Annex: A Past Year Exam Paper

Semester I: 1996/97

Q.1.(a). State the conditions under which the system of linear equation

$$\mathbf{Ax} = \mathbf{b}$$

has (i) no solution, (ii) exactly one solution, and (iii) more than one solutions.

Solution: Let the augmented matrix of the given system be

$$\tilde{\mathbf{A}} = [\mathbf{A} \quad \mathbf{b}]$$

(i) if the above augmented matrix and $\mathbf{A}$ have different ranks, the given system has no solution.

(ii) if the augmented matrix and $\mathbf{A}$ have a same rank, which is equal to the number of unknowns, then the system has exactly one solution.

(iii) if the augmented matrix and $\mathbf{A}$ have a same rank, which is less than the number of unknowns, then the system has infinitely many solutions.

Q.1.(b). Reduce the following augmented matrix to its (i) echelon form, (ii) row echelon form, and (iii) reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 7 & 3 & 2 \\ 2 & 5 & 11 & 22 \end{bmatrix}$$

Solution: The echelon form can be found as the following

$$\begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 7 & 3 & 2 \\ 2 & 5 & 11 & 22 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & -12 & -16 \\ 0 & 1 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & -12 & -16 \\ 0 & 0 & 13 & 26 \end{bmatrix}$$

The row echelon form and reduced row echelon form are respectively

$$\begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & -12 & -16 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & -20 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Q.1.(c). Compute the determinant of the coefficient matrix $\mathbf{A}$ associated with the augmented matrix in part (b). Does the inverse of $\mathbf{A}$ exist (do not compute it)? Justify.

Solution: $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & 3 \\ 2 & 5 & 11 \end{bmatrix}$$

Its determinant is given by

$$\det(\mathbf{A}) = 1\cdot 7\cdot 11 + 2\cdot 3\cdot 2 + 5\cdot 5\cdot 3 - 5\cdot 7\cdot 2 - 2\cdot 3\cdot 11 - 1\cdot 5\cdot 3 = 13$$

which is nonzero. Hence its inverse exists.

Q.1.(d). Compute the rank of the augmented matrix in part (b).

Solution: The augmented matrix and its echelon form was computed as the following:

$$\begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 7 & 3 & 2 \\ 2 & 5 & 11 & 22 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & -12 & -16 \\ 0 & 1 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & -12 & -16 \\ 0 & 0 & 13 & 26 \end{bmatrix}$$

Hence, the rank of the augmented matrix is equal to 3.

Q.1.(e). Consider the system

$$\begin{array}{rcrcrcr} 2x & & & + & 6z & = & 12 \\ & & 3y & + & 2z & = & 8 \\ 5x & + & 3y & + & 3z & = & 24 \\ 8x & + & 6y & + & 14z & = & 50 \end{array}$$

Solve the linear system for x, y and z. Is the solution unique? Justify.

Solution: The augmented matrix and its echelon form,

$$\begin{bmatrix} 2 & 0 & 6 & 12 \\ 0 & 3 & 2 & 8 \\ 5 & 3 & 3 & 24 \\ 8 & 6 & 14 & 50 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 6 & 12 \\ 0 & 3 & 2 & 8 \\ 0 & 3 & -12 & -6 \\ 0 & 6 & -10 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 6 & 12 \\ 0 & 3 & 2 & 8 \\ 0 & 0 & -14 & -14 \\ 0 & 0 & -14 & -14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 6 & 12 \\ 0 & 3 & 2 & 8 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The ranks of the augmented matrix and A are both equal to 3, which is the number of unknowns. Hence, the solution is unique and is given by

$$x = 3 \qquad y = 2 \qquad z = 1$$

Q.2. Consider the vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 2 & 4 & 1 & 1 \end{pmatrix}$$
$$\mathbf{v}_2 = \begin{pmatrix} 4 & 7 & 2 & 2 \end{pmatrix}$$
$$\mathbf{v}_3 = \begin{pmatrix} 6 & 8 & 7 & 5 \end{pmatrix}$$

(a) Are these vectors linearly independent? Justify.

Solution: Arrange these vectors as rows of a matrix and reduce it to echelon form as the following:

$$\begin{bmatrix} 2 & 4 & 1 & 1 \\ 4 & 7 & 2 & 2 \\ 6 & 8 & 7 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -4 & 4 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

which has a rank of 3. Hence, the given vectors are linearly independent.

Q.2.(b). Let these vectors span a vector space $\mathbf{V}$. Find the the dimension of $\mathbf{V}$ and a set of basis vectors for $\mathbf{V}$.

Solution: Clearly, the dimension of $\mathbf{V}$ is 3. The following are two sets of bases for $\mathbf{V}$:

$$\mathbf{v}_1 = \begin{pmatrix} 2 & 4 & 1 & 1 \end{pmatrix}$$
$$\mathbf{v}_2 = \begin{pmatrix} 4 & 7 & 2 & 2 \end{pmatrix}$$
$$\mathbf{v}_3 = \begin{pmatrix} 6 & 8 & 7 & 5 \end{pmatrix}$$

or

$$\mathbf{w}_1 = \begin{pmatrix} 2 & 4 & 1 & 1 \end{pmatrix}$$
$$\mathbf{w}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$
$$\mathbf{w}_3 = \begin{pmatrix} 0 & 0 & 2 & 1 \end{pmatrix}$$

Q.2.(c). Does the vector $\mathbf{v} = [\,2 \;\; 1 \;\; 1 \;\; 1\,]$ belong to the vector space $\mathbf{V}$. Justify.

Solution: Assume that $\mathbf{v}$ can be expressed as

$$\begin{aligned}
\mathbf{v} &= [2 \quad 1 \quad 1 \quad 1] = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \\
&= a_1[2 \quad 4 \quad 1 \quad 1] + a_2[4 \quad 7 \quad 2 \quad 2] + a_3[6 \quad 8 \quad 7 \quad 5] \\
&= [2a_1 + 4a_2 + 6a_3 \quad 4a_1 + 7a_2 + 8a_3 \quad a_1 + 2a_2 + 7a_3 \quad a_1 + 2a_2 + 5a_3]
\end{aligned}$$

or

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 7 & 8 \\ 1 & 2 & 7 \\ 1 & 2 & 5 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} \qquad \text{answer} = \text{yes!}$$

Hence,

$$\begin{bmatrix} 2 & 4 & 6 & 2 \\ 4 & 7 & 8 & 1 \\ 1 & 2 & 7 & 1 \\ 1 & 2 & 5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 6 & 2 \\ 0 & -1 & -4 & -3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 & -10 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q.2.(d). Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be arranged as the first, second and third rows of a 3×4 matrix $\mathbf{A}$. Let the null space of $\mathbf{A}$ be the vector space W. Determine the dimension of W.

Solution: W is the solution space of

$$\begin{bmatrix} 2 & 4 & 1 & 1 \\ 4 & 7 & 2 & 2 \\ 6 & 8 & 7 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix

$$\begin{bmatrix} 2 & 4 & 1 & 1 & 0 \\ 4 & 7 & 2 & 2 & 0 \\ 6 & 8 & 7 & 5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -4 & 4 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0.25 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 \end{bmatrix}$$

The dimension of W is equal to 1 as there is only one free variable.

Q.2.(e). Find a set of basis vector for W.

Solution: From the previous part, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -0.25x_4 \\ 0 \\ -0.5x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} -0.25 \\ 0 \\ -0.5 \\ 1 \end{pmatrix} x_4$$

Hence,

$$\begin{pmatrix} -0.25 \\ 0 \\ -0.5 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 \\ 0 \\ -2 \\ 4 \end{pmatrix}$$

is a basis vector for W.

Q.3. Consider the matrix given by

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 18 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

(a) Compute the eigenvalues and the eigenvector matrix, $\mathbf{P}$, of the matrix $\mathbf{A}$.

Solution: The characteristic equation of the given matrix,

$$\det(\mathbf{A} - l\mathbf{I}) = \det\begin{bmatrix} 1-l & 3 & 0 \\ 3 & 18-l & 3 \\ 0 & 3 & 1-l \end{bmatrix} = (1-l)^2(18-l) - 18(1-l)$$

$$= (1-l)[(1-l)(18-l) - 18] = (1-l)(l^2 - 19) = l(1-l)(l - 19)$$

Thus, the eigenvalues are

$$l_1 = 0 \qquad l_2 = 1 \qquad l_3 = 19$$

For λ_1,
$$\begin{bmatrix} 1 & 3 & 0 \\ 3 & 18 & 3 \\ 0 & 3 & 1 \end{bmatrix}\begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{e}_1 = \begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \end{pmatrix} = \frac{1}{\sqrt{19}}\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$$

For λ_2,
$$\begin{bmatrix} 0 & 3 & 0 \\ 3 & 17 & 3 \\ 0 & 3 & 0 \end{bmatrix}\begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{e}_2 = \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For λ_3,
$$\begin{bmatrix} -18 & 3 & 0 \\ 3 & -1 & 3 \\ 0 & 3 & -18 \end{bmatrix}\begin{pmatrix} e_{31} \\ e_{32} \\ e_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{e}_3 = \begin{pmatrix} e_{31} \\ e_{32} \\ e_{33} \end{pmatrix} = \frac{1}{\sqrt{38}}\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}$$

The normalized eigenvector matrix

$$\mathbf{P} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] = \frac{1}{\sqrt{38}}\begin{bmatrix} 3\sqrt{2} & -\sqrt{19} & 1 \\ -\sqrt{2} & 0 & 6 \\ 3\sqrt{2} & \sqrt{19} & 1 \end{bmatrix}$$

Q.3.(b). Compute the inverse of $\mathbf{P}$. What is the determinant of $\mathbf{P}$?

Solution:

$$\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}} = \frac{1}{\sqrt{38}}\begin{bmatrix} 3\sqrt{2} & -\sqrt{19} & 1 \\ -\sqrt{2} & 0 & 6 \\ 3\sqrt{2} & \sqrt{19} & 1 \end{bmatrix}^{\mathrm{T}} = \frac{1}{\sqrt{38}}\begin{bmatrix} 3\sqrt{2} & -\sqrt{2} & 3\sqrt{2} \\ -\sqrt{19} & 0 & \sqrt{19} \\ 1 & 6 & 1 \end{bmatrix}$$

and

$$\det(\mathbf{P}) = 1 \quad \text{or} \quad -1 \quad \text{as it is an orthogonal matrix.}$$

Q.3.(c). For the matrix $\mathbf{A}$, compute $\mathbf{A}^{148}$. Leave your answer in terms of the eigenvalues and the eigenvector matrix.

Solution: It is known that

$$\mathbf{A}=\mathbf{PDP}^{\mathrm{T}}=\frac{1}{\sqrt{38}}\begin{bmatrix} 3\sqrt{2} & -\sqrt{2} & 3\sqrt{2} \\ -\sqrt{19} & 0 & \sqrt{19} \\ 1 & 6 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 19 \end{bmatrix}\frac{1}{\sqrt{38}}\begin{bmatrix} 3\sqrt{2} & -\sqrt{19} & 1 \\ -\sqrt{2} & 0 & 6 \\ 3\sqrt{2} & \sqrt{19} & 1 \end{bmatrix}^{\mathrm{T}}$$

and hence

$$\mathbf{A}^{148}=\mathbf{PD}^{148}\mathbf{P}^{\mathrm{T}}$$

$$=\frac{1}{\sqrt{38}}\begin{bmatrix} 3\sqrt{2} & -\sqrt{2} & 3\sqrt{2} \\ -\sqrt{19} & 0 & \sqrt{19} \\ 1 & 6 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 19^{148} \end{bmatrix}\frac{1}{\sqrt{38}}\begin{bmatrix} 3\sqrt{2} & -\sqrt{19} & 1 \\ -\sqrt{2} & 0 & 6 \\ 3\sqrt{2} & \sqrt{19} & 1 \end{bmatrix}^{\mathrm{T}}$$

Q.3.(d). Using the result in part (a). Show that the quadratic form

$$Q = x_1^2 + 18x_2^2 + x_3^2 + 6x_1x_2 + 6x_2x_3$$

is positive semi-definite.

Solution: The given quadratic form can be re-written as

$$Q = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 18 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and $\mathbf{A}$ is symmetric with all its eigenvalues being non-negative. Hence, Q is positive semi-definite.

Q.3.(e). is skipped as the topic was not covered in the lectures.

Q.4.(a). Assume that the true value of $\alpha = \sin(\pi/5) = 0.587785$ to six decimal places. Determine the absolute and relative errors in approximating α by $\overline{\alpha}=0.588$. Also, find the absolute and relative errors in approximating 50α by $50\overline{\alpha}=29.39$.

Solution: The absolute error in approximating α by $\overline{\alpha}$ is

$$\left|a - \overline{a}\right| \cong 0.000215$$

and its relative error is

$$\frac{\left|a - \overline{a}\right|}{a} \cong 3.66 \times 10^{-4}$$

The absolute error in approximating 50α by $50\overline{\alpha}$ is

$$\left|50a - 50\overline{a}\right| \cong 0.00075$$

and its relative error is

$$\frac{\left|50a - 50\overline{a}\right|}{50a} \cong 2.55 \times 10^{-5}$$

Q.4.(b). Use Newton's method to design an iterative equation that computes

$$\sqrt{a-1}$$

for $a \geq 1$. Derive its associated absolute iterative error expression.

Solution: The solution is equivalent to the positive zero of the function,

$$f(x) = x^2 - a + 1$$

The Newton's iteration scheme is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a + 1}{2x_n} = \frac{1}{2}\left(x_n + \frac{a-1}{x_n}\right)$$

Then the absolute iterative error

$$e_{n+1} = \sqrt{a-1} - x_{n+1} = \sqrt{a-1} - \frac{1}{2}\left(x_n + \frac{a-1}{x_n}\right)$$

$$= -\frac{1}{2x_n}(x_n^2 - 2\sqrt{a-1}\,x_n + a - 1) = -\frac{1}{2x_n}\left(\sqrt{a-1} - x_n\right)^2 = -\frac{1}{2x_n}e_n^2$$

Q.4.(c). Perform the formula of part (b) with $x_0 = 1$ for six steps to obtain an approximation of

$$\sqrt{5}$$

(its exact value is equal to 2.236068). Show the details of all calculations.

Solution:

$$x_0 = 1, \quad a = 6$$

$$x_1 = \frac{1}{2}\left(x_0 + \frac{5}{x_0}\right) = \frac{1}{2}\left(1 + \frac{5}{1}\right) = 3$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right) = \frac{1}{2}\left(3 + \frac{5}{3}\right) = 2.3333333$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{5}{x_2}\right) = 2.2380952$$

$$x_4 = \frac{1}{2}\left(x_3 + \frac{5}{x_3}\right) = 2.2360688$$

$$x_5 = \frac{1}{2}\left(x_4 + \frac{5}{x_4}\right) = 2.2360679$$

$$x_6 = \frac{1}{2}\left(x_5 + \frac{5}{x_5}\right) = 2.2360679$$

Q.5. The following is a set of measured data for the function $f(x)=\cos\left(x/2\right)$

x_i	:	-3	-2	-1	0	1	2	3
$f(x_i)$	:	0.0707	0.5403	0.8776	1	0.8776	0.5403	0.0707

(a) Find the least square quadratic fit to the given measured data and compute its average error, $\bar{e}=\left(\sum e_i\right)/N$

Solution: The least square quadratic fit has the form

$$y=a_0+a_1x+a_2x^2$$

with coefficients being given by the solution of the linear system

$$\begin{bmatrix} \sum(x_i)^0 & \sum(x_i)^1 & \sum(x_i)^2 \\ \sum(x_i)^1 & \sum(x_i)^2 & \sum(x_i)^3 \\ \sum(x_i)^2 & \sum(x_i)^3 & \sum(x_i)^4 \end{bmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}=\begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}\Rightarrow\begin{bmatrix} 7 & 0 & 28 \\ 0 & 28 & 0 \\ 28 & 0 & 196 \end{bmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}=\begin{bmatrix} 3.9772 \\ 0 \\ 7.3502 \end{bmatrix}$$

which leads to a solution

$$a_0=0.9757 \qquad a_1=0 \qquad a_2=-0.1019$$

Hence, the least square quadratic fit is

$$y = 0.9757 - 0.1019x^2$$

The errors at each points are

$$e_1 = 0.0121 \quad e_2 = -0.0278 \quad e_3 = 0.0038 \quad e_4 = 0.0243$$
$$e_5 = 0.0121 \quad e_6 = -0.0278 \quad e_7 = 0.0038$$

The average error

$$\bar{e} = \left(\sum e_i\right) \Big/ N = 7.14 \times 10^{-5}$$

Q.5.(b). Compute directly the exact value of

$$\int_{-3}^{3} f(x)dx$$

to four decimal places.

Solution:

$$\int_{-3}^{3} f(x)dx = \int_{-3}^{3} \cos\left(\frac{x}{2}\right)dx = 2\sin\left(\frac{x}{2}\right)\Big|_{-3}^{3}$$

$$= 2\sin(1.5) - 2\sin(-1.5)$$

$$= 3.9900$$

Q.5.(c). Compute directly the approximate value of

$$\int_{-3}^{3} f(x)dx$$

using the given measured data and the trapezoidal rule.

Solution:

$$\int_{-3}^{3} f(x)dx \approx \frac{1}{2}[0.0707 + 2\times 0.5403 + 2\times 0.8776 + 2$$
$$+ 2\times 0.8776 + 2\times 0.5403 + 0.0707]$$
$$= 3.9065$$

Q.5.(d). Compute directly the approximate value of

$$\int_{-3}^{3} f(x)dx$$

using the given measured data and the Simpson's $1/3$ rule.

Solution:

$$\int_{-3}^{3} f(x)dx \approx \frac{1}{3}[0.0707 + 4\times 0.5403 + 2\times 0.8776 + 4$$
$$+ 2\times 0.8776 + 4\times 0.5403 + 0.0707]$$
$$= 3.9914$$

The above result is much closer to its true value than the previous one.

Q.6. Consider a boundary value problem

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \cos(x) \quad \left(0 < x < \frac{p}{3}, \quad 0 < y < 0.6\right)$$

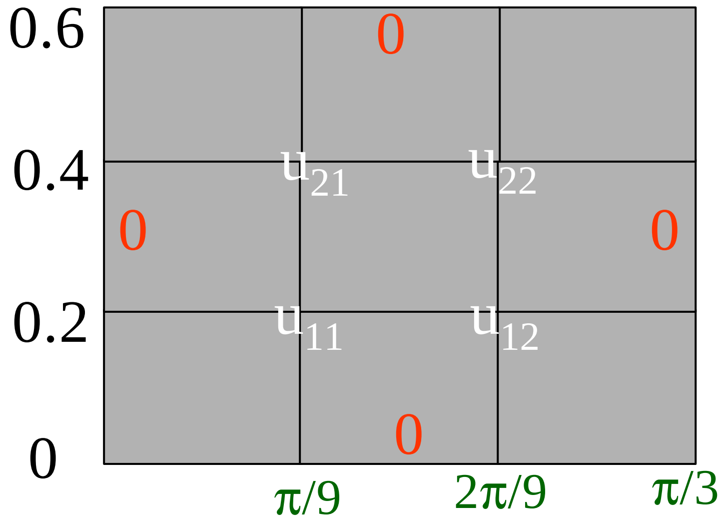
$$u(0, y) = u(p/3, y) = 0 \qquad (0 < y < 0.6)$$

$$u(x,0) = u(x,0.6) = 0 \qquad \left(0 < x < \frac{p}{3}\right)$$

To find an approximation of the solution $u(x,y)$, place a grid with horizontal mesh spacing $h=\pi/9$ and vertical mesh spacing $k=0.2$ on the region:

$$R = \left\{ (x, y) \middle| \, 0 \le x \le p/3, \; 0 \le y \le 0.6 \right\}$$

(a) Draw a grid on R with appropriate mesh points.



Q.6.(b). Use the central difference approximation to obtain a corresponding difference equation.

Solution: Recall that

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \quad h = \frac{p}{9}$$

$$u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}, \quad k = 0.2$$

Thus,

$$u_{xx} = u_{yy} + \cos(x)$$

$$\Rightarrow 8.2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) = 25(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) + \cos(i \cdot h)$$

and hence the difference equation is given by

$$\boxed{8.2u_{i+1,j} + 8.2u_{i-1,j} - 25u_{i,j+1} - 25u_{i,j-1} + 33.6u_{i,j} = \cos\left(i \cdot \frac{p}{9}\right)}$$

Q.6.(c). Use the obtained difference equation in part (b) and the given boundary conditions to derive a linear system that needs to be solved.

Solution: For u_{11},

$$8.2u_{21} - 25u_{12} + 33.6u_{11} = 0.9397$$

For u_{12},

$$8.2u_{22} - 25u_{11} + 33.6u_{12} = 0.9397$$

For u_{21},

$$8.2u_{11} - 25u_{22} + 33.6u_{21} = 0.7660$$

For u_{22},

$$8.2u_{12} - 25u_{21} + 33.6u_{22} = 0.7660$$

$$\begin{bmatrix} 33.6 & -25 & 8.2 & 0 \\ -25 & 33.6 & 0 & 8.2 \\ 8.2 & 0 & 33.6 & -25 \\ 0 & 8.2 & -25 & 33.6 \end{bmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{pmatrix} = \begin{pmatrix} 0.9397 \\ 0.9397 \\ 0.7660 \\ 0.7660 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{pmatrix} = \begin{pmatrix} 0.2679 \\ 0.2679 \\ -0.1664 \\ -0.1664 \end{pmatrix} \quad \text{not required!}$$