DTS5322-SC01: Control Systems

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Part 1: Course Outline (Part 1)

- Introduction to control systems; ordinary differential equations; Laplace transform; basic principle of feedback
- Modeling of physical systems
- Control system design — stability issues
- Control system design time domain specifications: steady state errors, overshoot, rise time and settling time.
- Control system design using Proportional-Integral-Derivative (PID) control technique.

Lectures and Tutorials:

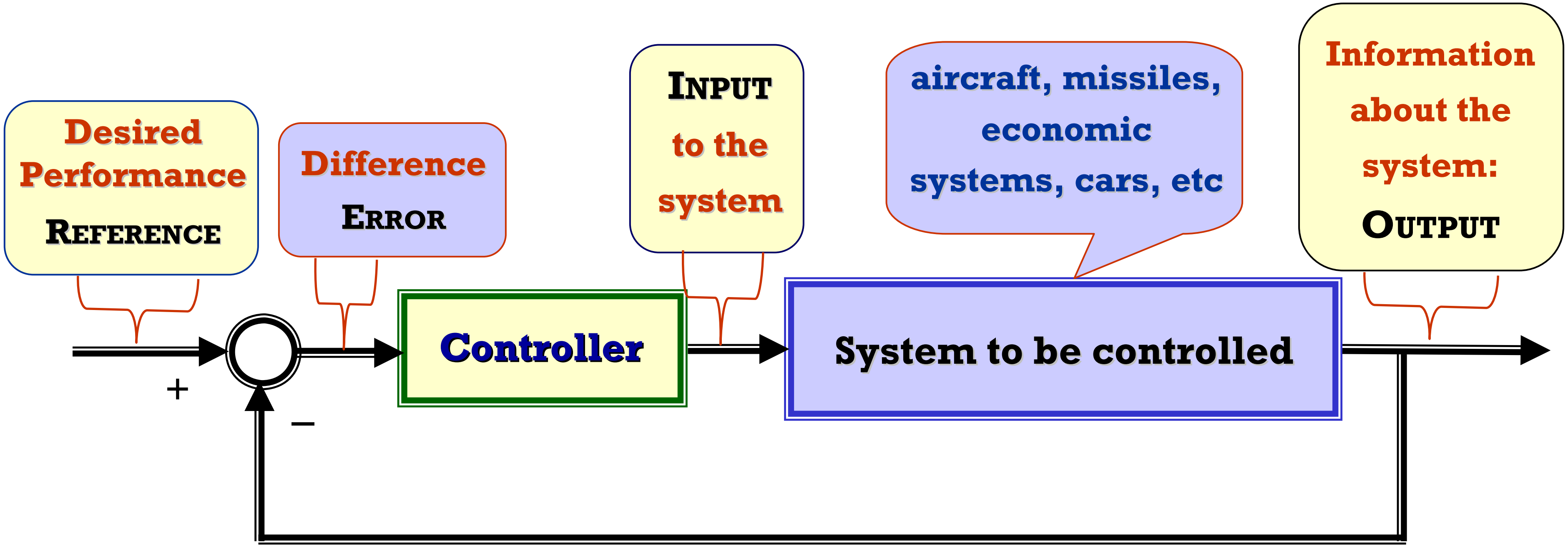
- Total lecture hours for the 1st part: 5.5 hours
- Total tutorial hours for the 1st part: 1.5 hours
- An overview on UAV helicopter systems: 1.0 hour
- In-class test for the 1st part: 1.0 hour

Reference Text:

- G. F. Franklin, J. D. Powell and A. Emami-Naeimi, *Feedback Control of Dynamic Systems*, 3rd Edition, Addison Wesley, New York, 1994.

Introduction

What is a control system?



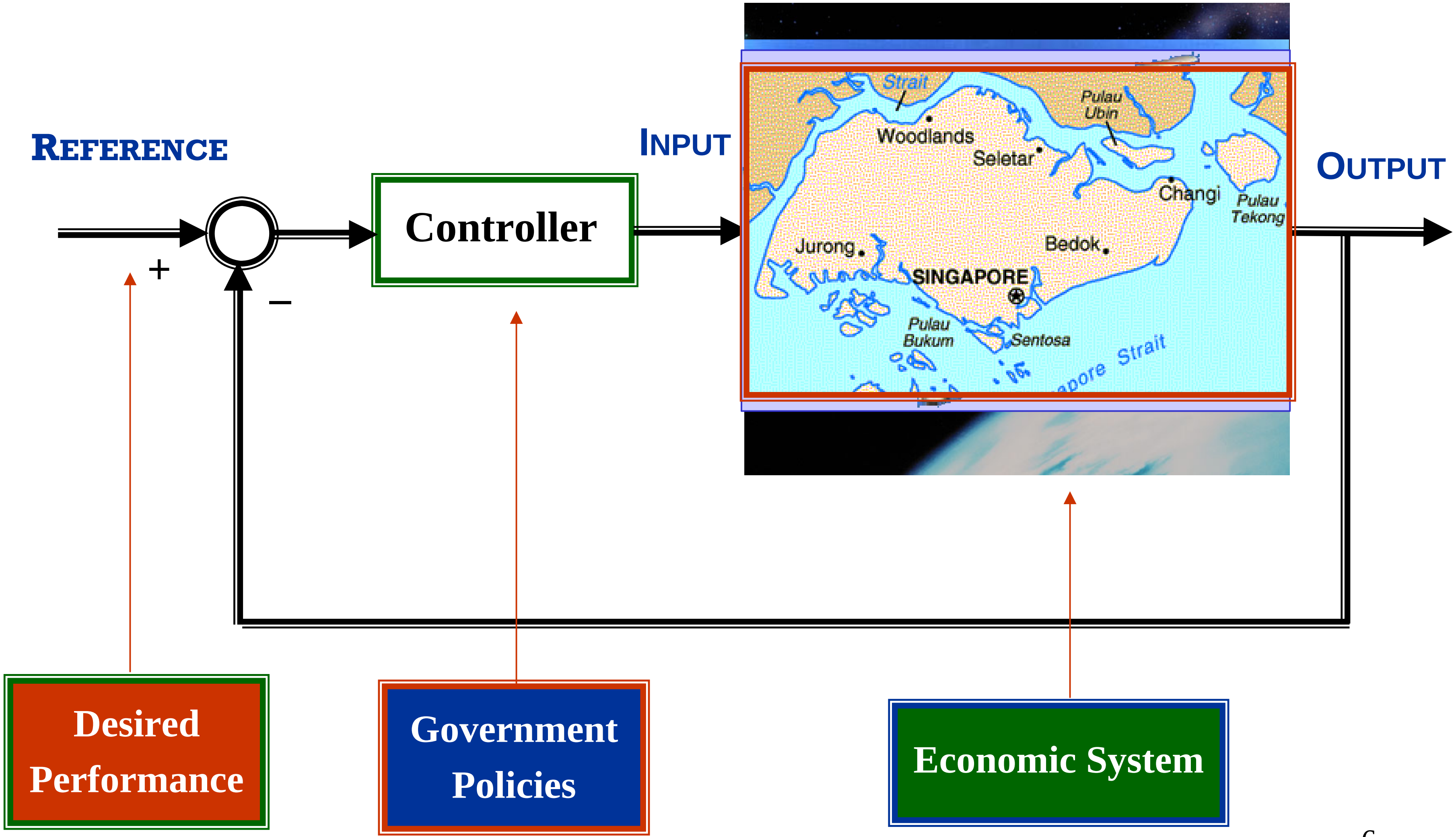
Objective: To make the system **OUTPUT** and the desired **REFERENCE** as close as possible, i.e., to make the **ERROR** as small as possible.

Key Issues:

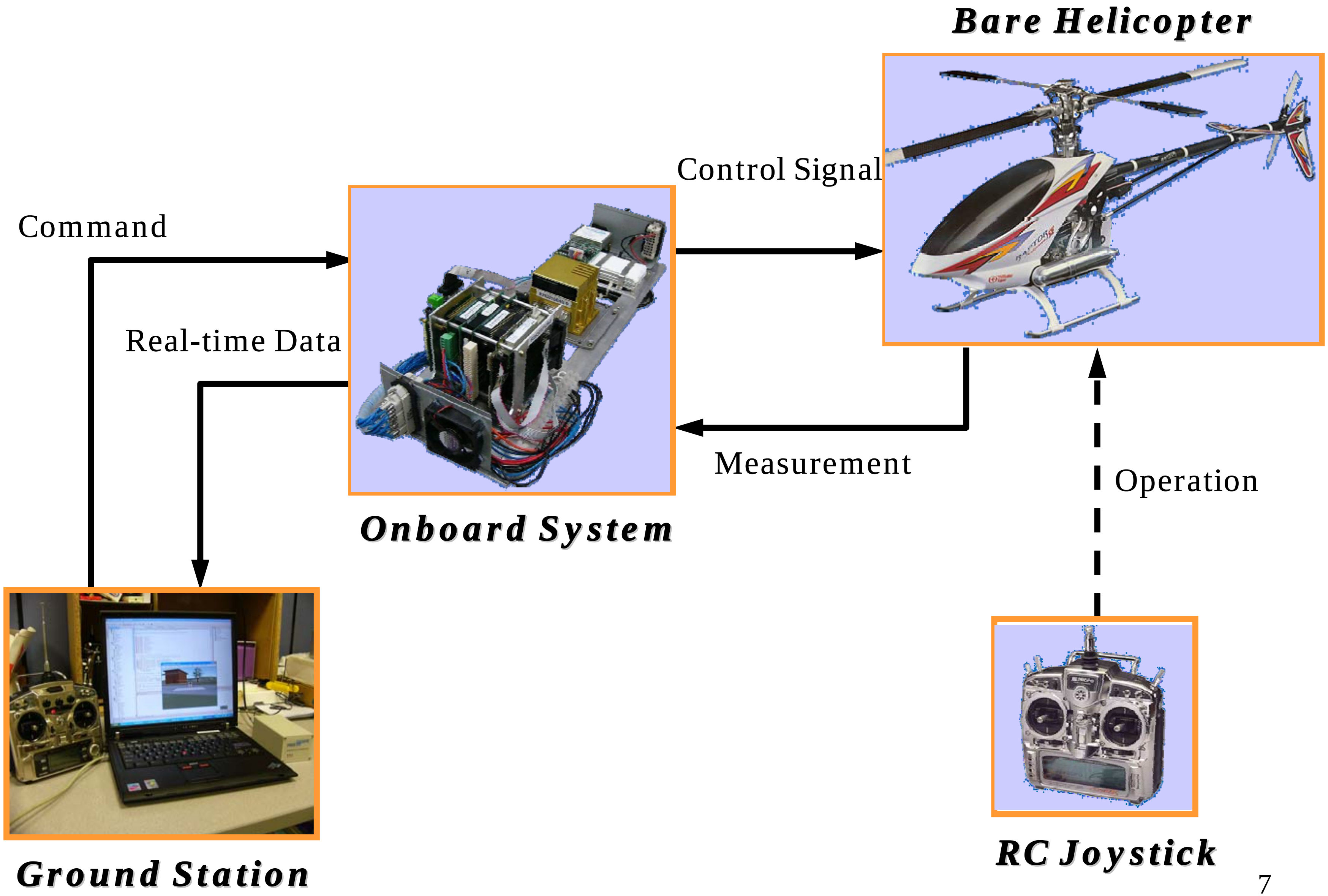
1) How to describe the system to be controlled? **(Modeling)**

2) How to design the controller? **(Control)**

Some Control Systems Examples:



An Overview on a UAV Helicopter System to be given later…



Background

Background Material: Differentiation

Given a function of time, say $f(t)$, its differentiation is the rate of change of the function.

Mathematically,

$$\dot{f}(t) = \frac{df(t)}{dt} = \frac{f(t+\Delta t) - f(t)}{\Delta t}, \quad \Delta t \text{ is small}$$

Example: Consider $f(t) = 1$. Obviously, the rate of change of a constant is zero, i.e., it does not change at all.

$$\dot{f}(t) = \frac{df(t)}{dt} = \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{1-1}{\Delta t} = \frac{0}{\Delta t} = 0$$

Example: Consider $f(t) = t$. The rate of change of this function is

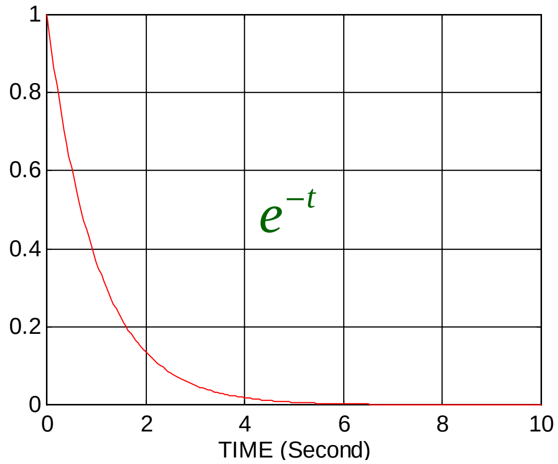
$$\dot{f}(t) = \frac{df(t)}{dt} = \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{(t+\Delta t) - t}{\Delta t} = \frac{\Delta t}{\Delta t} = 1$$

The rate of change of this function is constant.

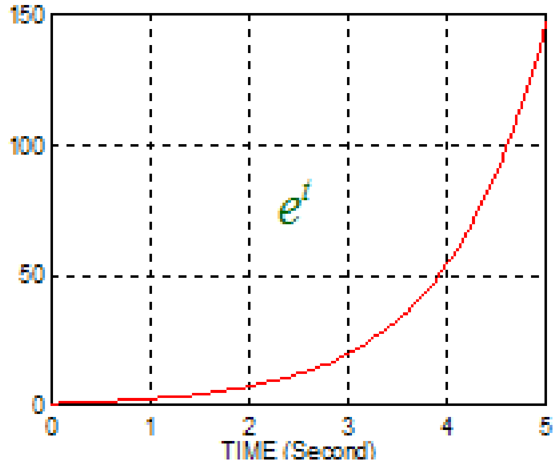
More examples for the differentiation (or derivative or the rate of change):

- The rate of change of displacement (x) is called speed (v), i.e., $v(t) = \dot{x}(t) = \dfrac{dx(t)}{dt}$
- The rate of change of speed (v) is called acceleration (a), i.e., $a(t) = \dot{v}(t) = \ddot{x}(t) = \dfrac{d^2 x(t)}{dt^2}$
- The derivative of a sine function is cosine, i.e., $\dfrac{d \sin \omega t}{dt} = \omega \cos \omega t$
- The derivative of a cosine function is minus sine, i.e., $\dfrac{d \cos \omega t}{dt} = -\omega \sin \omega t$
- The derivative of an exponential function is an exponential function, i.e., $\dfrac{de^{kt}}{dt} = ke^{kt}$

$e^0 = 1$



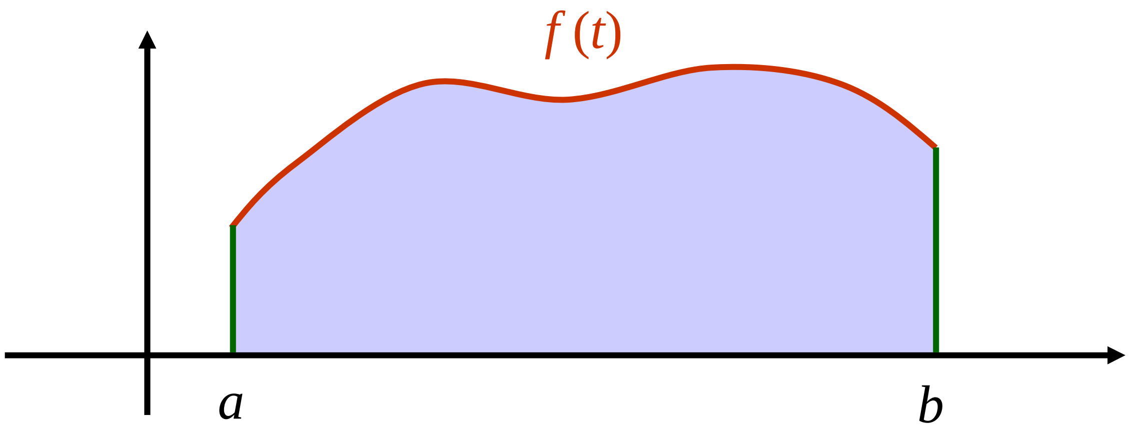
$e^{-\infty} = 0$



$e^{\infty} = \infty$

Background Material: Integration

The integration of a function $f(t)$ over a certain interval is the total area of the function within the interval, e.g.,



Mathematically, we write it as $\int_a^b f(t)dt = F(t)\Big|_a^b = F(b) - F(a)$, where $\dot{F}(t) = f(t)$.

Examples: 1) $f(t) = 1 \quad \Leftrightarrow \quad F(t) = t$

$$\int_a^b 1 \cdot dt = F(t)\Big|_a^b = F(b) - F(a) = b - a$$

2) $f(t) = e^{kt} \quad \Leftrightarrow \quad F(t) = \frac{1}{k}e^{kt}$

$$\int_a^b e^{kt}dt = \frac{1}{k}e^{kt}\Big|_a^b = \frac{1}{k}e^{kb} - \frac{1}{k}e^{ka}$$

Modeling

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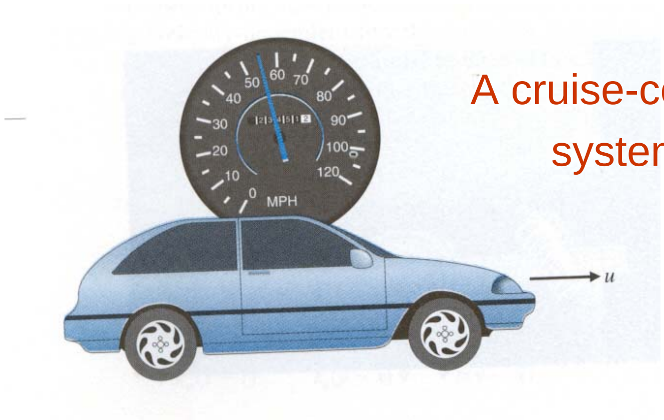
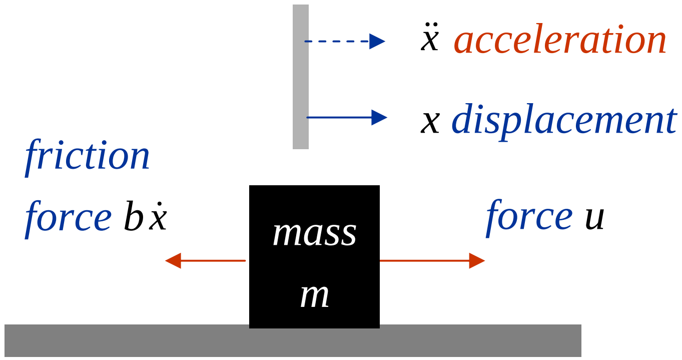
Modeling of Some Physical Systems

A simple mechanical system:



A cruise-control system



By the well-known Newton's Law of motion: $f = m\,a$, where f is the total force applied to an object with a mass m and a is the acceleration, we have

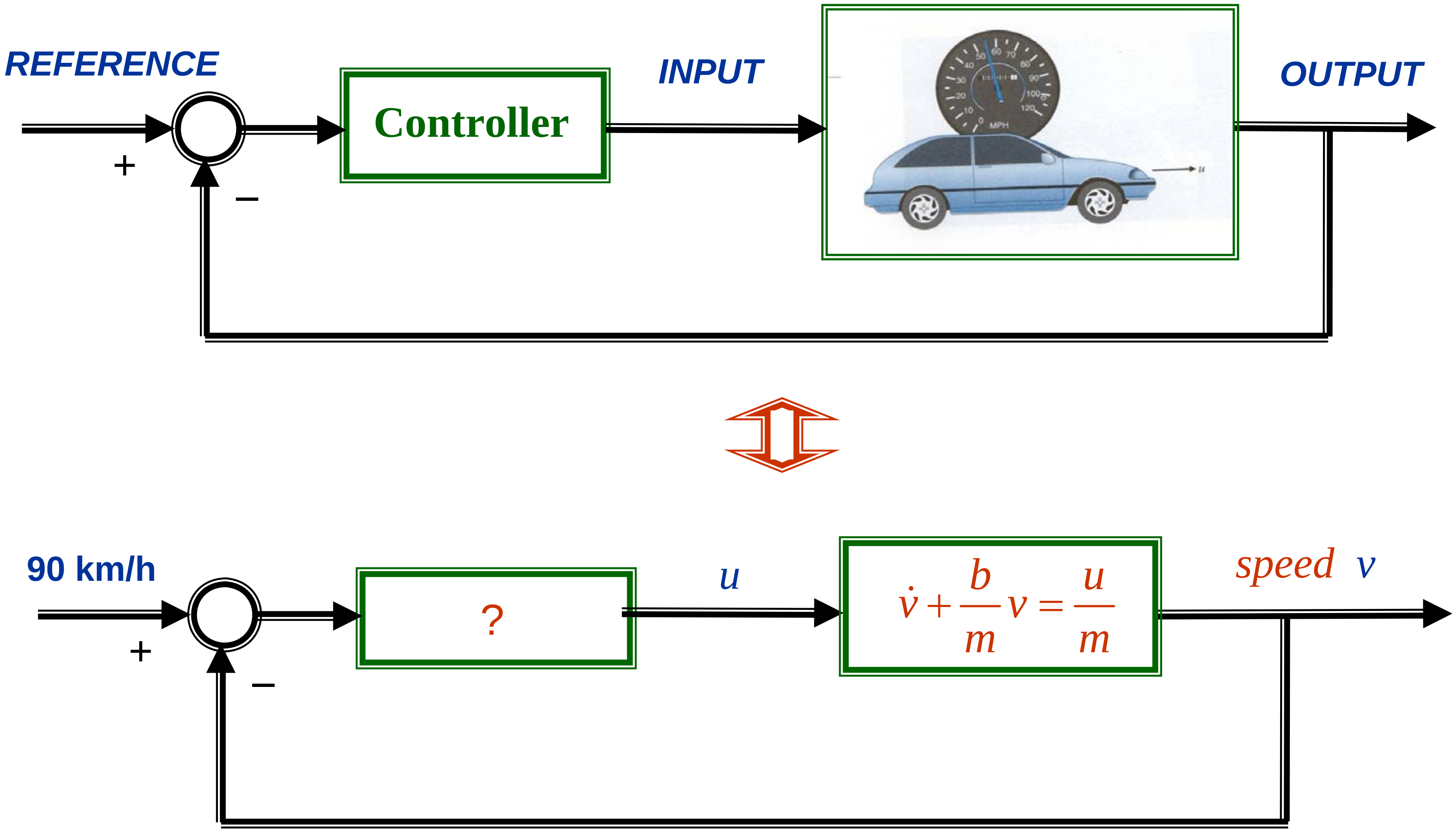
$$u - b\dot{x} = m\ddot{x} \quad \Leftrightarrow \quad \ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}$$

This a 2nd order *Ordinary Differential Equation* with respect to displacement x. It can be written as a 1st order *ODE* with respect to speed $v = \dot{x}$:

$$\boxed{\dot{v} + \frac{b}{m}v = \frac{u}{m}}$$

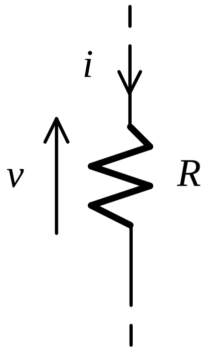
← model of the cruise control system, u is input force, v is output.

A cruise-control system:



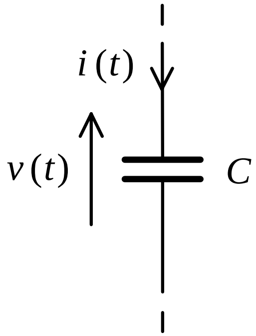
Basic electrical systems:

resistor



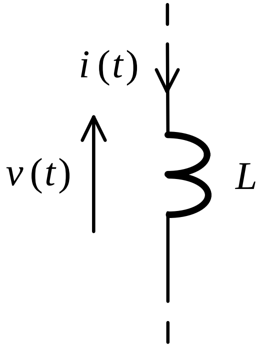
$$v = i\,R$$

capacitor



$$i = C\frac{dv}{dt}$$

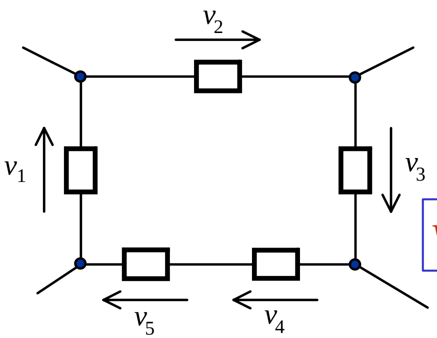
inductor



$$v = L\frac{di}{dt}$$

Kirchhoff's Voltage Law (KVL):

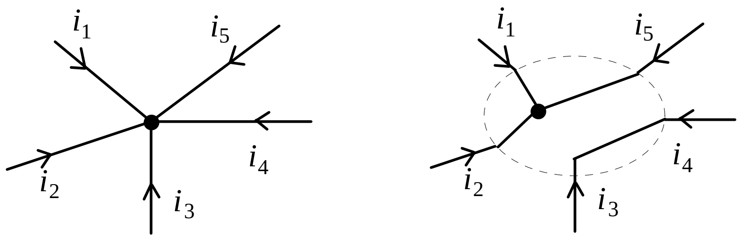
The sum of voltage drops around any close loop in a circuit is 0.



$$v_1 + v_2 + v_3 + v_4 + v_5 = 0$$

Kirchhoff's Current Law (KCL):

The sum of currents entering/leaving a node/closed surface is 0.

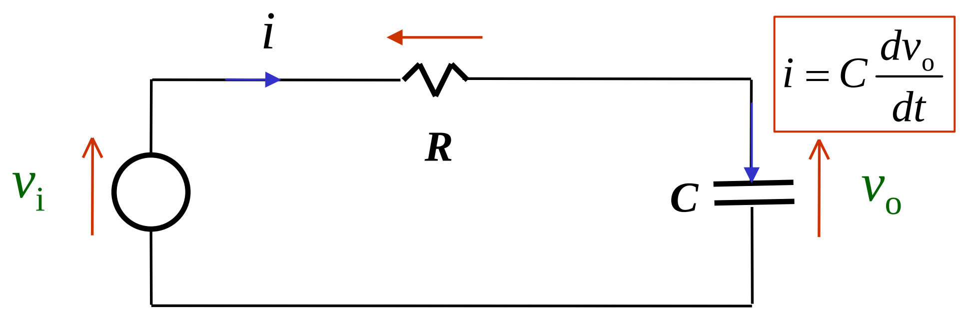


$$i_1 + i_2 + i_3 + i_4 + i_5 = 0$$

Modeling of a simple electrical system:

To find out relationship between the input (v_i) and the output (v_o) for the circuit:

$$v_R = Ri = RC\frac{dv_o}{dt}$$



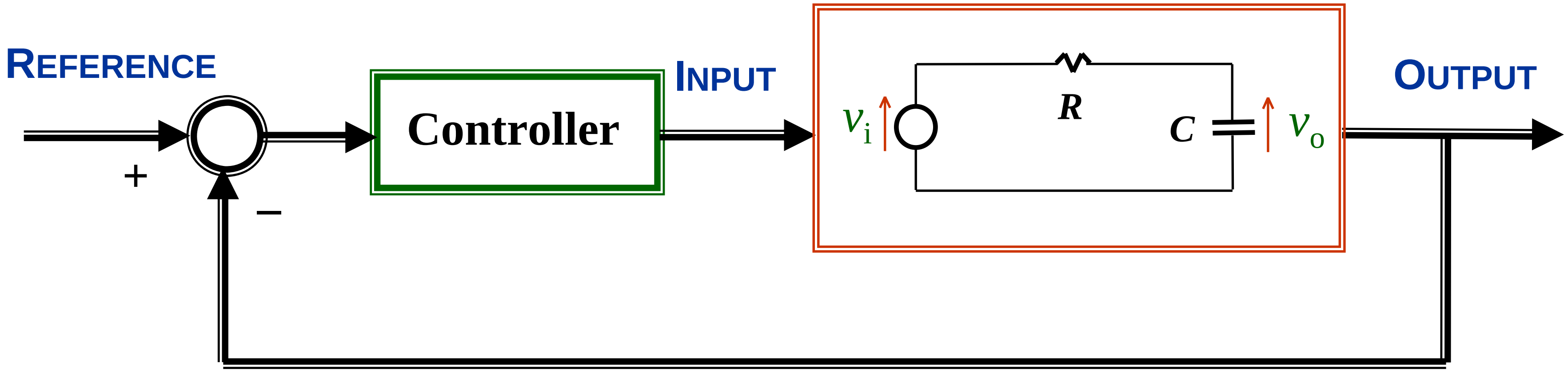
By KVL, we have $v_o + v_R - v_i = 0$

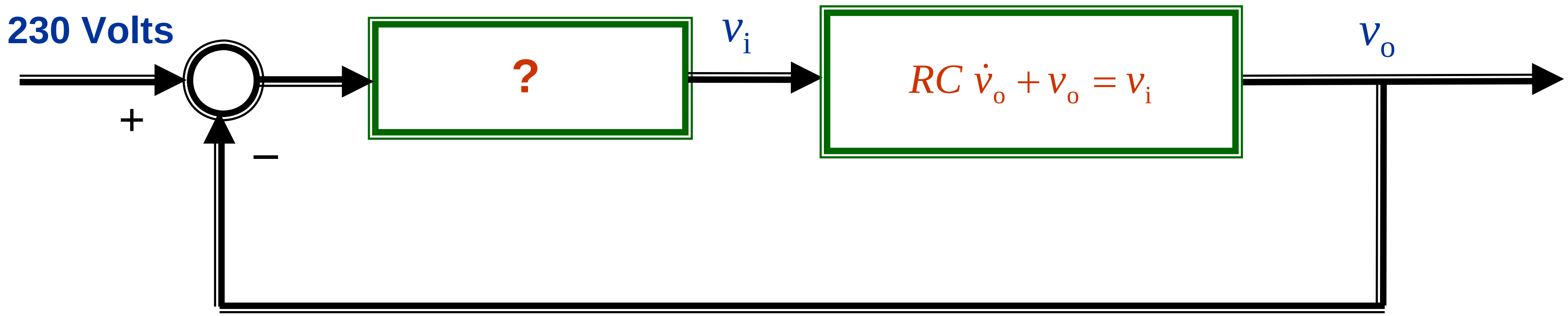
$$v_o + v_R - v_i = v_o + RC\frac{dv_o}{dt} - v_i = 0$$

$$RC\frac{dv_o}{dt} + v_o = v_i \quad \Leftrightarrow \quad RC\,\dot{v}_o + v_o = v_i$$

A dynamic model of the circuit

Control the output voltage of the electrical system:





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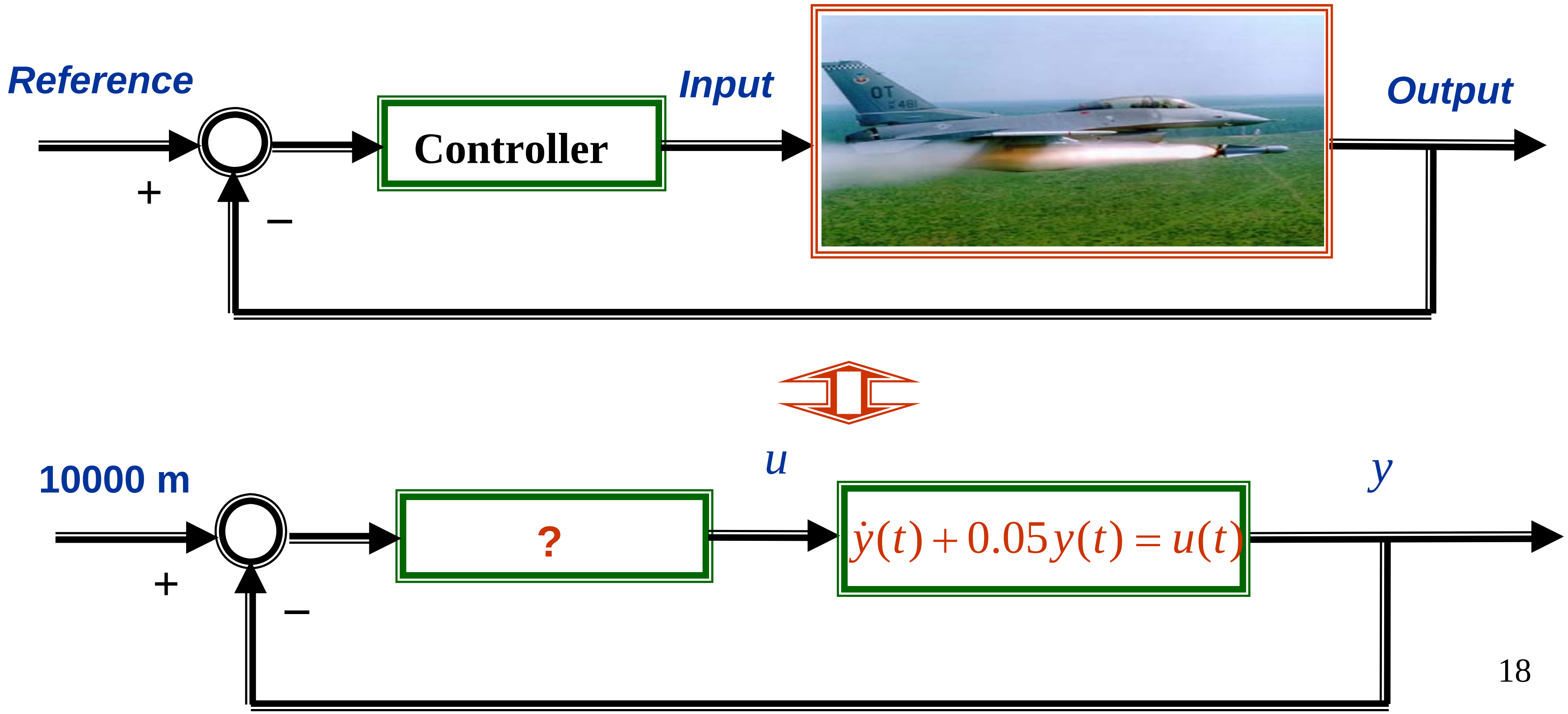
Control of a Fighter Aircraft

It can be showed that the vertical position of a fighter aircraft can be approximated by the following equation:

$$\dot{y}(t) + 0.05y(t) = u(t)$$

where $y(t)$ is the vertical position (in meters) above the sea level and $u(t)$ is the thrust force.



Ordinary Differential Equations

Ordinary Differential Equations

Many real life problems can be modelled as an ODE of the following form:

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = u(t)$$

This is called a 2nd order ODE as the highest order derivative in the equation is 2. The ODE is said to be homogeneous if $u(t) = 0$. In fact, many systems can be modelled or approximated as a 1st order ODE, i.e.,

$$\dot{y}(t) + a_0 y(t) = u(t)$$

An ODE is also called the time-domain model of the system, because it can be seen the above equations that $y(t)$ and $u(t)$ are functions of time t. The key issue associated with ODE is: how to find its solution? That is: how to find an explicit expression for $y(t)$ from the given equation?

Solutions to a 1st order ODE for a fighter aircraft:

$$\dot{y}(t) + 0.05y(t) = 0, \qquad y(0) = 5000, \text{ the initial condition}$$

The ODE is homogeneous. Replace the ODE with

$$\dot{y}(t) \Leftrightarrow s^1 = s \quad \& \quad y(t) \Leftrightarrow s^0 = 1 \longrightarrow \boxed{s + 0.05 = 0 \Rightarrow s = -0.05}$$

The solution is then given by $y(t) = ke^{st} = ke^{-0.05t}$, where k is a constant to be determined.

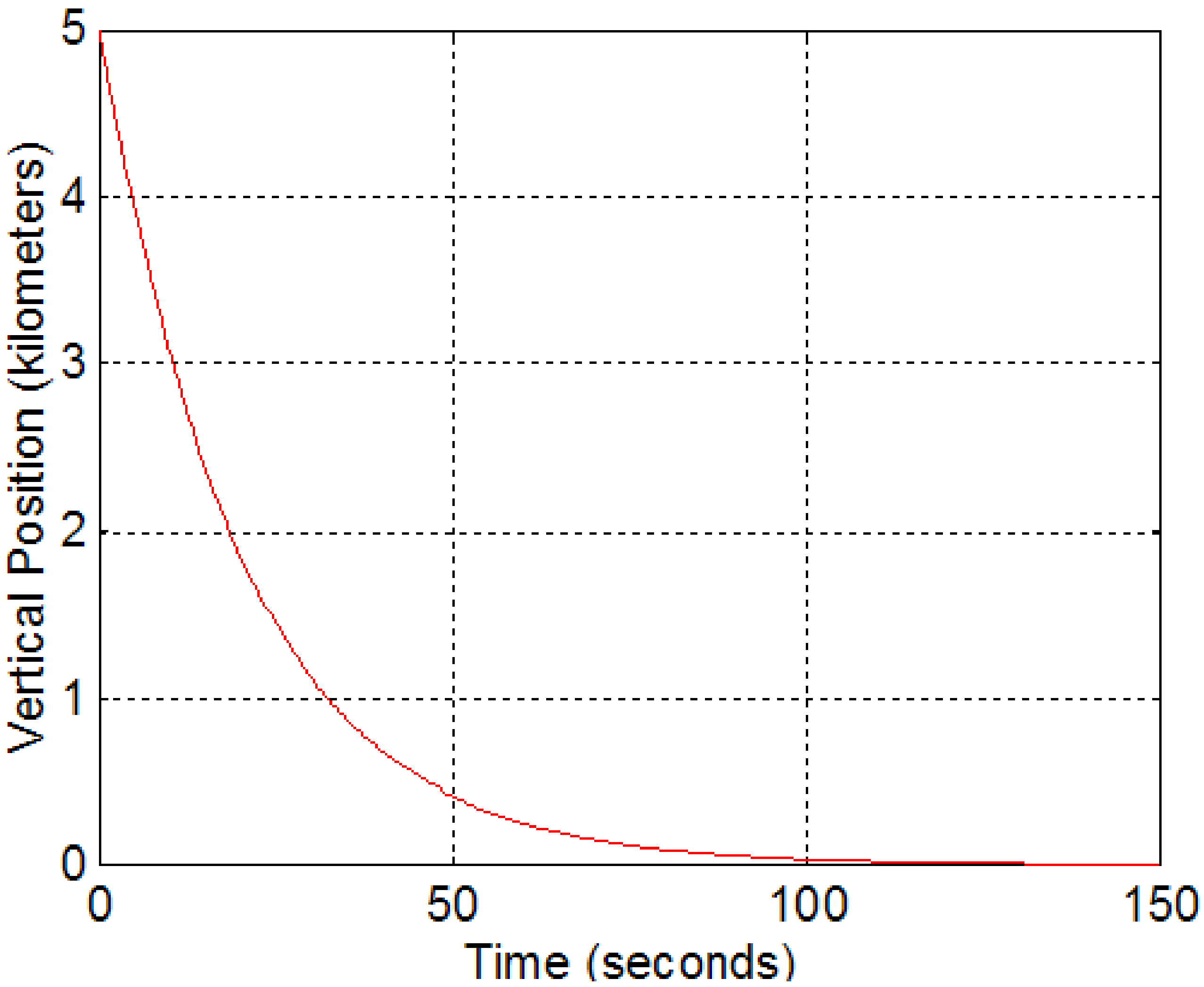
Substitute this function into the ODE,

$$\dot{y}(t) + 0.05y(t) = \frac{d}{dt}\left(ke^{-0.05t}\right) + 0.05\left(ke^{-0.05t}\right) = -0.05ke^{-0.05t} + 0.05ke^{-0.05t} = 0$$

It is indeed a solution. To find k, we use initial condition

$$y(0) = ke^{-0.05t}\Big|_{t=0} = ke^0 = k = 5000 \Longrightarrow \boxed{y(t) = 5000e^{-0.05t}}$$

Vertical Positions of the Fighter Aircraft:



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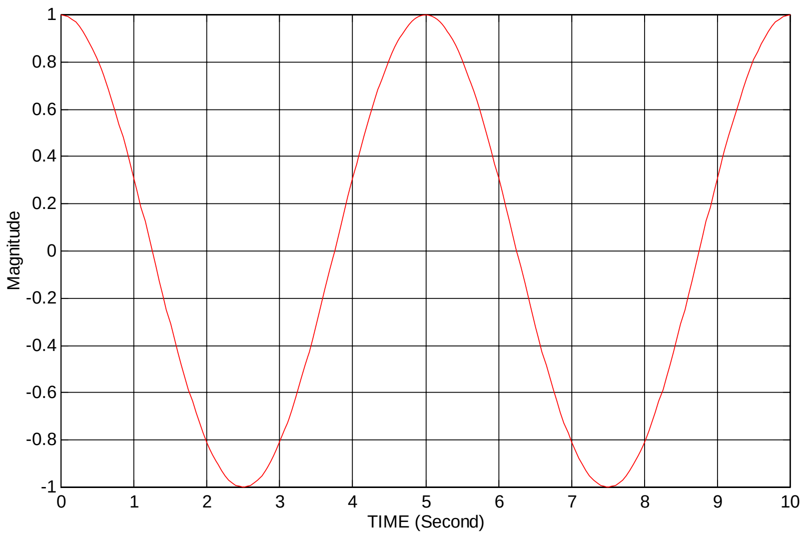
Laplace Transform

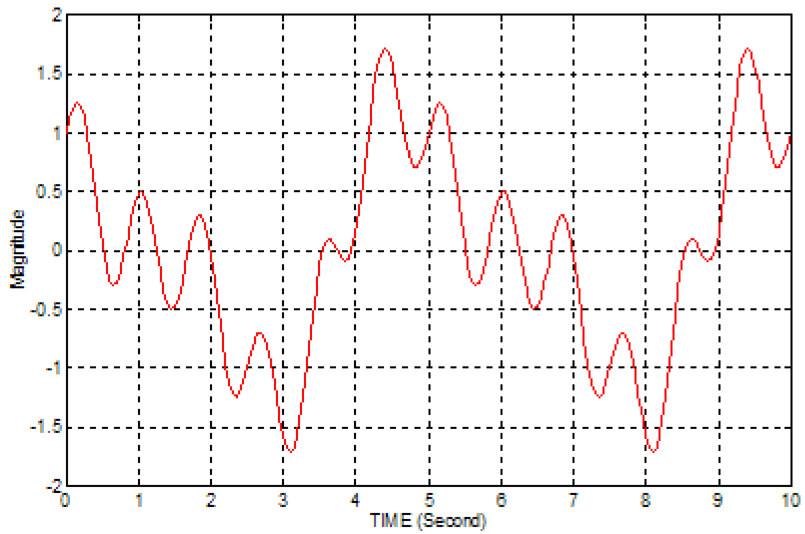
Laplace Transform and Inverse Laplace Transform

Let us first examine the following time-domain functions:



A cosine function with a frequency $f = 0.2$ Hz.

Note that it has a period $T = 5$ seconds.



$$x(t) = \cos(0.4\pi t) + \sin(0.8\pi t)\cos(1.6\pi t)$$

What are frequencies of this function?

Laplace transform is a tool to convert a time-domain function into a frequency-domain one in which information about frequencies of the function can be captured. It is often much easier to solve problems in frequency-domain with the help of Laplace transform.

Laplace Transform:

Given a time-domain function $f(t)$, its Laplace transform is defined as follows:

$$F(s) = L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$$

Example 1: Find the Laplace transform of a constant function $f(t) = 1$.

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty} = -\frac{1}{s}e^{-\infty} - \left(-\frac{1}{s}e^{0}\right) = -\frac{1}{s}\cdot 0 - \left(-\frac{1}{s}\cdot 1\right) = \frac{1}{s}$$

Example 2: Find the Laplace transform of an exponential function $f(t) = e^{-at}$.

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} e^{-at}e^{-st}dt = \int_0^{\infty} e^{-(s+a)t}dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}$$

Inverse Laplace Transform

Given a frequency-domain function $F(s)$, the inverse Laplace transform is to convert it back to its original time-domain function $f(t)$.

Here are some very useful Laplace and inverse Laplace transform pairs:

$f(t)$	$\Leftrightarrow$	$F(s)$
1	$\Leftrightarrow$	$\frac{1}{s}$
t	$\Leftrightarrow$	$\frac{1}{s^2}$
e^{-at}	$\Leftrightarrow$	$\frac{1}{s+a}$
te^{-at}	$\Leftrightarrow$	$\frac{1}{(s+a)^2}$

$f(t)$	$\Leftrightarrow$	$F(s)$
$\sin at$	$\Leftrightarrow$	$\frac{a}{s^2+a^2}$
$\cos at$	$\Leftrightarrow$	$\frac{s}{s^2+a^2}$
$e^{-at}\sin bt$	$\Leftrightarrow$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos bt$	$\Leftrightarrow$	$\frac{s+a}{(s+a)^2+b^2}$

Some useful properties of Laplace transform:

1. Superposition:

$$L\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 L\{f_1(t)\} + a_2 L\{f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$$

2. Differentiation: Assume that $f(0) = 0$.

$$L\left\{\frac{df(t)}{dt}\right\} = L\{\dot{f}(t)\} = sL\{f(t)\} = sF(s)$$

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = L\{\ddot{f}(t)\} = s^2 L\{f(t)\} = s^2 F(s)$$

3. Integration:

$$L\left\{\int_0^t f(\zeta)d\zeta\right\} = \frac{1}{s} L\{f(t)\} = \frac{1}{s} F(s)$$

Re-express ODE Models using Laplace Transform

Recall that the mechanical system in the cruise-control problem with $m = 1$ can be represented by an ODE:

$$\dot{v} + bv = u$$

Taking Laplace transform on both sides of the equation, we obtain

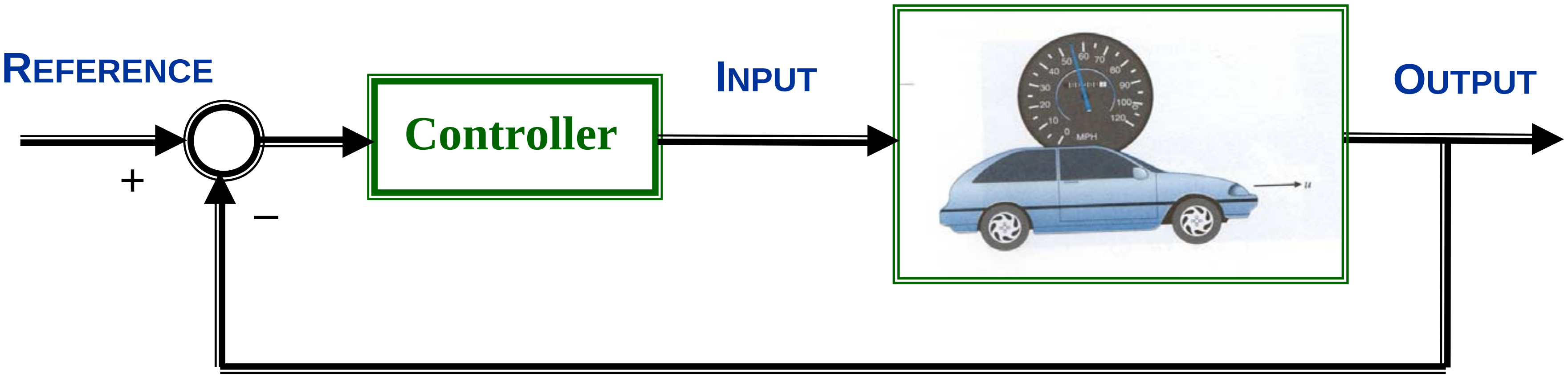
$$L\{\dot{v} + bv\} = L\{u\} \quad \Rightarrow \quad L\{\dot{v}\} + L\{bv\} = L\{u\}$$

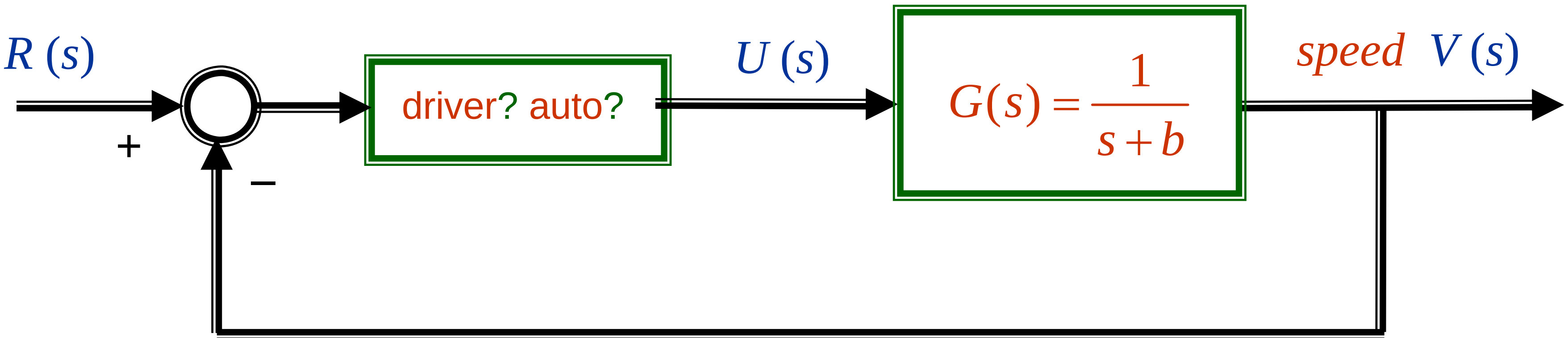
$$\Rightarrow \quad sL\{v\} + bL\{v\} = L\{u\} \quad \Rightarrow \quad sV(s) + bV(s) = U(s)$$

$$\Rightarrow \quad (s+b)V(s) = U(s) \quad \Rightarrow \quad \frac{V(s)}{U(s)} = \frac{1}{s+b} = G(s)$$

This is called the transfer function of the system model

A cruise-control system in frequency domain:





Recall that the fighter aircraft vertical positioning system can be represented by an ODE:

$$\dot{y} + 0.05y = u$$

Taking Laplace transform on both sides of the equation, we obtain

$$L\{\dot{y} + 0.05y\} = L\{u\} \quad \Rightarrow \quad L\{\dot{y}\} + 0.05L\{y\} = L\{u\}$$

$$\Rightarrow \quad sY(s) + 0.05Y(s) = U(s) \quad \Rightarrow \quad (s + 0.05)Y(s) = U(s)$$

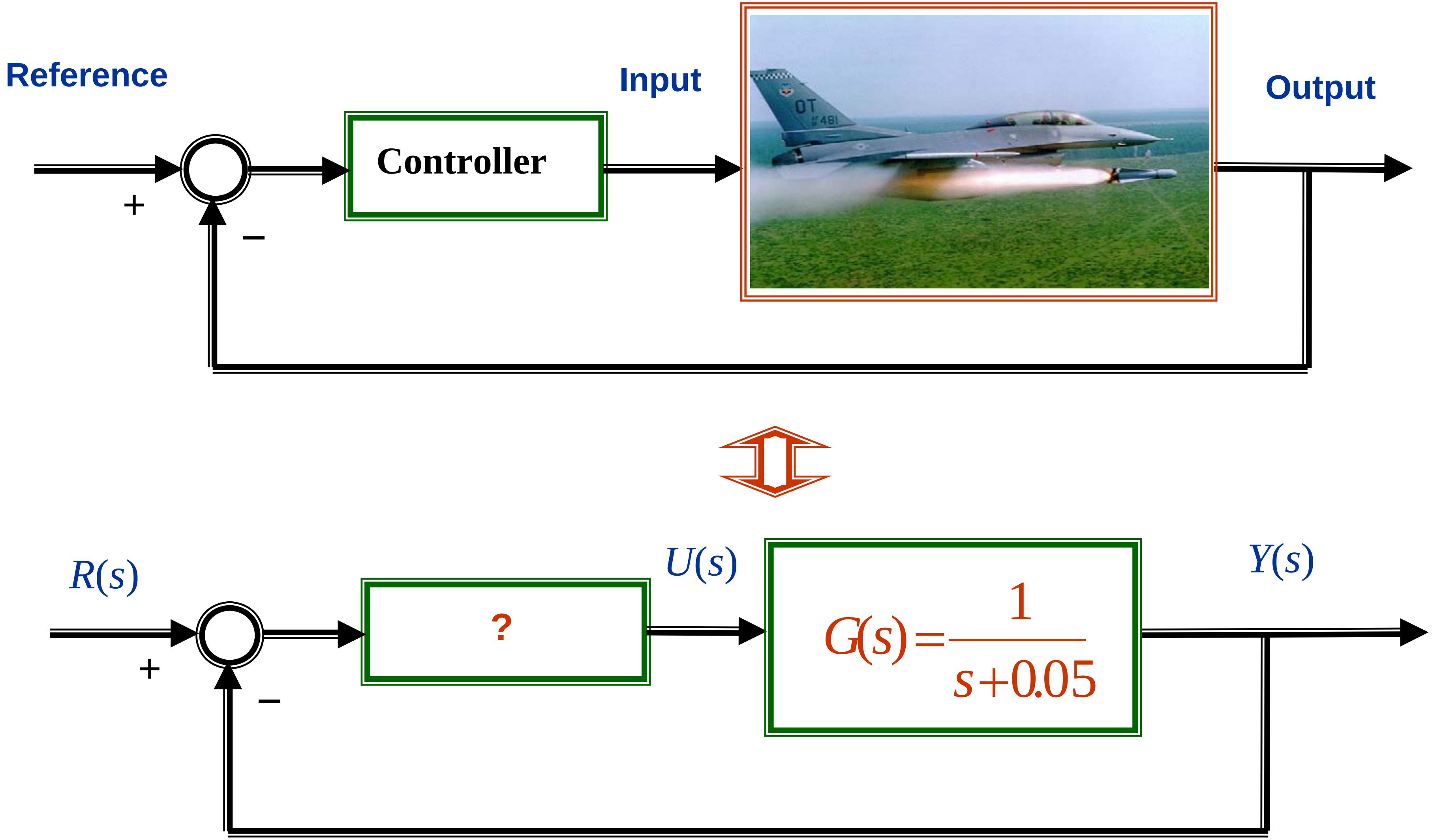
$$\Downarrow$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s + 0.05}$$

$$\Uparrow$$

This is the transfer function of the fighter aircraft

Control the fighter aircraft in frequency domain:



Feedback Control

Why do we need a feedback controller?

To answer this question, let us consider the fighter aircraft vertical positioning system, i.e.,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+0.05}$$

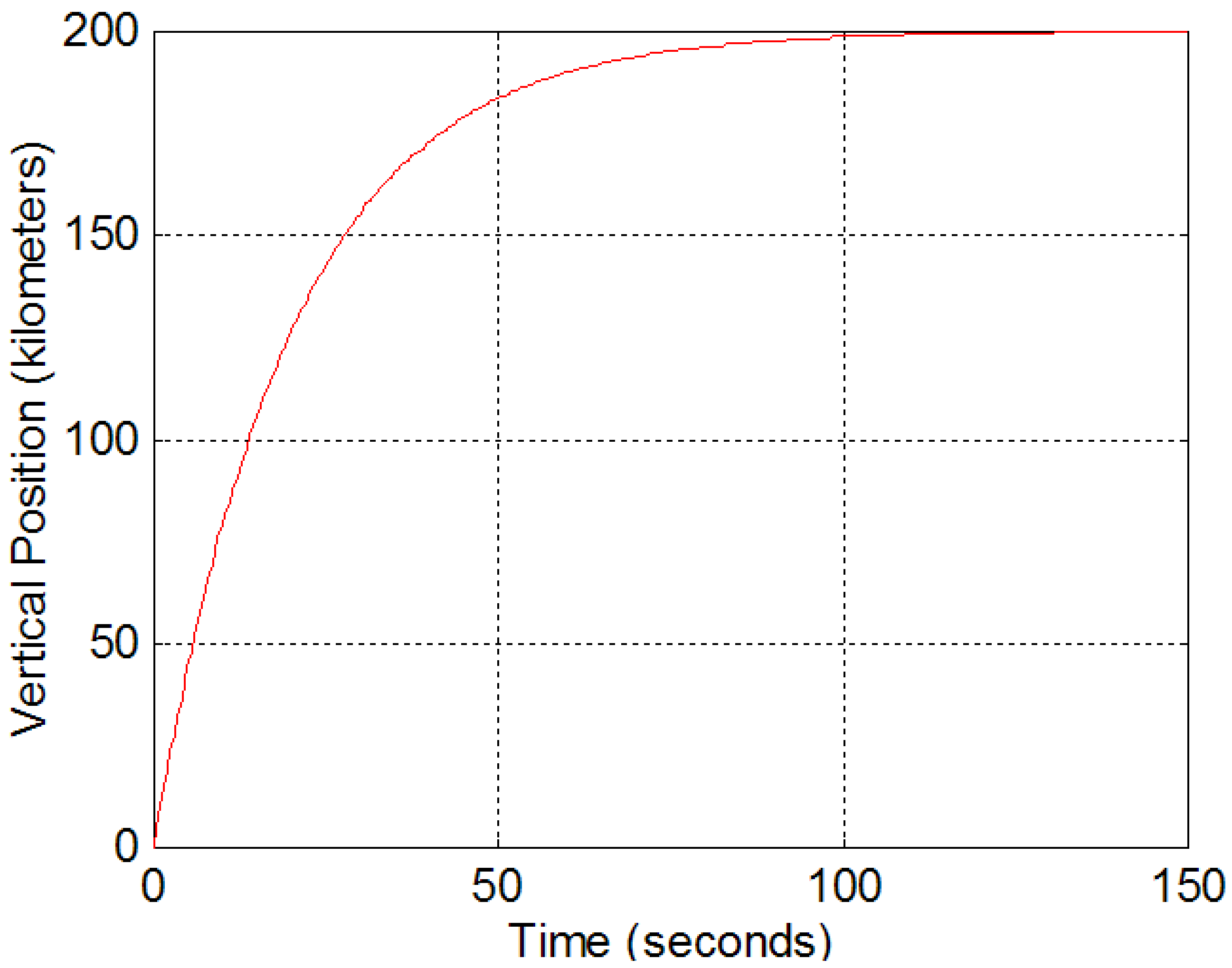
If we want the aircraft to reach 10000 m above the sea level without a controller, one might guess that we need to apply an input force $u(t) = 10000$. Let us try to see whether this works or not. From the Laplace transform table, we have

$$U(s) = L\{10000\} = \frac{10000}{s}$$

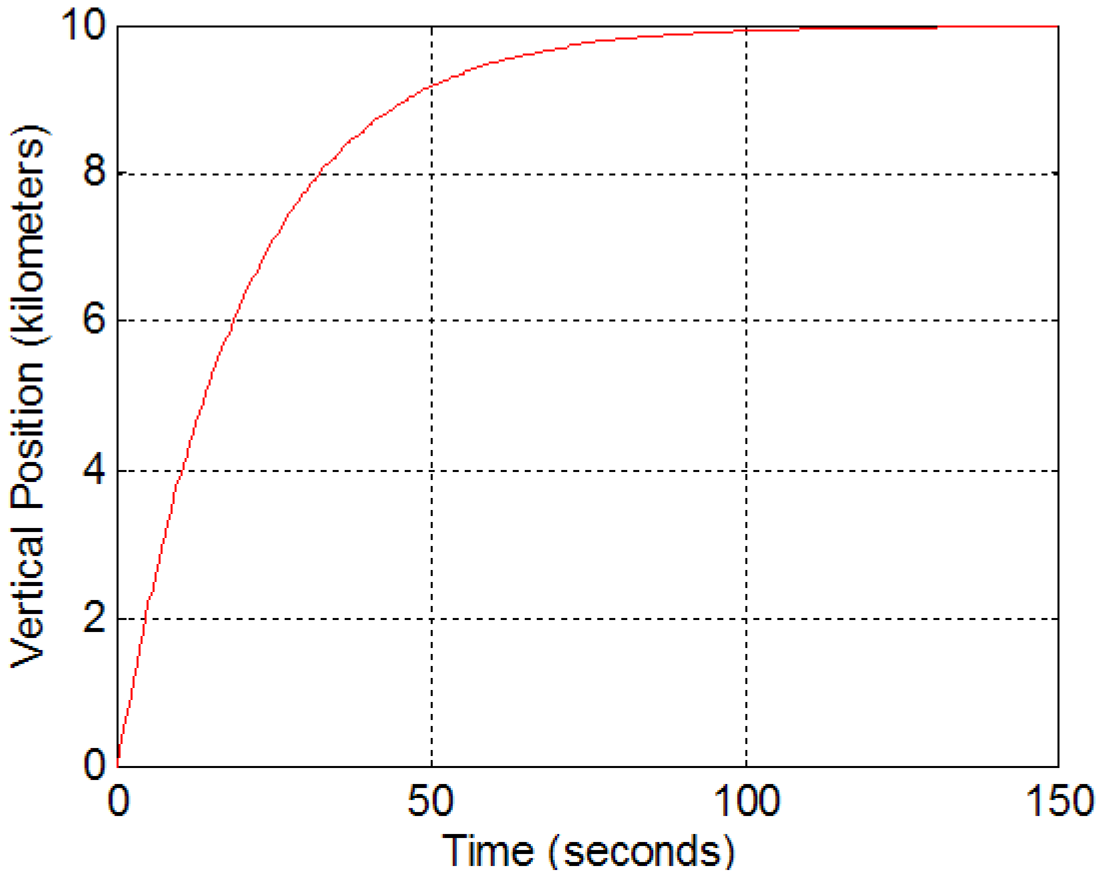
$$\Longrightarrow Y(s) = G(s)U(s) = \frac{1}{s+0.05}\frac{10000}{s} = 200000\left(\frac{1}{s} - \frac{1}{s+0.05}\right)$$

Taking inverse Laplace transform, we obtain $y(t) = 200000\left(1 - e^{-0.05t}\right)$

The vertical position of the aircraft: It reaches 200000 m instead of the desired 10000 m.



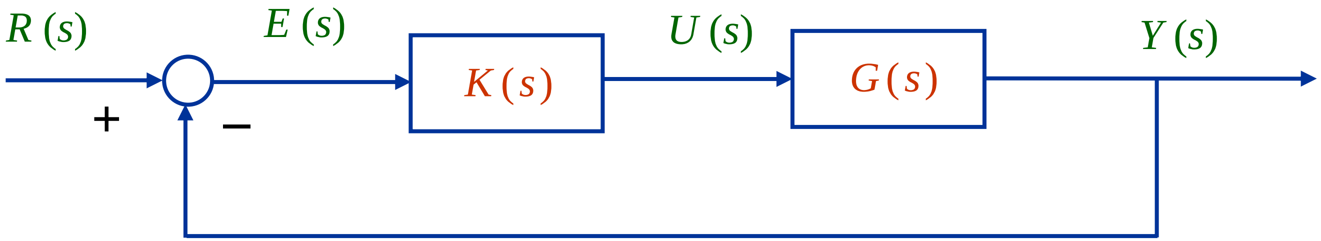
If we choose u = 500, the resulting position will be $y(t) = 10000\left(1 - e^{-0.05t}\right)$



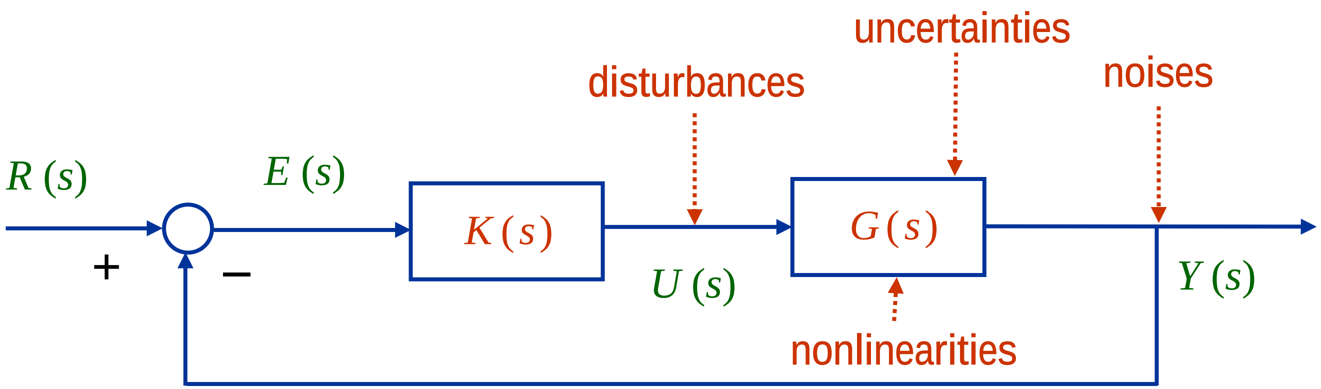
The aircraft will reach the desired level in about 130 seconds. Can we improve this?

The answer is yes and the solution is to use a feedback controller.

In general, a feedback control system can be represented by the following block diagram:



Given a system represented by $G(s)$ and a reference $R(s)$, the objective of control system design is to find a control law (or controller) $K(s)$ such that the resulting output $Y(s)$ is as close to reference $R(s)$ as possible, or the error $E(s) = R(s) - Y(s)$ is as small as possible. However, many other factors of life have to be carefully considered when dealing with real-life problems. These factors include:

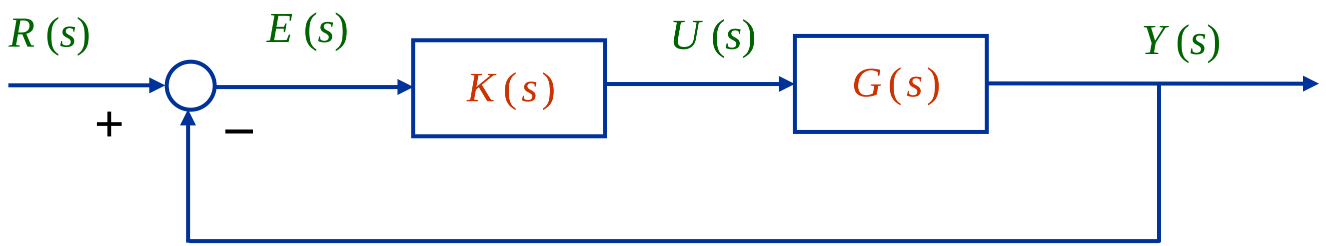


Control Techniques:

There are tons of research published in the literature on how to design control laws for various purposes. These can be roughly classified as the following:

- Classical control: Proportional-integral-derivative (PID) control, developed in 1940s and used for control of industrial processes. Examples: chemical plants, commercial aeroplanes.
- Optimal control: Linear quadratic regulator control, Kalman filter, H_2 control, developed in 1960s to achieve certain optimal performance and boomed by *NASA Apollo Project*.
- Robust control: H_∞ control, developed in 1980s & 90s to handle systems with uncertainties and disturbances and with high performances. Example: military systems.
- Nonlinear control: Currently hot research topics, developed to handle nonlinear systems with high performances. Examples: military systems such as aircraft, missiles.
- Intelligent control: Knowledge-based control, adaptive control, neural and fuzzy control, etc., researched heavily in 1990s, developed to handle systems with unknown models. Examples: economic systems, social systems, human systems.

Let us examine the following block diagram of control system:



Recall that the objective of control system design is trying to match the output $Y(s)$ to the reference $R(s)$. Thus, it is important to find the relationship between them. Recall that

$$G(s) = \frac{Y(s)}{U(s)} \quad \Rightarrow \quad Y(s) = G(s)U(s)$$

Similarly, we have $U(s) = K(s)E(s)$, and $E(s) = R(s) - Y(s)$. Thus,

$$Y(s) = G(s)U(s) = G(s)K(s)E(s) = G(s)K(s)\left[R(s) - Y(s)\right]$$

$$Y(s) = G(s)K(s)R(s) - G(s)K(s)Y(s) \;\Rightarrow\; \left[1 + G(s)K(s)\right]Y(s) = G(s)K(s)R(s)$$

$$\Rightarrow \boxed{H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)}}$$

Closed-loop transfer function from R to Y.

Thus, the block diagram of the control system can be simplified as,

$$R(s) \longrightarrow \boxed{H(s) = \frac{G(s)K(s)}{1+G(s)K(s)}} \longrightarrow Y(s)$$

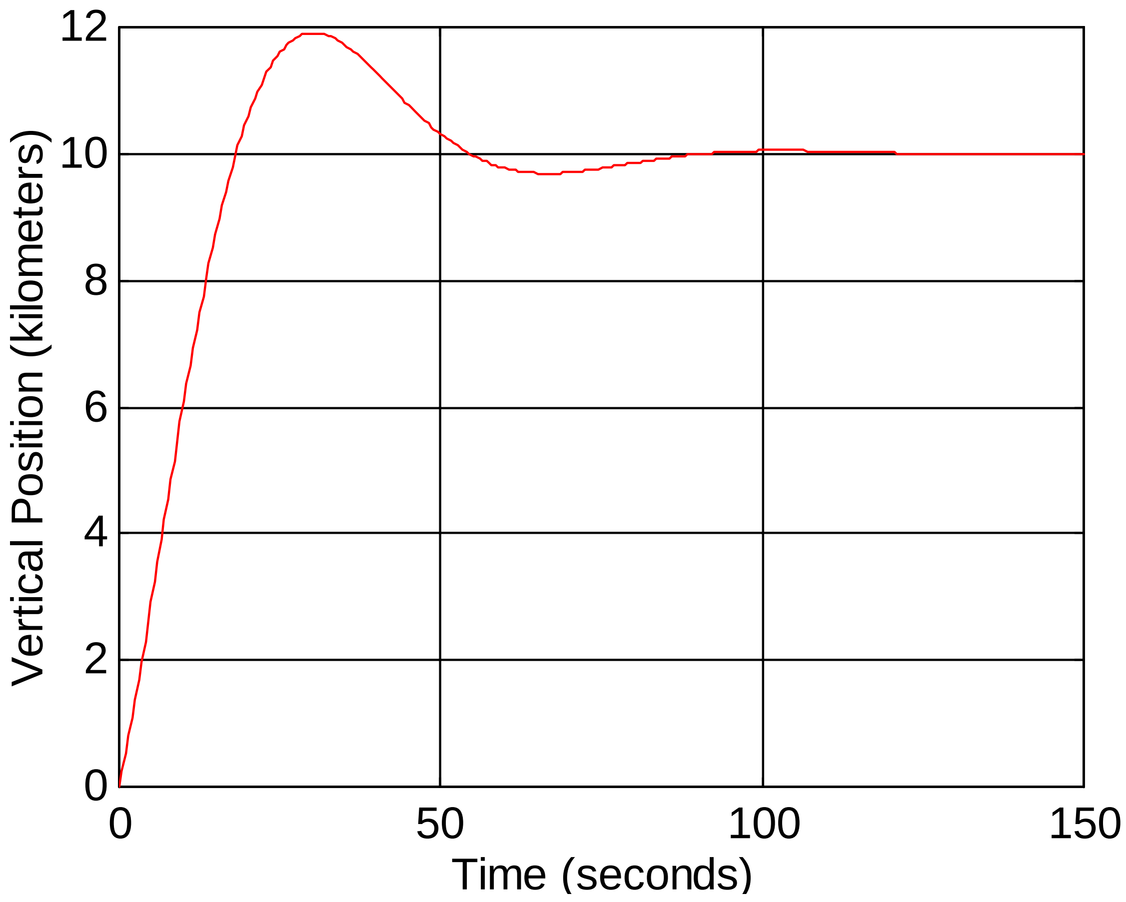
The whole control problem becomes how to choose an appropriate $K(s)$ such that the resulting $H(s)$ would yield desired properties between R and Y.

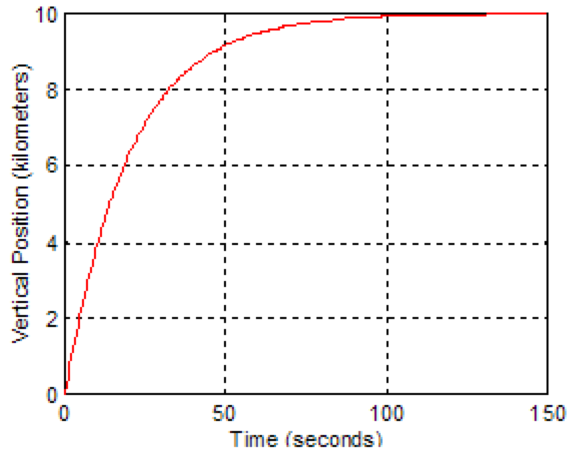
We'll focus on control system design of some first order systems $G(s) = \dfrac{b}{s+a}$ with a proportional-integral (PI) controller, $K(s) = k_p + \dfrac{k_i}{s} = \dfrac{k_p s + k_i}{s}$. This implies

$$H(s) = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{bk_p s + bk_i}{s^2 + (a + bk_p)\,s + bk_i}$$

The closed-loop system $H(s)$ is a second order system as its denominator is a polynomial s of degree 2.

Example 1: The performance of the fighter aircraft with a PI controller ($k_p = 0.05$, $k_i = 0.01$)

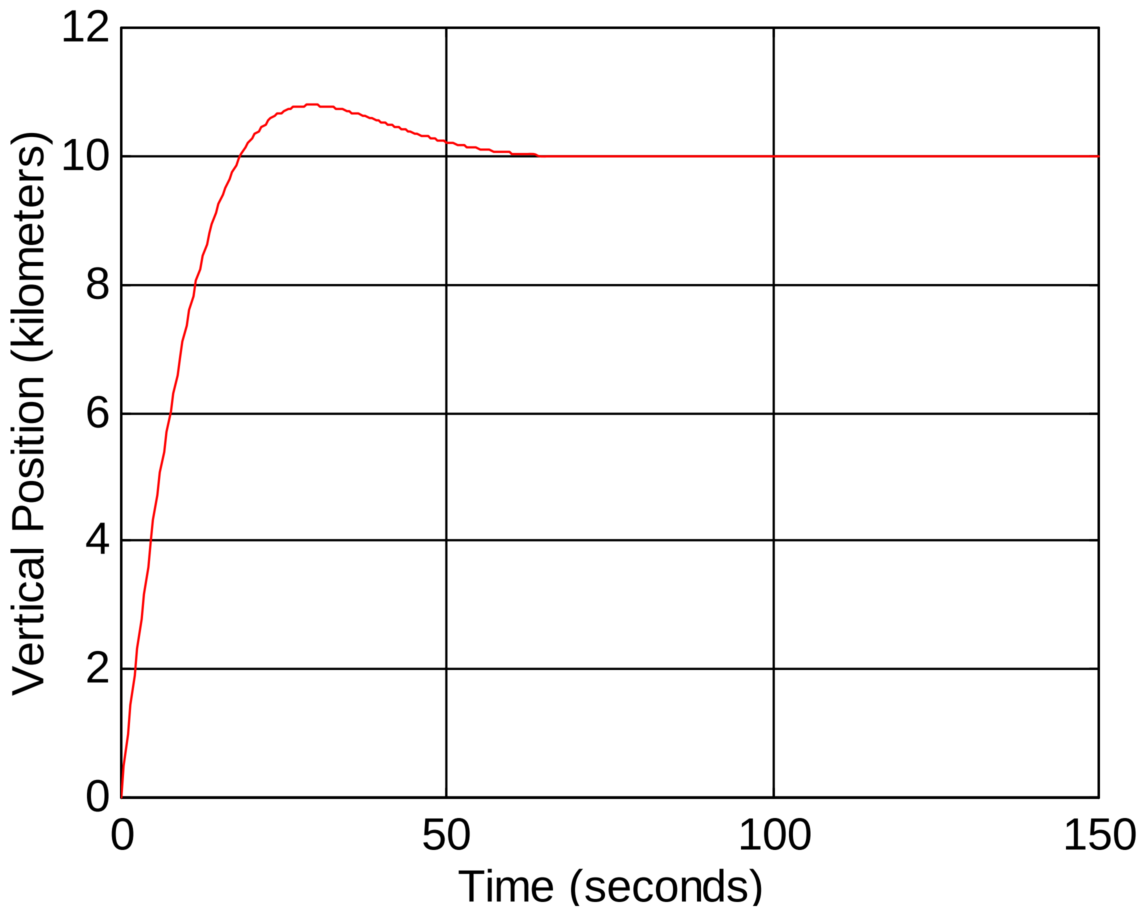


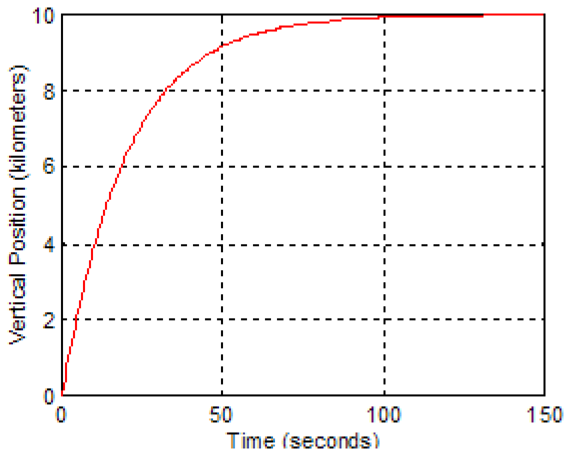


Performance of the system without a controller (see Page 35)

The response is much faster compared to the system without a controller. The drawback is that there is a 20% overshoot.

Example 2: The performance of the fighter aircraft with a PI controller ($k_p = 0.1,\ k_i = 0.01$)





Performance of the system without a controller (see Page 35)

The response is faster and the overshoot is smaller. The controller does improve the overall system performance a great deal.

System Stability

Stability of Control Systems

Example 1: Consider a closed-loop system with,

$R(s) = 1$ → $H(s) = \dfrac{1}{s^2 - 1}$ → $Y(s)$

We have

$$Y(s) = H(s)R(s) = \frac{1}{s^2-1} = \frac{1}{(s+1)(s-1)} = \frac{0.5}{s-1} - \frac{0.5}{s+1}$$

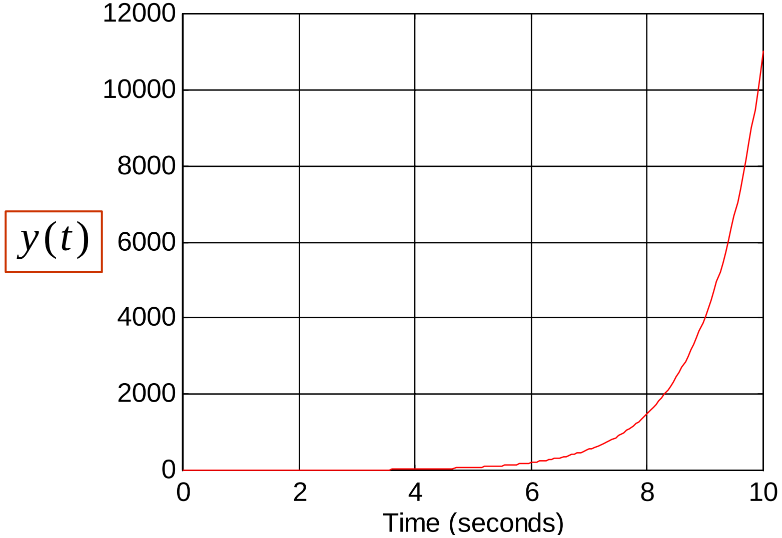
$e^{-at} \Leftrightarrow \dfrac{1}{s+a}$

$0.5e^{-t} \Leftrightarrow \dfrac{0.5}{s+1}$

&

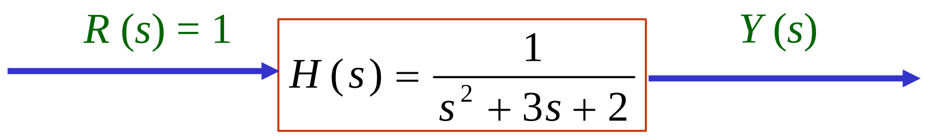
$0.5e^{t} \Leftrightarrow \dfrac{0.5}{s-1}$

Using the Laplace transform table on Page 26, we obtain $y(t) = 0.5(e^{t} - e^{-t})$



This system is said to be *unstable* because the output response $y(t)$ goes to infinity as time t is getting larger and large. This happens because the denominator of $H(s)$ has one positive root at $s = 1$.

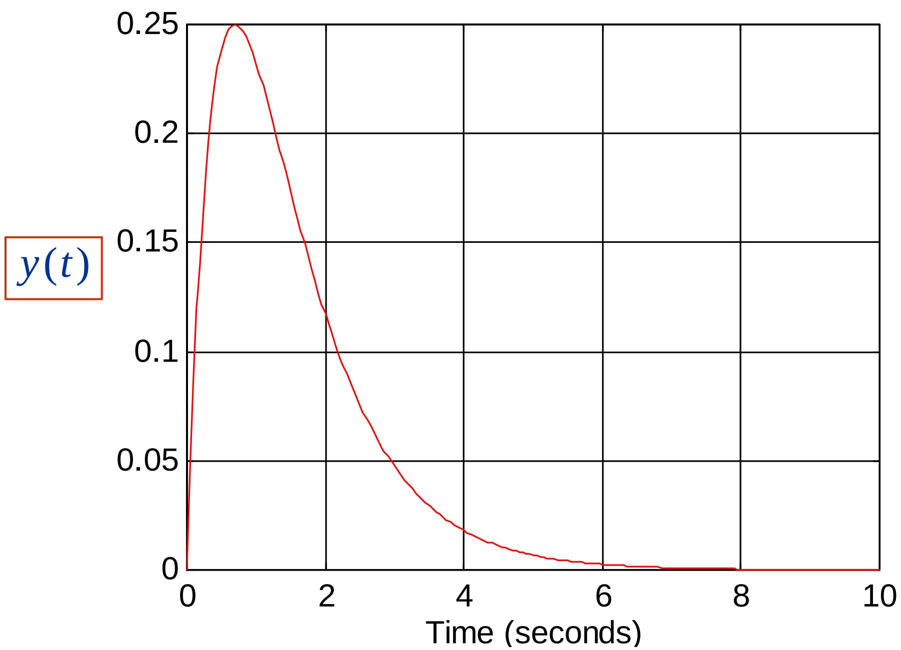
Example 2: Consider a closed-loop system with,



We have

$$Y(s) = H(s)R(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Using the Laplace transform table on Page 26, we obtain $y(t) = e^{-t} - e^{-2t}$

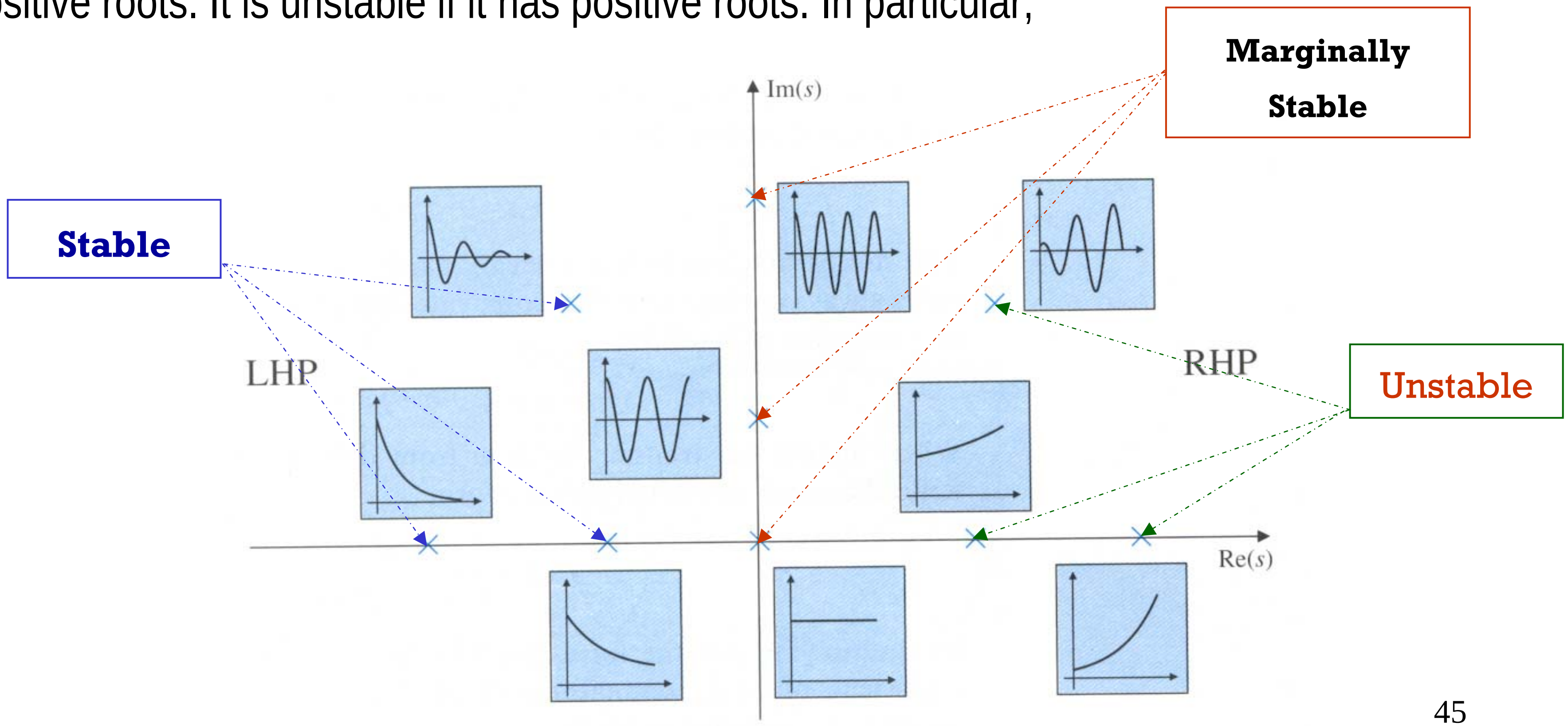


This system is said to be ***stable*** because the output response $y(t)$ goes to 0 as time t is getting larger and large. This happens because the denominator of $H(s)$ has no positive roots.

We consider a general 2nd order system,

$R(s) = 0$ → $$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$ → $Y(s)$

The system is stable if the denominator of the system, i.e., $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, has no positive roots. It is unstable if it has positive roots. In particular,



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Behavior of Systems

Behavior of Second Order Systems with a Step Inputs

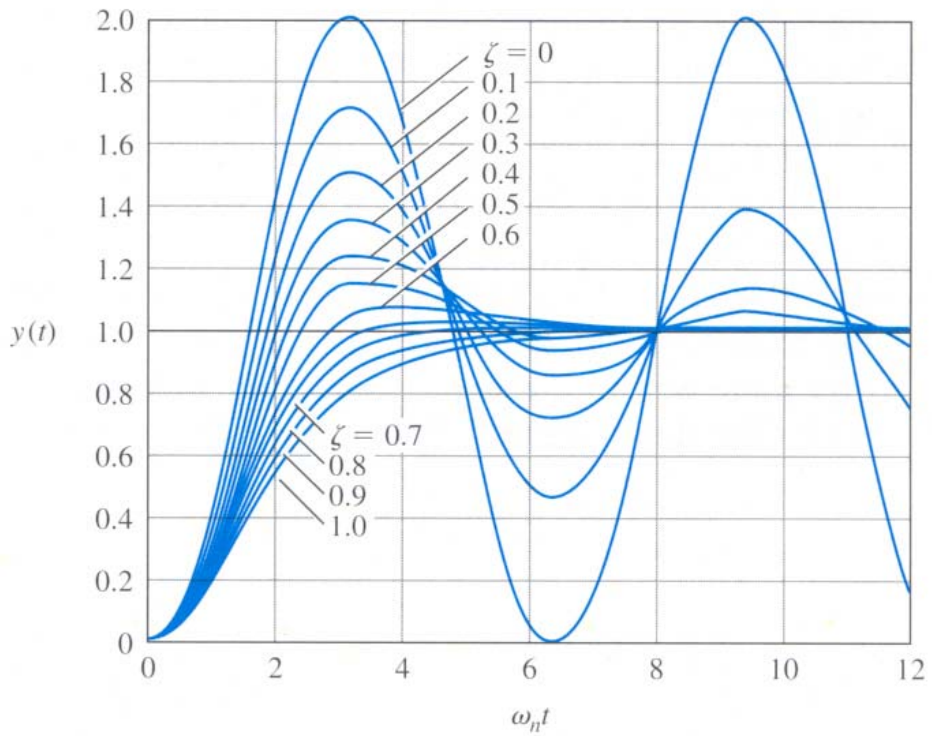
Again, consider the following block diagram with a standard 2nd order system,

$R(s) = 1/s$, $r = 1$ →

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

→ $Y(s)$

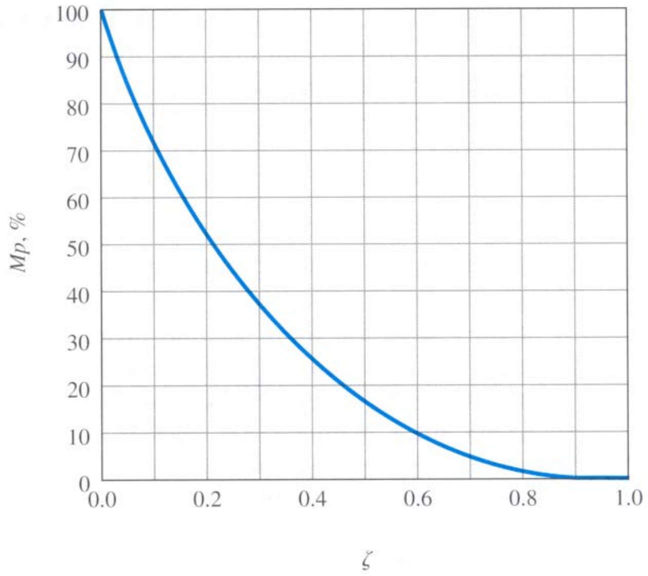
The behavior of the system is as follows:

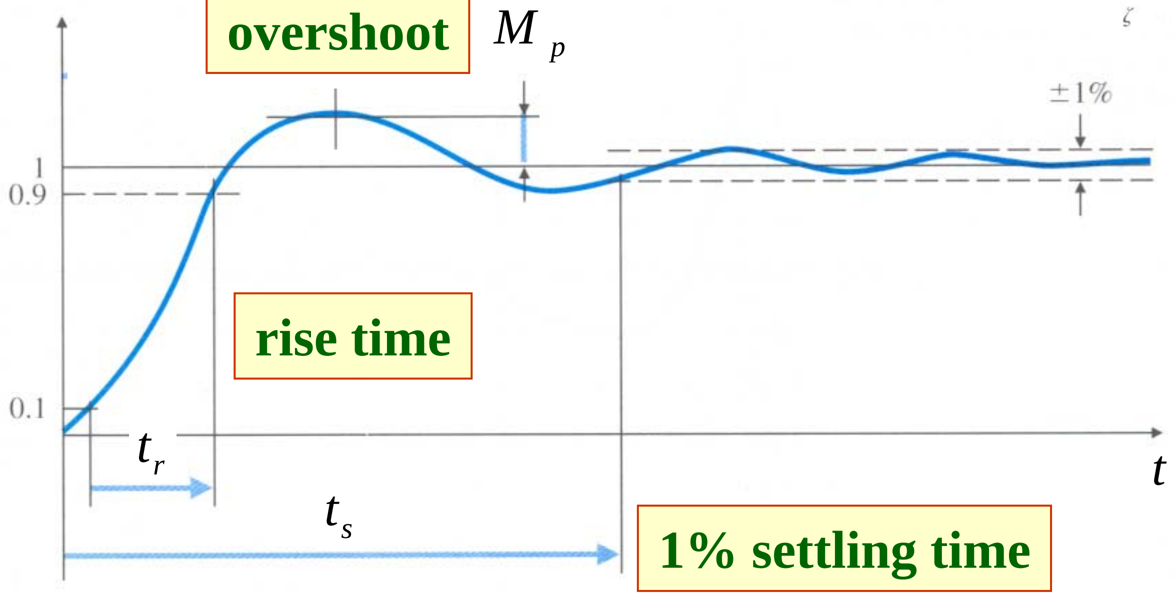


The behavior of the system is fully characterized by ζ, which is called the *damping ratio*, and ω_n, which is called the *natural frequency*.

Control System Design with Time-domain Specifications

$R(s) = 1/s$, $r = 1$ → $H(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ → $Y(s)$



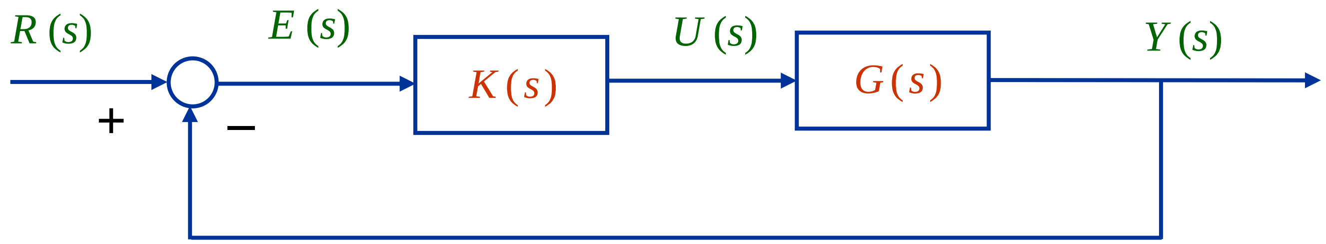


$$t_r \cong \frac{1.8}{\omega_n}$$

$$t_s \cong \frac{4.6}{\zeta\omega_n}$$

Control System Design

Recall that



with $G(s) = \dfrac{b}{s+a}$ and $K(s) = k_p + \dfrac{k_i}{s} = \dfrac{k_p s + k_i}{s}$ results a closed-loop system:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)} = \frac{bk_p s + bk_i}{s^2 + (a + bk_p)\,s + bk_i}$$

Compare this with the standard 2nd order system:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$2\zeta\omega_n = a + bk_p \qquad \omega_n^2 = bk_i$$

$$\Rightarrow \quad k_p = \frac{2\zeta\omega_n - a}{b} \qquad k_i = \frac{\omega_n^2}{b}$$

The key issue now is to choose parameters k_p and k_i such that the above resulting system has desired properties, such as prescribed settling time and overshoot.

Fighter Aircraft Control System Design

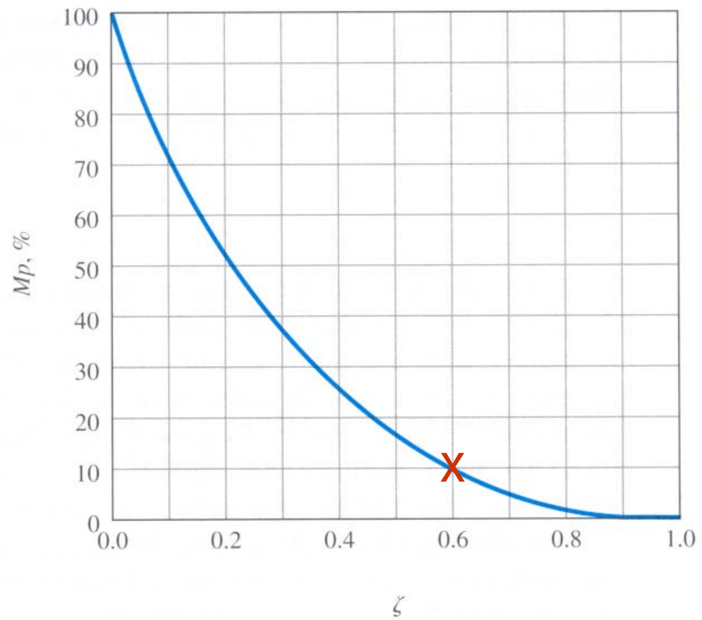
We have seen earlier that it would take more than 2 minutes for the fighter aircraft to reach 10000 m if it is without a feedback controller. Let us design a PI controller for it such that the aircraft will reach the desired vertical level in 30 seconds (i.e., the settling time is 30 sec) and the maximum overshoot is less than 10%.

To achieve an overshoot less than 10%, we obtain from the figure on the right that $\zeta > 0.6$

To be safe, we choose $\zeta = 0.8$

To achieve a settling time of 30 sec., we use

$$t_s = \frac{4.6}{\zeta\omega_n} \quad \Rightarrow \quad \omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.8 \times 30} = 0.192$$



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Recall that the fighter aircraft has a transfer function,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b}{s+a} = \frac{1}{s+0.05} \quad \Rightarrow \quad a = 0.05, \; b = 1$$

Then, using the formulae we have just derived, we obtain

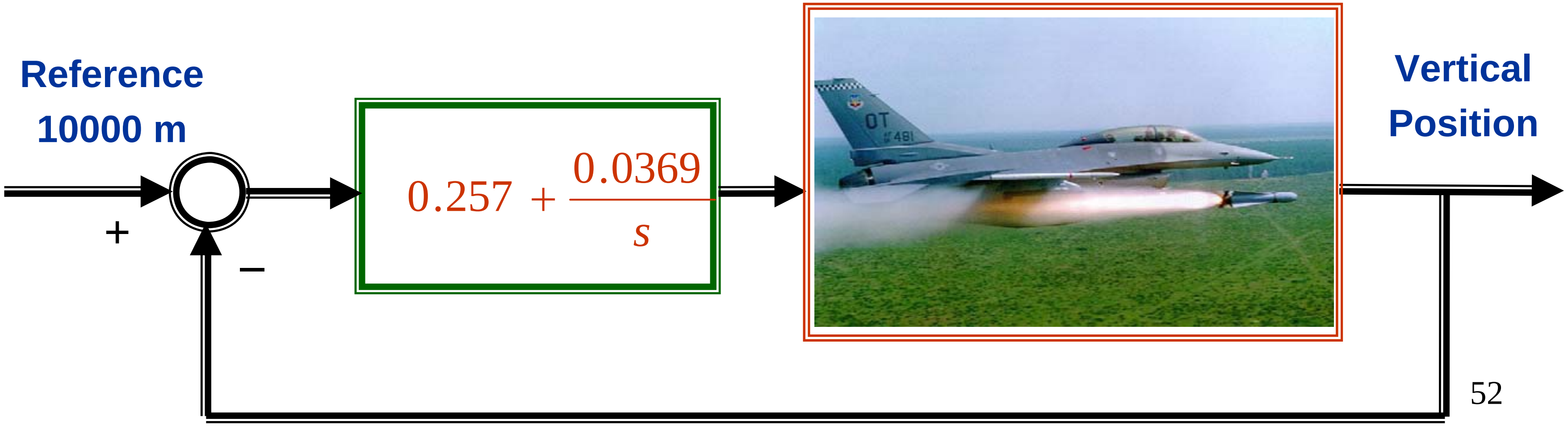
$$k_p = \frac{2\zeta\omega_n - a}{b} \qquad k_i = \frac{\omega_n^2}{b}$$

$$\Longrightarrow$$

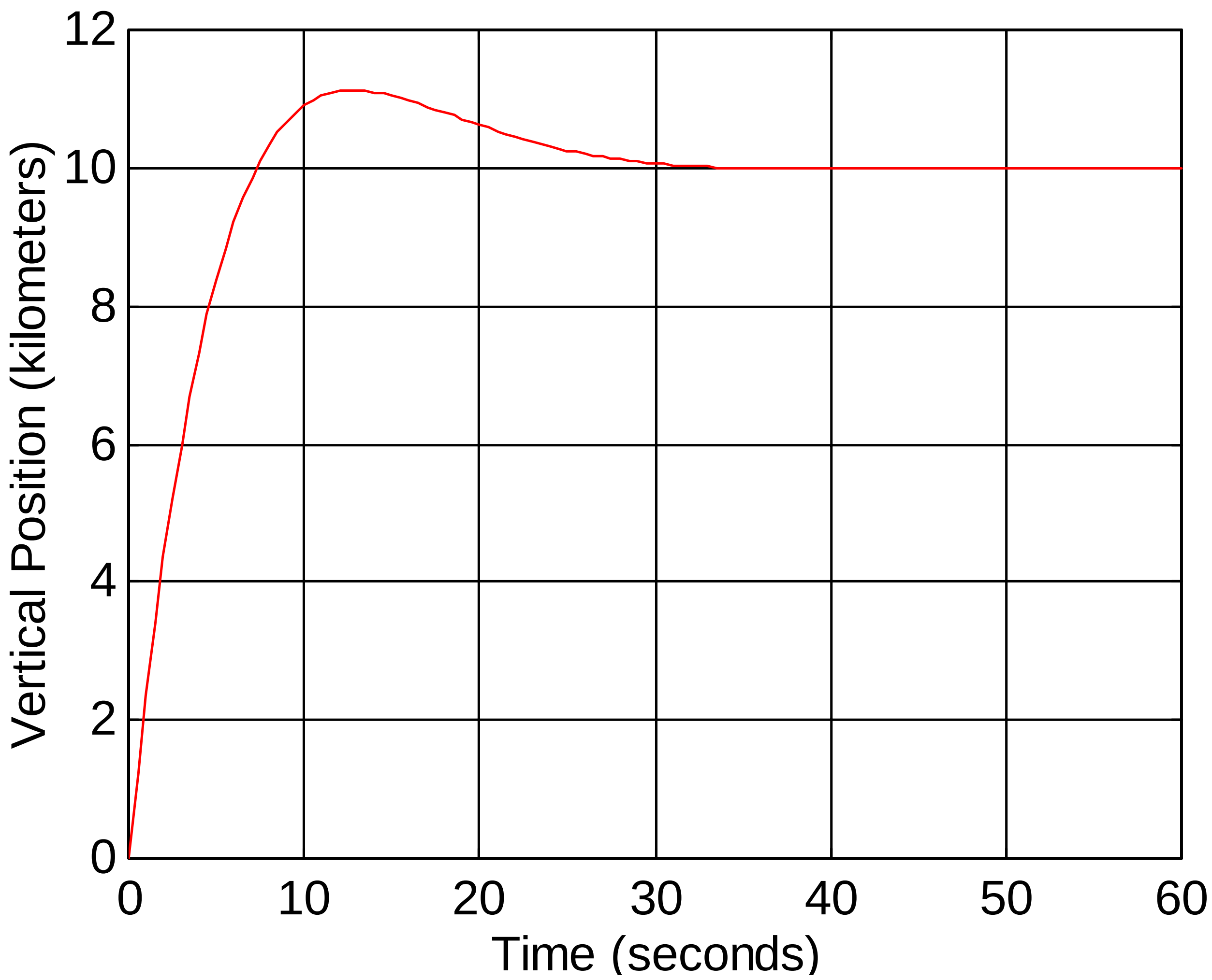
$$k_p = \frac{2\zeta\omega_n - a}{b} = \frac{2 \times 0.8 \times 0.192 - 0.05}{1} = 0.257$$

$$k_i = \frac{\omega_n^2}{b} = \frac{0.192^2}{1} = 0.0369$$

The final flight control system:



Simulation Result:



The resulting overshoot is about 10% and the settling time is about 30 seconds.

Thus, our design goal is achieved.

Cruise-Control System Design

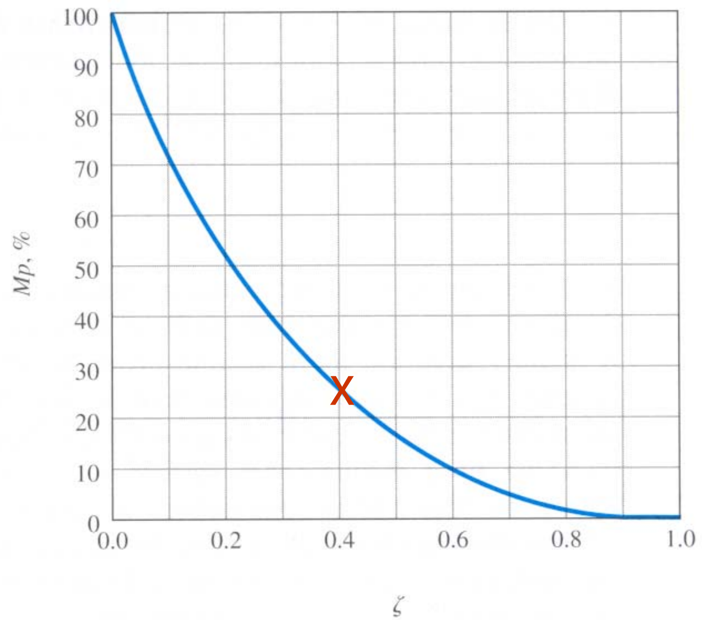
Recall the model for the cruise-control system, i.e., $\dfrac{V(s)}{U(s)} = \dfrac{1/m}{s + b/m}$. Assume that the mass of the car is 3000 kg and the friction coefficient $b = 1$. Design a PI controller for it such that the speed of the car will reach the desired speed 90 km/h in 10 seconds (i.e., the settling time is 10 s) and the maximum overshoot is less than 25%.

To achieve an overshoot less than 25%, we obtain from the figure on the right that $\zeta > 0.4$

To be safe, we choose $\zeta = 0.6$

To achieve a settling time of 10 s, we use

$$t_s = \frac{4.6}{\zeta\omega_n} \quad \Rightarrow \quad \omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.6 \times 10} = 0.767$$



The transfer function of the cruise-control system,

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1/m}{s+b/m}=\frac{1/3000}{s+1/3000} \quad \Rightarrow \quad a=b=1/3000=0.000333$$

Again, using the formulae derived,

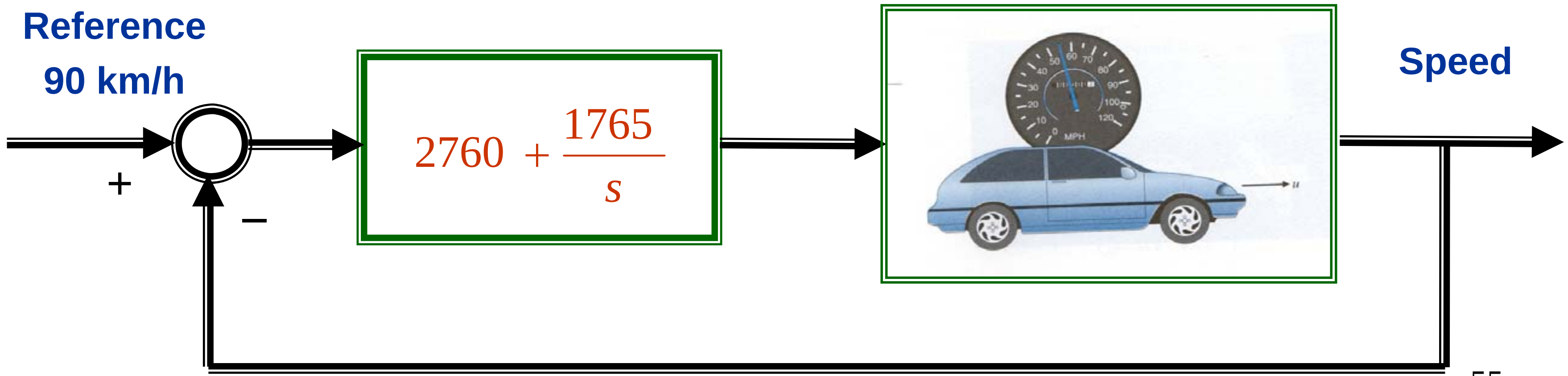
$$k_p=\frac{2\zeta\omega_n-a}{b}, \qquad k_i=\frac{\omega_n^2}{b}$$

$$\Longrightarrow$$

$$k_p=\frac{2\zeta\omega_n-a}{b}=\frac{2\times 0.6\times 0.767-1/3000}{1/3000}=2760$$

$$k_i=\frac{\omega_n^2}{b}=\frac{0.767^2}{1/3000}=1765$$

The final cruise-control system:



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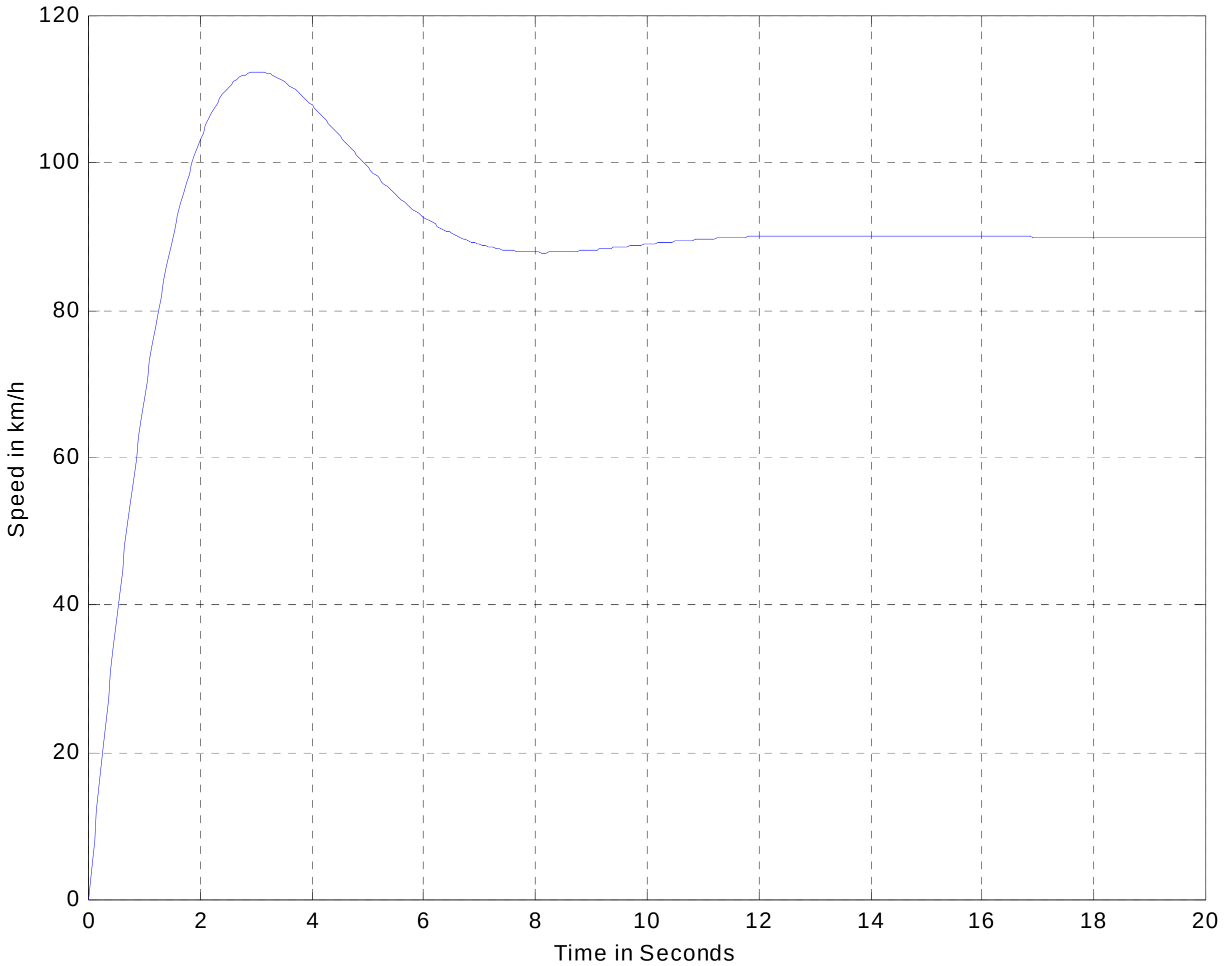
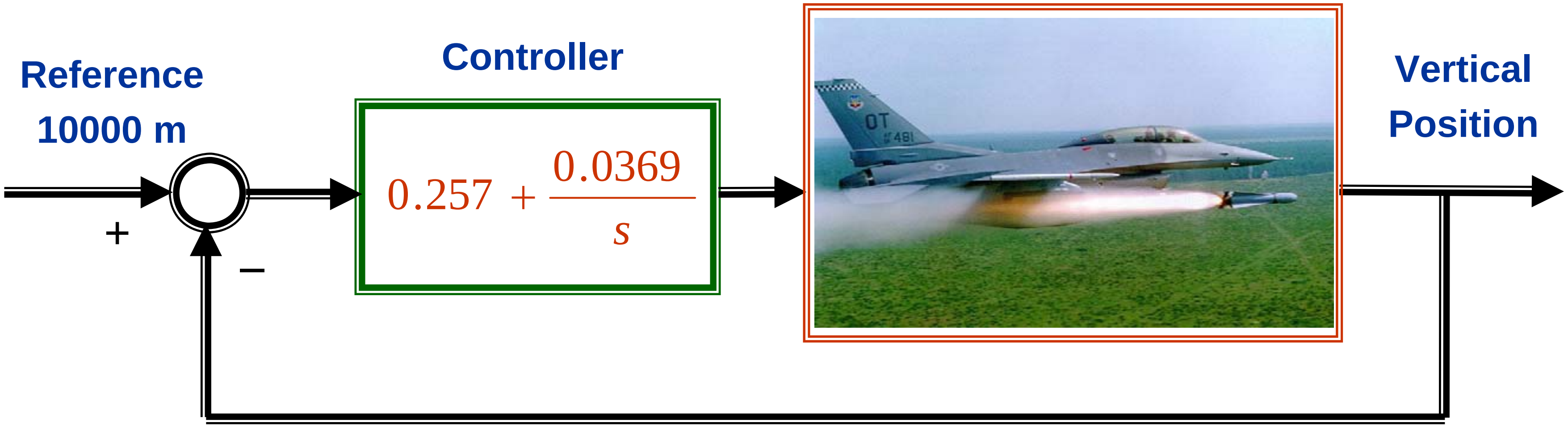
Simulation Result:



The resulting overshoot is less than 25% and the settling time is about 10 seconds.

Thus, our design goal is achieved.

Final Remarks on the Implementation of Controllers



The implementation of the above controller can be done using analog electronic devices such as resistors, inductors, capacitors, and operational amplifiers. However, it is more common nowadays to implement controllers using computers, as it is simple, low cost and reliable. More importantly, it is much easier to be re-programmed.

Tutorials

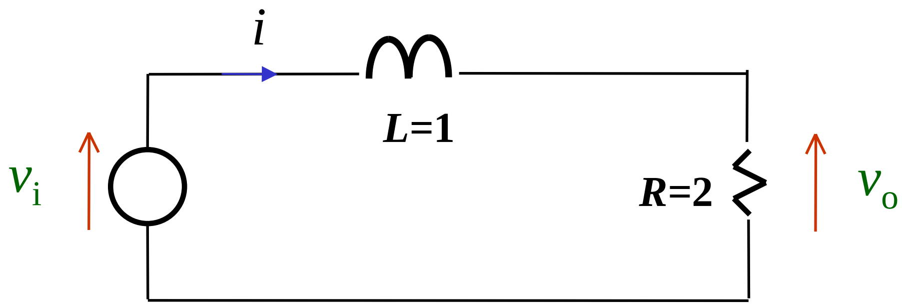
Tutorial One - Modeling and Laplace Transforms

Q.1. Consider a simple RL circuit below.

a) Find an ordinary differential equation in terms of the current i to characterize the dynamics of the circuit.

b) Find the transfer function from the input v_i to the output v_o.

c) Given that v_i is a unit step input, find the output voltage v_o.



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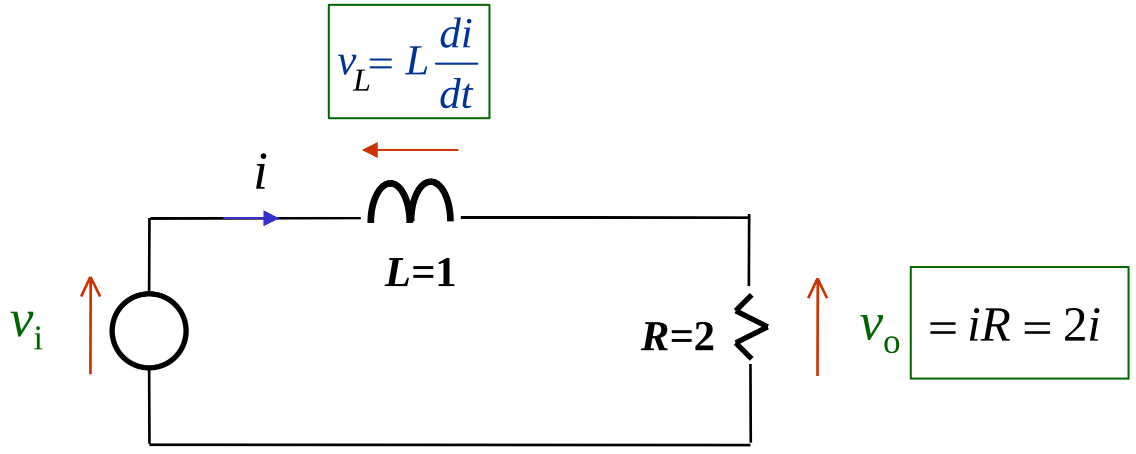
Solution: a)



By KVL,

$$v_i = v_L + v_o = L\frac{di}{dt} + iR = \frac{di}{dt} + 2i \quad \Rightarrow \quad \boxed{\frac{di}{dt} + 2i = v_i}$$ ODE

b) Taking Laplace transform on the both sides of the above ODE, we have

$$sI(s) + 2I(s) = V_i(s) \quad \Rightarrow \quad (s+2)I(s) = V_i(s) \quad \Rightarrow \quad \frac{I(s)}{V_i(s)} = \frac{1}{s+2}$$

$$\Rightarrow \quad \boxed{\frac{V_o(s)}{V_i(s)} = \frac{2I(s)}{V_i(s)} = \frac{2}{s+2}}$$ Transfer Function

c) The Laplace transform of the unit step input is given by $1/s$ (see p. 26).

$$\frac{V_o(s)}{V_i(s)} = \frac{2}{s+2} \quad \Rightarrow \quad V_o(s) = \frac{2}{s+2} V_i(s) = \boxed{\frac{2}{s+2}\cdot\frac{1}{s} = \frac{a}{s+2} + \frac{b}{s}}$$

Partial fraction

$$\frac{2}{s+2}\cdot\frac{1}{s} = \frac{a}{s+2} + \frac{b}{s} = \frac{as + bs + 2b}{(s+2)s} = \frac{(a+b)s + 2b}{(s+2)s}$$

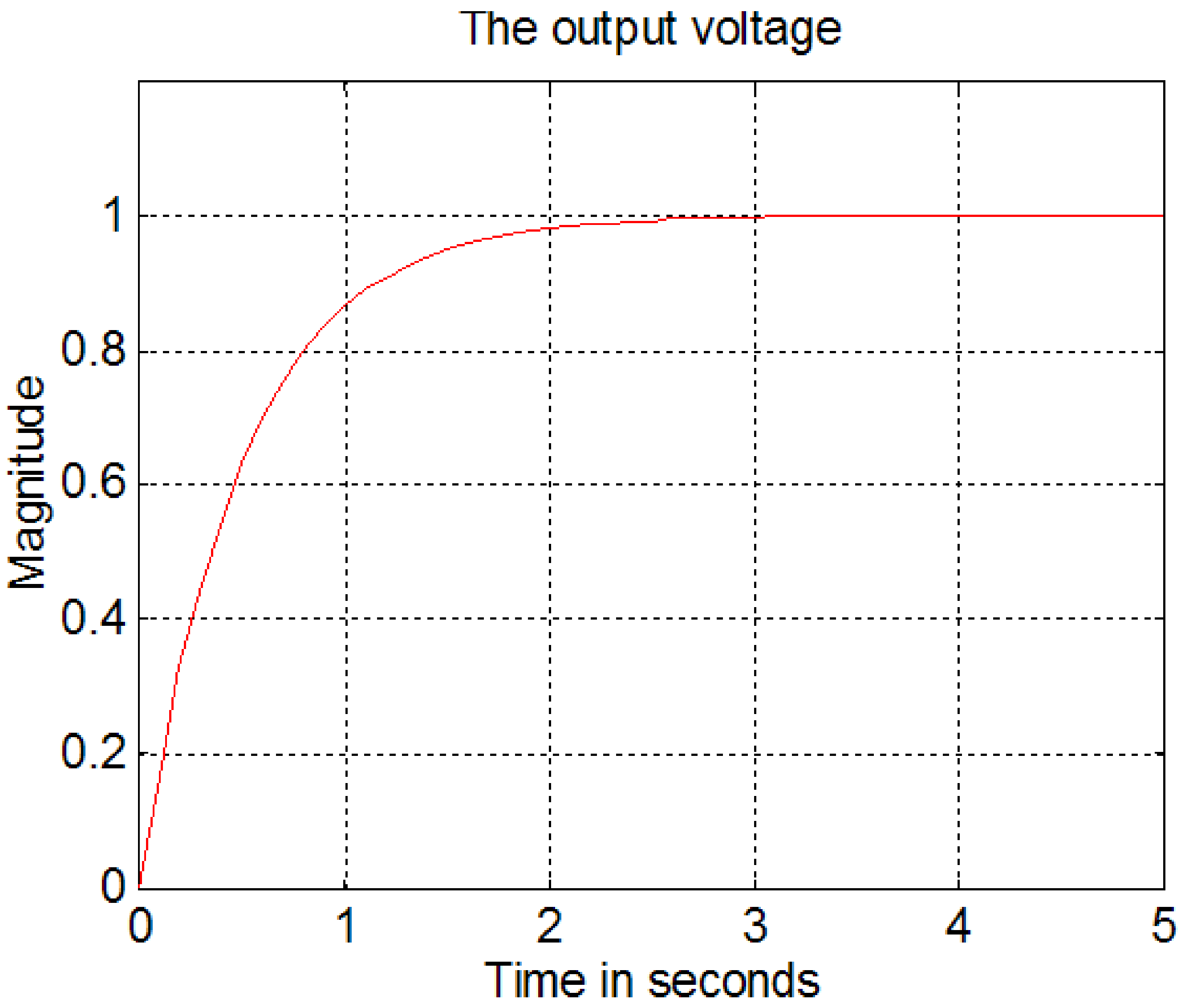
$$\Rightarrow \quad a + b = 0, \quad 2b = 2 \quad \Rightarrow b = 1, \quad a = -b = -1$$

$$\Rightarrow \quad V_o(s) = \frac{a}{s+2} + \frac{b}{s} = \frac{1}{s} - \frac{1}{s+2}$$

Taking inverse Laplace transform, we obtain

$$v_o(t) = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\} = 1 - e^{-2t}$$

The output voltage of the circuit:

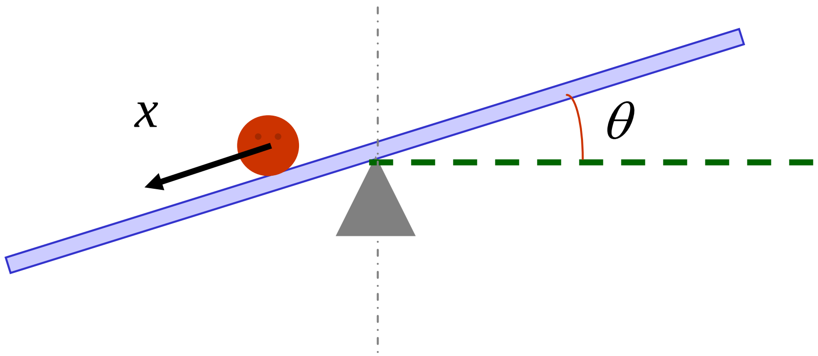


Q.2. Consider a ball balancing mechanical system below.

a) Find an ordinary differential equation in terms of the position x to characterize the dynamics of the system.

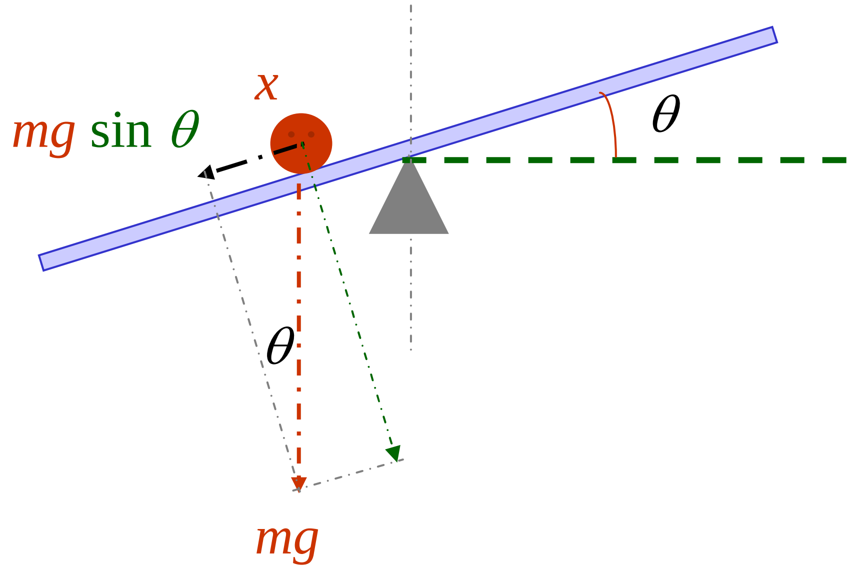
b) Find the transfer function from the input θ to the output x.

c) Given that θ is a unit impulse input, find the response of the position x.



Assume the angle θ is changing in a small range. For simplicity, assume that the gravity $g = 9.8 \approx 10$.

Solution: a) The only force acts on the system is the weight of the ball



By Newton's law of motion, i.e., $F = ma$, where a is the acceleration, m the mass and F is the force acting on the object in the direction of the motion,

$$F = ma \quad \Rightarrow \quad mg \sin\theta = ma = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = g\sin\theta$$

Since θ is assumed to be small, it can be shown that $\sin\theta \approx \theta$. Thus,

$$\ddot{x} = g\sin\theta \quad \Rightarrow \quad \ddot{x} = g\theta = 10\theta$$

b) The transfer function from the input θ to the output x can be obtained by

$$L\{\ddot{x}\} = L\{10\theta\} \quad \Rightarrow \quad s^2 X(s) = 10\,\Theta(s) \quad \Rightarrow$$

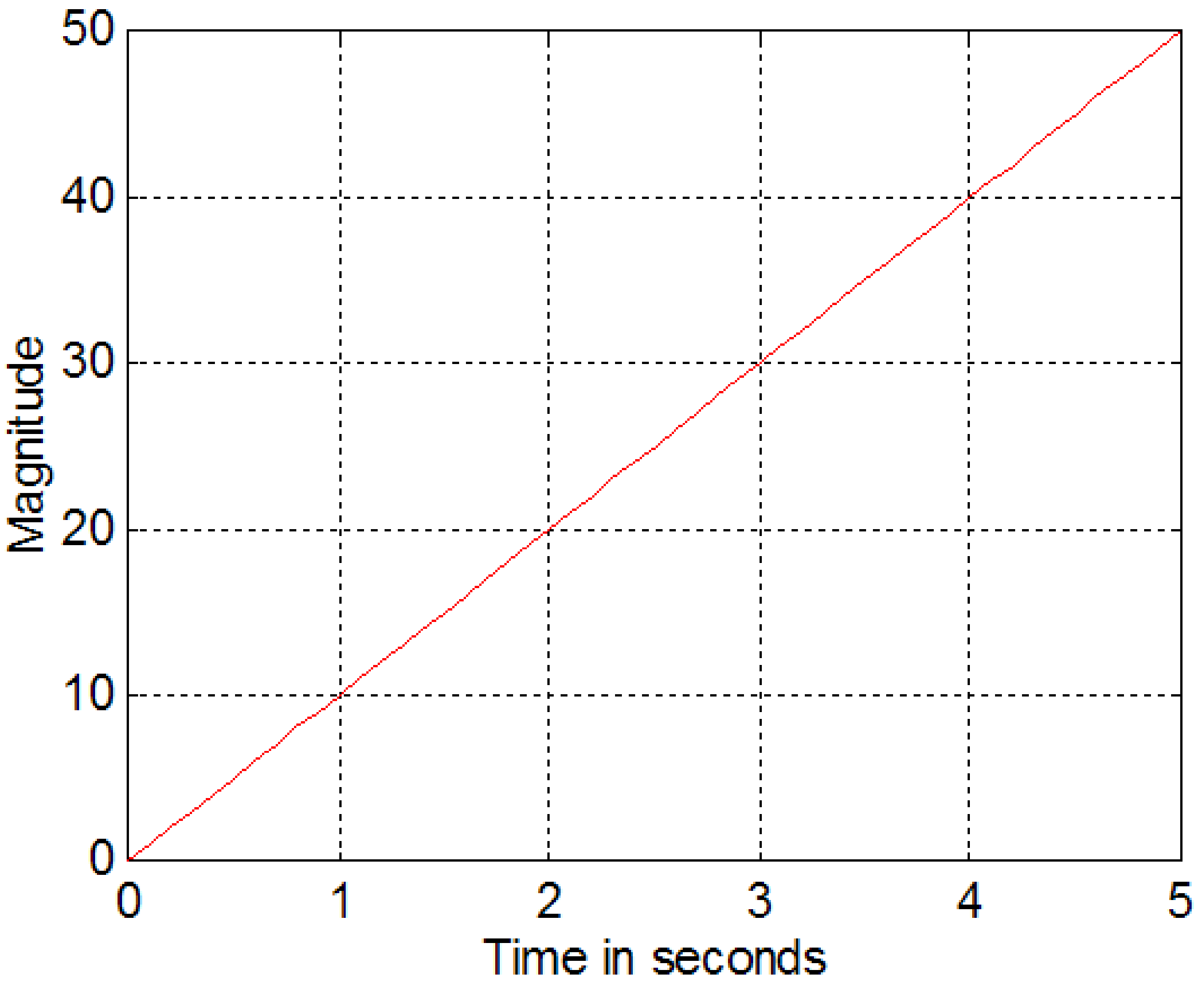
$$G(s) = \frac{X(s)}{\Theta(s)} = \frac{10}{s^2}$$

c) The Laplace transform of the unit impulse input is 1. From the above transfer function, we obtain

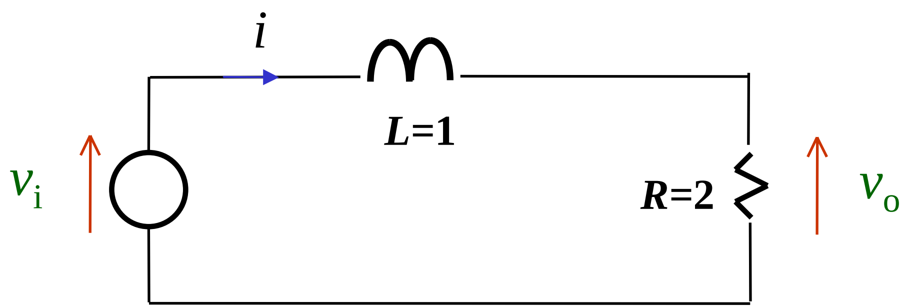
$$X(s) = \frac{10}{s^2}\Theta(s) = \frac{10}{s^2} \quad \Rightarrow \quad x(t) = L^{-1}\left\{\frac{10}{s^2}\right\} = 10\,L^{-1}\left\{\frac{1}{s^2}\right\} = 10\,t$$

Obviously, the ball will roll off the beam after an impulse force acting on it. It is an unstable system. A controller is definitely needed if one wish to balance the ball at the center of the beam.

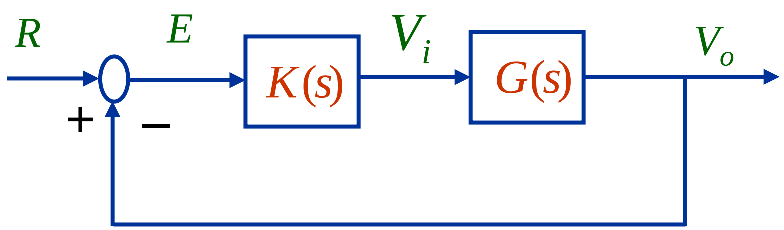
The position of the ball due to an impulse input:



Q.1. Consider a simple RL circuit below. It was shown that in *Tutorial One* that the settling time of the output response due to a step input is about 2.5 seconds. Design a PI controller (see the second figure) such that when it is applied to the circuit, the resulting settling time due to a unit step input is less than 1 second with a 10% overshoot.

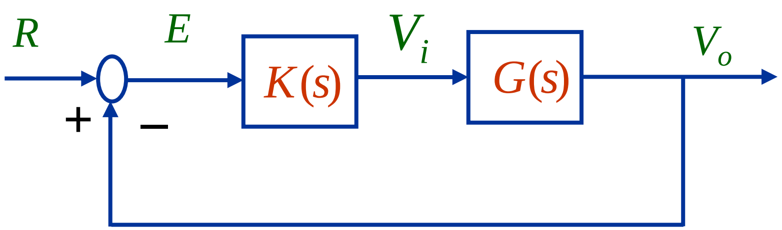


- *RL Circuit*



- *Control System*

Solution: Consider



$$V_o(s) = G(s)V_i(s) = G(s)K(s)E(s) = G(s)K(s)\left[R(s) - V_o(s)\right]$$

$$\Rightarrow \quad \left[1 + G(s)K(s)\right]V_o(s) = G(s)K(s)R(s) \quad \Rightarrow \quad \boxed{\frac{V_o(s)}{R(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)}}$$

- Recall from Q.1 of Tutorial One that

$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{2}{s+2}$$

and the PI controller has the form

$$K(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}$$

$$\frac{V_o(s)}{R(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)} = \frac{\left(\frac{2}{s+2}\right)\left(\frac{k_p s + k_i}{s}\right)}{1 + \left(\frac{2}{s+2}\right)\left(\frac{k_p s + k_i}{s}\right)}$$

$$\frac{V_o(s)}{R(s)} = \frac{\left(\frac{2}{s+2}\right)\left(\frac{k_p s + k_i}{s}\right)}{1+\left(\frac{2}{s+2}\right)\left(\frac{k_p s + k_i}{s}\right)} = \frac{\frac{2(k_p s + k_i)}{s(s+2)}}{1+\frac{2(k_p s + k_i)}{s(s+2)}} = \frac{2(k_p s + k_i)}{s(s+2)+2(k_p s + k_i)}$$

$$\Rightarrow \quad \frac{V_o(s)}{R(s)} = \frac{2k_p s + 2k_i}{s^2 + 2(1+k_p)s + 2k_i}$$

Comparing this with the standard 2nd order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

we can match almost all parameters by selecting:

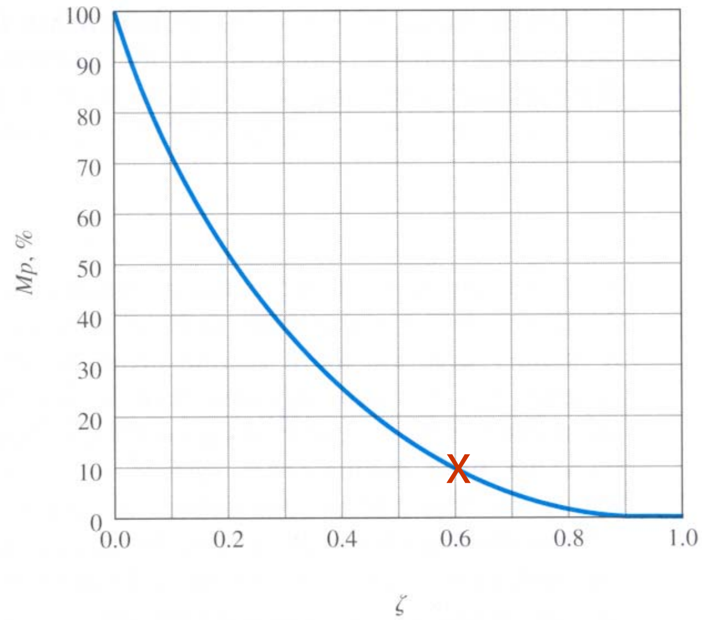
$$2\zeta\omega_n = 2(1+k_p) \quad \Rightarrow \quad k_p = \zeta\omega_n - 1 \qquad \omega_n^2 = 2k_i \quad \Rightarrow \quad k_i = \frac{\omega_n^2}{2}$$

To achieve an overshoot less than 10%, we obtain from the figure on the right that $\zeta > 0.6$

To be safe, we choose $\zeta = 0.8$

To achieve a settling time of 1 sec., we use

$$t_s = \frac{4.6}{\zeta\omega_n} \quad \Rightarrow \quad \omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.8 \times 1} = 5.75$$



Thus, we have

$$k_p = \zeta\omega_n - 1 = 4.6 - 1 = 3.6$$

$$k_i = \frac{\omega_n^2}{2} = \frac{5.75^2}{2} = 16.5$$

a resistor

$$K(s) = 3.6 + \frac{16.5}{s}$$

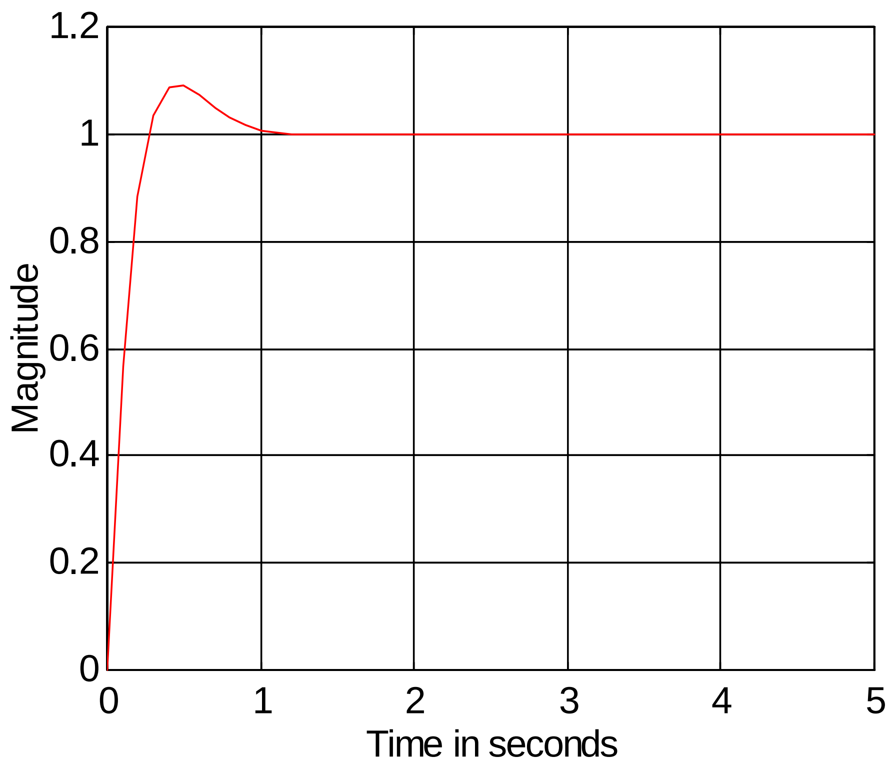
a capacitor

Prepared by Ben M. Chen

- Output response of controlled system:

  settling time = 1 second

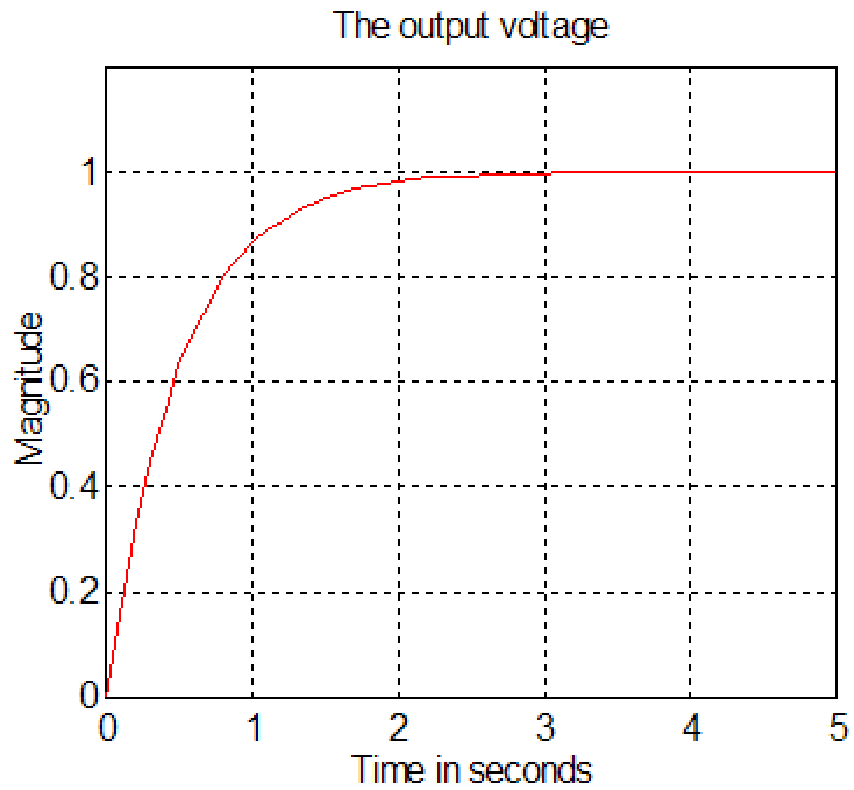
  overshoot = 9 % < 10 %



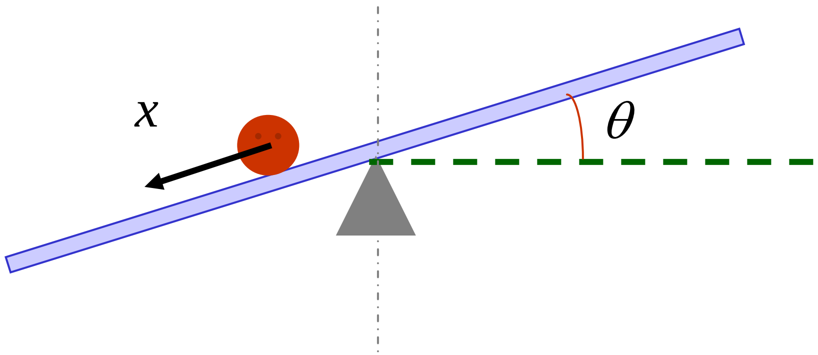
- Output response of uncontrolled system:

  settling time = 2.5 seconds

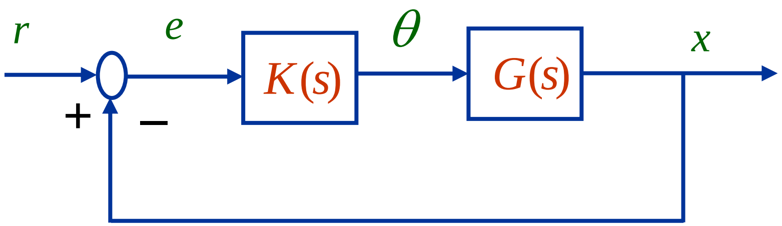
  overshoot = 0 %



Q.2. Consider a ball balancing mechanical system below. It was shown that in *Tutorial One* that the system is unstable. Design a PD controller (see the second figure) such that when it is applied to the system, the resulting settling time due to a unit step input is less than 4 second with a 25% overshoot.

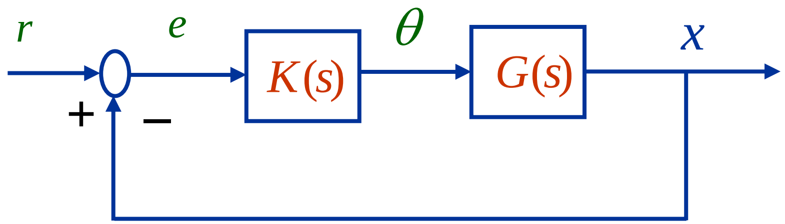


• Ball Balancing System



• Control System

Solution: Consider



Following exactly the same procedure as in Q.1, we can derive

$$\frac{X(s)}{R(s)} = \frac{G(s)K(s)}{1+G(s)K(s)}$$

- Recall from Q.2 of Tutorial One that

$$G(s) = \frac{10}{s^2}$$

and the PD controller has the form

$$K(s) = k_p + k_d s$$

$$\frac{X(s)}{R(s)} = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{\left(\frac{10}{s^2}\right)(k_p + k_d s)}{1 + \left(\frac{10}{s^2}\right)(k_p + k_d s)}$$

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$$\frac{X(s)}{R(s)}=\frac{\left(\dfrac{10}{s^2}\right)(k_p+k_d s)}{1+\left(\dfrac{10}{s^2}\right)(k_p+k_d s)}=\frac{10(k_p+k_d s)}{s^2+10(k_p+k_d s)}=\boxed{\frac{10k_d s+10k_p}{s^2+10k_d s+10k_p}}$$

Again, comparing this with the standard 2nd order system

$$\boxed{H(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

we can match almost all parameters by selecting:

$$2\zeta\omega_n=10k_d \Rightarrow \quad k_d=0.2\zeta\omega_n \qquad\qquad \omega_n^2=10k_p \quad \Rightarrow \quad k_p=0.1\omega_n^2$$

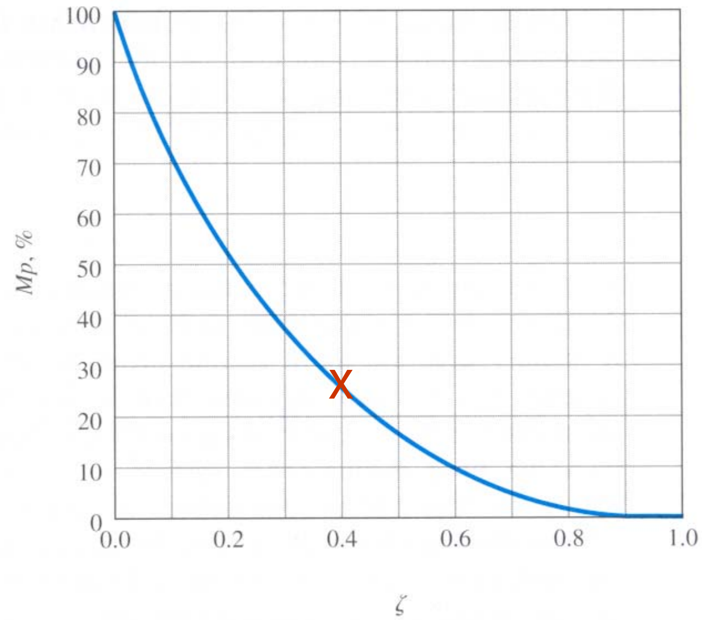
As usual, we have ignored the first term in the numerator of the transfer function from *R* to *X*. This is because that it does not affect much the overall response.

To achieve an overshoot less than 25%, we obtain from the figure on the right that $\zeta > 0.4$

To be safe, we choose $\zeta = 0.6$

To achieve a settling time of 4 sec., we use

$$t_s = \frac{4.6}{\zeta\omega_n} \quad\Rightarrow\quad \omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.6\times 4} = 1.92$$



Thus, we have

$$k_d = 0.2\zeta\omega_n = 0.2\times 0.6\times 1.92 = 0.23$$

$$k_p = 0.1\omega_n^2 = 0.1\times 1.92^2 = 0.37$$

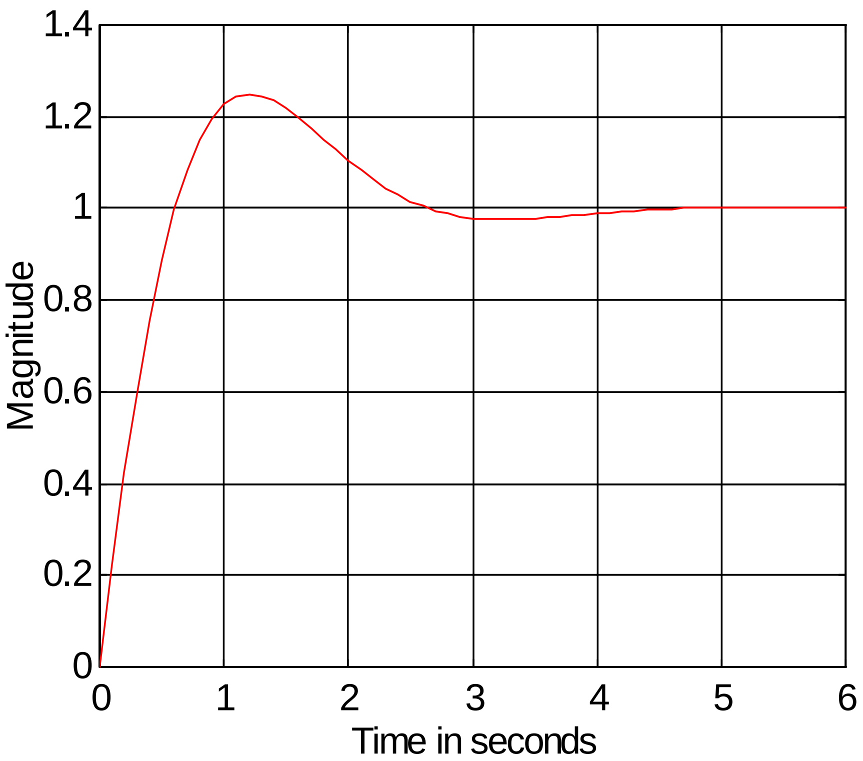
$$K(s) = 0.37 + 0.23s$$

PD Controller

- Output response of controlled system:

settling time = 4 seconds

overshoot = 25 %



- Output response of uncontrolled system:

An unstable system

The ball will roll off the beam