldsymbol{u}_2 + a_3\boldsymbol{u}_3 \\
&= b_1\boldsymbol{u}_1 + b_2\boldsymbol{u}_2 + b_3\boldsymbol{u}_3
\end{aligned}$$

   for some scalars b_1, b_2 and b_3. Hence, any vector in V can be expressed as a linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$, i.e., property (a) is showed. Next, to prove that they are linearly independent, we consider

$$a\boldsymbol{u}_1 + b\boldsymbol{u}_2 + c\boldsymbol{u}_3 = 0$$

   $\Longrightarrow$

$$a\boldsymbol{v}_1 + b(\boldsymbol{v}_1 + \boldsymbol{v}_2) + c(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) = 0$$

   $\Longrightarrow$

$$(a + b + c)\boldsymbol{v}_1 + (b + c)\boldsymbol{v}_2 + c\boldsymbol{v}_3 = 0$$

   Since $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are a basis of V, hence they are linearly independent and

$$a + b + c = 0, \quad b + c = 0, \quad c = 0.$$

   $\Longrightarrow$

$$a = 0, \quad b = 0, \quad c = 0.$$

   Thus, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are linearly independent, i.e., property (b) is showed. Therefore, they form a basis of V as well.

3. (a) A vector space V consists of the vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ and all their linear combinations. Determine the dimension and the basis for V if

$$\boldsymbol{v}_1 = [\,1 \quad 2 \quad 3 \quad 4 \quad 5\,], \quad \boldsymbol{v}_2 = [\,1 \quad 3 \quad 5 \quad 6 \quad 7\,],$$

and

$$\boldsymbol{v}_3 = [\,1 \quad 3 \quad 6 \quad 8 \quad 9\,], \quad \boldsymbol{v}_4 = [\,1 \quad 3 \quad 6 \quad 10 \quad 12\,].$$

Is the basis unique? Justify your answer.

(b) Express $\boldsymbol{v} = [\,5 \quad 4 \quad 3 \quad 2 \quad 1\,]$ as a linear combination of the basis vectors found in (a) if this vector belongs to the vector space V.

Solution: (a) We need to find the largest number of linearly independent vectors in V that span V. Using the procedure in the lecture note, we first form

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \boldsymbol{v}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 6 & 7 \\ 1 & 3 & 6 & 8 & 9 \\ 1 & 3 & 6 & 10 & 12 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix}$$

Clearly, the dimension of V is 4 and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ form a basis for V, as they are linearly independent. Obviously, basis is non-unique. The following is another basis for V:

$$\boldsymbol{u}_1 = [\,1 \quad 2 \quad 3 \quad 4 \quad 5\,], \quad \boldsymbol{u}_2 = [\,0 \quad 1 \quad 2 \quad 2 \quad 2\,],$$

$$\boldsymbol{u}_3 = [\,0 \quad 0 \quad 1 \quad 2 \quad 2\,], \quad \boldsymbol{u}_4 = [\,0 \quad 0 \quad 0 \quad 2 \quad 3\,].$$

(b) Let

$$\boldsymbol{v} = [\,5 \quad 4 \quad 3 \quad 2 \quad 1\,] = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 + a_4\boldsymbol{v}_4$$

The augmented matrix is then given by

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 5 \\ 2 & 3 & 3 & 3 & 4 \\ 3 & 5 & 6 & 6 & 3 \\ 4 & 6 & 8 & 10 & 2 \\ 5 & 7 & 9 & 12 & 1 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & -6 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

Clearly, no solution exists to this linear system, or equivalently, $\boldsymbol{v}$ does not belong to the vector space V.

4. (a) If $\boldsymbol{A}$ is a $m \times p$ matrix, what is the largest possible value of its rank and the smallest possible value of its nullity.

   (b) Are there values of r and s for which the rank of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

   is one or two. If so, find those values.

Solution: (a)

$$\text{maximal rank of an } m \times p \text{ matrix } \boldsymbol{A} = \min(m, p)$$

Using the property,

$$\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) = p$$

$\Longrightarrow$

$$\text{nullity}(\boldsymbol{A}) \geq p - \min(m, p).$$

(b) From (a), we know that

$$\text{maximal rank of } \boldsymbol{A} = \min(4, 3) = 3$$

Observing that the first row and the fourth row are always linearly independent regardless the choices of r and s. Thus, the given matrix has at least a rank of 2. The only choice to make its rank equal to 2 is to choose $r = 2$ and $s = 1$ such that the second column is identically zero. Otherwise, its rank is always equal to 3.

5. (a) Show that the following set of linear equations has a unique solution and then find it:

$$\begin{array}{rcrcrcr} 4x_1 & - & x_2 & & & = & 1 \\ & & x_2 & + & 3x_3 & = & 0 \\ x_1 & & & - & 4x_3 & = & -2 \end{array}$$

(b) Show that the following homogeneous system has a non-trivial solution. Find the solution space and its dimensions:

$$\begin{array}{rcrcrcrcr} & & 9x_2 & + & x_3 & - & 5x_4 & = & 0 \\ x_1 & + & x_2 & & & - & 4x_4 & = & 0 \\ & & & & x_3 & + & 8x_4 & = & 0 \end{array}$$

Proof: (a) The augmented matrix for the system is given by

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 4 & -1 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & -4 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 4 & -2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 19 & 9 \end{bmatrix}$$

$\Longrightarrow$ rand$(\boldsymbol{A})$ = rank$(\tilde{\boldsymbol{A}}) = 3$. Hence, the given system has at least one solution. Also, the number of unknowns is equal to 3, which implies that the system has a unique solution and it is given by

$$x_1 = -\frac{2}{19}, \quad x_2 = -\frac{27}{19}, \quad x_3 = \frac{9}{19}.$$

(b) The coefficient matrix of the system is

$$\boldsymbol{A} = \begin{bmatrix} 0 & 9 & 1 & -5 \\ 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 1 & 1/9 & -5/9 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

$\Longrightarrow$ rank $(\boldsymbol{A}) = 3 < 4 =$ the number of unknowns. $\Longrightarrow$ there exist non-trivial solutions. The augmented matrix,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 9 & 1 & -5 & 0 \\ 1 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 8 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 0 & -4 & 0 \\ 0 & 1 & 1/9 & -5/9 & 0 \\ 0 & 0 & 1 & 8 & 0 \end{bmatrix}$$

$\Longrightarrow x_1$, x_2 and x_3 are leading variables (= rank$(\boldsymbol{A})$); x_4 is free variable ($= n-$rank$(\boldsymbol{A})$). The dimension of the solution space is equal to 1. This space is completely characterized by 1 linearly independent solution vector,

$$\text{solution vector} = \begin{pmatrix} \frac{23}{9} & \frac{13}{9} & -8 & 1 \end{pmatrix} x_4,$$

where x_4 is totally free.

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TE 2401 Linear Algebra and Numerical Methods

Solutions to Tutorial Set 3

1. (a) Find the characteristic equation, eigenvalues and the corresponding eigenvectors for the following matrix,

$$\boldsymbol{A} = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

(b) From the characteristic equation, find the determinant of $\boldsymbol{A}$. What is the rank of $\boldsymbol{A}$?

(c) Now consider $\boldsymbol{A}^{\mathrm{T}}$. Find its eigenvalues and eigenvectors. Let the eigenvalues be λ_1, λ_2 and λ_3 and the corresponding eigenvectors of $\boldsymbol{A}$ be $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ and those of $\boldsymbol{A}^{\mathrm{T}}$ be $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$. Verify that for $i \neq j$, $\boldsymbol{x}_i$ and $\boldsymbol{y}_j$ are orthogonal.

Solution: (a) The characteristic equation of $\boldsymbol{A}$ is given by

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0 \quad \text{or} \quad \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

or

$$\begin{vmatrix} \lambda - 5 & -6 & -2 \\ 0 & \lambda + 1 & 1 \\ -1 & 0 & \lambda - 2 \end{vmatrix} = 0 \implies \lambda^3 - 6\lambda^2 + \lambda + 14 = 0$$

$\implies$ The eigenvalues of $\boldsymbol{A}$ are then given by $\lambda_1 = 2$, $\lambda_2 = 2 + \sqrt{11}$ and $\lambda_3 = 2 - \sqrt{11}$.

$$(\lambda_1 \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}_1 = \begin{bmatrix} -3 & -6 & -2 \\ 0 & 3 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\implies$ One possible solution is $\boldsymbol{x}_1 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$. Any its multiple is also a solution. Similarly, one can obtain $\boldsymbol{x}_2 = \begin{pmatrix} \sqrt{11} \\ (3 - \sqrt{11})/2 \\ 1 \end{pmatrix}$ and $\boldsymbol{x}_3 = \begin{pmatrix} -\sqrt{11} \\ (3 + \sqrt{11})/2 \\ 1 \end{pmatrix}$.

(b) Noting that

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

we have

$$\det(0 \cdot \boldsymbol{I} - \boldsymbol{A}) = \det(-\boldsymbol{A}) = (-1)^n \det(\boldsymbol{A}) = a_n$$

For our problem, we have $a_n = 14$ and $n = 3$. Thus, $\det(\boldsymbol{A}) = -14$, which implies that rank $(\boldsymbol{A}) = 3$.

(c) The eigenvalues of $\boldsymbol{A}^{\mathrm{T}}$ are the same as the eigenvalues of $\boldsymbol{A}$. The eigenvectors of $\boldsymbol{A}^{\mathrm{T}}$ can be found in the usual way, i.e.,

$$(\lambda_1 \boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}})\boldsymbol{y}_1 = \begin{bmatrix} -3 & 0 & -1 \\ -6 & 3 & 0 \\ -2 & 1 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \boldsymbol{y}_1 = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}.$$

Similarly, we can obtain

$$\boldsymbol{y}_2 = \begin{pmatrix} 1 \\ 3\sqrt{11} - 9 \\ -3 + \sqrt{11} \end{pmatrix} \quad \text{and} \quad \boldsymbol{y}_3 = \begin{pmatrix} 1 \\ -3\sqrt{11} - 9 \\ -3 - \sqrt{11} \end{pmatrix}$$

It is simple to check that

$$\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{y}_2 = \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{y}_3 = \boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{y}_1 = \boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{y}_3 = \boldsymbol{x}_3^{\mathrm{T}}\boldsymbol{y}_1 = \boldsymbol{x}_3^{\mathrm{T}}\boldsymbol{y}_2 = 0.$$

This property holds in general.

2. (a) Consider the symmetric matrix $\boldsymbol{A}$, where

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Find its eigenvalues and normalized eigenvectors.

(b) Show that these eigenvectors are orthogonal to each other.

(c) Arrange these eigenvectors as the columns of a matrix $\boldsymbol{P}$. Find the inverse of $\boldsymbol{P}$. Verify that in this case $\boldsymbol{P}^{-1} = \boldsymbol{P}^{\mathrm{T}}$.

Solution: (a)

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 = 0$$

$\implies \lambda_1 = 2$ and $\lambda_2 = 4$.

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{x}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \boldsymbol{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\implies \|\boldsymbol{x}_1\| = \sqrt{2}$. Thus, its normalized eigenvector is given by $\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Similarly, the normalized eigenvector corresponding to λ_2 is given by $\boldsymbol{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) Clearly, $\boldsymbol{e}_1^{\mathrm{T}}\boldsymbol{e}_2 = 0$. Hence, they are orthogonal.

(c) Let

$$\boldsymbol{P} = [\,\boldsymbol{e}_1 \quad \boldsymbol{e}_2\,] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Compute

$$\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\boldsymbol{P}^{-1} = \boldsymbol{P}^{\mathrm{T}}$. This property is true in general for symmetric matrices.

3. Show that the determinant of a Vandermonde matrix $\boldsymbol{V}$ given by

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix},$$

is non-zero if a, b and c are distinct.

Proof:

$$\det(\boldsymbol{V}) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$\begin{aligned}
&= bc^2 + ab^2 + a^2c - a^2b - b^2c - c^2a \\
&= a^2(c-b) + bc(c-b) + a(b^2 - c^2) \\
&= (a^2 + bc)(c-b) + a(b+c)(b-c) \\
&= (b-c)(ab + ac - a^2 - bc) \\
&= (b-c)\Big(b(a-c) - a(a-c)\Big) \\
&= (b-c)(b-a)(a-c) \\
&\neq 0, \qquad \text{if } a,\ b \text{ and } c \text{ are distinct.}
\end{aligned}$$

4. Given a matrix

$$A = \begin{bmatrix} -2 & 2 & 1 \\ -1 & 1 & 1 \\ -2 & 2 & 1 \end{bmatrix},$$

compute A^{314150}, $\cos(A\pi)$ and $e^{\mathbf{A}}$.

Solution: To compute A^{314150}, one might multiply it directly (which will take about 20 years) or find some easy ways. Let us diagonalize the given matrix A first by finding its eigenvalues and eigenvectors.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+2 & -2 & -1 \\ 1 & \lambda-1 & -1 \\ 2 & -2 & \lambda-1 \end{vmatrix} = \lambda^3 - \lambda = 0$$

Hence, the eigenvalues of A are given by $\lambda_1 = -1$, $\lambda_2 = 0$, and $\lambda_3 = 1$. Their corresponding eigenvectors are

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let

$$P = [\, x_1 \quad x_2 \quad x_3 \,] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

and

$$D = P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad A = PDP^{-1}.$$

Noting that

$$\begin{aligned} A^{314150} &= A \cdot A \cdot A \cdots A \qquad \text{(314150 times)} \\ &= PDP^{-1} \cdot PDP^{-1} \cdot PDP^{-1} \cdots PDP^{-1} \\ &= PD^{314150}P^{-1} \\ &= P \begin{bmatrix} (-1)^{314150} & 0 & 0 \\ 0 & 0^{314150} & 0 \\ 0 & 0 & 1^{314150} \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Similarly,

$$\cos(\boldsymbol{A}\pi) = \boldsymbol{P}\cos(\boldsymbol{D}\pi)\boldsymbol{P}^{-1} = \boldsymbol{P}\begin{bmatrix} \cos(-\pi) & 0 & 0 \\ 0 & \cos(0) & 0 \\ 0 & 0 & \cos(\pi) \end{bmatrix}\boldsymbol{P}^{-1}$$

$\Longrightarrow$

$$\cos(\boldsymbol{A}\pi) = \boldsymbol{P}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\boldsymbol{P}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix},$$

and

$$e^{\mathbf{A}} = \boldsymbol{P}\begin{bmatrix} e^{-1} & 0 & 0 \\ 0 & e^{0} & 0 \\ 0 & 0 & e \end{bmatrix}\boldsymbol{P}^{-1} = \begin{bmatrix} 1-e+1/e & e-1/e & e-1 \\ 1-e & e & e-1 \\ 1/e-e & e-1/e & e \end{bmatrix}.$$

5. Show that $Q = 3x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_3 + 2kx_2x_3$ can be expressed as $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$. Find the symmetric matrix $\boldsymbol{A}$. Find all values of k for which the quadratic form Q is positive definite.

Proof: (a) In general,

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \boldsymbol{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$$

In our case, we have $n = 3$ and

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_ix_j = 3x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_3 + 2kx_2x_3$$

Comparing similar terms on both sides, we obtain

$$a_{11} = 3, \quad a_{22} = 1, \quad a_{33} = 2,$$

and

$$a_{13} + a_{31} = 2, \quad a_{23} + a_{32} = 2k, \quad a_{12} + a_{21} = 0,$$

For a symmetric $\boldsymbol{A}$, we should choose

$$a_{13} = a_{31} = 1, \quad a_{23} = a_{32} = k, \quad a_{12} = a_{21} = 0$$

and hence

$$\boldsymbol{A} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & k \\ 1 & k & 2 \end{bmatrix}.$$

Note that a matrix $\boldsymbol{A}$ is positive definite if and only if the determinant of every leading principal submatrix of $\boldsymbol{A}$ is positive. For our case,

$$\det(3) = 3 > 0, \quad \det\left(\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\right) = 3 > 0, \quad \det(\boldsymbol{A}) = 5 - 3k^2$$

For positive definiteness of $\boldsymbol{A}$, we need $5 - 3k^2 > 0$ or $-\sqrt{5/3} < k < \sqrt{5/3}$.

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TE 2401 Linear Algebra and Numerical Methods

Solutions to Tutorial Set 4

1. Assume that the true value of π is 3.14159265 to eight decimal places.

   (a) Find the absolute and relative errors in approximating π by $\bar{\pi} = 3.1416$

   (b) Find the absolute and relative errors in approximating 100π by $100\bar{\pi} = 314.16$

Solution: (a) The absolute error is

$$|\pi - \bar{\pi}| \approx 0.00000735$$

and the relative error is

$$\frac{|\pi - \bar{\pi}|}{\pi} \approx 2.34e - 6$$

Since the absolute error is less than $5e - 5$ but greater than $5e - 6$, we say that the approximation is accurate to four decimal places. Similarly, from the relative error, we say that the accuracy is five significant figures.

(b) The absolute error is now

$$|100\pi - 100\bar{\pi}| \approx 0.000735$$

So, the approximation is accurate to two decimal places. For the relative error, we again have

$$\frac{100|\pi - 100\bar{\pi}|}{100\pi} \approx 2.34e - 6$$

which is the same as in (a) and is five significant figures of accuracy.

2. Discuss the convergence of the following two iterative schemes for possible solution of $f(x) = e^x - 1 - 2x = 0$

   (a) $x_{n+1} = (e^{x_n} - 1)/2$

   (b) $x_{n+1} = \ln(1 + 2x_n)$

Solution: Observe that $f(1) < 0 < f(2)$, so the equation has a solution in the interval $[1, 2]$.

(a) In this case $g(x) = (e^x - 1)/2$, thus $g'(x) = e^x/2$ which is greater than unity throughout the interval $[1, 2]$. This iteration scheme will then fail to converge. Let us take $x_0 = 1.5$. Then we obtain iteration results:

$$1.7408445, \quad 2.3510785, \quad 4.7484424, \quad 57.2021964, \quad \cdots\cdots$$

It does fail.

(b) In this case, $g(x) = \ln(1 + 2x)$, thus $g'(x) = 2/(1 + 2x)$ which is less than unity in the interval $[1, 2]$. If one start an initial point x_0 within the interval $[1, 2]$, then by the convergent theorem for fixed point method, we know that the iteration scheme will converge. One of its iteration with $x_0 = 1.5$ will be given in the next question.

3. Solve the equation $e^x - 1 - 2x = 0$ by Newton's method. The solution is expected to be computed up to 7 decimal places. Compare the speed of convergence with that using 2(b), assuming the same starting point of $x_0 = 1.5$.

Solution: First, we note that

$$f'(x) = (e^x - 1 - 2x)' = e^x - 2$$

Hence, the Newton iteration becomes

$$x_{n+1} = x_n - \frac{e^{x_n} - 2x_n - 1}{e^{x_n} - 2}$$

and the convergence of the above formula can be easily verified in the interval $[1, 2]$. Starting at $x_0 = 1.5$, the first five iterations yield

$$1.3059027, \quad 1.2590587, \quad 1.2564392, \quad 1.2564312, \quad 1.2564312$$

If 2(b) is used instead, the first 20 iterations starting at $x_0 = 1.5$ are as follows

$$1.3862944, \quad 1.3277614, \quad 1.2962391, \quad 1.2788423, \quad 1.2691099$$

$$1.2636237, \quad 1.2605178, \quad 1.2587552, \quad 1.2577534, \quad 1.2571837$$

$$1.2568595, \quad 1.2566750, \quad 1.2565700, \quad 1.2565102, \quad 1.2564762$$

$$1.2564568, \quad 1.2564458, \quad 1.2564395, \quad 1.2464359, \quad 1.2564339$$

Obviously, Newton's method is much faster.

4. Consider the problem of computing $\sqrt{a}$ for a positive real number a.

   (a) Use Newton's method to design an iterative equation for this purpose.

   (b) Derive the absolute and relative iterative error formulas for the equation obtained in (a).

Solution: (a) Let $f(x) = x^2 - a$. Then the root for $f(x)$ in $(0, +\infty)$is what we want. Since $f'(x) = 2x$, the Newton's iteration scheme or equation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 0, 1, 2, \cdots$$

(b) Subtracting $\sqrt{a}$ from both sides of the above equation, the absolute error formula is obtained as

$$\begin{aligned} e_{n+1} &= \sqrt{a} - x_{n+1} \\ &= \sqrt{a} - \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ &= -\frac{1}{2x_n}\left(a - 2\sqrt{a}x_n + x_n^2\right) \\ &= -\frac{1}{2x_n}\left(\sqrt{a} - x_n\right)^2 \\ &= -\frac{1}{2x_n}e_n^2 \end{aligned}$$

For the relative error formula

$$r_{n+1} = \frac{e_{n+1}}{\sqrt{a}} = -\frac{\sqrt{a}}{2x_n}\left(\frac{\sqrt{a} - x_n}{\sqrt{a}}\right)^2 = -\frac{\sqrt{a}}{2x_n}r_n^2$$

5. **Solution to (a):**

(a) Complete Newton's Divided Deference table for $\sin(x)$ with $x_j = 0,\ 0.1\pi,\ 0.2\pi,\ \cdots,\ 0.5\pi$.

x_j	$f_0^{[0]}$	$f_0^{[1]}$	$f_0^{[2]}$	$f_0^{[3]}$	$f_0^{[4]}$	$f_0^{[5]}$
0	0					
		0.98364				
0.1π	0.30902		-0.15325			
		0.88735		-0.14668		
0.2π	0.58779		-0.29149		0.02411	
		0.70420		-0.11638		0.00572
0.3π	0.80902		-0.40118		0.03310	
		0.45312		-0.07478		
0.4π	0.95106		-0.47166			
		0.15578				
0.5π	1					

(b) From this table, compute $\sin(\pi/4)$ using linear and quadratic interpolation from the intervals: $(0.2\pi, \cdots)$.

Solution: Using linear interpolation, we obtain

$$\sin\left(\frac{\pi}{4}\right) = 0.58779 + 0.05\pi(0.70420) = 0.69841$$

Using quadratic interpolation, we obtain

$$\sin\left(\frac{\pi}{4}\right) = 0.58779 + 0.05\pi(0.70420) + (0.05\pi)(-0.05\pi)(-0.40118) = 0.70831$$

(c) The error of the n-th order polynomial $p_n(x)$ interpolating $\{x_i\}$ is given by

$$e_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\eta)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where η is a point in the smallest interval containing $\{x_i\}$. Also note that x_0 in the above formula is the actual starting point of the intervals for interpolation while in case (b), $x_0 = 0.2\pi$.

Use the above formula to compute error estimates for the linear and quadratic interpolations in (b) and compare these estimates with the actual errors.

Solution: The actual errors are

$$e_1\left(\frac{\pi}{4}\right) = 0.070711 - 0.69841 = 8.7e - 3$$

for linear interpolation and

$$e_2\left(\frac{\pi}{4}\right) = 0.070711 - 0.70831 = -1.2e - 3$$

for quadratic interpolation.

To compute the error estimates, note that $f''(x) = -\sin(x)$ and $f'''(x) = -\cos(x)$. Use the given formula for linear case

$$e_1(x)|_{x=0.25\pi} = -\frac{1}{2!}\sin(x^*)(x - 0.2\pi)(x - 0.3\pi)|_{x=0.25\pi} = (1.2337e - 2)\sin(x^*)$$

with $0.2\pi \le x^* \le 0.3\pi$. Thus,

$$7.25e - 3 \le e_1\left(\frac{\pi}{4}\right) \le 9.98e - 3$$

with the middle value $8.6e - 3$, which is quite close to the actual error.

For quadratic case

$$\begin{aligned} e_2(x)|_{x=0.25\pi} &= -\frac{1}{3!}\cos(x^*)(x - 0.2\pi)(x - 0.3\pi)(x - 0.4\pi)|_{x=0.25\pi} \\ &= (-1.9379e - 3)\cos(x^*) \end{aligned}$$

with $0.2\pi \le x^* \le 0.4\pi$. Thus,

$$-1.57e - 3 \le e_2\left(\frac{\pi}{4}\right) \le -0.60e - 3$$

with the middle value $-1.1e - 3$, which is also quite close to the actual error.

6. (a) Find the least square quadratic fit to the following shifted data for the function:

$$f(x) = \frac{x+2}{4} e^{(x+2)/4} - 1$$

x_i	-3	-2	-1	0	1	2	3
$f(x_i)$	-1.195	-1	-0.679	-0.1756	0.5878	1.718	3.363

Solution: The least square fit has the form

$$\hat{y} = a_0 + a_1 x + a_2 x^2$$

The coefficients can be determined by solving the following linear equations

$$\begin{bmatrix} \sum(x_i)^0 & \sum(x_i)^1 & \sum(x_i)^2 \\ \sum(x_i)^1 & \sum(x_i)^2 & \sum(x_i)^3 \\ \sum(x_i)^2 & \sum(x_i)^3 & \sum(x_i)^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}$$

Filling in the given data, we have

$$\begin{bmatrix} 7 & 0 & 28 \\ 0 & 28 & 0 \\ 28 & 0 & 196 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2.6192 \\ 20.3768 \\ 22.2928 \end{bmatrix}$$

Solving for the coefficients,

$$a_0 = -0.1885, \quad a_1 = 0.7277, \quad a_3 = 0.1407$$

(b) Compute the error at the data points and describe how they are distributed?

Solution: The error at each point can be computed as $e_i = f(x_i) - \hat{y}_i$:

$$e_1 = -0.0897, \quad e_2 = 0.0811, \; e_3 = 0.0965, \quad e_4 = 0.0129,$$

$$e_5 = -0.0921, \quad e_6 = -0.1120, \quad e_7 = 0.1020$$

The average error $\bar{e} = (\sum e_i)/N = -1.86e{-}4$ can be used to compute the variance $\sigma_e = \sqrt{\sum(e_i - \bar{e})^2/N}$ which measures the distribution of e_i with respect to $\bar{e}$. The variance is obtained as $\sigma_e = 0.0891$, which is relatively large in comparison with the maximum error. This is because N is small in this exercise.

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TE 2401 Linear Algebra and Numerical Methods

Solutions to Tutorial Set 5

1. Compute $J = \int_0^1 f(x)dx$ with $f(x) = x^3$ using the following numerical approximations:

   (a) The Trapezoidal Rule:

   $$J_n = \frac{1}{2n}\left[f(0) + f(1) + 2\sum_{j=1}^{n-1} f\left(\frac{j}{n}\right)\right], \quad n = 1, 2, 4$$

   and compute $J - J_i$, $i = 1, 2, 4$.

   Solution: Applying the above rule, we obtain

   $$J_1 = \frac{1}{2}\left[0^3 + 1^3\right] = 0.5$$

   $$J_2 = \frac{1}{4}\left[0 + 1 + 2(0.5)^3\right] = 0.3125$$

   and

   $$J_4 = \frac{1}{8}\left[0 + 1 + 2(0.25)^3 + 2(0.5)^3 + 2(0.75)^3\right] = 0.265625$$

   Also note that the exact value

   $$J = \int_0^1 x^3 dx = \frac{1}{4}x^4 \Big|_0^1 = 0.25$$

   Hence,

   $$J - J_1 = -0.25, \quad J - J_2 = -0.0625, \quad J - J_4 = -0.015625$$

   Errors are decreasing as n is increasing.

(b) The simpler (midpoint) Rectangular Rule

$$I_n = \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{2j-1}{2n}\right), \quad n = 1, 2, 4$$

and compute $J - I_i$, $i = 1, 2, 4$.

Solution: From the above formula, we have

$$I_1 = f(0.5) = 0.125$$

$$I_2 = \frac{1}{2}\Big[f(0.25) + f(0.75)\Big] = 0.21875$$

and

$$I_4 = \frac{1}{4}\Big[f(0.125) + f(0.375) + f(0.625) + f(0.875)\Big] = 0.2422$$

Hence,

$$I - I_1 = 0.125, \quad I - I_2 = 0.03125, \quad I - I_4 = 0.0078$$

(c) Simpson's $\frac{1}{3}$ Rule with $h = 0.5$,

$$K_1 = \frac{h}{3}\Big[f(0) + 4f(h) + f(1)\Big]$$

and with $h = 0.25$

$$K_2 = \frac{h}{3}\Big[f(0) + 4f(h) + 2f(2h) + 4f(3h) + f(1)\Big]$$

and compute $J - K_i$, $i = 1, 2$.

Solution: We have

$$K_1 = \frac{0.5}{3}\Big[f(0) + 4f(0.5) + f(1)\Big] = \frac{0.5}{3}\Big[0 + 4 \times 0.125 + 1\Big] = 0.25$$

and

$$\begin{aligned} K_2 &= \frac{0.25}{3}\Big[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)\Big] \\ &= \frac{0.25}{3}\Big[0 + 4 \times 0.015625 + 2 \times 0.125 + 4 \times 0.421875 + 1\Big] \\ &= 0.25 \end{aligned}$$

Hence, $J - K_1 = J - K_2 = 0$. Clearly, for this problem, Simpson's $\frac{1}{3}$ rule yields the best result.

2. Evaluate the following integrals numerically as indicated.

$$\mathrm{Si}(x) = \int_0^x \frac{\sin x^*}{x^*} dx^*, \quad \mathrm{C}(x) = \int_0^x \cos(x^{*2}) dx^*$$

These are non-elementary integrals. $\mathrm{Si}(x)$ is called the sine integral and $\mathrm{C}(x)$ the Fresnel integral.

(a) $\mathrm{Si}(1)$ by the trapezoidal rule with $n = 5$ and $n = 10$.

(b) $\mathrm{Si}(1)$ by the Simpson's $\frac{1}{3}$ rule with $h = 0.5$ and $h = 0.1$.

(c) $\mathrm{C}(2)$ by the trapezoidal rule with $n = 10$.

(d) $\mathrm{C}(2)$ by the Simpson's $\frac{1}{3}$ rule with $h = 0.2$.

Solution: (a) The trapezoidal rule with $n = 5$ $(h = 1/5 = 0.2)$ gives

$$\begin{aligned}
\mathrm{Si}(1) &\approx \frac{h}{2}\Big[f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + f_5\Big] \\
&= \frac{0.2}{2}\Big[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)\Big] \\
&= 0.1\left[1 + 2\frac{\sin 0.2}{0.2} + 2\frac{\sin 0.4}{0.4} + 2\frac{\sin 0.6}{0.6} + 2\frac{\sin 0.8}{0.8} + \sin 1.0\right] \\
&= 0.1[1 + 1.98669 + 1.94709 + 1.88214 + 1.79339 + 0.84147] \\
&= 0.94508
\end{aligned}$$

The trapezoidal rule with $n = 10$ $(h = 1/10 = 0.1)$ gives

$$\begin{aligned}
\mathrm{Si}(1) &\approx \frac{h}{2}\Big[f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + 2f_5 + 2f_6 + 2f_7 + 2f_8 + 2f_9 + f_{10}\Big] \\
&= \frac{0.1}{2}\left[f(0) + 2\sum_{k=1}^{9} f(0.1k) + f(1)\right] \\
&= 0.05\left[1 + 2\sum_{k=1}^{9} \frac{\sin(0.1k)}{0.1k} + \sin 1.0\right] \\
&= 0.94583
\end{aligned}$$

The exact value of $\mathrm{Si}(1) = 0.94608$.

(b) The Simpson's $\frac{1}{3}$ rule with $h = 0.5$ gives

$$\begin{aligned}
\mathrm{Si}(1) &\approx \frac{h}{3}\Big[f_0 + 4f_1 + f_2\Big] \\
&= \frac{0.5}{3}\Big[1 + 4\frac{\sin 0.5}{0.5} + \sin 1.0\Big] \\
&= \frac{0.5}{3}\Big[1 + 3.83540 + 0.84147\Big] \\
&= 0.94615
\end{aligned}$$

The Simpson's $\frac{1}{3}$ rule with $h = 0.1$ gives

$$\begin{aligned}
\mathrm{Si}(1) &\approx \frac{h}{3}\Big[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + 2f_6 + 4f_7 + 2f_8 + 4f_9 + f_{10}\Big] \\
&= \frac{0.1}{3}\Big[1 + 4\sum_{k=1}^{5}\frac{\sin[0.1(2k-1)]}{0.1(2k-1)} + 2\sum_{k=1}^{4}\frac{\sin(0.2k)}{0.2k} + \sin 1.0\Big] \\
&= 0.94608
\end{aligned}$$

Clearly, Simpson's rule yields better results.

(c) The trapezoidal rule with $n = 10$ ($h = 2/10 = 0.2$) gives

$$\begin{aligned}
\mathrm{C}(2) &\approx \frac{h}{2}\Big[f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + 2f_5 + 2f_6 + 2f_7 + 2f_8 + 2f_9 + f_{10}\Big] \\
&= \frac{0.2}{2}\Big[f(0) + 2\sum_{k=1}^{9} f(0.2k) + f(2)\Big] \\
&= 0.1\left[1 + 2\sum_{k=1}^{9}\cos[(0.2k)^2] + \cos 4\right] \\
&= 0.4716
\end{aligned}$$

The exact value of $\mathrm{C}(2) = 0.4615$.

(d) The Simpson's $\frac{1}{3}$ rule with $h = 0.2$ gives

$$\begin{aligned}
\mathrm{C}(2) &\approx \frac{h}{3}\Big[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + 2f_6 + 4f_7 + 2f_8 + 4f_9 + f_{10}\Big] \\
&= \frac{0.2}{3}\Big[1 + 4\sum_{k=1}^{5}\cos\{[0.2(2k-1)]^2\} + 2\sum_{k=1}^{4}\cos\{(0.4k)^2\} + \cos 4\Big] \\
&= 0.4612
\end{aligned}$$

Clearly, Simpson's rule again yields better results.

3. Given the following initial condition problem

$$y'(x) = y(x), \quad y(0) = 1$$

(a) Show that the solution of Euler's method with the step-size h can be written as

$$y_n = c(h)^{x_n}, \quad n = 0, 1, 2, \cdots$$

where $c(h) = (1+h)^{1/h}$ (note that $\lim_{h\to 0} c(h) = e$) and $x_n = nh$.

Solution: Euler's method leads to

$$y_{n+1} = (1+h)y_n, \quad y_0 = 1$$

Its solution is

$$y_n = (1+h)^n, n = 0, 1, 2, 3, \cdots$$

which can be written as

$$y_n = [(1+h)^{1/h}]^{nh} = c(h)^{x_n}$$

(b) Show that

$$\max_{0\le x\le 1} |y(x_n) - y_n| = e - c(h) \approx \frac{h}{2}e$$

where $e = 2.718281828\cdots$, the natural number.

Solution: Note that

$$y(x_n) - y_n = e^{x_n} - c(h)^{x_n}$$

As $e > c(h)$, the above function increases as x_n increases. Thus

$$\max_{0\le x\le 1} |y(x_n) - y_n| = e - c(h)$$

Also note that

$$c(h) = \exp\left\{\frac{1}{h}\ln(1+h)\right\} = \exp\left\{1 - \frac{h}{2} + \frac{h^2}{3} - \cdots\right\}$$

Hence

$$e - c(h) = e\left[1 - \exp\left\{-\frac{h}{2} + \frac{h^2}{3} - \cdots\right\}\right] = e\left(\frac{h}{2} - \frac{1}{2}\cdot\frac{1}{4}h^2 - \cdots\right) \approx \frac{h}{2}e$$

4. Two second order methods for solving numerically the initial condition problem:

$$\frac{dy}{dx} = f(x, y)$$

with $x_0 = x(0)$ and $y_0 = y(0)$ are:

Method I	Method II
$x_0 = x(0), \quad y_0 = y(0)$	$x_0 = x(0), \quad y_0 = y(0)$
$x_{n+1} = x_n + h$	$x_{n+1} = x_n + h$
$k_1 = h \cdot f(x_n, y_n)$	$k_1 = h \cdot f(x_n, y_n)$
$k_2 = h \cdot f(x_{n+1}, y_n + k_1)$	$k_3 = h \cdot f(x_{n+1} + h, y_n + 2k_1)$
$y_{n+1} = y_n + (k_1 + k_2)/2$	$y_{n+1} = y_n + (3k_1 + k_3)/4$

(a) Show that these methods give identical results for $f(x, y) = x + y$.

Solution: Let $f(x, y) = x + y$. Then $k_1 = h(x_n + y_n)$ and

$$k_2 = h(x_{n+1} + y_n + k_1) = h\left[x_n + h + y_n + h(x_n + y_n)\right]$$

for Method I. Thus

$$\frac{k_1 + k_2}{2} = h\left[x_n + y_n + \frac{h}{2}(1 + x_n + y_n)\right]$$

Next, for Method II,

$$k_3 = h(x_{n+1} + h + y_n + 2k_1) = h\Big[x_n + y_n + 2h + 2h(x_n + y_n)\Big]$$

which implies

$$\begin{aligned}
\frac{3k_1 + k_3}{4} &= \frac{h}{4}\Big[3(x_n + y_n) + x_n + y_n + 2h(1 + x_n + y_n)\Big] \\
&= h\left[x_n + y_n + \frac{h}{2}(1 + x_n + y_n)\right] \\
&= \frac{k_1 + k_2}{2}
\end{aligned}$$

This proves the assertion.

(b) For $f(x,y) = y + 1 + e^x$, take four steps with each method, and compare your results with analytical solution,

$$y = xe^x - 1$$

Use $h = 0.05$ and initial conditions: $x(0) = 0$ and $y(0) = -1$.

Solution: The following table demonstrates the details

n	x_n	y_n	$y_{n\mathrm{I}}$	$y_{n\mathrm{II}}$	e_{I}	e_{II}
0	0	-1	-1	-1	0	0
1	0.05	-0.947436	-0.947468	-0.947435	$3.2e-5$	$-1e-6$
2	0.10	-0.889483	-0.889550	-0.889481	$6.7e-5$	$-2e-6$
3	0.15	-0.825725	-0.825833	-0.825724	$1.08e-4$	$-1e-6$
4	0.20	-0.755719	-0.755874	-0.755720	$1.55e-4$	$1e-6$

This demonstrates that the algorithms of the same order can give significantly different results for the same problem.

5. Find a numerical solution for a vibrating string which is characterized by a wave equation

$$u_{tt} = u_{xx}, \quad 0 \le x \le 1, \;\; t \ge 0$$

with an initial displacement

$$u(x, 0) = \sin(\pi x)$$

and an initial velocity

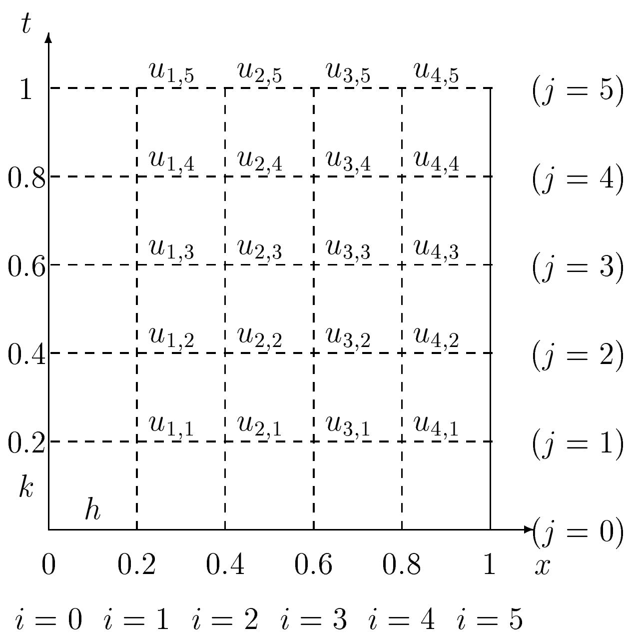
$$u_t(x, 0) = 0$$

and boundary conditions

$$u(0, t) = u(1, t) = 0$$

Let $h = k = 0.2$ and compute the solution $u(x, t)$ for t up to 1 second. Note that the exact solution to this problem is given by $u(x, t) = \sin(\pi x)\cos(\pi t)$. Compare your results with the exact solution.

Solution: Let us solve this problem step by step. The first step is to define a grid on the region of interests,



Step 2: Approximate derivatives at mesh points by central difference quotients, i.e.,

$$u_{xx}(ih, jk) = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

and

$$u_{tt}(ih, jk) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

where in this problem, we have $h = k = 0.2$. This brings the given PDE, $u_{tt} = u_{xx}$, to a difference equation (note that $r^* = c^2k^2/h^2 = 1$),

$$u_{i,j+1} = (2 - 2r^*) \cdot u_{i,j} + r^* u_{i-1,j} + r^* u_{i+1,j} - u_{i,j-1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

For $j = 0$, we would have to use equation ($\star$) derived on page 102 of the second part of our lecture notes, i.e.,

$$u_{i,1} = (1 - r^*) f_1(x_i) + \frac{1}{2} r^* f_1(x_{i-1}) + \frac{1}{2} r^* f_1(x_{i+1}) + k f_2(x_i)$$

which is reduced to

$$u_{i,1} = \frac{1}{2}\sin\Big(\pi \cdot (i-1) \cdot 0.2\Big) + \frac{1}{2}\sin\Big(\pi \cdot (i+1) \cdot 0.2\Big)$$

Step 3: The initial conditions are

$$u_{1,0} = \ \ 0.588, \quad u_{2,0} = \ \ 0.951, \quad u_{3,0} = \ \ 0.951, \quad u_{4,0} = \ \ 0.588$$

For $j = 0, 1, 2, 3, 4$, we obtain

$$u_{1,1} = \ \ 0.476, \quad u_{2,1} = \ \ 0.769, \quad u_{3,1} = \ \ 0.769, \quad u_{4,1} = \ \ 0.476$$

$$u_{1,2} = \ \ 0.182, \quad u_{2,2} = \ \ 0.294, \quad u_{3,2} = \ \ 0.294, \quad u_{4,2} = \ \ 0.182$$

$$u_{1,3} = -0.182, \quad u_{2,3} = -0.294, \quad u_{3,3} = -0.294, \quad u_{4,3} = -0.182$$

$$u_{1,4} = -0.476, \quad u_{2,4} = -0.769, \quad u_{3,4} = -0.769, \quad u_{4,4} = -0.476$$

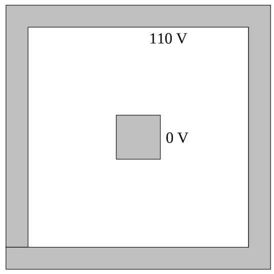
$$u_{1,5} = -0.588, \quad u_{2,5} = -0.951, \quad u_{3,5} = -0.951, \quad u_{4,5} = -0.588$$

It turns out that these values are exact as it can be easily verified by the exact solution of the problem. For example, the exact solutions of

$$u_{3,3} = u(3 \times 0.2, 3 \times 0.2) = u(0.6, 0.6) = \sin(0.6\pi)\cos(0.6\pi) = -0.294 \quad (\checkmark)$$

$$u_{4,5} = u(4 \times 0.2, 5 \times 0.2) = u(0.8, 1.0) = \sin(0.8\pi)\cos(1.0\pi) = -0.588 \quad (\checkmark)$$

6. The cross-section of a coaxial cable has an inner 1-cm-square conductor at the center of an outer conductor which has a 5-cm-square inner boundary, as shown in the figure. The inner conductor is kept at zero volts while the outer conductor is kept at 110 volts.



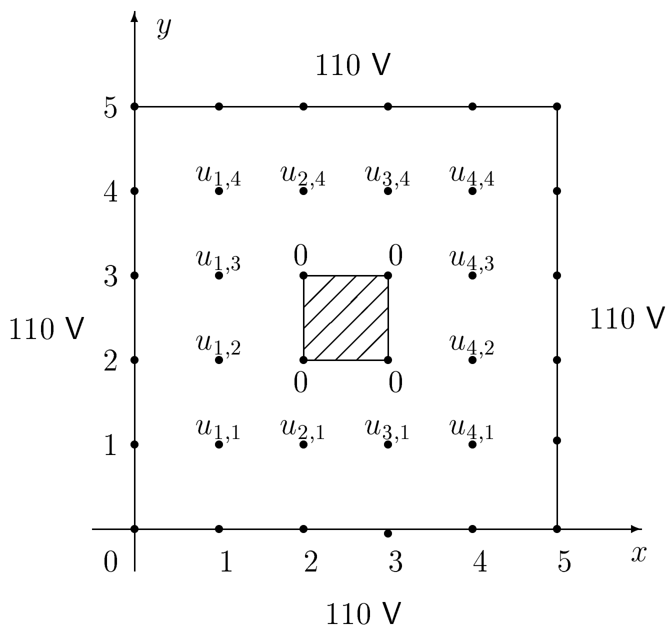
To find an approximation of the potential between the two conductors, place a grid with horizontal mesh spacing $h = 1$ cm and vertical mesh spacing $k = 1$ cm on the region:

$$R = \{(x, y) \mid 0 \leq x \leq 5,\ 0 \leq y \leq 5\}$$

where we want to approximate the solution of Laplace's equation $u_{xx} + u_{yy} = 0$. Use the central difference approximations and the given boundary conditions to derive the linear system that needs to be solved. Solve it using any software package.

Solution: Let us do it step-by-step as follows:

Step 1: Define a grid on R with mesh points



Step 2: Approximate derivatives at mesh points by central difference quotients, i.e.,

$$u_{xx}(ih, jk) = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

and

$$u_{yy}(ih, jk) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

where in this problem, we have $h = k = 1$ cm. This brings the given PDE, $u_{xx}+u_{yy} = 0$, to a difference equation,

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

Then for the point $u_{1,1}$, we have

$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = u_{2,1} + 110 + u_{1,2} + 110 - 4u_{1,1} = 0$$

or

$$u_{2,1} + u_{1,2} - 4u_{1,1} = -220 \tag{0.3}$$

Similarly, we obtain

$$u_{1,1} + u_{3,1} - 4u_{2,1} = -110 \tag{0.4}$$

$$u_{2,1} + u_{4,1} - 4u_{3,1} = -110 \tag{0.5}$$

$$\vdots$$

$$u_{3,4} + u_{4,3} - 4u_{4,4} = -220 \tag{0.6}$$

Let $u = [\, u_{14} \quad u_{24} \quad u_{34} \quad u_{44} \quad u_{13} \quad u_{43} \quad u_{12} \quad u_{42} \quad u_{11} \quad u_{21} \quad u_{31} \quad u_{41} \,]$. We have

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 \end{bmatrix} u = \begin{bmatrix} -220 \\ -110 \\ -110 \\ -220 \\ -110 \\ -110 \\ -110 \\ -110 \\ -220 \\ -110 \\ -110 \\ -220 \end{bmatrix}$$

The above equation can be solved to obtain $u_{1,4} = u_{4,4} = u_{1,1} = u_{4,1} = 88$ V and all the rest u's are equal to 66 V.

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