EE5105: Optimal Control Systems

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Optimal Control Systems, Part 2 - Course Outline

- Revision: Introduction to control systems; ordinary differential equations; state space representation; Laplace transform; principle of feedback; modelling; system stability; PID control; Bode and Nyquist plot; gain and phase margins.

- Properties of linear quadratic regulation (LQR) control; returned differences; guaranteed gain and phase margins; Kalman filter; linear quadratic Gaussian (LQG) design technique.

- Introduction to modern control system design; H_2 and H_∞ optimal control; solutions to regular and singular H_2 and H_∞ optimal control problems; solutions to some robust control problems.

- Loop transfer recovery (LTR) design technique; Issues on controller structures.

Reference Textbooks

- G. F. Franklin, J. D. Powell and A. Emami-Naeini, Feedback Control of Dynamic Systems, 3rd Edition, Addison Wesley, New York, 1994.
- B. D. O. Anderson and J. B. Moore, Optimal Control, Prentice Hall, London, 1989.
- F. L. Lewis, Applied Optimal Control and Estimation, Prentice Hall, Englewood Cliffs, New Jersey, 1992.
- A. Saberi, B. M. Chen and P. Sannuti, Loop Transfer Recovery: Analysis and Design, Springer, London, 1993.
- A. Saberi, P. Sannuti and B. M. Chen, H_2 Optimal Control, Prentice Hall, London, 1995.
- B. M. Chen, Robust and H_∞ Control, Springer, London, 2000.

Homework Assignments & Projects

There will be three (3) homework assignments for all students in this second part of the course. Some of them are practical problems and most of them require computer simulations. All students are expected to have knowledge in MATLAB™ (Control Toolbox and Robust Control Toolbox) and SIMULINK™ after completing these assignments. Homework assignments are to be marked and counted as a certain percentage in your final grade.

There will be one (1) design project for master of engineering and Ph.D. students. These students are required to complete a control system design for a coupled tank system using the techniques learnt in the class and implement it to the real system through a web-based experiment facility available at http://vlab.ee.nus.edu.sg/vlab. The project report is to be handed in to my office within one week after the completion of the course.

Final Grades for Part 2

1. For 5000 Level:

Final Grade = 70% $\times$ Final exam marks for Part 2 (max = 50) + …

30% $\times$ Homework assignments marks (max = 50)

2. For 6000 Level:

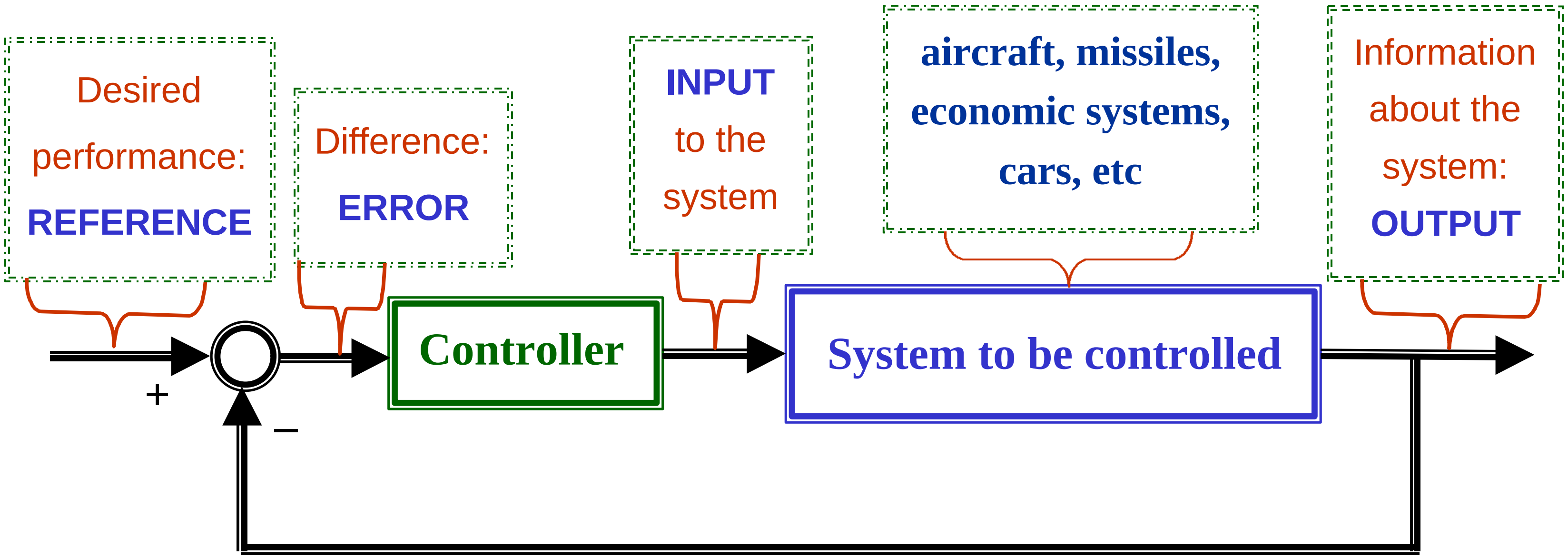
Final Grade = 70% $\times$ Final exam marks for Part 2 (max = 50) + …

20% $\times$ Homework assignments marks (max = 50) +

10% $\times$ Design project marks (max = 50)

Revision: Basic Concepts

What is a control system?

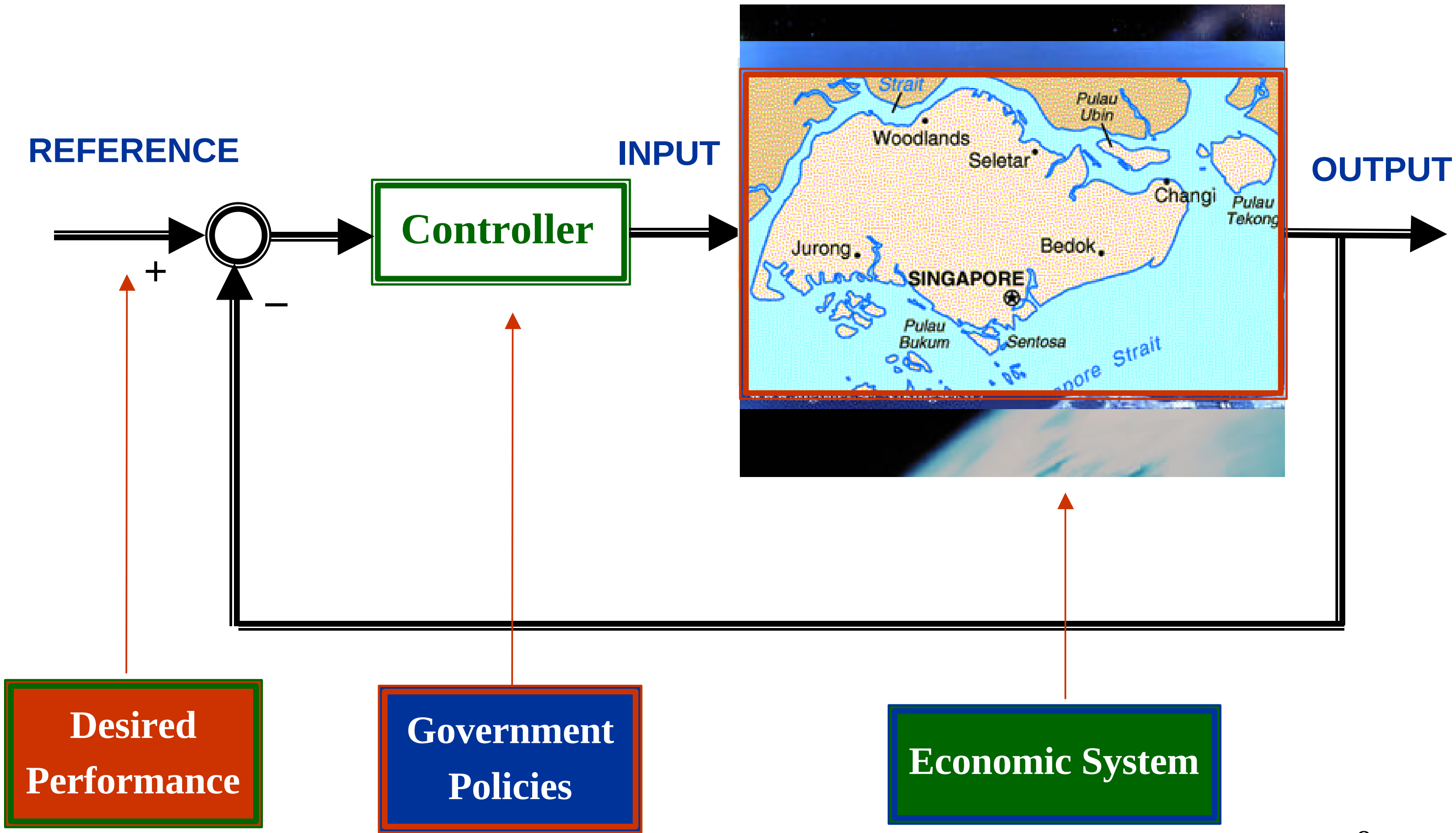


Objective: To make the system OUTPUT and the desired REFERENCE as close as possible, i.e., to make the ERROR as small as possible.

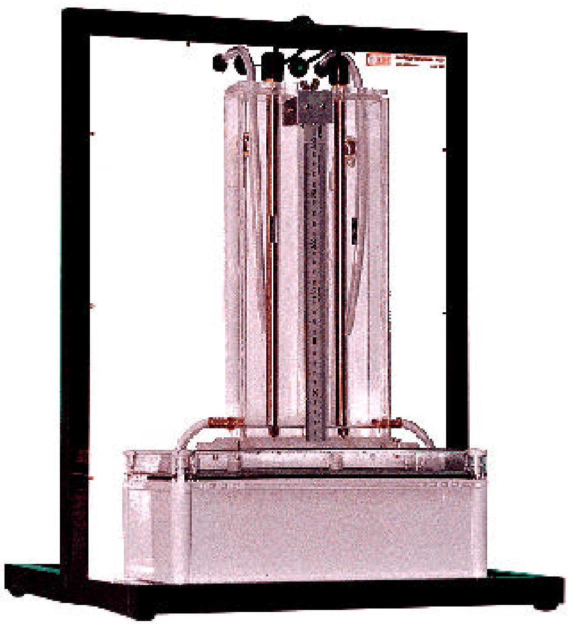
Key Issues: 1) How to describe the system to be controlled? (Modelling)

2) How to design the controller? (Control)

Some Control Systems Examples:



A Live Demonstration on Control of a Coupled-Tank System through Internet Based Virtual Laboratory Developed by NUS



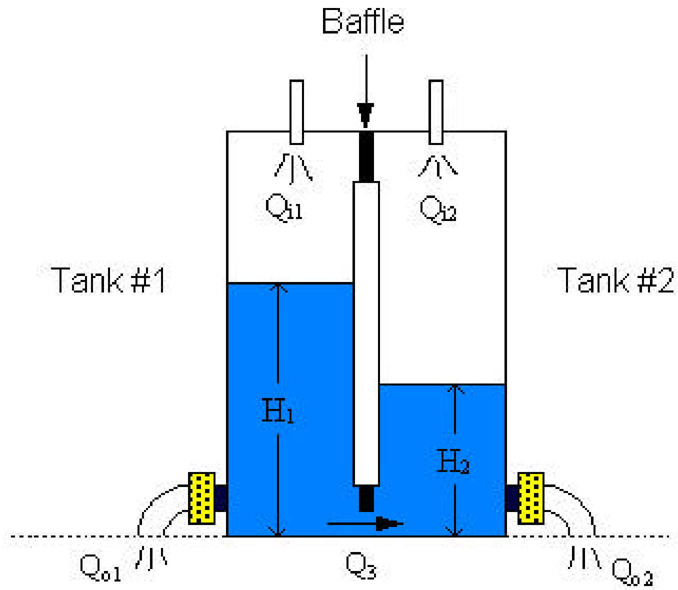
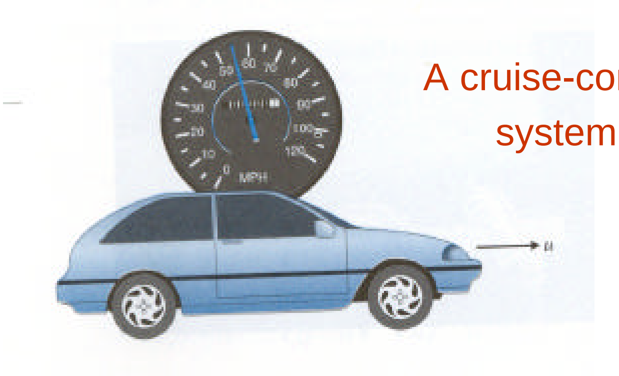


Figure 1 Diagram of coupled-tank control apparatus

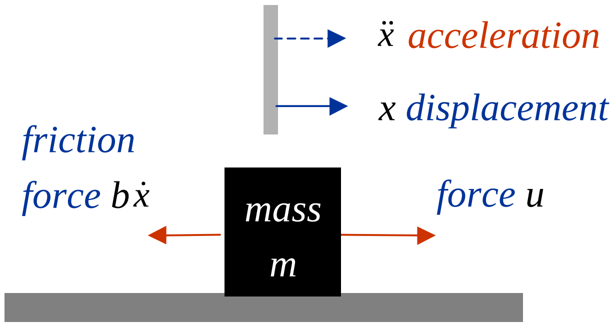
The objective is to control the flow levels of two coupled tanks. It is a reduced-scale model of some commonly used chemical plants.

Modelling of Some Physical Systems

A simple mechanical system:



A cruise-control system



By the well-known Newton's Law of motion: $f = m\,a$, where f is the total force applied to an object with a mass m and a is the acceleration, we have

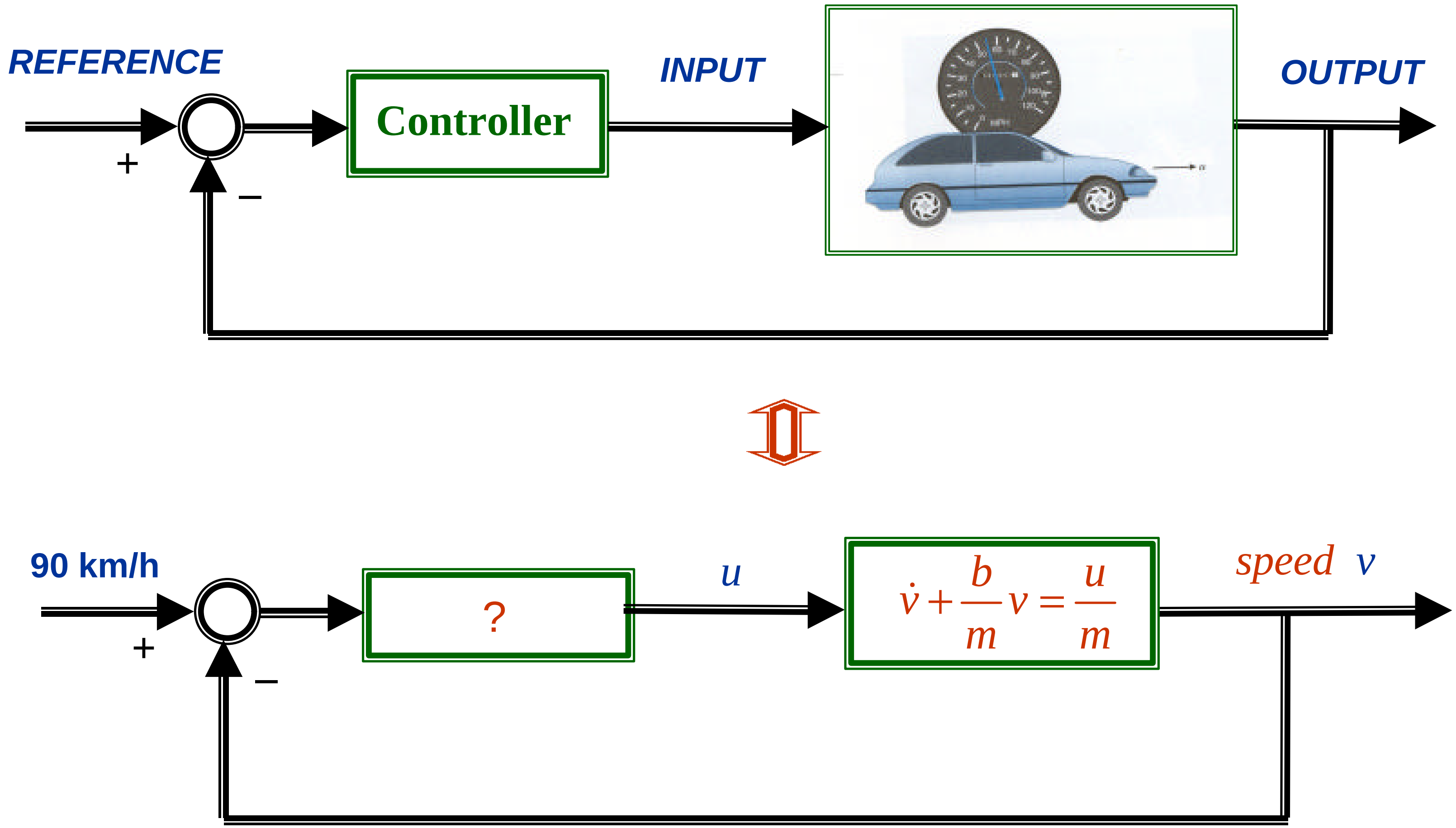
$$u - b\dot{x} = m\ddot{x} \qquad \Leftrightarrow \qquad \ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}$$

This a 2nd order *Ordinary Differential Equation* with respect to displacement x. It can be written as a 1st order *ODE* with respect to speed $v = \dot{x}$:

$$\boxed{\dot{v} + \frac{b}{m}v = \frac{u}{m}}$$

$\leftarrow$ model of the cruise control system, u is input force, v is output.

A cruise-control system:



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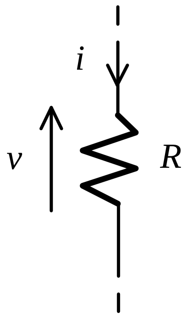
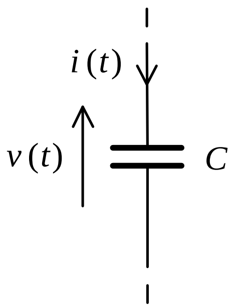
Basic electrical systems:

resistor



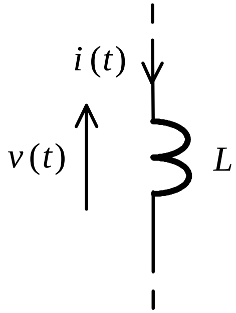
$$v = i\,R$$

capacitor



$$i = C\frac{dv}{dt}$$

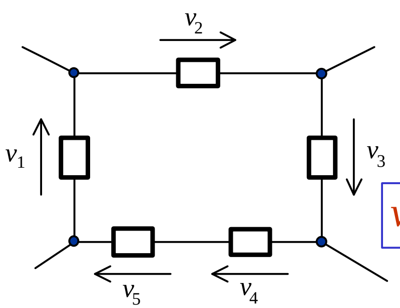
inductor



$$v = L\frac{di}{dt}$$

Kirchhoff's Voltage Law (KVL):

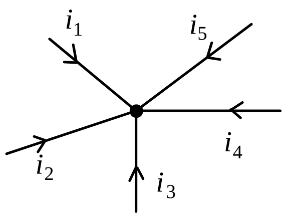
The sum of voltage drops around any close loop in a circuit is 0.

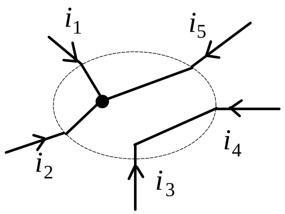


$$v_1+v_2+v_3+v_4+v_5 = 0$$

Kirchhoff's Current Law (KCL):

The sum of currents entering/leaving a note/closed surface is 0.



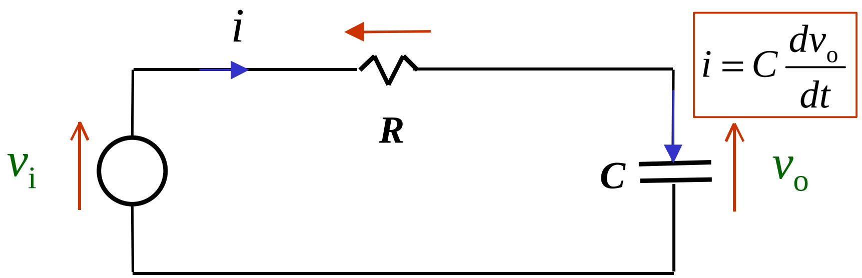


$$i_1+i_2+i_3+i_4+i_5 = 0$$

Modelling of a simple electrical system:

To find out relationship between the input (v_i) and the output (v_o) for the circuit:

$$\boxed{v_R = Ri = RC\frac{dv_o}{dt}}$$

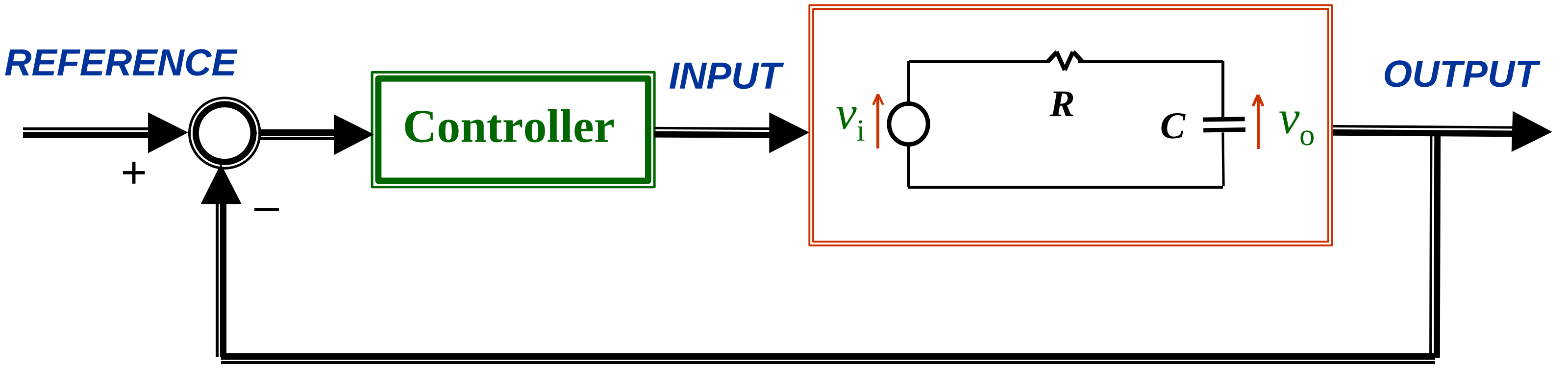


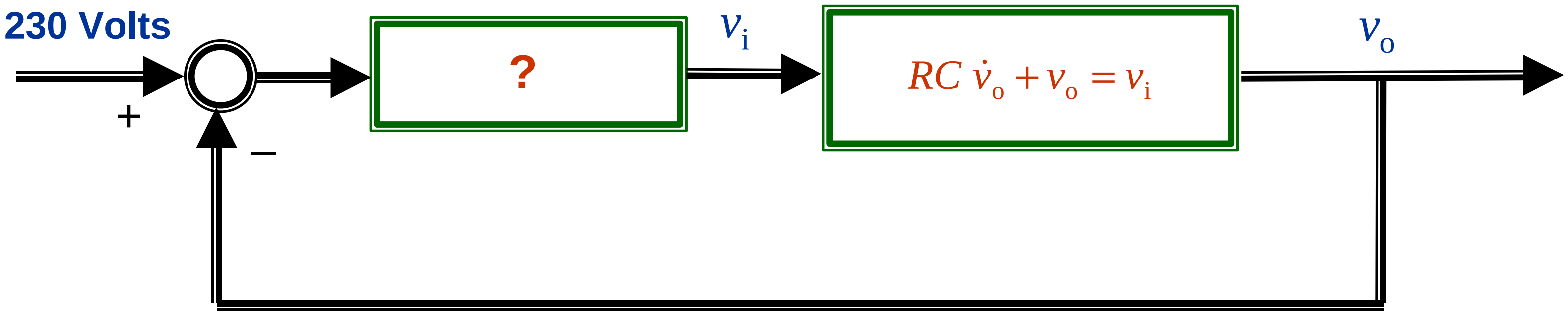
By KVL, we have $\boxed{v_o + v_R - v_i = 0}$ ➡

$$\boxed{v_o + v_R - v_i = v_o + RC\frac{dv_o}{dt} - v_i = 0}$$

➡ $RC\dfrac{dv_o}{dt} + v_o = v_i \quad \Leftrightarrow \quad \boxed{RC\,\dot{v}_o + v_o = v_i}$ → **A dynamic model of the circuit**

Control the output voltage of the electrical system:





Ordinary Differential Equations

Many real life problems can be modelled as an ODE of the following form:

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = u(t)$$

This is called a 2nd order ODE as the highest order derivative in the equation is 2. The ODE is said to be homogeneous if $u(t) = 0$. In fact, many systems can be modelled or approximated as a 1st order ODE, i.e.,

$$\boxed{\dot{y}(t) + a_0 y(t) = u(t)}$$

An ODE is also called the time-domain model of the system, because it can be seen the above equations that $y(t)$ and $u(t)$ are functions of time t. The key issue associated with ODE is: how to find its solution? That is: how to find an explicit expression for $y(t)$ from the given equation?

State Space Representation

Recall that many real life problems can be modelled as an ODE of the following form:

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = u(t)$$

If we define so-called **state variables**,

$$\left.\begin{array}{l} x_1 = y \\ x_2 = \dot{y} \end{array}\right\} \qquad \begin{array}{l} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = -a_1 \dot{y} - a_0 y + u = -a_1 x_2 - a_0 x_1 + u \end{array}$$

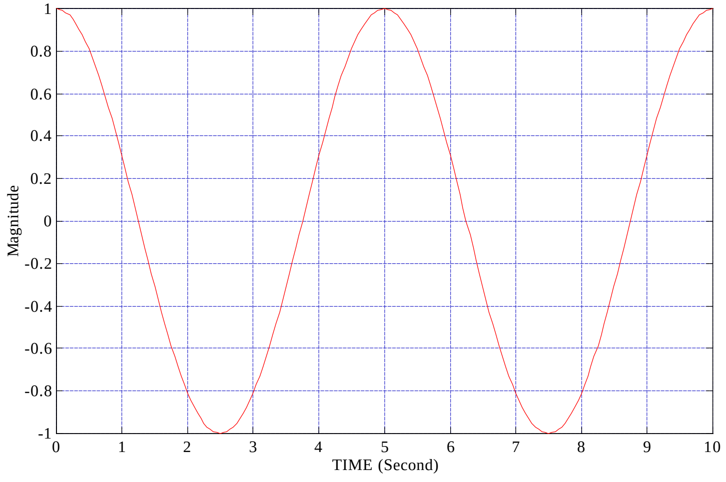
We can rewrite these equations in a more compact (matrix) form,

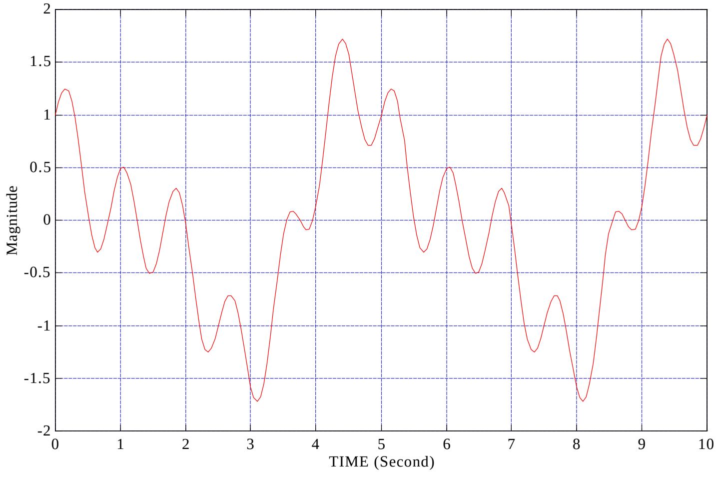
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This is called the **state space representation** of the ODE or the dynamic systems.

Laplace Transform and Inverse Laplace Transform

Let us first examine the following time-domain functions:





A cosine function with a frequency $f = 0.2$ Hz.

Note that it has a period $T = 5$ seconds.

$$x(t) = \cos(0.4\pi t) + \sin(0.8\pi t)\cos(1.6\pi t)$$

What are frequencies of this function?

Laplace transform is a tool to convert a time-domain function into a frequency-domain one in which information about frequencies of the function can be captured. It is often much easier to solve problems in frequency-domain with the help of Laplace transform.

Laplace Transform:

Given a time-domain function $f(t)$, its Laplace transform is defined as follows:

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

Example 1: Find the Laplace transform of a constant function $f(t) = 1$.

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\bigg|_0^\infty = -\frac{1}{s}e^{-\infty} - \left(-\frac{1}{s}e^{0}\right) = -\frac{1}{s}\cdot 0 - \left(-\frac{1}{s}\cdot 1\right) = \frac{1}{s}, \quad \mathrm{Re}(s) > 0$$

Example 2: Find the Laplace transform of an exponential function $f(t) = e^{-at}$.

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = -\frac{1}{s+a}e^{-(s+a)t}\bigg|_0^\infty = \frac{1}{s+a}, \quad \mathrm{Re}(s) > -a$$

Inverse Laplace Transform

Given a frequency-domain function $F(s)$, the inverse Laplace transform is to convert it back to its original time-domain function $f(t)$.

Here are some very useful Laplace and inverse Laplace transform pairs:

$f(t)$	$\Leftrightarrow$	$F(s)$
1	$\Leftrightarrow$	$\dfrac{1}{s}$
t	$\Leftrightarrow$	$\dfrac{1}{s^2}$
e^{-at}	$\Leftrightarrow$	$\dfrac{1}{s+a}$
te^{-at}	$\Leftrightarrow$	$\dfrac{1}{(s+a)^2}$

$f(t)$	$\Leftrightarrow$	$F(s)$
$\sin at$	$\Leftrightarrow$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\Leftrightarrow$	$\dfrac{s}{s^2+a^2}$
$e^{-at}\sin bt$	$\Leftrightarrow$	$\dfrac{b}{(s+a)^2+b^2}$
$e^{-at}\cos bt$	$\Leftrightarrow$	$\dfrac{s+a}{(s+a)^2+b^2}$

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Some useful properties of Laplace transform:

1. Superposition:

$$L\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 L\{f_1(t)\} + a_2 L\{f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$$

2. Differentiation: Assume that $f(0) = 0$.

$$L\left\{\frac{df(t)}{dt}\right\} = L\{\dot{f}(t)\} = sL\{f(t)\} = sF(s)$$

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = L\{\ddot{f}(t)\} = s^2 L\{f(t)\} = s^2 F(s)$$

3. Integration:

$$L\left\{\int_0^t f(z)\,dz\right\} = \frac{1}{s} L\{f(t)\} = \frac{1}{s} F(s)$$

Re-express ODE Models using Laplace Transform (Transfer Function)

Recall that the mechanical system in the cruise-control problem with $m = 1$ can be represented by an ODE:

$$\boxed{\dot{v} + bv = u}$$

Taking Laplace transform on both sides of the equation, we obtain

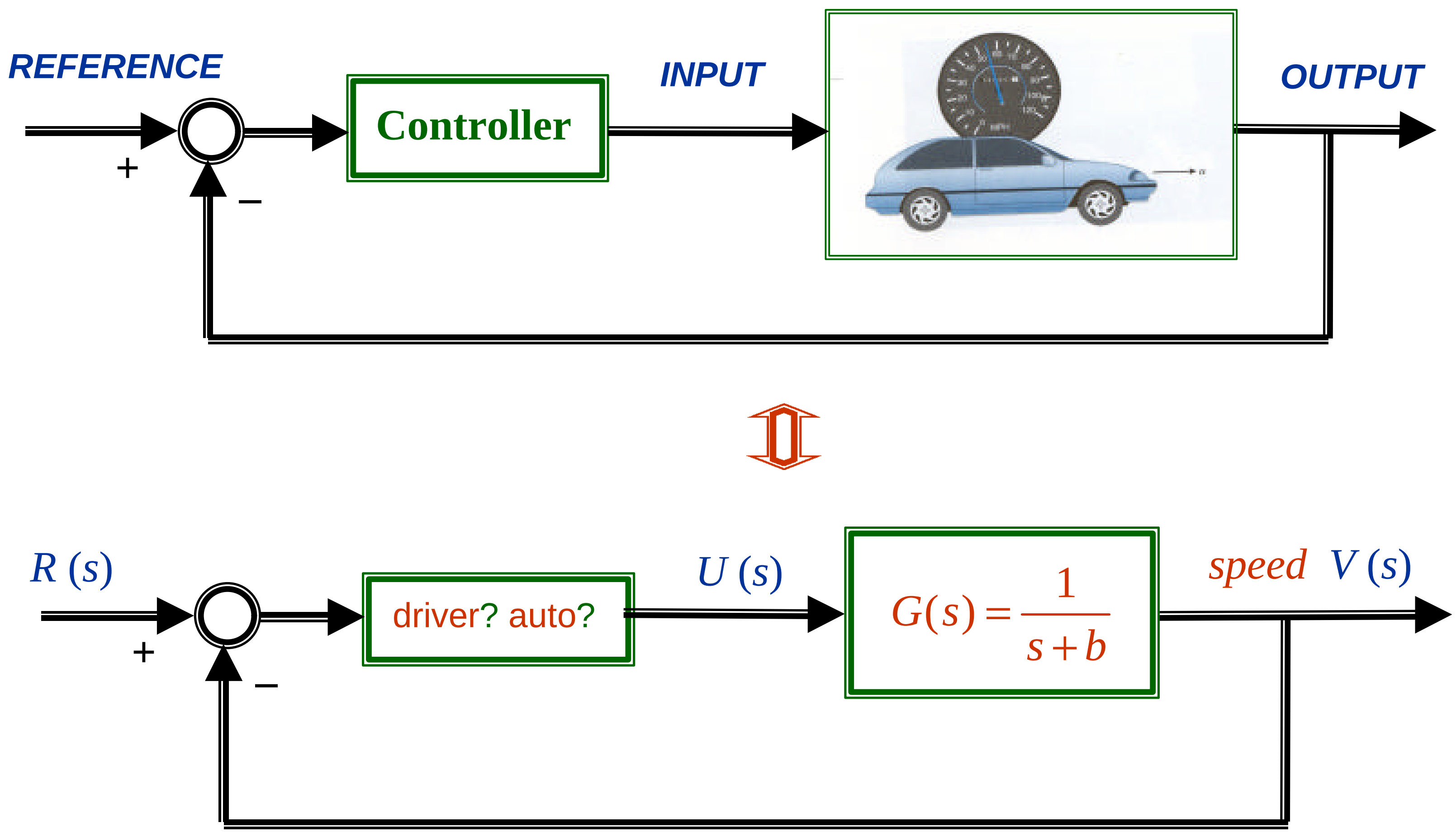
$$L\{\dot{v} + bv\} = L\{u\} \quad \Rightarrow \quad L\{\dot{v}\} + L\{bv\} = L\{u\}$$

$$\Rightarrow \quad sL\{v\} + bL\{v\} = L\{u\} \quad \Rightarrow \quad sV(s) + bV(s) = U(s)$$

$$\Rightarrow \quad (s+b)V(s) = U(s) \quad \Rightarrow \quad \boxed{\frac{V(s)}{U(s)} = \frac{1}{s+b} = G(s)}$$

This is called the transfer function of the system model

A cruise-control system in frequency domain:



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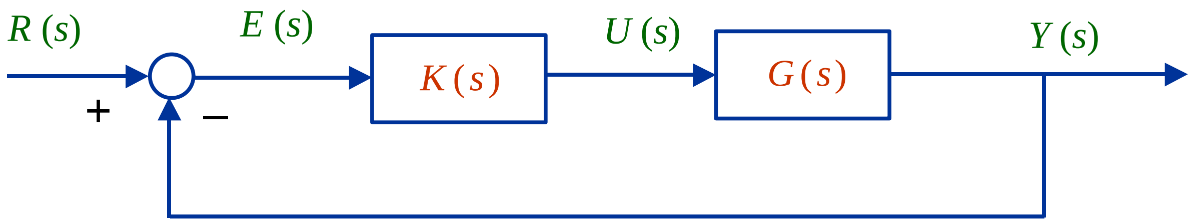
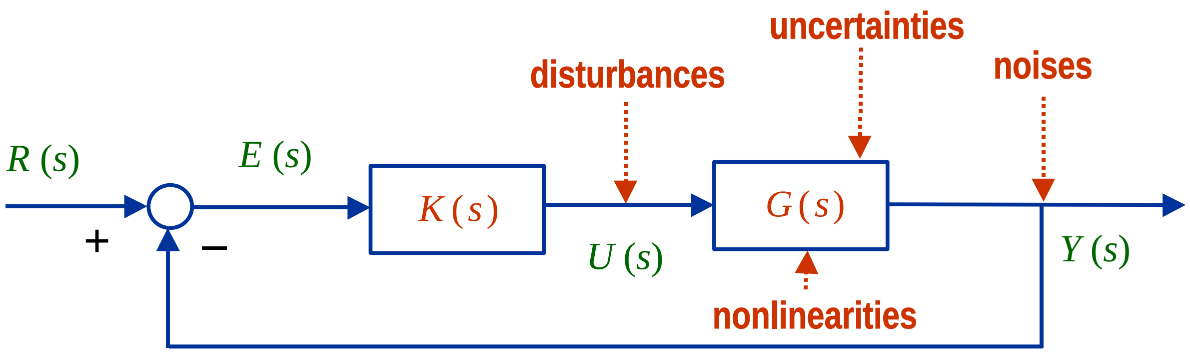
In general, a feedback control system can be represented by the following block diagram:



Given a system represented by $G(s)$ and a reference $R(s)$, the objective of control system design is to find a control law (or controller) $K(s)$ such that the resulting output $Y(s)$ is as close to reference $R(s)$ as possible, or the error $E(s) = R(s) - Y(s)$ is as small as possible. However, many other factors of life have to be carefully considered when dealing with real-life problems. These factors include:



Control Techniques – A Brief View:

There are tons of research published in the literature on how to design control laws for various purposes. These can be roughly classified as the following:

- Classical control: Proportional-integral-derivative (PID) control, developed in 1940s and used for control of industrial processes. Examples: chemical plants, commercial aeroplanes.
- Optimal control: Linear quadratic regulator control, Kalman filter, H_2 control, developed in 1960s to achieve certain optimal performance and boomed by *NASA Apollo Project*.
- Robust control: H_∞ control, developed in 1980s & 90s to handle systems with uncertainties and disturbances and with high performances. Example: military systems.
- Nonlinear control: Currently hot research topics, developed to handle nonlinear systems with high performances. Examples: military systems such as aircraft, missiles.
- Intelligent control: Knowledge-based control, adaptive control, neural and fuzzy control, etc., researched heavily in 1990s, developed to handle systems with unknown models. Examples: economic systems, social systems, human systems.

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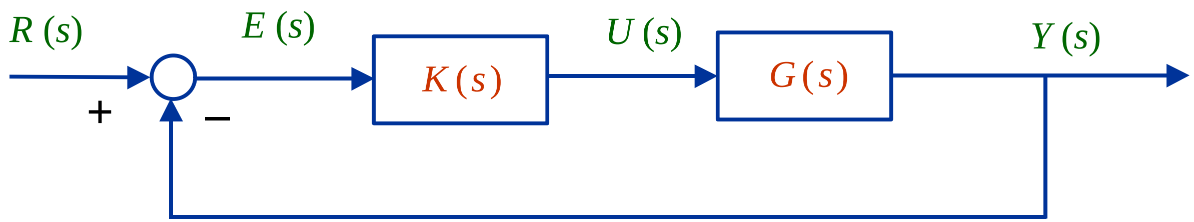
Classical Control

Let us examine the following block diagram of control system:



Recall that the objective of control system design is trying to match the output $Y(s)$ to the reference $R(s)$. Thus, it is important to find the relationship between them. Recall that

$$G(s) = \frac{Y(s)}{U(s)} \quad \Rightarrow \quad Y(s) = G(s)U(s)$$

Similarly, we have $U(s) = K(s)E(s)$, and $E(s) = R(s) - Y(s)$. Thus,

$$Y(s) = G(s)U(s) = G(s)K(s)E(s) = G(s)K(s)[R(s) - Y(s)]$$

$$Y(s) = G(s)K(s)R(s) - G(s)K(s)Y(s) \;\Rightarrow\; [1 + G(s)K(s)]Y(s) = G(s)K(s)R(s)$$

$$\Rightarrow \quad \boxed{H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)}}$$

Closed-loop transfer function from R to Y.

Thus, the block diagram of the control system can be simplified as,

$$R(s) \longrightarrow \boxed{H(s) = \frac{G(s)K(s)}{1+G(s)K(s)}} \longrightarrow Y(s)$$

The whole control problem becomes how to choose an appropriate $K(s)$ such that the resulting $H(s)$ would yield desired properties between R and Y.

We'll focus on control system design of some first order systems $G(s) = \dfrac{b}{s+a}$ with a proportional-integral (PI) controller, $K(s) = k_p + \dfrac{k_i}{s} = \dfrac{k_p s + k_i}{s}$. This implies

$$\boxed{H(s) = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{bk_p s + bk_i}{s^2 + (a + bk_p)\, s + bk_i}}$$

The closed-loop system $H(s)$ is a second order system as its denominator is a polynomial s of degree 2.

Stability of Control Systems

Example 1: Consider a closed-loop system with,

$$R(s) = 1 \longrightarrow \boxed{H(s) = \frac{1}{s^2 - 1}} \longrightarrow Y(s)$$

$$e^{-at} \Leftrightarrow \frac{1}{s+a}$$

$$\Downarrow$$

$$0.5e^{-t} \Leftrightarrow \frac{0.5}{s+1}$$

&

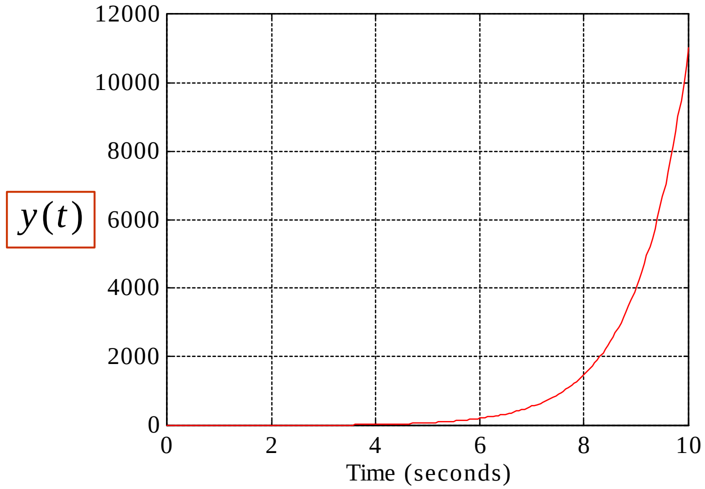
$$0.5e^{t} \Leftrightarrow \frac{0.5}{s-1}$$

We have

$$Y(s) = H(s)R(s) = \frac{1}{s^2 - 1} = \frac{1}{(s+1)(s-1)} = \frac{0.5}{s-1} - \frac{0.5}{s+1}$$

Using the Laplace transform table, we obtain

$$y(t) = 0.5(e^{t} - e^{-t})$$



This system is said to be ***unstable*** because the output response $y(t)$ goes to infinity as time t is getting larger and large. This happens because the denominator of $H(s)$ has one positive root at $s = 1$.

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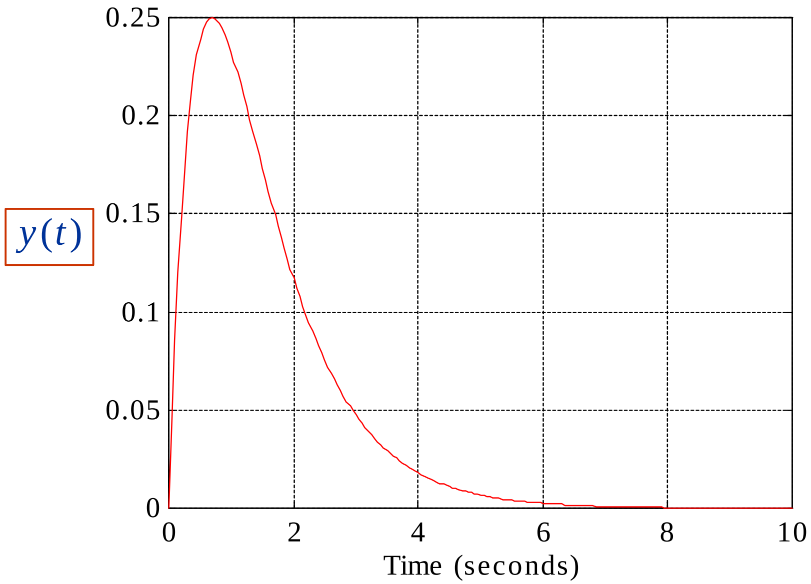
Example 2: Consider a closed-loop system with,

$$R(s) = 1 \longrightarrow \boxed{H(s) = \frac{1}{s^2 + 3s + 2}} \longrightarrow Y(s)$$

We have

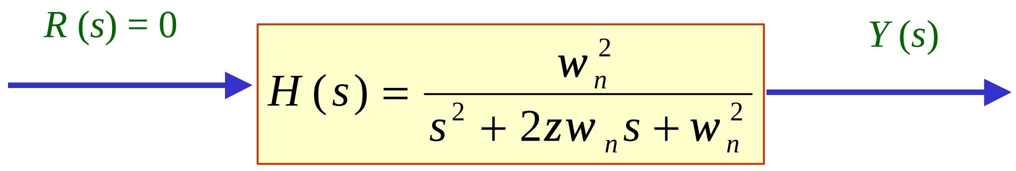
$$Y(s) = H(s)R(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Using the Laplace transform table, we obtain $y(t) = e^{-t} - e^{-2t}$

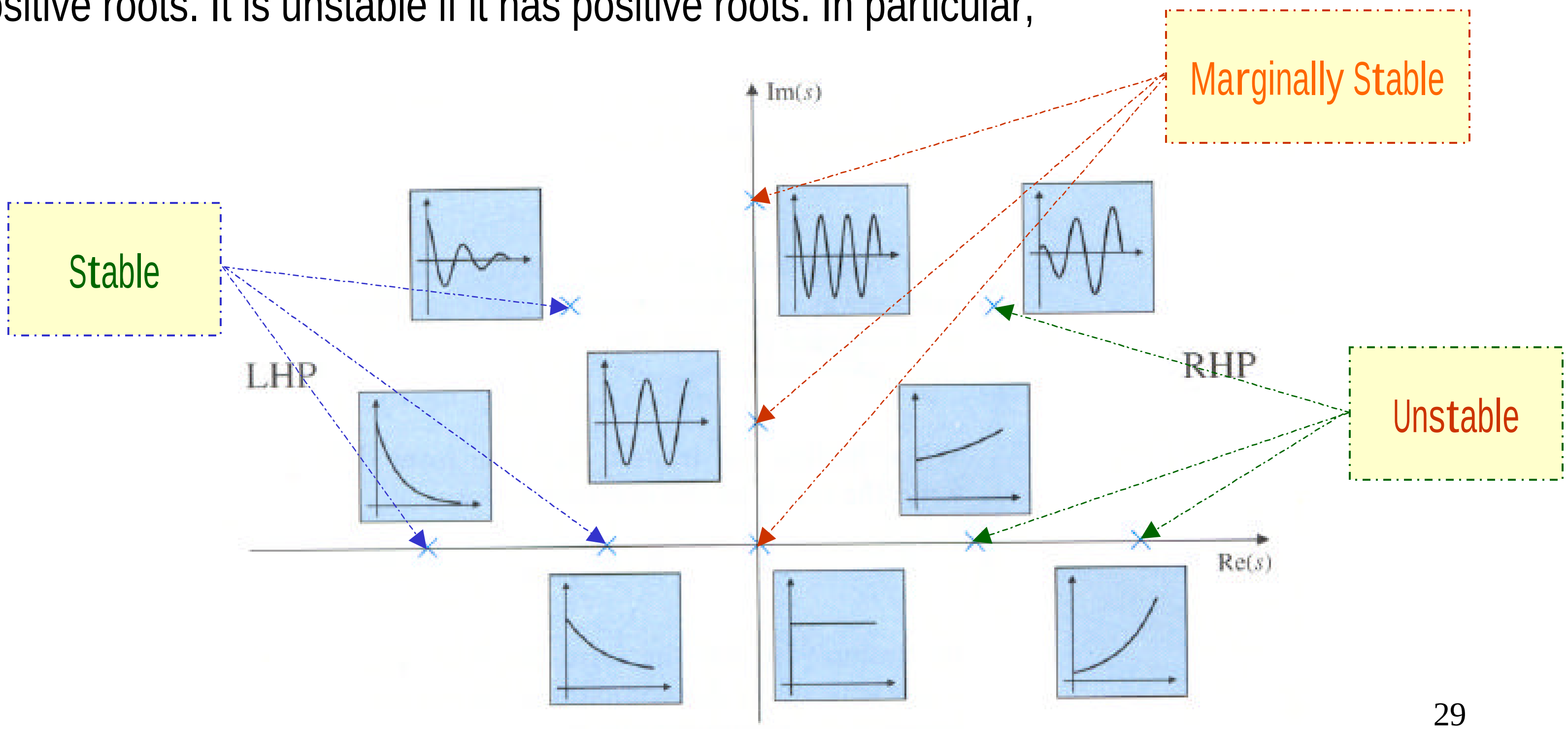


This system is said to be ***stable*** because the output response $y(t)$ goes to 0 as time t is getting larger and large. This happens because the denominator of $H(s)$ has no positive roots.

We consider a general 2nd order system,



The system is stable if the denominator of the system, i.e., $s^2+2\zeta\omega_n s+\omega_n^2=0$, has no positive roots. It is unstable if it has positive roots. In particular,



Stability in the State Space Representation

Consider a general linear system characterized by a state space form,

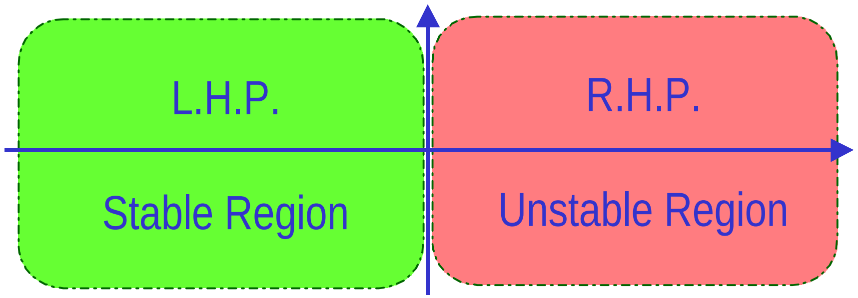
$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

Then,

1. It is stable if and only if all the eigenvalues of A are in the open left-half plane.

2. It is marginally stable if and only if A has eigenvalues are in the closed left-half plane with some (simple) on the imaginary axis.

3. It is unstable if and only if A has at least one eigenvalue in the right-half plane.



Lyapunov Stability

Consider a general dynamic system, $\dot{x} = f(x)$. If there exists a so-called Lyapunov function $V(x)$, which satisfies the following conditions:

1. $V(x)$ is continuous in x and $V(0) = 0$;

2. $V(x) > 0$ (positive definite);

3. $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0$ (negative definite),

then we can say that the system is asymptotically stable at $x = 0$. If in addition,

$$V(x) \to \infty, \quad \text{as } \|x\| \to \infty$$

then we can say that the system is globally asymptotically stable at $x = 0$. In this case, the stability is independent of the initial condition $x(0)$.

Lyapunov Stability for Linear Systems

Consider a linear system, $\dot{x} = Ax$. The system is asymptotically stable (i.e., the eigenvalues of matrix A are all in the open RHP) if for any given appropriate dimensional real positive definite matrix $Q = Q^{\mathrm{T}} > 0$, there exists a real positive definite solution $P = P^{\mathrm{T}} > 0$ for the following Lyapunov equation:

$$A^{\mathrm{T}} P + PA = -Q$$

Proof. Define a Lyapunov function $V(x) = x^{\mathrm{T}} P x$. Obviously, the first and second conditions on the previous page are satisfied. Now consider

$$\dot{V}(x) = \dot{x}^{\mathrm{T}} P x + x^{\mathrm{T}} P \dot{x} = (Ax)^{\mathrm{T}} P x + x^{\mathrm{T}} P A x = x^{\mathrm{T}} \left(A^{\mathrm{T}} P + PA \right) x = -x^{\mathrm{T}} Q x < 0$$

Hence, the third condition is also satisfied. The result follows.

Note that the condition, $Q = Q^{\mathrm{T}} > 0$, can be replaced by $Q = Q^{\mathrm{T}} \geq 0$ and $\left(A,\ Q^{1/2} \right)$ being detectable.

Behavior of Second Order Systems with a Step Inputs

Again, consider the following block diagram with a standard 2nd order system,

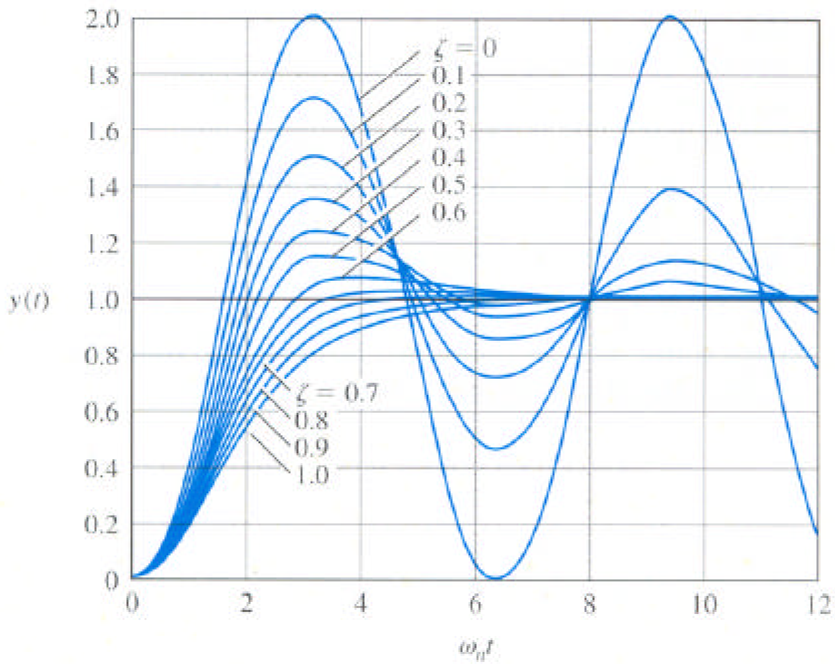
$R(s) = 1/s$

$r = 1$

$$H(s) = \frac{w_n^2}{s^2 + 2zw_n s + w_n^2}$$

$Y(s)$

The behavior of the system is as follows:



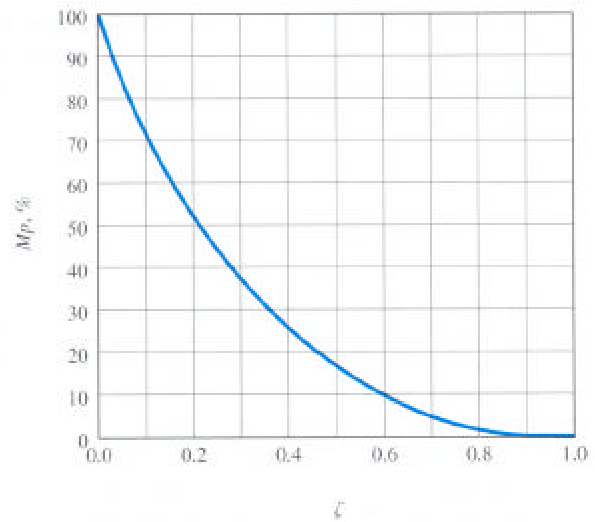
The behavior of the system is fully characterized by z, which is called the *damping ratio*, and w_n, which is called the *natural frequency*.

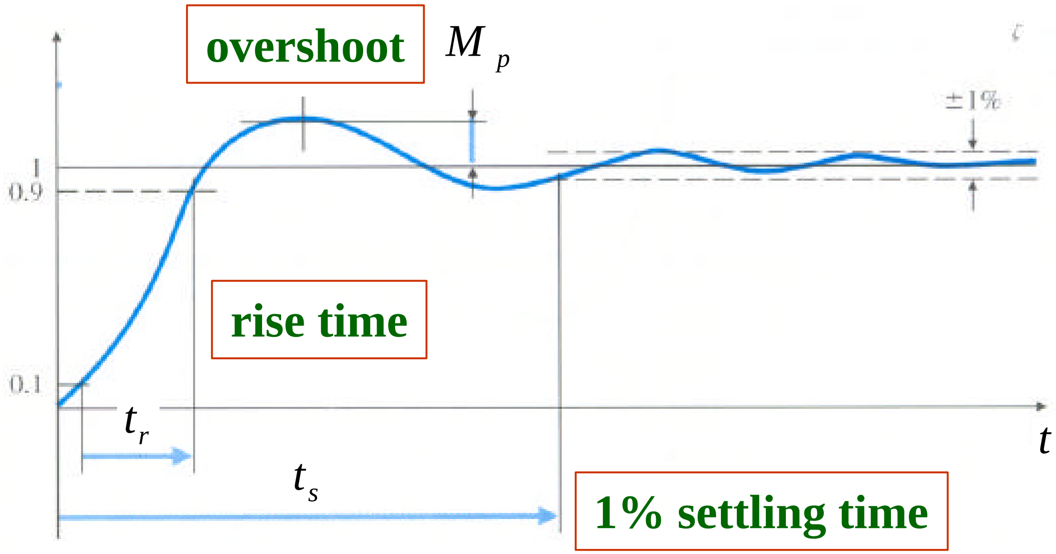
Control System Design with Time-domain Specifications

$R(s) = 1/s$, $r = 1$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$Y(s)$

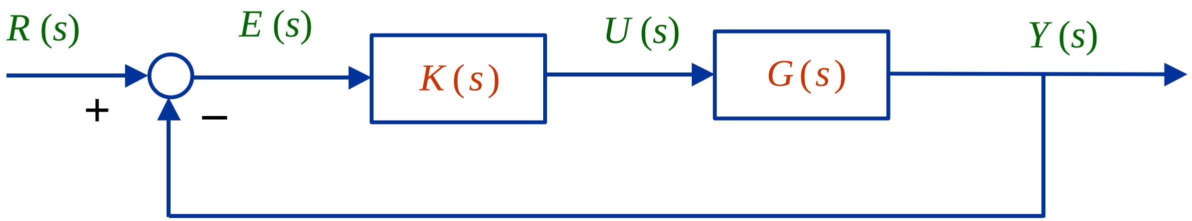




$$t_r \cong \frac{1.8}{\omega_n}$$

$$t_s \cong \frac{4.6}{\zeta\omega_n}$$

PID Design Technique:



with $G(s) = \dfrac{b}{s+a}$ and $K(s) = k_p + \dfrac{k_i}{s} = \dfrac{k_p s + k_i}{s}$ results a closed-loop system:

$$\boxed{H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{bk_p s + bk_i}{s^2 + (a+bk_p)\, s + bk_i}}$$

Compare this with the standard 2nd order system:

$$\boxed{H(s) = \frac{w_n^2}{s^2 + 2zw_n s + w_n^2}}$$

$$\boxed{\begin{aligned} 2zw_n &= a + bk_p \\ w_n^2 &= bk_i \end{aligned}} \rightarrow \boxed{\begin{aligned} k_p &= \frac{2zw_n - a}{b} \\ k_i &= \frac{w_n^2}{b} \end{aligned}}$$

The key issue now is to choose parameters k_p and k_i such that the above resulting system has desired properties, such as prescribed settling time and overshoot.

Cruise-Control System Design

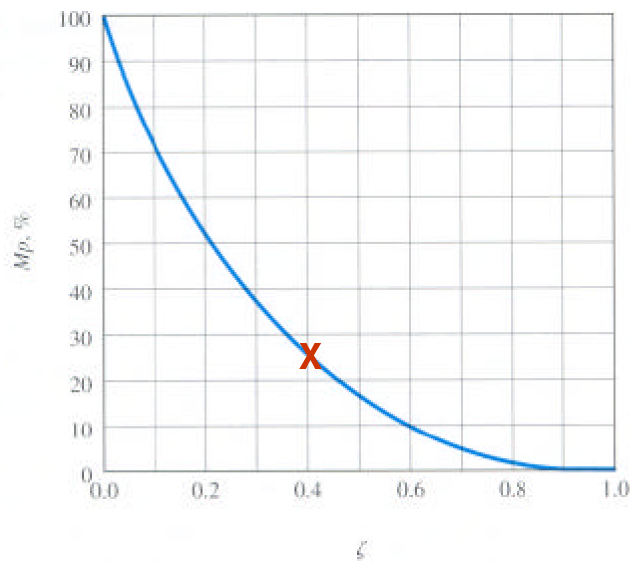
Recall the model for the cruise-control system, i.e., $\dfrac{V(s)}{U(s)} = \dfrac{1/m}{s + b/m}$. Assume that the mass of the car is 3000 kg and the friction coefficient $b = 1$. Design a PI controller for it such that the speed of the car will reach the desired speed 90 km/h in 10 seconds (i.e., the settling time is 10 s) and the maximum overshoot is less than 25%.

To achieve an overshoot less than 25%, we obtain from the figure on the right that $\boxed{\zeta > 0.4}$

To be safe, we choose $\boxed{\zeta = 0.6}$

To achieve a settling time of 10 s, we use

$$t_s = \frac{4.6}{\zeta \omega_n} \quad \Rightarrow \quad \omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.6 \times 10} = 0.767$$



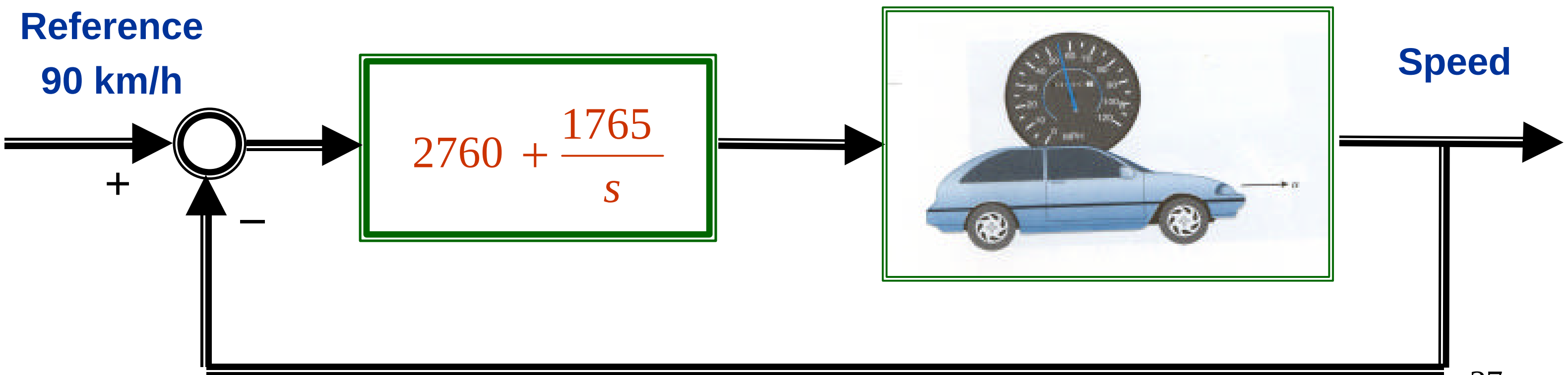
The transfer function of the cruise-control system,

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1/m}{s+b/m}=\frac{1/3000}{s+1/3000}\quad\Rightarrow\quad a=b=1/3000=0.000333$$

Again, using the formulae derived,

$$\boxed{\begin{aligned}k_p&=\frac{2zw_n-a}{b}\\ k_i&=\frac{w_n^2}{b}\end{aligned}}\quad\Longrightarrow\quad \begin{aligned}k_p&=\frac{2zw_n-a}{b}=\frac{2\times0.6\times0.767-1/3000}{1/3000}=2760\\ k_i&=\frac{w_n^2}{b}=\frac{0.767^2}{1/3000}=1765\end{aligned}$$

The final cruise-control system:



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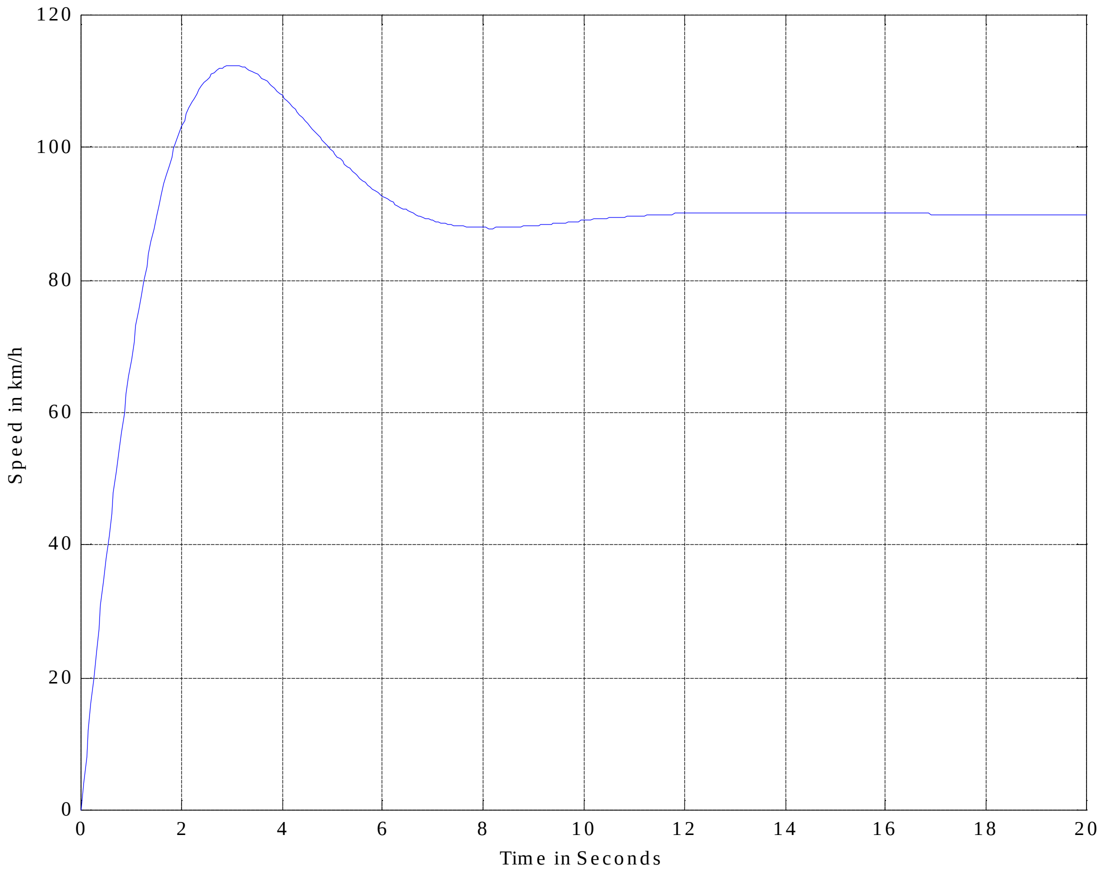
Simulation Result:

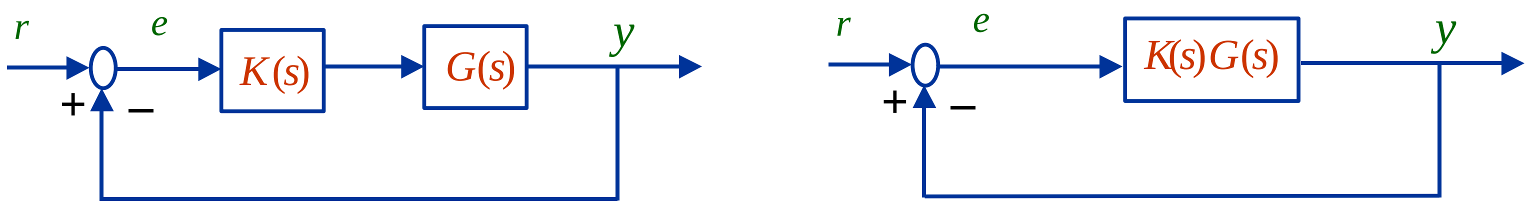


The resulting overshoot is less than 25% and the settling time is about 10 seconds.

Thus, our design goal is achieved.

Bode Plots

Consider the following feedback control system,

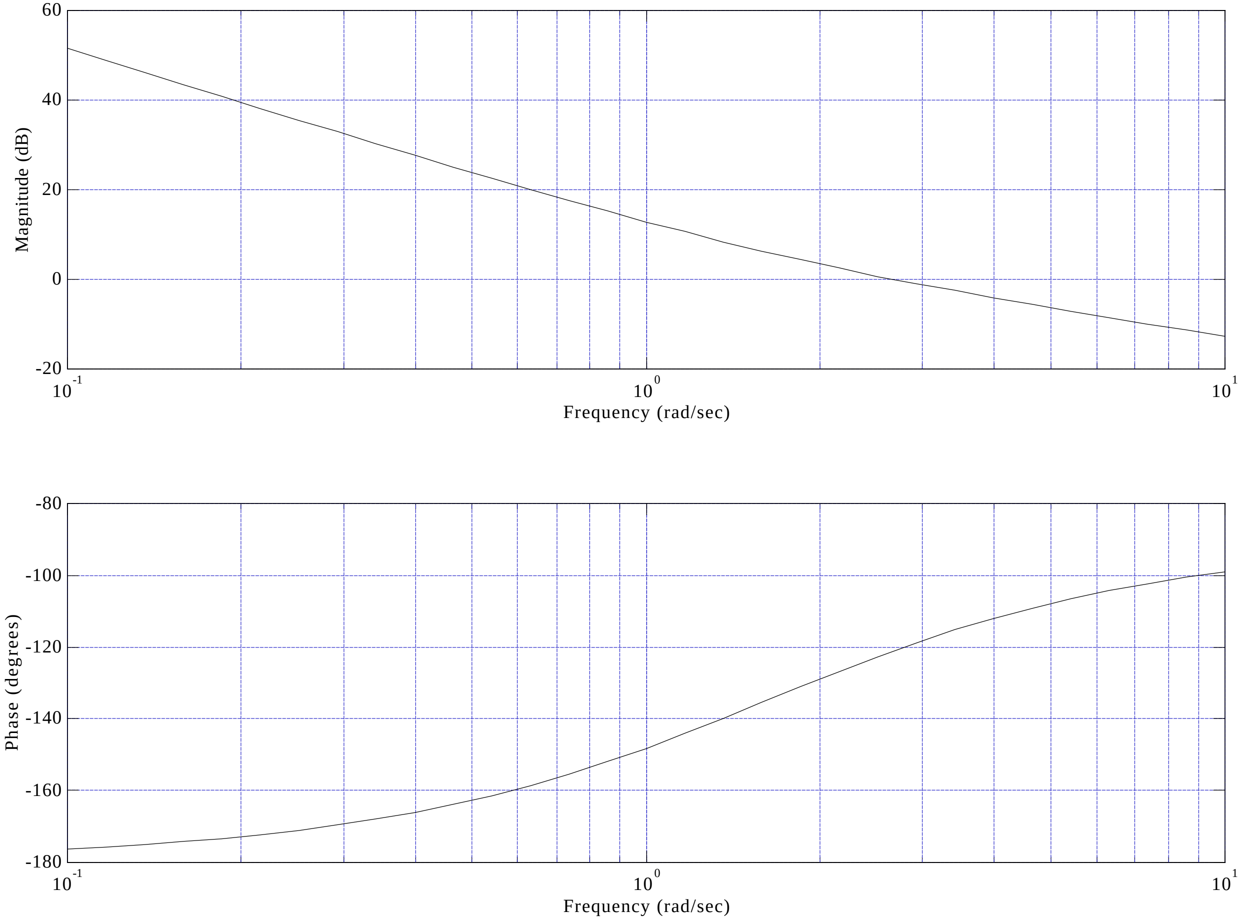


Bode Plots are the the magnitude and phase responses of the open-loop transfer function, i.e., $K(s)\ G(s)$, with s being replaced by jw. For example, for the ball and beam system we considered earlier, we have

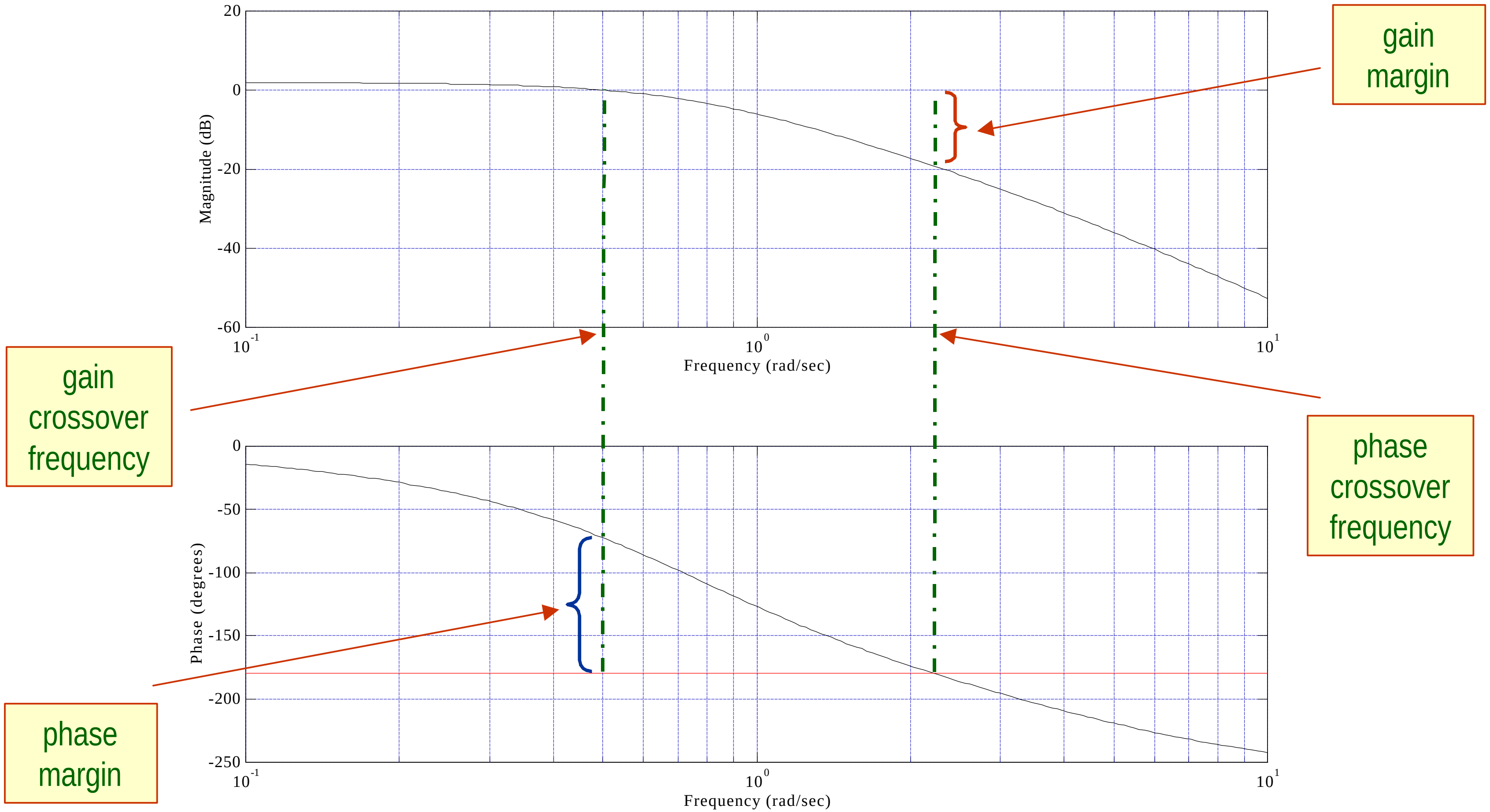
$$K(s)G(s)\Big|_{s=jw} = (0.37 + 0.23s)\frac{10}{s^2}\Big|_{s=jw} = \frac{3.7 + 2.3s}{s^2}\Big|_{s=jw} = \frac{3.7 + j2.3w}{-w^2}$$

$$\boxed{|K(jw)G(jw)| = \frac{\sqrt{3.7^2 + (2.3w)^2}}{w^2}, \quad \angle K(jw)G(jw) = \tan^{-1}\left(\frac{2.3w}{3.7}\right) - 180^\circ}$$

Bode magnitude and phase plots of the ball and beam system:

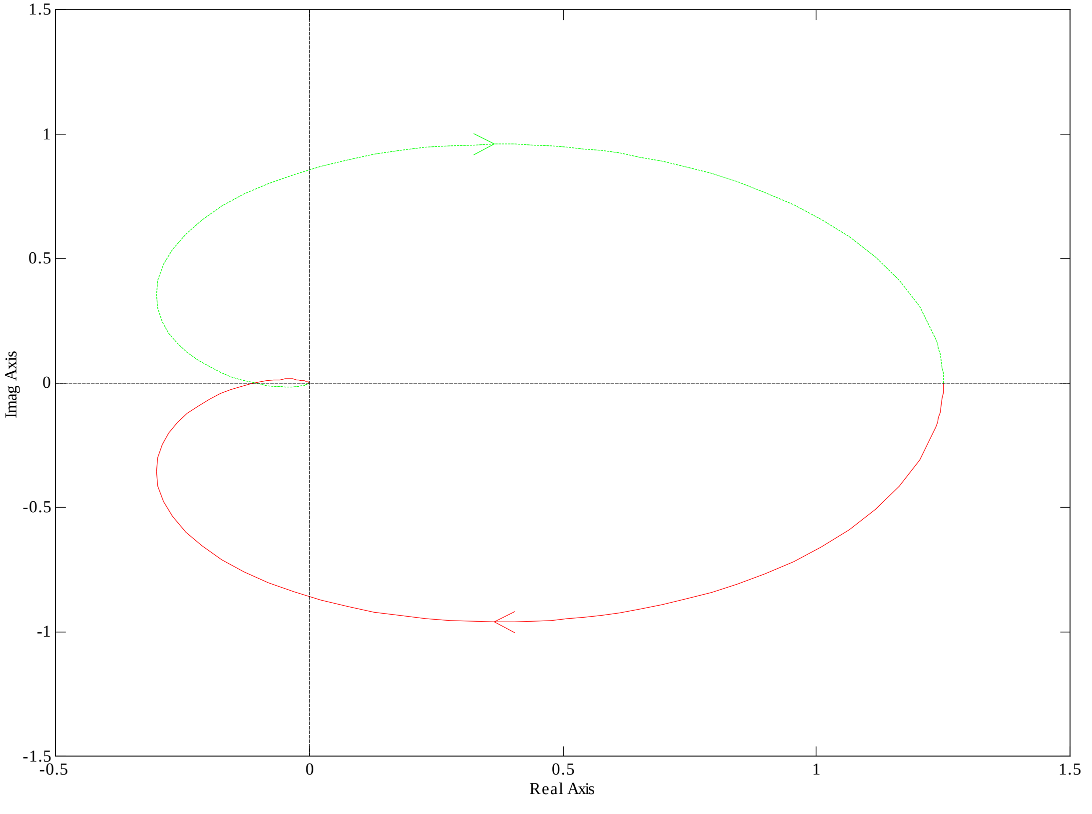


Gain and phase margins



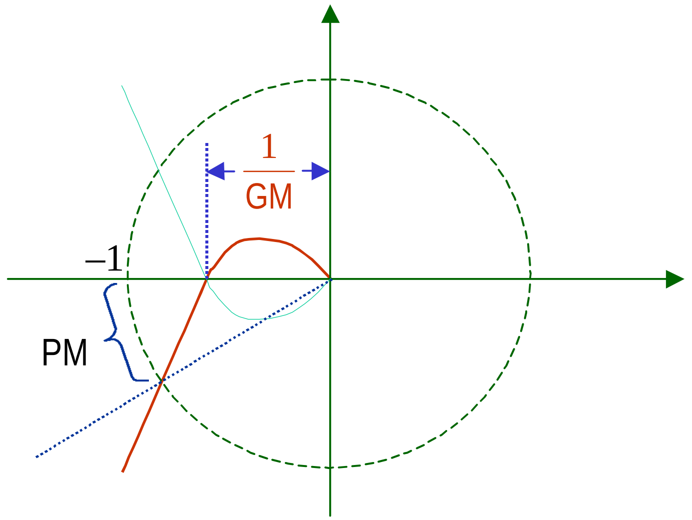
Nyquist Plot

Instead of separating into magnitude and phase diagrams as in Bode plots, Nyquist plot maps the open-loop transfer function $K(s)\ G(s)$ directly onto a complex plane, e.g.,



Gain and phase margins

The gain margin and phase margin can also be found from the Nyquist plot by zooming in the region in the neighbourhood of the origin.



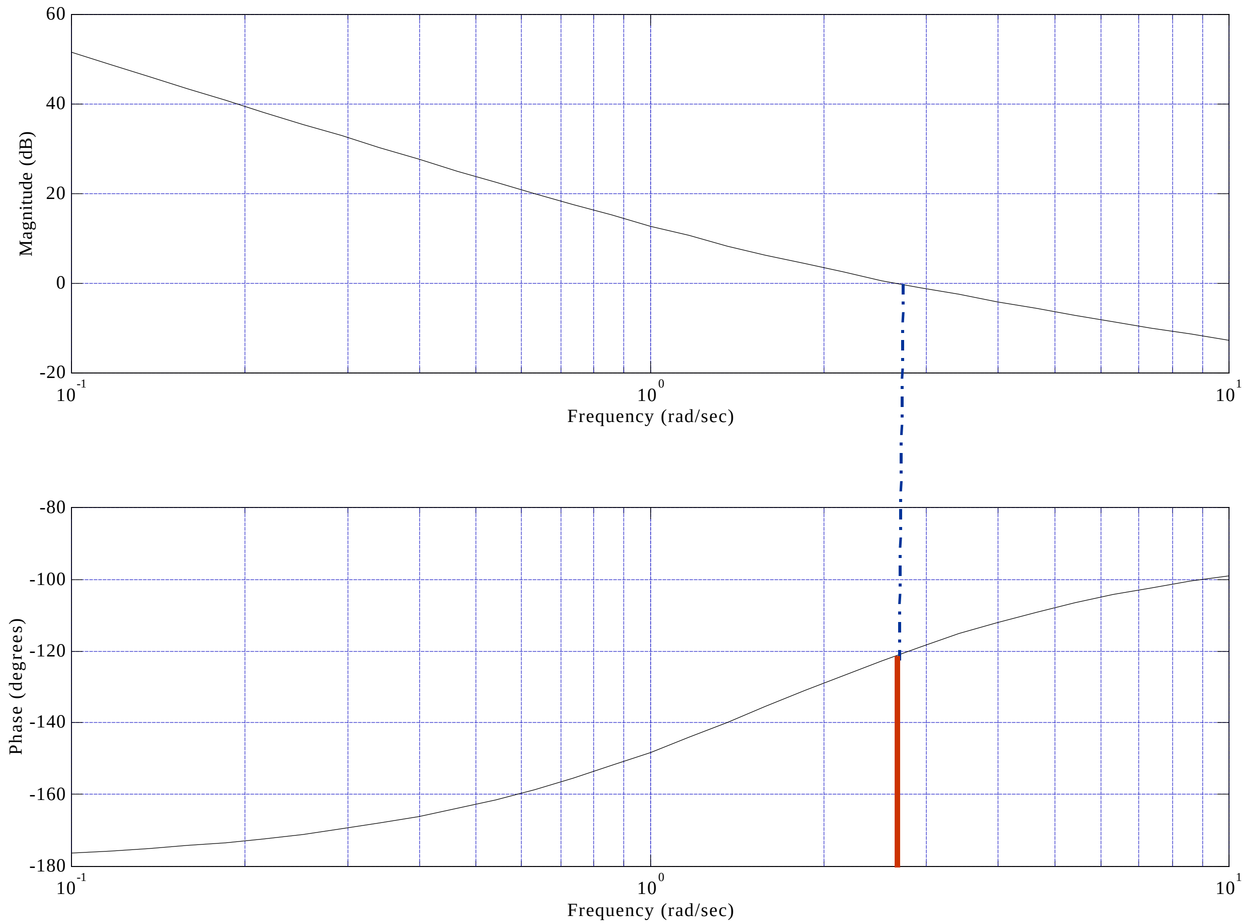
Remark: Gain margin is the maximum additional gain you can apply to the closed-loop system such that it will still remain stable. Similarly, phase margin is the maximum phase you can tolerate to the closed-loop system such that it will still remain stable.

Mathematically,

$$\text{GM} = \frac{1}{|K(jw_p)G(jw_p)|}, \quad \text{where } w_p \text{ is such that } \angle K(jw_p)G(jw_p) = 180^\circ$$

$$\text{PM} = \angle K(jw_g)G(jw_g) + 180^\circ, \quad \text{where } w_g \text{ is such that } |K(jw_g)G(jw_g)| = 1$$

Example: Gain and phase margins of the ball and beam system: $\boxed{PM = 58^\circ,\ GM = \infty}$



Properties of LQR Control

Linear Quadratic Regulator (LQR)

Consider a linear system characterized by

$$\dot{x} = A\,x + B\,u$$

where (A, B) is stabilizable. We define the cost index

$$J(x,u,Q,R) = \int_0^\infty (x^\top Q x + u^\top R u)\,dt, \qquad Q \ge 0,\ \ R > 0$$

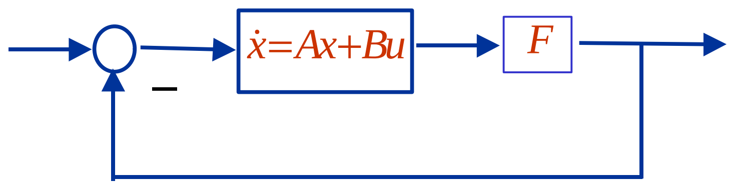
and $(A, Q^{1/2})$ is detectable. The linear quadratic regulation problem is to find a control law $u = -F\,x$ such that $(A - B\,F)$ is stable and J is minimized. It was shown in the first Part of this course that the solution is given by

$$F = R^{-1} B^\top P$$

with

$$PA + A^\top P - PBR^{-1}B^\top P + Q = 0$$

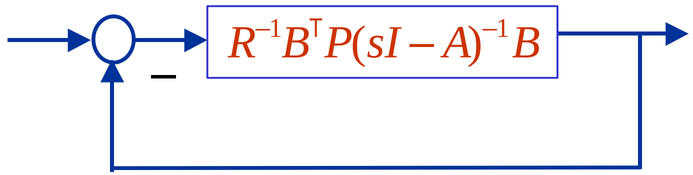
If we arrange the LQR control in the following block diagram,



we can find its gain margin and phase margin as we have done in classical control. It is clear that the open-loop transfer function,

$$\text{Open loop transfer function} = F(sI-A)^{-1}B = R^{-1}B^{\mathrm{T}}P(sI-A)^{-1}B$$

The block diagram can be re-drawn as follows,



Return Difference Equality and Inequality

Consider the LQR control law. The following so-called return difference equality hold:

$$R+B^{\mathrm{T}}(-jwI-A^{\mathrm{T}})^{-1}Q(jwI-A)B=[I+B^{\mathrm{T}}(-jwI-A^{\mathrm{T}})^{-1}F^{\mathrm{T}}]R[I+F(jwI-A)^{-1}B]$$

The following is called the return difference inequality:

$$\boxed{[I+B^{\mathrm{T}}(-jwI-A^{\mathrm{T}})^{-1}F^{\mathrm{T}}]R[I+F(jwI-A)^{-1}B]\geq R}$$

Proof. Recall that

$$\boxed{F=R^{-1}B^{\mathrm{T}}P} \qquad \boxed{PA+A^{\mathrm{T}}P-PBR^{-1}B^{\mathrm{T}}P+Q=0}$$

Then we have

$$-PjwI+PA+PjwI+A^{\mathrm{T}}P-(PBR^{-1})R(R^{-1}B^{\mathrm{T}}P)+Q=0$$

$$\Downarrow$$

$$P(jwI-A)+(-jwI-A^{\mathrm{T}})P+F^{\mathrm{T}}RF=Q$$

Multiplying it on the left by $B^{\top}(-jwI - A^{\top})^{-1}$ and on the right by $(jwI - A)^{-1}B$, we obtain,

$$B^{\top}(-jwI - A^{\top})^{-1}P(jwI - A)(jwI - A)^{-1}B + B^{\top}(-jwI - A^{\top})^{-1}(-jwI - A^{\top})P(jwI - A)^{-1}B$$
$$+ B^{\top}(-jwI - A^{\top})^{-1}F^{\top}RF(jwI - A)^{-1}B = B^{\top}(-jwI - A^{\top})^{-1}Q(jwI - A)^{-1}B$$

$$\Downarrow$$

$$B^{\top}(-jwI - A^{\top})^{-1}PB + B^{\top}P(jwI - A)^{-1}B + B^{\top}(-jwI - A^{\top})^{-1}F^{\top}RF(jwI - A)^{-1}B$$
$$= B^{\top}(-jwI - A^{\top})^{-1}Q(jwI - A)^{-1}B$$

Noting the fact that

$$\boxed{F = R^{-1}B^{\top}P \quad \Rightarrow \quad B^{\top}P = RF \quad \& \quad PB = F^{\top}R}$$

we have

$$B^{\top}(-jwI - A^{\top})^{-1}F^{\top}R + RF(jwI - A)^{-1}B + B^{\top}(-jwI - A^{\top})^{-1}F^{\top}RF(jwI - A)^{-1}B$$
$$= B^{\top}(-jwI - A^{\top})^{-1}Q(jwI - A)^{-1}B$$

$$\Downarrow$$

$$R + B^{\top}(-jwI - A^{\top})^{-1}Q(jwI - A)B = [I + B^{\top}(-jwI - A^{\top})^{-1}F^{\top}]R[I + F(jwI - A)^{-1}B]$$

Single Input Case

In the single input case, the transfer function

$$\boxed{\text{Open loop transfer function} = f(sI - A)^{-1} b}$$

is a scalar function. Let $Q = h\, h^{\mathrm{T}}$. Then, the return difference equation is reduced to

$$r + b^{\mathrm{T}}(-jwI - A^{\mathrm{T}})^{-1} h h^{\mathrm{T}} (jwI - A) b = r\,[1 + b^{\mathrm{T}}(-jwI - A^{\mathrm{T}})^{-1} f^{\mathrm{T}}][1 + f\,(jwI - A)^{-1} b]$$

$$\Downarrow$$

$$r + \left| h^{\mathrm{T}}(jwI - A)^{-1} b \right|^2 = r \left| 1 + f\,(jwI - A)^{-1} b \right|^2$$

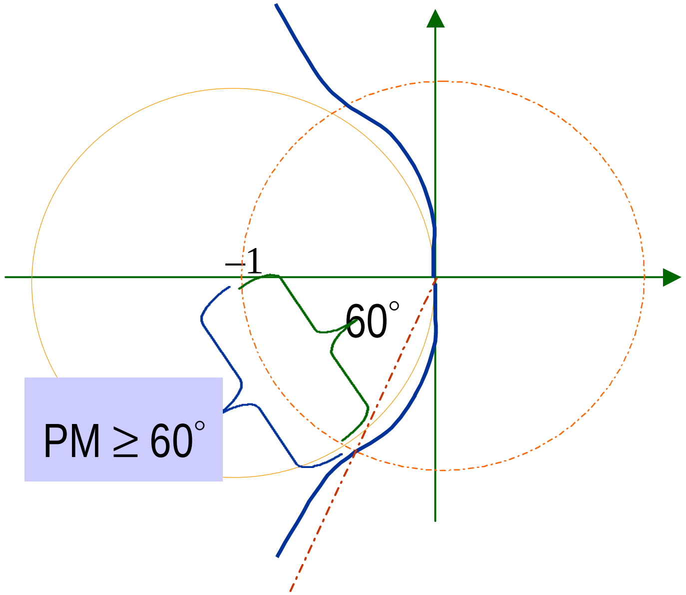
$$\Downarrow$$

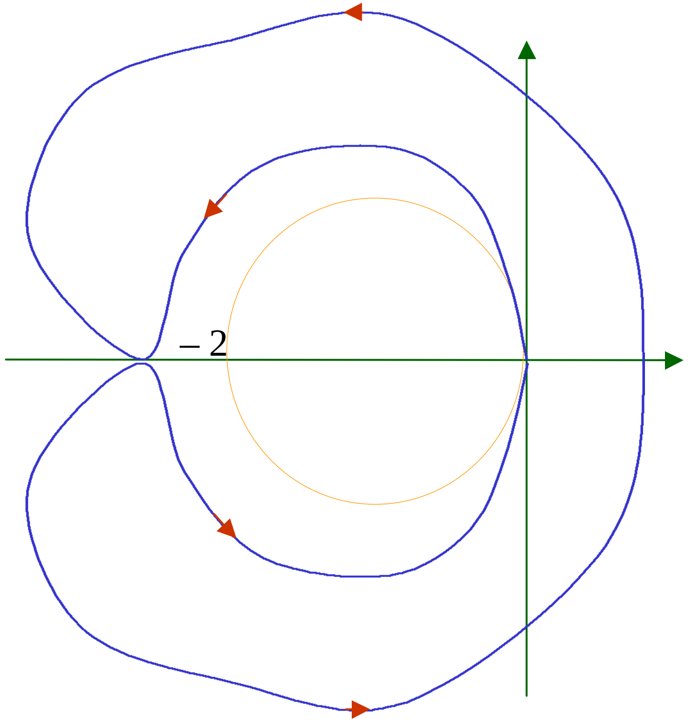
$$r \left| 1 + f\,(jwI - A)^{-1} b \right|^2 \ge r$$

$$\Downarrow$$

$$\boxed{\left| 1 + f\,(jwI - A)^{-1} b \right|^2 \ge 1 \quad \text{Return Difference Inequality.}}$$

Graphically, $\left|1+f(jwI-A)^{-1}b\right|^2 \geq 1 \quad \Rightarrow \quad \left|\, f(jwI-A)^{-1}b-(-1+j0)\,\right| \geq 1$ implies that





Clearly, the phase margin resulting from the LQR design is at least 60 degrees.

The gain margin is from [0.5, ¥).

Example: Consider a given plant characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

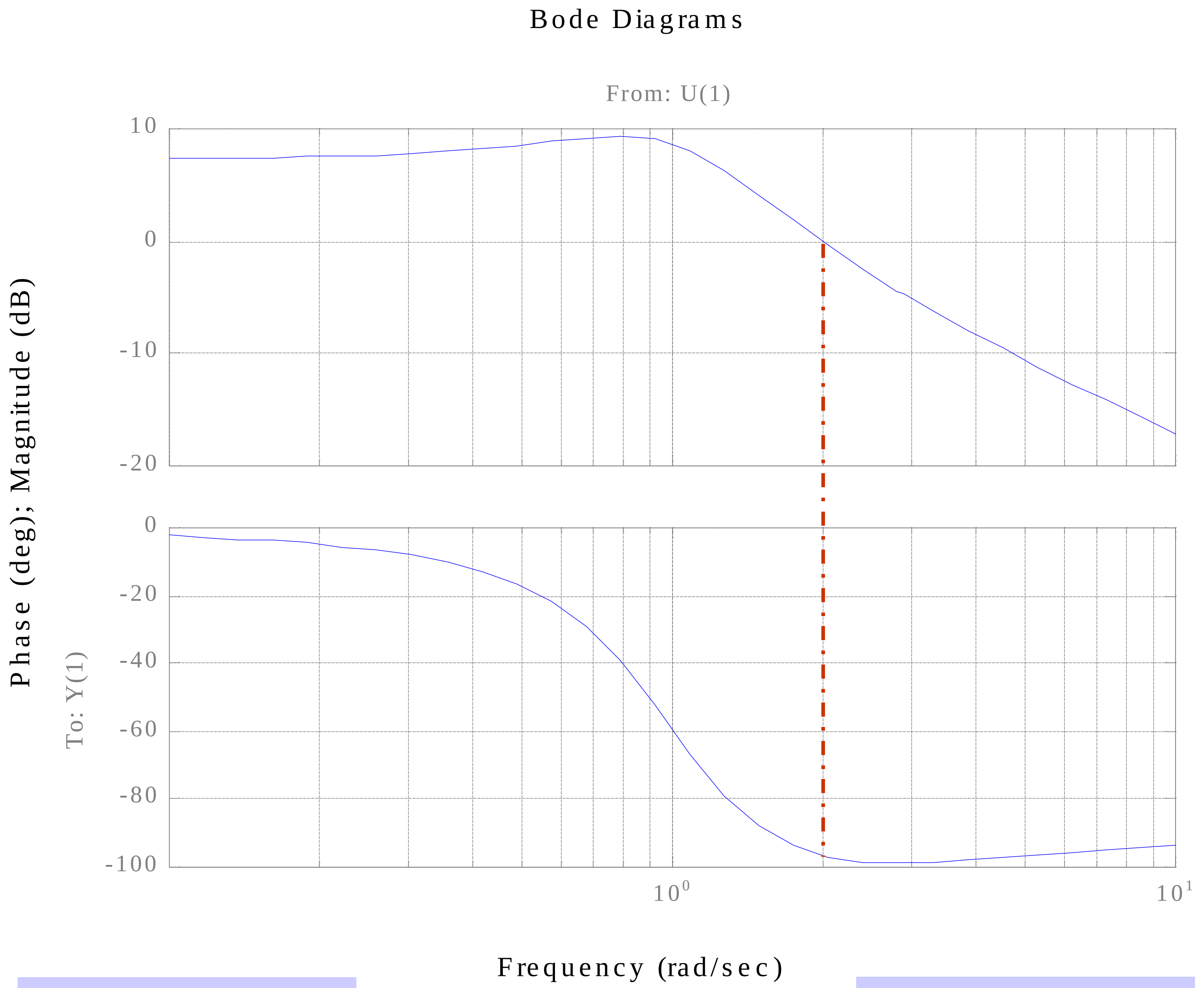
Solving the LQR problem which minimizes the following cost function

$$\boxed{J(x,u,Q,R) = \int_0^\infty (x^\mathrm{T} Q x + u^\mathrm{T} R u) dt, \qquad \text{with} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 0.1}$$

we obtain

$$P = \begin{bmatrix} 0.6872 & 0.2317 \\ 0.2317 & 0.1373 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 2.3166 & 1.3734 \end{bmatrix}$$

which results the closed-loop eigenvalues at $-1.1867 \pm j1.3814$. Clearly, the closed-loop system is asymptotically stable.



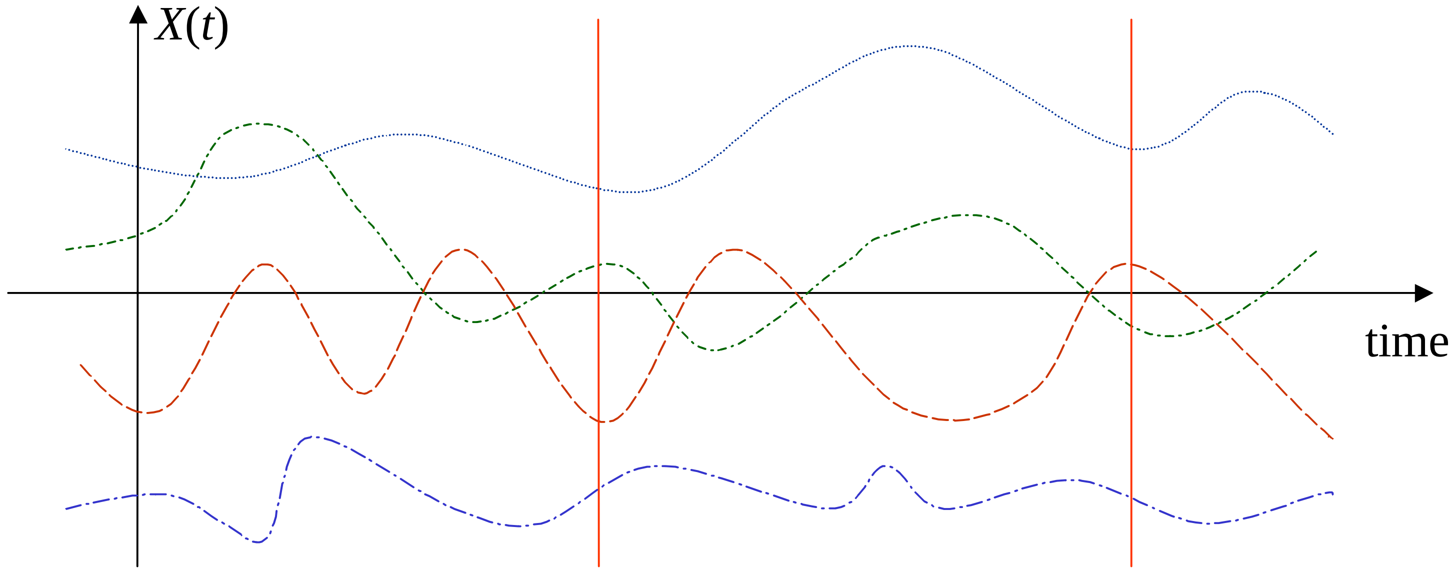
PM = 84°

GM = ∞

Kalman Filter

Review: Random Process

A **random variable** X is a mapping between the sample space and the real numbers. A **random process** (a.k.a **stochastic process**) is a mapping from the sample space into an ensemble of time functions (known as sample functions). To every member in the sample space, there corresponds a function of time (a sample function) $X(t)$.



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Mean, Moment, Variance, Covariance of Stationary Random Process

Let $f(x,t)$ be the **probability density function** (p.d.f.) associated with a random process $X(t)$. If the p.d.f. is independent of time t, i.e., $f(x,t) = f(x)$, then the corresponding random process is said to be **stationary**. We will focus our attention only on this class of random processes in this course. For this type of random processes, we define:

1) **mean** (or **expectation**):

$$m = E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

2) **moment** (j-th order moment)

$$E[X^j] = \int_{-\infty}^{\infty} x^j \cdot f(x) dx$$

3) **variance**

$$s^2 = E[(x-m)^2] = \int_{-\infty}^{\infty} (x-m)^2 f(x) dx$$

4) **covariance of two random processes**

$$\mathrm{con}(v,w) = E[(v - E[v])(w - E[w])]$$

Two random processes v and w are said to be **independent** if

$$\boxed{E[vw] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} vwf(v,w) dvdw = 0, \qquad f(v,w) \text{ is the joint p.d.f. of } v \text{ and } w.}$$

Autocorrelation Function and Power Spectrum

Autocorrelation function is used to describe the time domain property of a random process.

Given a random process v, its **autocorrelation function** is defined as follows:

$$R_x(t_1, t_2) = E\left[v(t_1)v(t_2)\right]$$

If v is a stationary process,

$$R_x(t_1, t_2) = R_x(t_2 - t_1) = R_x(t) = R_x(t, t+t) = E\left[v(t)v(t+t)\right]$$

Note that $R_x(0)$ is the time average of the power or energy of the random process.

Power spectrum of a random process is the Fourier transform of its autocorrelation function.

It is a frequency domain property of the random process. To be more specific, it is defined as

$$\boxed{S_x(w) = \frac{1}{2pj}\int_{-\infty}^{\infty} R_x(t)\, e^{jwt}\, dt}$$

White Noise, Color Noise and Gaussian Random Process

White Noise is a random process with a constant power spectrum, and an autocorrelation function

$$R_x(t) = q \cdot d(t)$$

which implies that a white noise has an infinite power and thus it is non existent in our real life. However, many noises (or the so-called color noises, or noises with finite energy and finite frequency components) can be modeled as the outputs of linear systems with an injection of white noise into their inputs, i.e., any **color noise** can be generated by a white noise

$$\text{while noise} \longrightarrow \boxed{\text{Linear System}} \longrightarrow \text{color noise}$$

Gaussian Process v is also known as normal process has a p.d.f.

$$f(v) = \frac{1}{s\sqrt{2p}} e^{-(v-m)^2 / 2s^2}, \qquad m = \text{mean}, \quad s^2 = \text{variance}$$

Kalman Filter for a Linear Time Invariant (LTI) System

Consider a LTI system characterized by

$$\begin{cases} \dot{x} = Ax + Bu + v(t) & v \text{ is the input noise} \\ y = Cx + w(t) & w \text{ is the measurement noise} \end{cases}$$

Assume: 1) (A, C) is observable

2) $v(t)$ and $w(t)$ are independent white noises with the following properties

$$E[v(t)] = 0, \quad E[w(t)] = 0$$

$$E[v(t)v^{\mathrm{T}}(t)] = Qd(t-t), \;\; Q = Q^{\mathrm{T}} \geq 0, \;\; E[w(t)w^{\mathrm{T}}(t)] = Rd(t-t), \;\; R = R^{\mathrm{T}} > 0$$

3) $\left(A, Q^{\frac{1}{2}} \right)$ is stabilizable (to guarantee closed-loop stability).

The problem of **Kalman Filter** is to design a state estimator to estimate the state $x(t)$ by $\hat{x}(t)$ such that the estimation error covariance is minimized, i.e., the following index is minimized:

$$J_e = E[\{x(t) - \hat{x}(t)\}^{\mathrm{T}}\{x(t) - \hat{x}(t)\}]$$

Construction of Steady State Kalman Filter

Kalman filter is a state observer with a specially selected observer gain (or Kalman filter gain). It has the dynamic equation:

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y}), \qquad \hat{x}(0) \text{ is given}$$
$$\hat{y} = C\hat{x}$$

with the Kalman filter gain K_e being given as

$$K_e = P_e C^{\mathrm{T}} R^{-1}$$

where P_e is the positive definite solution of the following Riccati equation,

$$P_e A^{\mathrm{T}} + AP_e - P_e C^{\mathrm{T}} R^{-1} C P_e + Q = 0$$

Let $e = x - \hat{x}$. We can show (see next) that such a Kalman filter has the following properties:

$$\lim_{t\to\infty} E\left[e(t)\right] = \lim_{t\to\infty} E\left[x(t) - \hat{x}(t)\right] = 0, \qquad \lim_{t\to\infty} J_e = \lim_{t\to\infty} E[e^{\mathrm{T}}(t)\, e(t)] = \text{trace } P_e$$

Kalman Filter and Linear Quadratic Regulator – They Are Dual

Recall the optimal regulator problem,

$$\dot{x} = Ax + Bu \qquad x(0) = x_0 \text{ given}$$

$$J = \int_0^\infty \left(x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u \right) dt, \quad Q = Q^{\mathrm{T}} \ge 0 \text{ and } R = R^{\mathrm{T}} > 0$$

The LQR problem is to find a state feedback law $u = -F\,x$ such that J is minimized. It was shown that the solution to the above problem is given by

$$F = R^{-1} B^{\mathrm{T}} P \qquad\qquad PA + A^{\mathrm{T}} P - PBR^{-1} B^{\mathrm{T}} P + Q = 0, \quad P = P^{\mathrm{T}} > 0$$

and the optimal value of J is given by $J = x_0^{\mathrm{T}} P\, x_0$. Note that x_0 is arbitrary. Let us consider a special case when x_0 is a random vector with .

$$E[x_0] = 0, \;\; E[x_0 x_0^{\mathrm{T}}] = I$$

Then, we have

$$E[J] = E[x_0^{\mathrm{T}} P x_0] = E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_{0i} x_{0j} \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} E[x_{0i} x_{0j}] = \sum_{i=1}^{n} p_{ii} = \text{trace } P$$

The Duality

§ Linear Quadratic Regulator

$$F = R^{-1}B^{\mathrm{T}}P$$

$$PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q = 0$$

$$J_{\text{optimal}} = \text{trace } P$$

[a] Kalman Filter

$$K_e = P_e C^{\mathrm{T}} R^{-1}$$

$$P_e A^{\mathrm{T}} + AP_e - P_e C^{\mathrm{T}} R^{-1} C P_e + Q = 0$$

$$J_{\text{optimal}} = \text{trace } P_e$$

These two problems are equivalent (or dual) if we let

$$A^{\mathrm{T}} \longleftrightarrow A$$

$$B^{\mathrm{T}} \longleftrightarrow C$$

$$F^{\mathrm{T}} \longleftrightarrow K_e$$

$$P \longleftrightarrow P_e$$

Proof of the Properties of Kalman Filter

Recall that the dynamics of the given plant and Kalman filter, i.e.,

$$\begin{aligned} \dot{x} &= Ax + Bu + v(t) \\ y &= Cx + w(t) \end{aligned} \qquad \& \qquad \begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K_e(y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned}$$

We have

$$\begin{aligned} \dot{e} = \dot{x} - \dot{\hat{x}} &= Ax + Bu + v(t) - A\hat{x} - Bu - K_e[Cx + w(t) - C\hat{x}] \\ &= (A - K_e C)(x - \hat{x}) + v(t) - K_e w(t) \\ &= \left(A - K_e C\right)e + [\, I \quad -K_e \,]\begin{pmatrix} v \\ w \end{pmatrix} = \bar{A}e + d(t) \end{aligned}$$

with

$$E[d(t)] = E\left[[I \quad -K_e] \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right] = [I \quad -K_e] \begin{pmatrix} E[v(t)] \\ E[w(t)] \end{pmatrix} = [I \quad -K_e] \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

Next, it is reasonable to assume that initial error $e(0)$ and $d(t)$ are independent, i.e.,

$$E[e(0)\, d^{\mathrm{T}}(t)] = 0$$

Furthermore,

$$
\begin{aligned}
E[d(t)d^{\mathrm{T}}(t)] &= [I \quad -K_e]\begin{pmatrix} E[v(t)v^{\mathrm{T}}(t)] & E[v(t)w^{\mathrm{T}}(t)] \\ E[w(t)v^{\mathrm{T}}(t)] & E[w(t)w^{\mathrm{T}}(t)] \end{pmatrix}\begin{bmatrix} I \\ -K_e^{\mathrm{T}} \end{bmatrix} \\
&= [I \quad -K_e]\begin{pmatrix} Qd(t-t) & 0 \\ 0 & Rd(t-t) \end{pmatrix}\begin{bmatrix} I \\ -K_e^{\mathrm{T}} \end{bmatrix} \\
&= \left(Q + K_e R K_e^{\mathrm{T}} \right) d(t-t) = \nabla d(t-t)
\end{aligned}
$$

We will next show $\bar{A}$ is asymptotically stable and

$$
\lim_{t\to\infty} E[\, e(t)\, e^{\mathrm{T}}(t)\,] = P_e
$$

and $P_e \bar{A}^{\mathrm{T}} + \bar{A} P_e = -\nabla$. Recall that the solution to the following state equation from your linear systems course notes:

$$
\dot{e} = \bar{A} e + d(t)
$$

i.e.,

$$
\boxed{e(t) = e^{\bar{A}t} \cdot e(0) + \int_0^t e^{\bar{A}(t-t)} d(t) \cdot dt}
$$

Also, recall that $K_e = P_e C^{\mathrm{T}} R^{-1}$ and

$$P_e A^{\mathrm{T}} + A P_e - P_e C^{\mathrm{T}} R^{-1} C P_e + Q = 0$$

We have

$$P_e A^{\mathrm{T}} - P_e C^{\mathrm{T}} R^{-1} C P_e + A P_e - P_e C^{\mathrm{T}} R^{-1} C P_e + P_e C^{\mathrm{T}} R^{-1} C P_e + Q = 0$$

$$\Rightarrow \quad P_e (A^{\mathrm{T}} - C^{\mathrm{T}} R^{-1} C P_e) + (A - P_e C^{\mathrm{T}} R^{-1} C) P_e + P_e C^{\mathrm{T}} R^{-1} C P_e + Q = 0$$

$$\Rightarrow \quad P_e (A^{\mathrm{T}} - C^{\mathrm{T}} K_e^{\mathrm{T}}) + (A - K_e C) P_e + P_e C^{\mathrm{T}} R^{-1} C P_e + Q = 0$$

$$\Rightarrow \quad P_e \bar{A}^{\mathrm{T}} + \bar{A} P_e = -P_e C^{\mathrm{T}} R^{-1} C P_e - Q = -\nabla \le 0$$

Since $Q = Q^{\mathrm{T}} \ge 0$ and $\left(A,\ Q^{1/2}\right)$ is assumed to be stabilizable, it follows from Lyapunov stability theory that matrix $\bar{A} = \left(A - K_e C\right)$ is asymptotically stable.

Noting that $e^{\bar{A}t}$ is deterministic, we have

$$P(t)=E[e(t)e^{\mathrm{T}}(t)]=E\left[\left(e^{\bar{A}t}\cdot e(0)+\int_0^t e^{\bar{A}(t-\tau)}d(\tau)\cdot d\tau\right)\cdot\left(e^{\bar{A}t}\cdot e(0)+\int_0^t e^{\bar{A}(t-\tau)}d(\tau)\cdot d\tau\right)^{\mathrm{T}}\right]$$

$$=e^{\bar{A}t}E[e(0)e^{\mathrm{T}}(0)]e^{\bar{A}^{\mathrm{T}}t}+\int_0^t e^{\bar{A}(t-\tau)}E[d(\tau)e^{\mathrm{T}}(0)]e^{\bar{A}^{\mathrm{T}}t}\cdot d\tau$$

$$+\int_0^t e^{\bar{A}t}E[e(0)d^{\mathrm{T}}(\tau)]e^{\bar{A}^{\mathrm{T}}(t-\tau)}\cdot d\tau+\int_0^t e^{\bar{A}(t-\tau)}d\tau\int_0^t E[d(\tau)d^{\mathrm{T}}(s)]e^{\bar{A}^{\mathrm{T}}(t-s)}\cdot ds$$

$$=e^{\bar{A}t}E[e(0)e^{\mathrm{T}}(0)]e^{\bar{A}^{\mathrm{T}}t}+\int_0^t e^{\bar{A}(t-\tau)}d\tau\int_0^t \nabla d(\tau-s)e^{\bar{A}^{\mathrm{T}}(t-s)}\cdot ds$$

$$=e^{\bar{A}t}E[e(0)e^{\mathrm{T}}(0)]e^{\bar{A}^{\mathrm{T}}t}+\int_0^t e^{\bar{A}(t-\tau)}\nabla e^{\bar{A}^{\mathrm{T}}(t-\tau)}\cdot d\tau=e^{\bar{A}t}E[e(0)e^{\mathrm{T}}(0)]e^{\bar{A}^{\mathrm{T}}t}+\int_0^t e^{\bar{A}h}\nabla e^{\bar{A}^{\mathrm{T}}h}\cdot dh$$

Since $\bar{A}$ is stable, we have $e^{\bar{A}t}\to 0,$ as $t\to\infty$. Thus,

$$\boxed{P(\infty)=\int_0^\infty e^{\bar{A}h}\nabla e^{\bar{A}^{\mathrm{T}}h}\cdot dh}$$

We next show that $P(\infty) = P_e$, i.e., the solution associated with the Kalman filter ARE. Let

$$\dot{z} = \bar{A}^{\mathrm{T}} z, \quad z(0) \text{ given} \quad \Rightarrow \quad z(t) = e^{\bar{A}t} z(0), \quad z(\infty) = 0$$

In view of $P_e \bar{A}^{\mathrm{T}} + \bar{A} P_e = -\nabla$, we have

$$z^{\mathrm{T}} [P_e \bar{A}^{\mathrm{T}} + \bar{A} P_e] z = -z^{\mathrm{T}} \nabla z \quad \Rightarrow \quad z^{\mathrm{T}} P_e \bar{A}^{\mathrm{T}} z + z^{\mathrm{T}} \bar{A} P_e z = -z^{\mathrm{T}} \nabla z$$

$$\Rightarrow \quad z^{\mathrm{T}} P_e \dot{z} + \dot{z}^{\mathrm{T}} P_e z = -z^{\mathrm{T}} \nabla z \quad \Rightarrow \quad \frac{d}{dt}(z^{\mathrm{T}} P_e z) = -z^{\mathrm{T}} \nabla z$$

Next, we have

$$\int_0^\infty z^{\mathrm{T}} \nabla z dt = \int_0^\infty z^{\mathrm{T}}(0) e^{\bar{A}^{\mathrm{T}} t} \nabla e^{\bar{A}t} z(0) dt = z^{\mathrm{T}}(0) \left[\int_0^\infty e^{\bar{A}^{\mathrm{T}} t} \nabla e^{\bar{A}t} dt \right] z(0) = z^{\mathrm{T}}(0) P(\infty) z(0)$$

$$\int_0^\infty \frac{d}{dt}(z^{\mathrm{T}} P_e z) dt = z^{\mathrm{T}}(t) P_e z(t) \Big|_0^\infty = z^{\mathrm{T}}(\infty) P_e z(\infty) - z^{\mathrm{T}}(0) P_e z(0) = 0 - z^{\mathrm{T}}(0) P_e z(0)$$

Thus, we have for every given $z(0)$,

$$\boxed{z^{\mathrm{T}}(0) P_e z(0) = \quad z^{\mathrm{T}}(0) P(\infty) z(0) \quad \Rightarrow \quad P_e = P(\infty) = \int_0^\infty e^{\bar{A}^{\mathrm{T}} h} \nabla e^{\bar{A}h} dh}$$

It is now simple to see that

$$\lim_{t\to\infty} E[e(t)e^{\mathrm{T}}(t)] = P(\infty) = P_e \quad \Rightarrow \quad \lim_{t\to\infty} E[e^{\mathrm{T}}(t)e(t)] = \text{trace } \; P_e$$

Finally, we have

$$\boxed{E[e(t)] = e^{\bar{A}t} \cdot E[e(0)] + \int_0^t e^{\bar{A}(t-t)} E[d(t)] \cdot dt = 0}$$

Example: Consider a given plant characterized by the following state space model,

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + v(t), \quad E[v(t)v^{\mathrm{T}}(t)] = Qd(t-t) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix} d(t-t) \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + w(t), \quad E[w(t)w^{\mathrm{T}}(t)] = Rd(t-t) = 0.2d(t-t) \end{cases}$$

Solving the Kalman filter ARE, we obtain

$$P_e = \begin{bmatrix} 0.0792 & -0.0343 \\ -0.0343 & 0.0314 \end{bmatrix}, \quad K_e = \begin{bmatrix} 0.3962 \\ -0.1715 \end{bmatrix}$$

$$\boxed{\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}}$$

Linear Quadratic Gaussian (LQG)

Problem Statement

It is very often in control system design for a real life problem that one cannot measure all the state variables of the given plant. Thus, the linear quadratic regulator, although it has a very impressive gain and phase margins (GM = ∞ and PM = 60 degrees), is impractical as it utilizes all state variables in the feedback, i.e., $u = -F\,x$. In most of practical situations, only partial information of the state of the given plant is accessible or can be measured for feedback. The natural questions one would ask:

- Can we recover or estimate the state variables of the plant through the partially measurable information? The answer is yes. The solution is Kalman filter.
- Can we replace x the control law in LQR, i.e., $u = -F\,x$, by the estimated state to carry out a meaningful control system design? The answer is yes. The solution is called LQG.
- Do we still have impressive properties associated with LQG? The answer is no. Any solution? Yes. It is called **loop transfer recovery** (LTR) technique (to be covered later).

Linear Quadratic Gaussian Design

Consider a given plant characterized by

$$\begin{cases} \dot{x} = Ax + Bu + v(t) & v \text{ is the input noise} \\ y = Cx + w(t) & w \text{ is the measurement noise} \end{cases}$$

where $v(t)$ and $w(t)$ are white with zero means. $v(t)$, $w(t)$ and $x(0)$ are independent, and

$$E[v(t)v^{\mathrm{T}}(t)] = Q_e d(t-t), \;\; Q_e \ge 0, \;\; E[w(t)w^{\mathrm{T}}(t)] = R_e d(t-t), \;\; R_e > 0, \quad E[x(0)] = x_0$$

The performance index has to be modified as follows:

$$J = \lim_{T \to \infty} \frac{1}{T} E\left[\int_0^T (x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u) dt \right], \qquad Q \ge 0, \;\; R > 0$$

The **Linear Quadratic Gaussian** (LQG) control is to design a control law that only requires the measurable information such that when it is applied to the given plant, the overall system is stable and the performance index is minimized.

Solution to the LQG Problem – Separation Principle

Step 1. Design an LQR control law $u = -F\,x$ which solves the following problem,

$$\dot{x} = A\,x + B\,u \qquad J(x,u,Q,R) = \int_0^\infty (x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)dt, \qquad Q \ge 0,\ R > 0$$

i.e., compute

$$PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q = 0, \quad P > 0, \qquad F = R^{-1}B^{\mathrm{T}}P.$$

Step 2. Design a Kalman filter for the given plant, i.e.,

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$

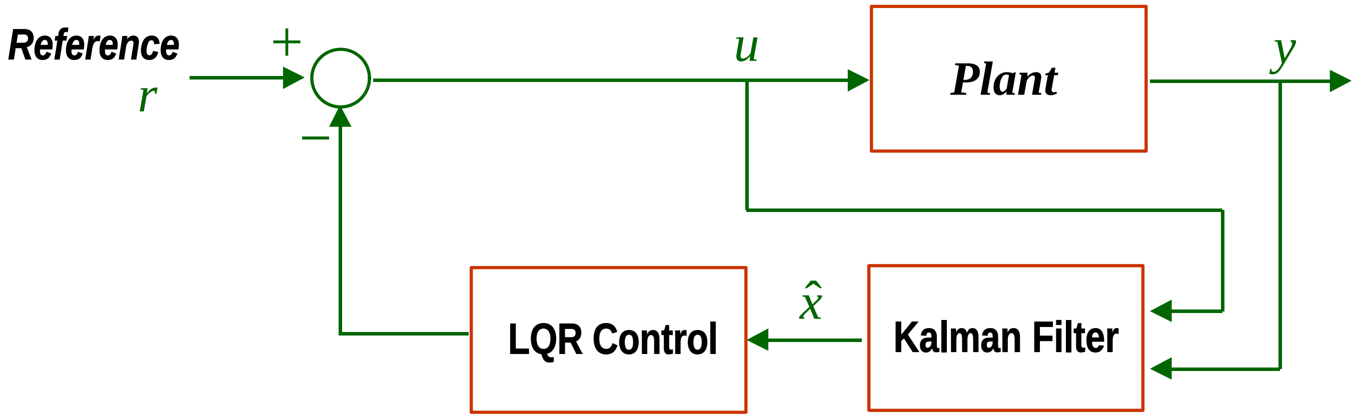
where

$$K_e = P_eC^{\mathrm{T}}R^{-1}, \qquad P_eA^{\mathrm{T}} + AP_e - P_eC^{\mathrm{T}}R_e^{-1}CP_e + Q_e = 0, \quad P_e > 0.$$

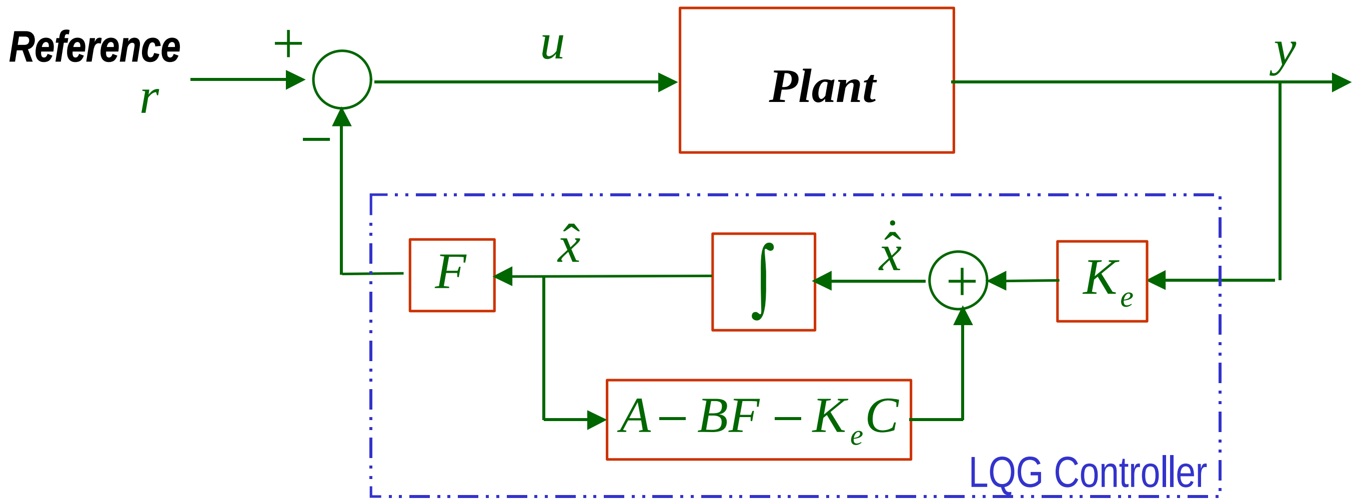
Step 3. The LQG control law is given by $u = -F\ \hat{x}$, i.e.,

$$\begin{cases} \dot{\hat{x}} = A\,\hat{x} + B\,u + K_e(y - C\,\hat{x}) \\ u = -F\ \hat{x} \end{cases} \Rightarrow \boxed{\begin{cases} \dot{\hat{x}} = (A - BF - K_eC)\,\hat{x} + K_e\,y \\ u = -F\ \hat{x} \end{cases}}$$

Block Diagram Implementation of LQG Control Law



More Detailed Block Diagram



Closed-Loop Dynamics of the Given Plant together with LQG Controller

Recall the plant: $\begin{cases} \dot{x} = Ax + Bu + v(t) \\ y = Cx + w(t) \end{cases}$ and the controller $\begin{cases} \dot{\hat{x}} = (A - BF - K_e C)\,\hat{x} + K_e\, y \\ u = -F\,\hat{x} + r \end{cases}$

We define a new variable $e = x - \hat{x}$ and thus

$$\begin{aligned} \dot{e} = \dot{x} - \dot{\hat{x}} &= Ax - BF\hat{x} + Br + v(t) - A\hat{x} + BF\hat{x} + K_e C\hat{x} - K_e Cx - K_e w(t) \\ &= A(x - \hat{x}) - K_e C(x - \hat{x}) + Br + v(t) - K_e w(t) = (A - K_e C)e + Br + v(t) - K_e w(t) \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= Ax + Bu + v(t) = Ax - BF\hat{x} + Br + v(t) = Ax - BF\,(x - e) + Br + v(t) \\ &= (A - BF\,)x + BFe + Br + v(t) \end{aligned}$$

Clearly, the closed-loop system is characterized by the following state space equation,

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{bmatrix} A - BF & BF \\ 0 & A - K_e C \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r + \tilde{v}, \quad \tilde{v} = \begin{pmatrix} v \\ v - K_e w \end{pmatrix} \\ y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + w \end{cases}$$

The closed-loop poles are given by $l\,(A - BF\,) \cup l\,(A - K_e C)$, which are stable.

Homework Assignment 1 (Hand in your solutions next week)

Recall the dynamical model for the cruise-control system,

$$\ddot{x}_1 + \frac{b}{m}\dot{x}_1 = \frac{u}{m}$$

where x_1 is the displacement of the car and u is the input force. For simplicity but no loss of generality, we assume that $m = 1$ and $b = 1$. However, in practical situations, there are always disturbances (due to rough road surfaces, etc.) presenting in the system. Thus, a more realistic model should be the following,

$$\ddot{x}_1 + \dot{x}_1 = u + \text{some noise}$$

Assume that only the displacement of the car can be measured, i.e., the measurement output

$$y = x_1 + w(t)$$

where $w(t)$ is the measurement noise and is assumed to be white and independent of the system noise in the ODE.

- Convert the ODE model of the system into a state space form $\dot{x} = Ax + Bu + v(t)$.

- Assume that $v(t)$ is nonexistent and all states of the plant are available for feedback. Find an LQR control law, which minimizes the following performance index:

$$J(x,u,Q,R) = \int_0^\infty (x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u) dt, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad R = 0.01$$

  What are the gain and phase margins resulting from your LQR design?

- Design a Kalman filter for the plant. Assume that both $v(t)$ and $w(t)$ have zero means and

$$E[v(t)v^{\mathrm{T}}(\tau)] = Q_e \delta(t-\tau), \;\; Q_e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \;\; E[w(t)w^{\mathrm{T}}(\tau)] = R_e \delta(t-\tau), \;\; R_e = 0.2$$

- Design an LQG control law, which minimizes the following performance index:

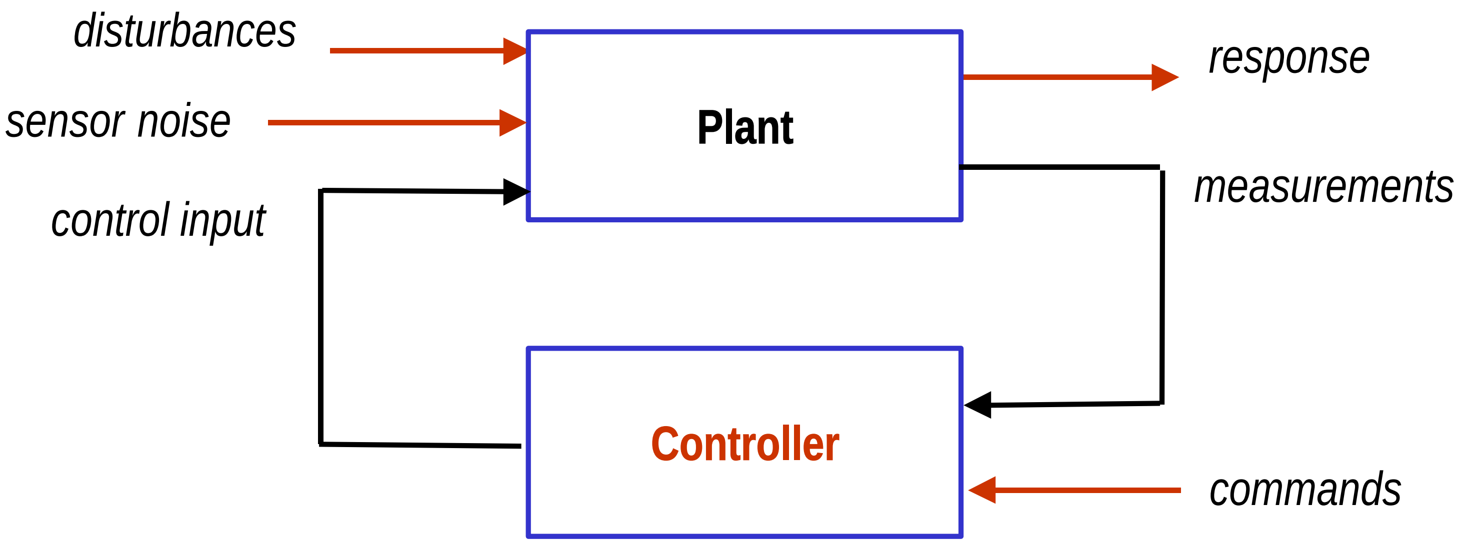
$$J = \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T (x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u) dt \right], \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \;\; R = 0.1$$

  What are the closed-loop eigenvalues? Simulate your design using SIMULINK with

$$r = 0, \quad x(0) = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \quad \hat{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Introduction to Robust Control

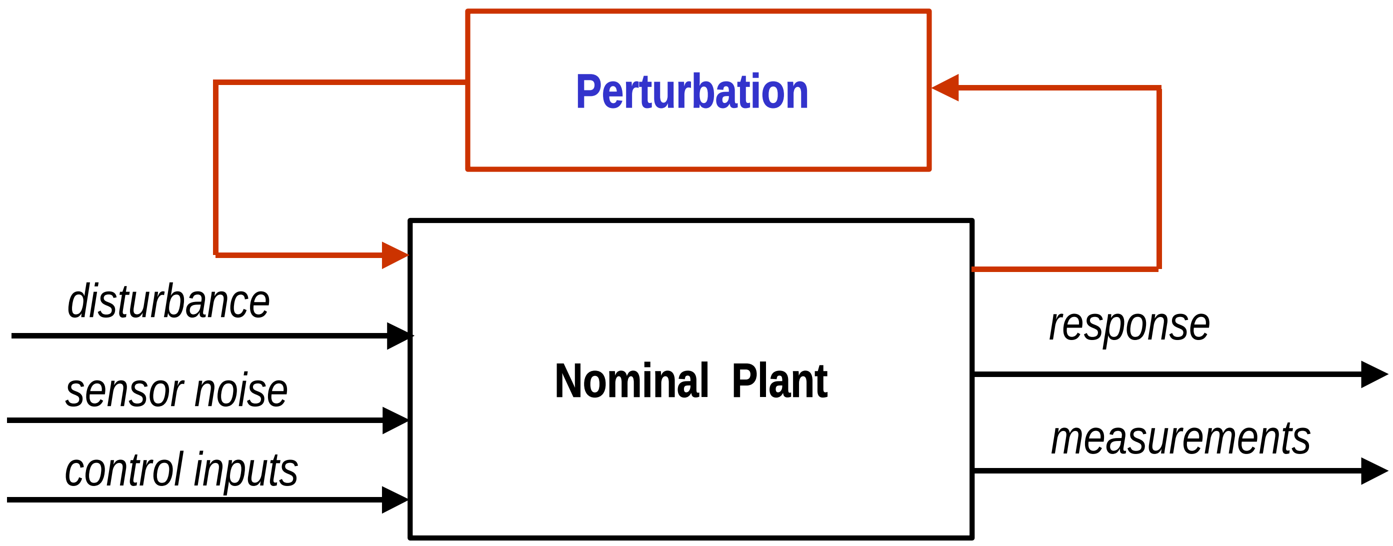
A Real Control Problem



Controller Objective: To provide desired responses in face of

- Uncertain plant dynamics + External inputs { *disturbances*, *sensor noise*, *control input* }

Representation of Uncertain Plant Dynamics



- Nominal Plant is a FDLTI System
- Perturbation is Member of Set of Possible Perturbations

Analysis Objectives

- **Nominal Performance Question (H_2 Optimal Control):**

  Are closed loop responses acceptable for disturbances? sensor noise? commands?

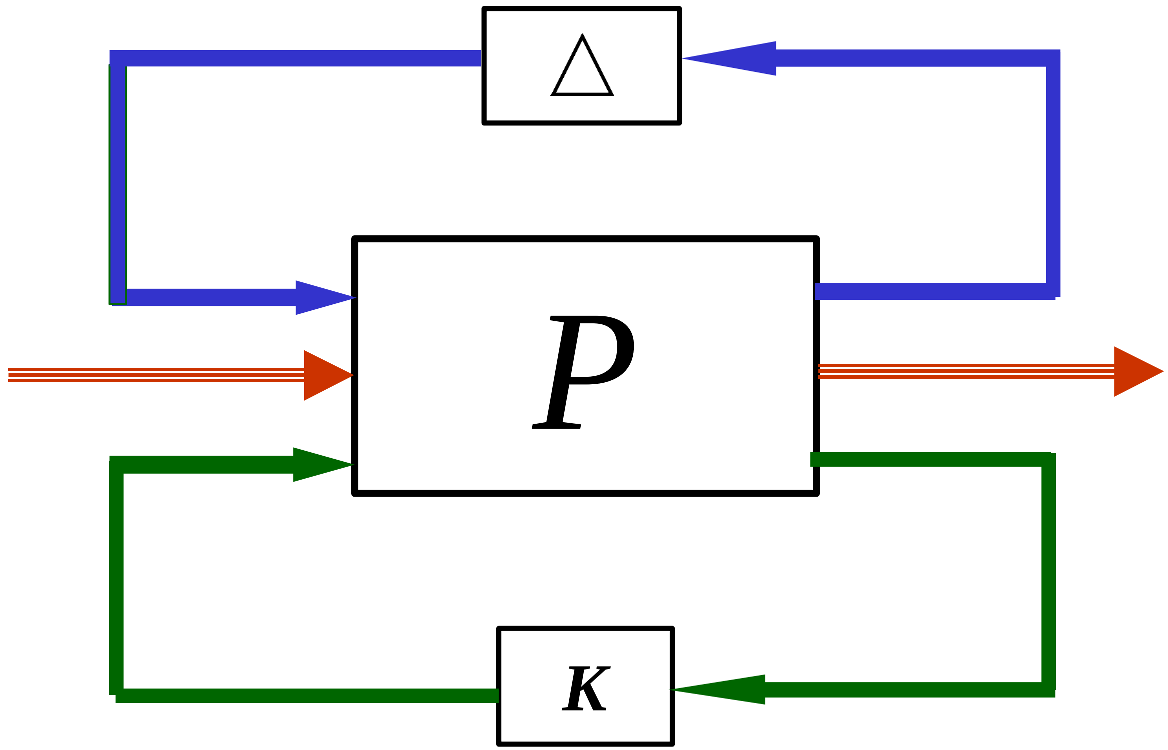
- **Robust Stability Question (H_∞ Optimal Control):**

  Is closed loop system stable for nominal plant? for all possible perturbations?

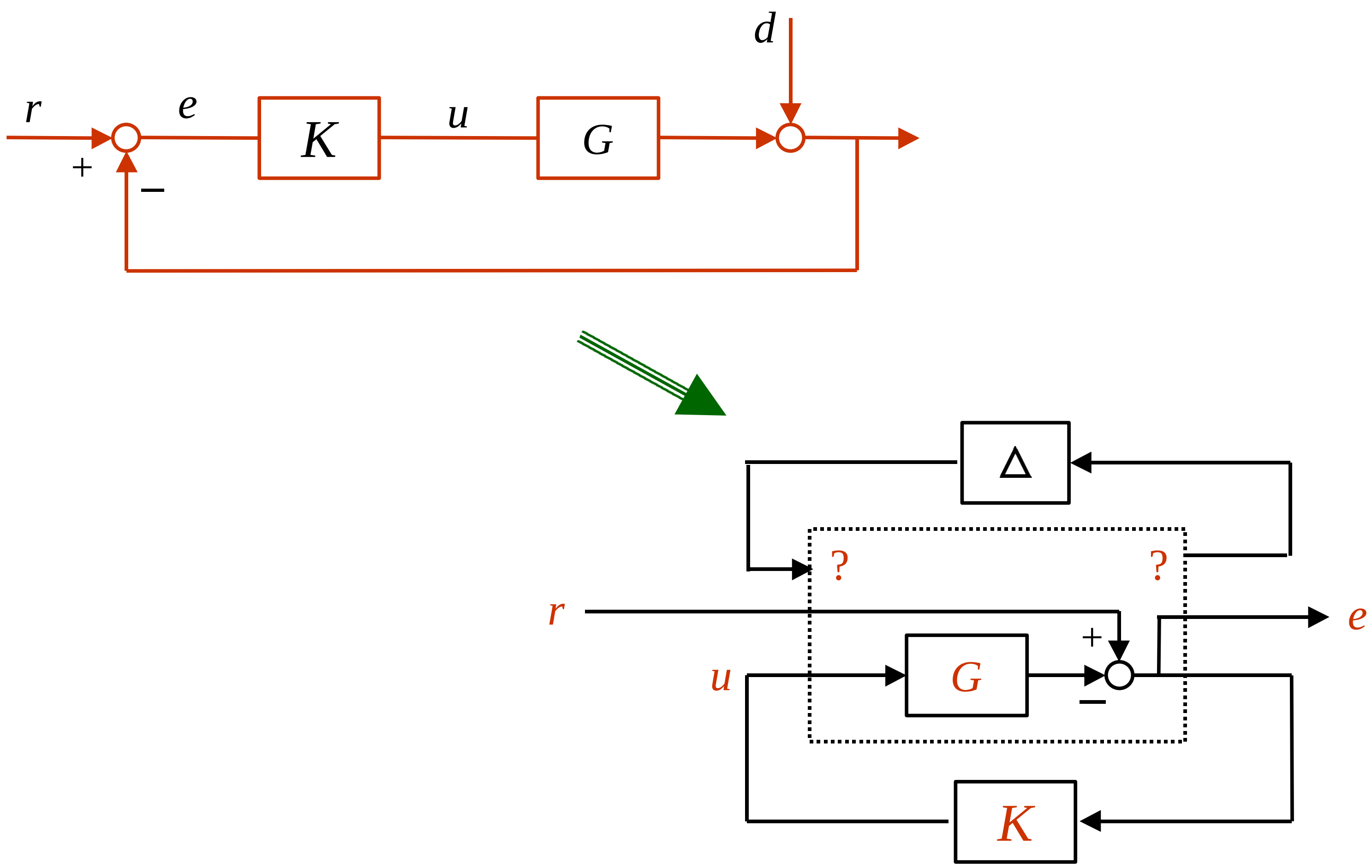
- **Robust Performance Question (Mixed H_2/H_∞ Optimal Control):**

  Are closed loop responses acceptable for all possible perturbations and all external inputs? Simultaneously?

Complete Picture of Robust Control Problem



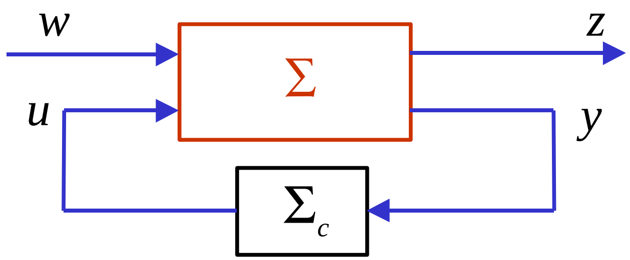
Standard Feedback Loops in Terms of General Interconnection Structure



H_2 and H_∞ Optimal Control

Introduction to the Problems

Consider a stabilizable and detectable linear time-invariant system Σ with a proper controller Σ_c



where

$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x + 0\,u + D_1\,w \\ z = C_2\,x + D_2\,u + 0\,w \end{cases} \qquad \Sigma_c : \begin{cases} \dot{v} = A_c\,v + B_c\,y \\ u = C_c\,v + D_c\,y \end{cases}$$

$\begin{cases} x \in \Re^n \Leftrightarrow \text{state variable} \\ y \in \Re^p \Leftrightarrow \text{measurement} \\ z \in \Re^q \Leftrightarrow \text{controlled output} \end{cases}$ & $\begin{array}{l} u \in \Re^m \Leftrightarrow \text{control input} \\ w \in \Re^l \Leftrightarrow \text{disturbance} \\ v \in \Re^k \Leftrightarrow \text{controller state} \end{array}$

The problems of H_2 and H_∞ optimal control are to design a proper control law Σ_c such that when it is applied to the given plant with disturbance, i.e., Σ, we have

- The resulting closed loop system is internally stable (this is necessary for any control system design).

- The resulting closed-loop transfer function from the disturbance w to the controlled output z, say, $T_{zw}(s)$, is as small as possible, i.e., the effect of the disturbance on the controlled output is minimized.

  - H_2 optimal control: the H_2-norm of $T_{zw}(s)$ is minimized.

  - H_∞ optimal control: the H_∞-norm of $T_{zw}(s)$ is minimized.

Note: A transfer function is a function of frequencies ranging from 0 to ∞. It is hard to tell if it is large or small. The common practice is to measure its norms instead. H_2-norm and H_∞-norm are two commonly used norms in measuring the sizes of a transfer function.

The Closed Loop Transfer Function from Disturbance to Controlled Output

Recall that

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x + 0\,u + D_1\,w \\ z = C_2\,x + D_2\,u + 0\,w \end{cases}$$

and

$$\Sigma_c: \begin{cases} \dot{v} = A_c\,v + B_c\,y \\ u = C_c\,v + D_c\,y \end{cases}$$

$$\Rightarrow \quad \dot{v} = A_c v + B_c(C_1 x + D_1 w) = A_c v + B_c C_1 x + B_c D_1 w$$

$$\Rightarrow \begin{cases} \dot{x} = A\,x + B\,(C_c v + D_c y) + E\,w \\ y = C_1\,x + D_1\,w \\ z = C_2\,x + D_2\,(C_c v + D_c y) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = Ax + BC_c v + BD_c(C_1 x + D_1 w) + Ew \\ z = C_2 x + D_2 C_c v + D_2 D_c (C_1 x + D_1 w) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = Ax + BC_c v + BD_c y + Ew \\ z = C_2 x + D_2 C_c v + D_2 D_c y \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = (A + BD_c C_1)x + BC_c v + (E + BD_c D_1)w \\ z = (C_2 + D_2 D_c C_1)x + D_2 C_c v + D_2 D_c D_1 w \end{cases}$$

$$\Rightarrow \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} A + BD_c C_1 & BC_c \\ B_c C_1 & A_c \end{bmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{bmatrix} E + BD_c D_1 \\ B_c D_1 \end{bmatrix} w = A_{\mathrm{cl}}\,\tilde{x} + B_{\mathrm{cl}} w \\ z = [C_1 + D_2 D_c C_1 \quad D_2 C_c] \begin{pmatrix} x \\ v \end{pmatrix} + D_2 D_c D_1\, w = C_{\mathrm{cl}}\,\tilde{x} + D_{\mathrm{cl}} w \end{cases}$$

Prepared by Ben M. Chen

Thus, the closed-loop transfer function from w to z is given by

$$T_{zw}(s) = C_{\mathrm{cl}}(sI - A_{\mathrm{cl}})^{-1}B_{\mathrm{cl}} + D_{\mathrm{cl}}$$

The resulting closed-loop system is internally stable if and only if the eigenvalues of

$$A_{\mathrm{cl}} = \begin{bmatrix} A + BD_c C_1 & BC_c \\ B_c C_1 & A_c \end{bmatrix}$$

are all in open left half complex plane.

Remark: For the state feedback case, $C_1 = I$ and $D_1 = 0$, i.e., all the states of the given system can be measured, Σ_c can then be reduced to $u = F\,x$ and the corresponding closed-loop transfer function is reduced to

$$T_{zw}(s) = (C_2 + D_2 F)(sI - A - BF)^{-1}E$$

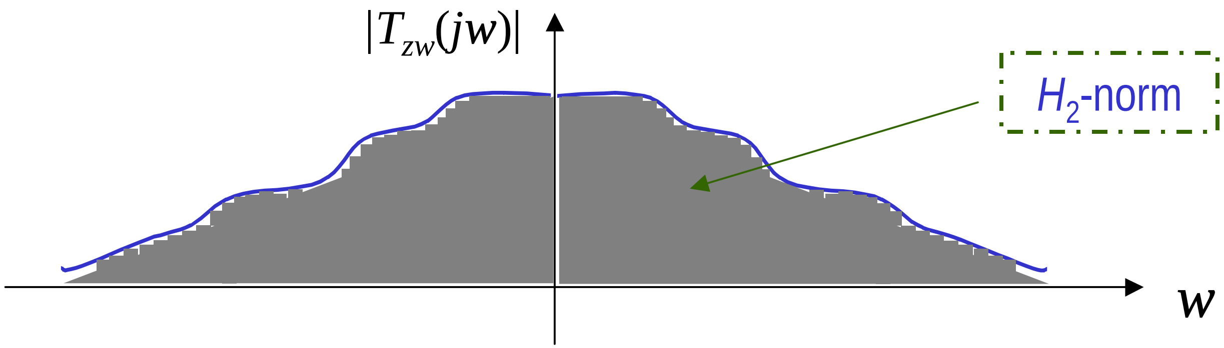
The closed-loop stability implies and is implied that $A + B\,F$ has stable eigenvalues.

H_2-norm and H_∞-norm of a Transfer Function

Definition: (H_2-norm) Given a stable and proper transfer function $T_{zw}(s)$, its H_2-norm is defined as

$$\|T_{zw}\|_2 = \left(\frac{1}{2\pi} \operatorname{trace} \left[\int_{-\infty}^{+\infty} T_{zw}(j\omega) T_{zw}(j\omega)^{\mathrm{H}} d\omega \right] \right)^{\frac{1}{2}}$$

Graphically,

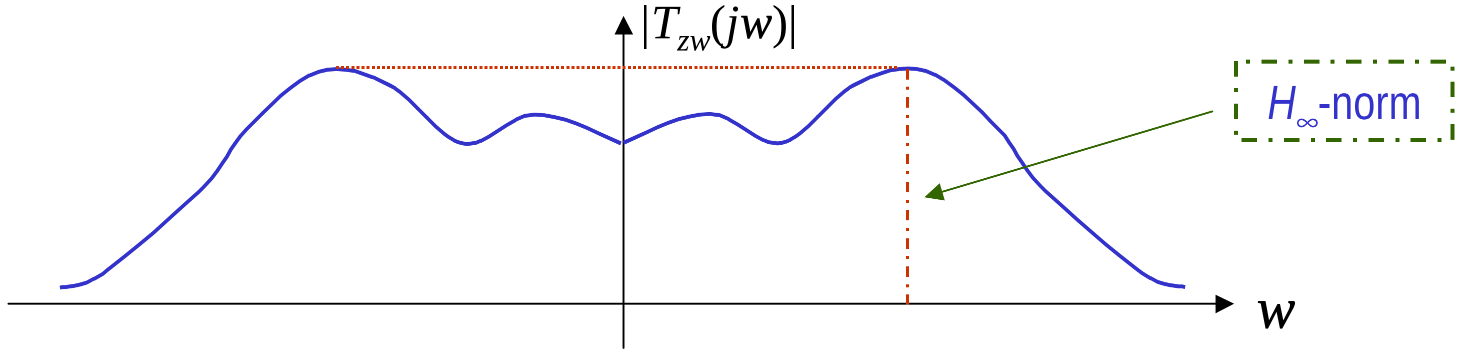


Note: The H_2-norm is the total energy corresponding to the impulse response of $T_{zw}(s)$. Thus, minimization of the H_2-norm of $T_{zw}(s)$ is equivalent to the minimization of the total energy from the disturbance w to the controlled output z.

Definition: (H_∞-norm) Given a stable and proper transfer function $T_{zw}(s)$, its H_∞-norm is defined as

$$\boxed{\|T_{zw}\|_\infty = \sup_{0 \le w < \infty} \sigma_{\max}\left[T_{zw}(jw)\right]}$$

where $\sigma_{\max}[T_{zw}(jw)]$ denotes the maximum singular value of $T_{zw}(jw)$. For a single-input-single-output transfer function $T_{zw}(s)$, it is equivalent to the magnitude of $T_{zw}(jw)$. Graphically,



Note: The H_∞-norm is the worst case gain in $T_{zw}(s)$. Thus, minimization of the H_∞-norm of $T_{zw}(s)$ is equivalent to the minimization of the worst case (gain) situation on the effect from the disturbance w to the controlled output z.

Infima and Optimal Controllers

Definition: (The infimum of H_2 optimization) The infimum of the H_2 norm of the closed-loop transfer matrix $T_{zw}(s)$ over all stabilizing proper controllers is denoted by γ_2^*, that is

$$\gamma_2^* := \inf\left\{ \| T_{zw} \|_2 \;\middle|\; \Sigma_c \text{ internally stabilizes } \Sigma \right\}.$$

Definition: (The H_2 optimal controller) A proper controller Σ_c is said to be an H_2 optimal controller if it internally stabilizes Σ and $\| T_{zw} \|_2 = \gamma_2^*$.

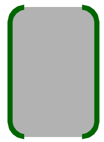
Definition: (The infimum of H_∞ optimization) The infimum of the H_∞-norm of the closed-loop transfer matrix $T_{zw}(s)$ over all stabilizing proper controllers is denoted by γ_∞^*, that is

$$\gamma_\infty^* := \inf\left\{ \| T_{zw} \|_\infty \;\middle|\; \Sigma_c \text{ internally stabilizes } \Sigma \right\}.$$

Definition: (The H_∞ γ-suboptimal controller) A proper controller Σ_c is said to be an H_∞ γ-suboptimal controller if it internally stabilizes Σ and $\| T_{zw} \|_\infty < \gamma \; (> \gamma_\infty^*)$

Critical Assumptions - Regular Case vs Singular Case

Most results in H_2 and H_∞ optimal control deal with a so-called a regular problem or regular case because it is simple. An H_2 or H_∞ optimal problem is said to be **regular** if the following conditions are satisfied,

1. D_2 is of maximal column rank, i.e., D_2 is a tall and full rank matrix 
2. The subsystem (A,B,C_2,D_2) has no invariant zeros on the imaginary axis;
3. D_1 is of maximal row rank, i.e., D_1 is a fat and full rank matrix 
4. The subsystem (A,E,C_1,D_1) has no invariant zeros on the imaginary axis;

An H_2 or H_∞ optimal problem is said to be **singular** if it is not regular, i.e., at least one of the above 4 conditions is satisfied.

Solutions to the State Feedback Problems - the Regular Case

The state feedback H_2 and H_∞ control problems are referred to the problems in which all the states of the given plant Σ are available for feedback. That is the given system is

$$\Sigma: \quad \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = \quad\; x \\ z = C_2\,x + D_2\,u \end{cases}$$

where (A, B) is stabilizable, D_2 is of maximal column rank and (A, B, C_2, D_2) has no invariant zeros on the imaginary axis.

In the state feedback case, we are looking for a static control law

$$\boxed{u = F\,x}$$

Solution to the Regular H_2 State Feedback Problem

Solve the following algebraic Riccati equation (H_2-ARE)

$$A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 - \left(PB + C_2^{\mathrm{T}}D_2\right)\left(D_2^{\mathrm{T}}D_2\right)^{-1}\left(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P\right) = 0$$

for a unique positive semi-definite solution $P \ge 0$. The H_2 optimal state feedback law is then given by

$$u = F\ x = -\left(D_2^{\mathrm{T}}D_2\right)^{-1}\left(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P\right)x$$

It can be showed that the resulting closed-loop system $T_{zw}(s)$ has the following property:

$$\| T_{zw} \|_2 = g_2^* .$$

It can also be showed that $g_2^* = \left[\mathrm{trace}\ (E^{\mathrm{T}}PE)\right]^{1/2}$. Note that the trace of a matrix is defined as the sum of all its diagonal elements.

Example: Consider a system characterized by

$$
\Sigma: \begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{E} w \\ y = x \\ z = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C_2} x + \underbrace{1}_{D_2} \cdot u \end{cases}
$$

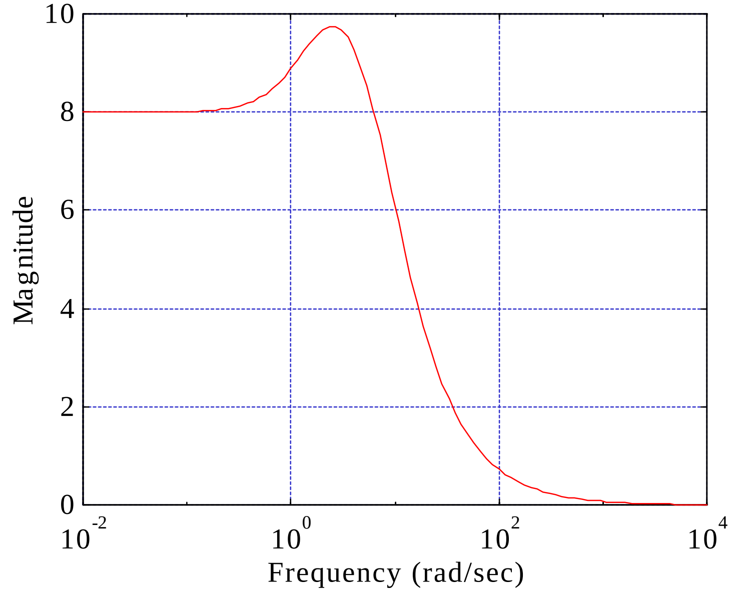
Solving the following H_2-ARE using MATLAB, we obtain a positive definite solution

$$
P = \begin{bmatrix} 144 & 40 \\ 40 & 16 \end{bmatrix}
$$

and

$$
F = \begin{bmatrix} -41 & -17 \end{bmatrix}
$$

The closed-loop magnitude response from the disturbance to the controlled output:



The optimal performance or infimum is given by

$$
\boxed{g_2^* = 19.1833}
$$

Classical Linear Quadratic Regulation (LQR) Problem is a Special Case of H_2 Control

It can be shown that the well-known LQR problem can be re-formulated as an H_2 optimal control problem. Consider a linear system,

$$\dot{x} = A\,x + B\,u, \quad x(0) = X_0$$

The LQR problem is to find a control law $u = F\,x$ such that the following index is minimized:

$$J = \int_0^\infty \left(x^\top Q\,x + u^\top R u\right) dt \ ,$$

where $Q \geq 0$ is a positive semi-definite matrix and $R > 0$ is a positive definite matrix. The problem is equivalent to finding a static state feedback H_2 optimal control law $u = F\,x$ for

$$\begin{cases} \dot{x} = A\,x + B\,u + X_0 w \\ y = x \\ z = \begin{bmatrix} 0 \\ Q^{1/2} \end{bmatrix} x + \begin{bmatrix} R^{1/2} \\ 0 \end{bmatrix} u \end{cases}$$

Solution to the Regular H_∞ State Feedback Problem

Given $\gamma > \gamma_\infty^*$, solve the following algebraic Riccati equation (H_∞-ARE)

$$A^\top P + PA + C_2^\top C_2 + PEE^\top P/\gamma^2 - (PB + C_2^\top D_2)\left(D_2^\top D_2\right)^{-1}(D_2^\top C_2 + B^\top P) = 0$$

for a unique positive semi-definite solution $P \geq 0$. The H_∞ γ-suboptimal state feedback law is then given by

$$u = F\,x = -(D_2^\top D_2)^{-1}\left(D_2^\top C_2 + B^\top P\right)x$$

The resulting closed-loop system $T_{zw}(s)$ has the following property: $\|T_{zw}\|_\infty < \gamma$.

Remark: The computation of the best achievable H_∞ attenuation level, i.e., γ_∞^*, is in general quite complicated. For certain cases, γ_∞^* can be computed exactly. There are cases in which γ_∞^* can only be obtained using some iterative algorithms. One method is to keep solving the H_∞-ARE for different values of γ until it hits γ_∞^* for which and any $\gamma < \gamma_\infty^*$, the H_∞-ARE does not have a solution. Please see the reference textbook by Chen (2000) for details.

Example: Again, consider the following system

$$\Sigma: \begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{E} w \\ y = x \\ z = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C_2} x + \underbrace{1}_{D_2} \cdot u \end{cases}$$

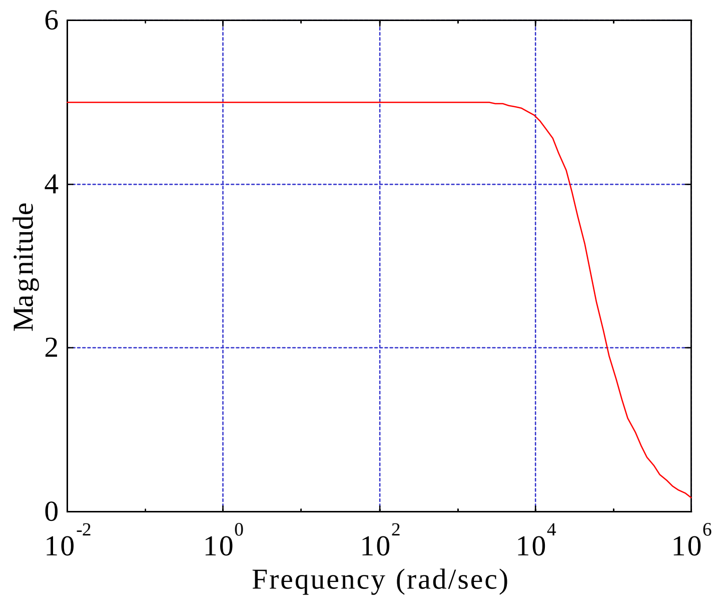
It can be showed that the best achievable H_∞ performance for this system is $\boxed{g_\infty^* = 5}$. Solving the following H_∞-ARE using MATLAB with $\gamma = 5.001$, we obtain a positive definite solution

$$P = \begin{bmatrix} 330111.5 & 110028.8 \\ 110028.8 & 26679.1 \end{bmatrix}$$

and

$$F = \begin{bmatrix} -110029.8 & -36680.1 \end{bmatrix}$$

The closed-loop magnitude response from the disturbance to the controlled output:



Clearly, the worse case gain, occurred at the low frequency is roughly equal to 5 (actually between 5 and 5.001)

Solutions to the State Feedback Problems - the Singular Case

Consider the following system again,

$$\Sigma: \quad \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = \quad x \\ z = C_2\,x + D_2\,u \end{cases}$$

where (A, B) is stabilizable, D_2 is not necessarily of maximal rank and (A, B, C_2, D_2) might have invariant zeros on the imaginary axis.

Solution to this kind of problems can be done using the following trick (or so-called a perturbation approach): Define a new controlled output

$$\tilde{z} = \begin{bmatrix} z \\ e\,x \\ e\,u \end{bmatrix} = \begin{bmatrix} C_2 \\ e\,I \\ 0 \end{bmatrix} x + \begin{bmatrix} D_2 \\ 0 \\ e\,I \end{bmatrix} u$$

small perturbations

Clearly, $z \propto \tilde{z}$ if $e = 0$.

Now let us consider the perturbed system

$$\tilde{\Sigma} \quad : \quad \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = x \\ \tilde{z} = \tilde{C}_2 x + \tilde{D}_2 u \end{cases} \qquad \text{where} \quad \tilde{C}_2 := \begin{bmatrix} C_2 \\ e\,I \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{D}_2 := \begin{bmatrix} D_2 \\ 0 \\ e\,I \end{bmatrix}$$

Obviously, $\tilde{D}_2$ is of maximal column rank and $(A, B, \tilde{C}_2, \tilde{D}_2)$ is free of invariant zeros for any $e > 0$. Thus, $\tilde{\Sigma}$ satisfies the conditions of the regular state feedback case, and hence we can apply the procedures for regular cases to $\tilde{\Sigma}$ find the H_2 and H_∞ control laws.

Example:

$$\Sigma: \quad \begin{aligned} \dot{x} &= \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y &= x \\ z &= \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0 \cdot u \end{aligned} \quad \Longrightarrow \quad \tilde{\Sigma}: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ z = \begin{bmatrix} 1 & 1 \\ e & 0 \\ 0 & e \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ e \end{bmatrix} u \end{cases}$$

Solution to the General H_2 State Feedback Problem

Given a small $e > 0$, Solve the following algebraic Riccati equation (H_2-ARE)

$$A^{\mathrm{T}}\tilde{P} + \tilde{P}A + \tilde{C}_2^{\mathrm{T}}\tilde{C}_2 - \left(\tilde{P}B + \tilde{C}_2^{\mathrm{T}}\tilde{D}_2\right)\left(\tilde{D}_2^{\mathrm{T}}\tilde{D}_2\right)^{-1}\left(\tilde{D}_2^{\mathrm{T}}\tilde{C}_2 + B^{\mathrm{T}}\tilde{P}\right) = 0$$

for a unique positive semi-definite solution $\tilde{P} \geq 0$. Obviously, $\tilde{P}$ is a function of e. The H_2 optimal state feedback law is then given by

$$u = \tilde{F}\ x = -\left(\tilde{D}_2^{\mathrm{T}}\tilde{D}_2\right)^{-1}\left(\tilde{D}_2^{\mathrm{T}}\tilde{C}_2 + B^{\mathrm{T}}\tilde{P}\right)x$$

It can be showed that the resulting closed-loop system $T_{zw}(s)$ has the following property:

$$\| T_{zw} \|_2 \rightarrow g_2^* \quad \text{as } e \rightarrow 0$$

It can also be showed that

$$\left[\mathrm{trace}(E^{\mathrm{T}}\tilde{P}E)\right]^{1/2} \rightarrow g_2^* \quad \text{as } e \rightarrow 0.$$

Example: Consider a system characterized by

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0 \cdot u \end{cases}$$

Solving the following H_2-ARE using MATLAB

with $e = 1$, we obtain

$$\tilde{P} = \begin{bmatrix} 186.1968 & 46.2778 \\ 46.2778 & 18.2517 \end{bmatrix}, \quad F = \begin{bmatrix} -46.2778 & -18.2517 \end{bmatrix}$$

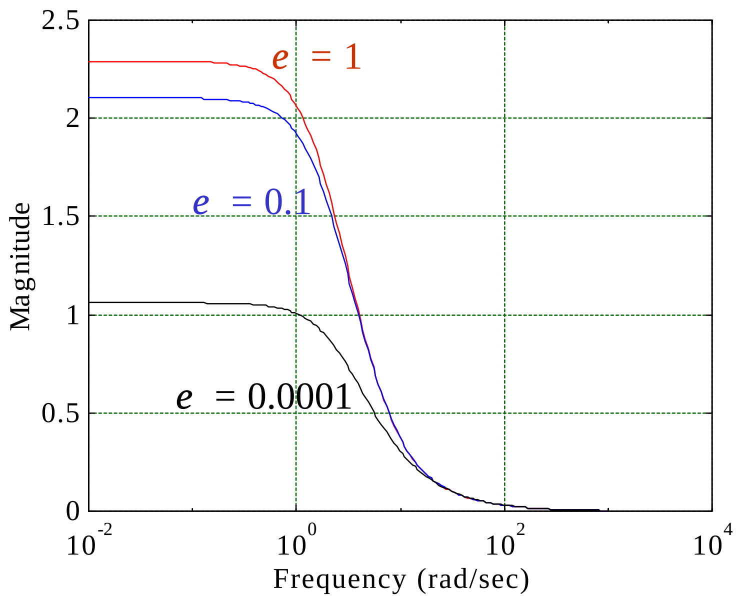
- $e = 0.1$

$$\tilde{P} = \begin{bmatrix} 21.2472 & 4.9311 \\ 4.9311 & 1.8975 \end{bmatrix}, \quad F = \begin{bmatrix} -49.3111 & -18.9748 \end{bmatrix}$$

- $e = 0.0001$

$$\tilde{P} = \begin{bmatrix} 1.6701 & 0.0424 \\ 0.0424 & 0.0112 \end{bmatrix}, \quad F = \begin{bmatrix} -423.742 & -112.222 \end{bmatrix}$$

The closed-loop magnitude response from the disturbance to the controlled output:



The optimal performance or infimum is given by

$$\boxed{\gamma_2^* = 1.225}$$

Solution to the General $H_{¥}$ State Feedback Problem

$$\tilde{C}_2 := \begin{bmatrix} C_2 \\ e\, I \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{D}_2 := \begin{bmatrix} D_2 \\ 0 \\ e\, I \end{bmatrix}$$

Step 1: Given a $\gamma > \gamma_\infty^*$, choose $e = 1$.

Step 2: Define the corresponding $\tilde{C}_2$ and $\tilde{D}_2$

Step 3: Solve the following algebraic Riccati equation (H_∞-ARE)

$$A^{\mathrm{T}}\tilde{P} + \tilde{P}A + \tilde{C}_2^{\mathrm{T}}\tilde{C}_2 + \tilde{P}EE^{\mathrm{T}}\tilde{P}/g^2 - (\tilde{P}B + \tilde{C}_2^{\mathrm{T}}\tilde{D}_2)\left(\tilde{D}_2^{\mathrm{T}}\tilde{D}_2\right)^{-1}(\tilde{D}_2^{\mathrm{T}}\tilde{C}_2 + B^{\mathrm{T}}\tilde{P}) = 0$$

for $\tilde{P}$.

Step 4: If $\tilde{P} > 0$, go to Step 5. Otherwise, reduce the value of e and go to Step 2.

Step 5: Compute the required state feedback control law

$$u = \tilde{F}\, x = -(\tilde{D}_2^{\mathrm{T}}\tilde{D}_2)^{-1}\left(\tilde{D}_2^{\mathrm{T}}\tilde{C}_2 + B^{\mathrm{T}}\tilde{P}\right)x$$

It can be showed that the resulting closed-loop system $T_{zw}(s)$ has: $\|T_{zw}\|_\infty < g$.

Example: Again, consider the following system

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0 \cdot u \end{cases}$$

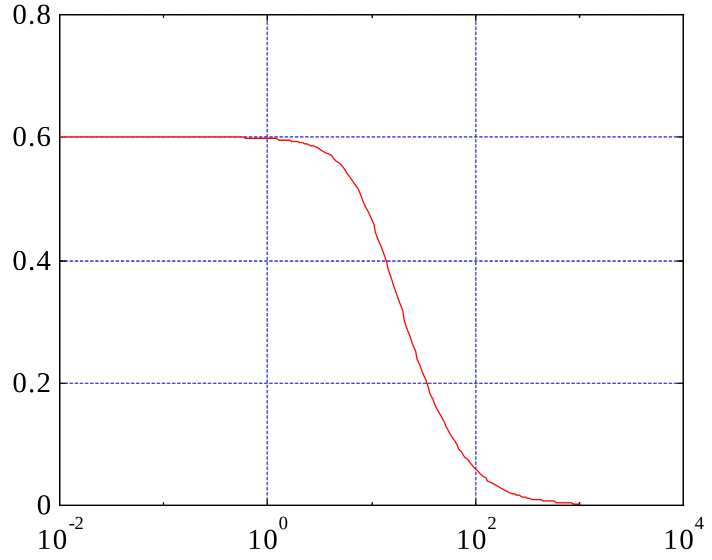
It can be showed that the best achievable H_∞ performance for this system is $\boxed{g_\infty^* = 0.5}$. Solving the following H_∞-ARE using MATLAB with $\gamma = 0.6$ and $e = 0.001$, we obtain a positive definite solution

$$P = \begin{bmatrix} 15.1677 & 0.9874 \\ 0.9874 & 0.0981 \end{bmatrix}$$

and

$$F = \begin{bmatrix} -987.363 & -98.1161 \end{bmatrix}$$

The closed-loop magnitude response from the disturbance to the controlled output:



Clearly, the worse case gain, occurred at the low frequency is slightly less than 0.6. The design specification is achieved.

Solutions to Output Feedback Problems - the Regular Case

Recall the system with measurement feedback, i.e.,

$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x \qquad\quad + D_1\,w \\ z = C_2\,x + D_2\,u \end{cases}$$

where (A, B) is stabilizable and (A, C_1) is detectable. Also, it satisfies the following regularity assumptions:

1. D_2 is of maximal column rank, i.e., D_2 is a tall and full rank matrix
2. The subsystem (A,B,C_2,D_2) has no invariant zeros on the imaginary axis;
3. D_1 is of maximal row rank, i.e., D_1 is a fat and full rank matrix
4. The subsystem (A,E,C_1,D_1) has no invariant zeros on the imaginary axis;

Solution to the Regular H_2 Output Feedback Problem

Solve the following algebraic Riccati equation (H_2-ARE)

$$A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 - \left(PB + C_2^{\mathrm{T}}D_2\right)\left(D_2^{\mathrm{T}}D_2\right)^{-1}\left(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P\right) = 0$$

for a unique positive semi-definite solution $P \geq 0$, and the following ARE

$$QA^{\mathrm{T}} + AQ + EE^{\mathrm{T}} - \left(QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}}\right)\left(D_1D_1^{\mathrm{T}}\right)^{-1}\left(D_1E^{\mathrm{T}} + C_1Q\right) = 0$$

for a unique positive semi-definite solution $Q \geq 0$. The H_2 optimal output feedback law is then given by

$$\Sigma_c : \begin{cases} \dot{v} = \left(A + BF + KC_1\right)v - K\ y \\ u = F\ v \end{cases}$$

where $F = -(D_2^{\mathrm{T}}D_2)^{-1}\left(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P\right)$ and $K = -\left(QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}}\right)\left(D_1D_1^{\mathrm{T}}\right)^{-1}$

Furthermore,

$$g_2^* = \left\{\text{trace}\,(E^{\mathrm{T}}PE) + \text{trace}\left[\left(A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2\right)Q\ \right]\right\}^{1/2}$$

Example: Consider a system characterized by

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \qquad\qquad + 1 \cdot w \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 1 \cdot u \end{cases}$$

Solving the following H_2-AREs using MATLAB, we obtain

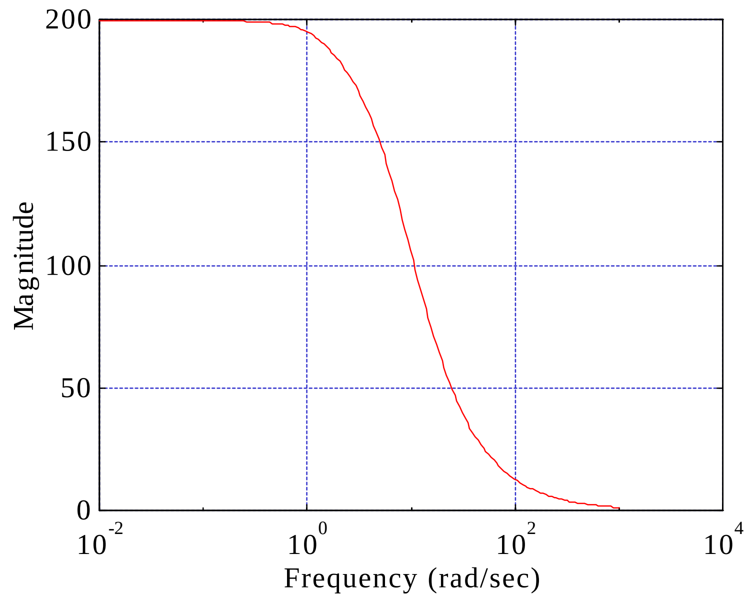
$$P = \begin{bmatrix} 144 & 40 \\ 40 & 16 \end{bmatrix} \qquad F = \begin{bmatrix} -41 & -17 \end{bmatrix}$$

$$Q = \begin{bmatrix} 49.7778 & 23.3333 \\ 23.3333 & 14.0000 \end{bmatrix} \qquad K = \begin{bmatrix} -24.3333 \\ -16.0000 \end{bmatrix}$$

and an output feedback control law,

$$\Sigma_c : \begin{cases} \dot{v} = \begin{bmatrix} 5 & -22.3333 \\ -38 & -29 \end{bmatrix} v + \begin{bmatrix} 24.3333 \\ 16 \end{bmatrix} y \\ u = \begin{bmatrix} -41 & -17 \end{bmatrix} v \end{cases}$$

The closed-loop magnitude response from the disturbance to the controlled output:



The optimal performance or infimum is given by

$$\boxed{g_2^* = 347.3}$$

Solution to the Regular $H_{¥}$ Output Feedback Problem

Given a $\gamma > \gamma_\infty^*$, solve the following algebraic Riccati equation (H_∞-ARE)

$$A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 + PEE^{\mathrm{T}}P/g^2 - (PB + C_2^{\mathrm{T}}D_2)\left(D_2^{\mathrm{T}}D_2\right)^{-1}(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P) = 0$$

for a unique positive semi-definite solution $P \geq 0$, and the following ARE

$$QA^{\mathrm{T}} + AQ + EE^{\mathrm{T}} + QC_2^{\mathrm{T}}C_2Q/g^2 - \left(QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}}\right)\left(D_1D_1^{\mathrm{T}}\right)^{-1}\left(D_1E^{\mathrm{T}} + C_1Q\right) = 0$$

for a unique positive semi-definite solution $Q \geq 0$. In fact, these P and Q satisfy the so-called coupling condition: $r(PQ) < g^2$. The H_∞ γ-suboptimal output feedback law is then given by

$$\Sigma_c : \begin{cases} \dot{v} = A_c\, v + B_c\, y \\ u = C_c\, v \end{cases}$$

where $A_c = A + g^{-2}EE^TP + BF + \left(I - g^{-2}QP\right)^{-1}K\left(C_1 + g^{-2}D_1E^TP\right)$

$$B_c = -\left(I - g^{-2}QP\right)^{-1}K, \qquad C_c = F.$$

and where $F = -(D_2^{\mathrm{T}}D_2)^{-1}\left(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P\right), \quad K = -\left(QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}}\right)\left(D_1D_1^{\mathrm{T}}\right)^{-1}.$

Example: Consider a system characterized by

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x + 1 \cdot w \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 1 \cdot u \end{cases}$$

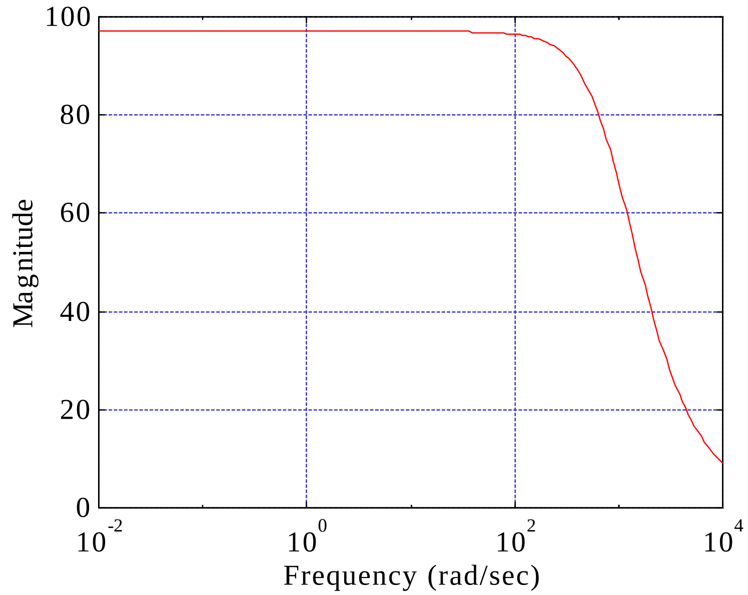
It can be showed that the best achievable H_∞ performance for this system is $\boxed{g_\infty^* = 96.32864}$.

Solving the following H_∞-AREs using MATLAB with $\gamma = 97$, we obtain

$$P = \begin{bmatrix} 144.353 & 40.1168 \\ 40.1168 & 16.0392 \end{bmatrix} \qquad Q = \begin{bmatrix} 49.8205 & 23.3556 \\ 23.3556 & 14.0118 \end{bmatrix}$$

$$\Sigma_c : \begin{cases} \dot{v} = \begin{bmatrix} -38.814 & -1848.66 \\ -59.414 & -914.112 \end{bmatrix} v + \begin{bmatrix} 1836.58 \\ 894.227 \end{bmatrix} y \\ u = \begin{bmatrix} -41.116 & -17.039 \end{bmatrix} v \end{cases}$$

The closed-loop magnitude response from the disturbance to the controlled output:



Clearly, the worse case gain, occurred at the low frequency is slightly less than 97. The design specification is achieved.

Solutions to the Output Feedback Problems - the Singular Case

For general systems for which the regularity conditions are not satisfied, it can be solved again using the so-called perturbation approach. We define a new controlled output:

$$\tilde{z} = \begin{bmatrix} z \\ e\,x \\ e\,u \end{bmatrix} = \begin{bmatrix} C_2 \\ e\,I \\ 0 \end{bmatrix} x + \begin{bmatrix} D_2 \\ 0 \\ e\,I \end{bmatrix} u$$

and new matrices associated with the disturbance inputs:

$$\tilde{E} = [\, E \quad e\,I \quad 0 \,] \qquad \text{and} \qquad \tilde{D}_1 = [\, D_1 \quad 0 \quad e\,I \,].$$

The H_2 and H_∞ control problems for singular output feedback case can be obtained by solving the following perturbed regular system with sufficiently small e :

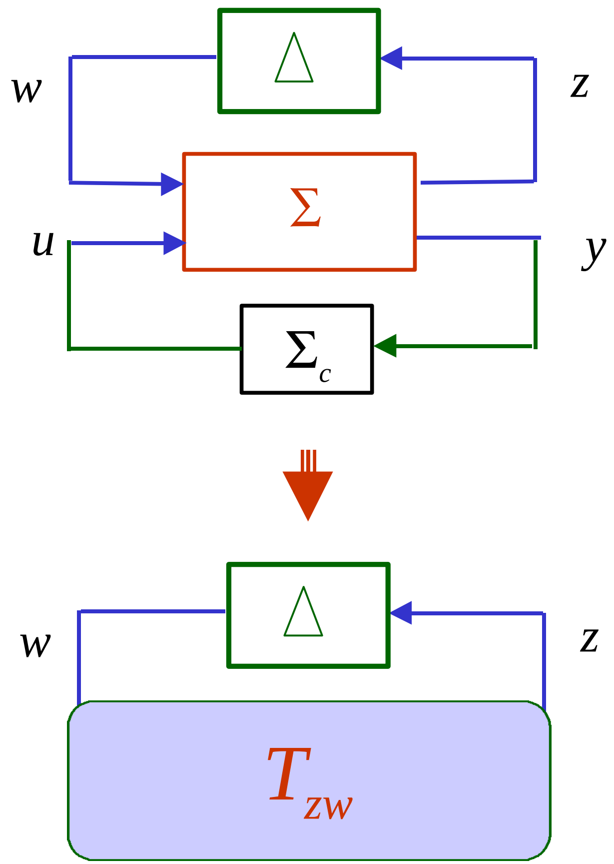
$$\tilde{\Sigma}: \quad \begin{cases} \dot{x} = A\,x + B\,u + \tilde{E}\,\tilde{w} \\ y = C_1\,x + \tilde{D}_1\,\tilde{w} \\ \tilde{z} = \tilde{C}_2\,x + \tilde{D}_2\,u \end{cases}$$

Remark: Perturbation approach might have serious numerical problems!

Some Robust Control Problems

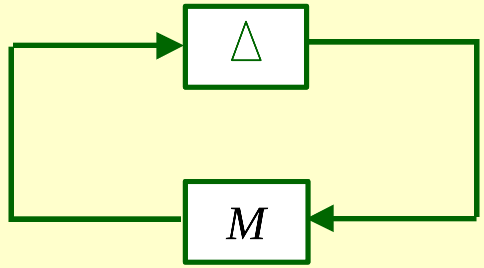
Robust Stabilization of Systems with Unstructured Uncertainties

Consider an uncertain plant with an unstructured perturbation,



Small Gain Theory (!)

If $\triangle$ is stable and $\|\Delta\|_\infty \cdot \|M\|_\infty < 1$, then the interconnected system is stable.

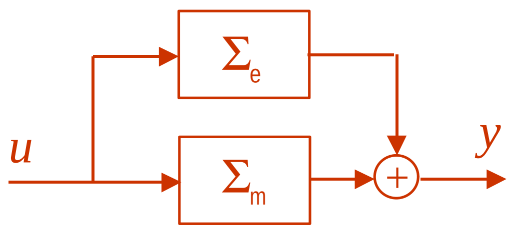


Assume $\| T_{zw} \|_\infty < g$. Then the system with unstructured uncertainty if

$$\| T_{zw} \|_\infty \cdot \| \Delta \|_\infty < g \cdot \| \Delta \|_\infty < 1 \quad \Rightarrow \quad \| \Delta \|_\infty < \frac{1}{g}$$

Robust Stabilization with Additive Perturbation

Consider an uncertain plant with additive perturbations,



Σ_m has a transfer function $G_m(s) = C_m(sI - A_m)^{-1} B_m + D_m$

Σ_e is an unknown perturbation.

Σ_m and $\Sigma_m + \Sigma_e$ have same number of unstable poles.

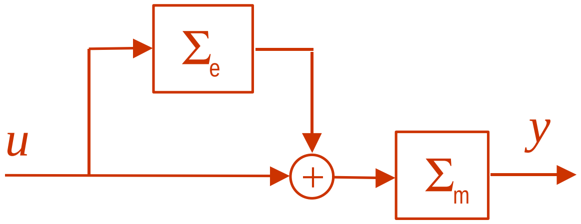
Given a $\gamma_a > 0$, the problem of robust stabilization for plants additive perturbations is to find a proper controller such that when it is applied to the uncertain plant, the resulting closed-loop system is stable for all possible perturbations with their L_∞-norm $\le \gamma_a$. (The definition of L_∞-norm is the same as that of H_∞-norm except for L_∞-norm, the system need not be stable.)

Such a problem is equivalent to find an H_∞ γ-suboptimal control law (with $\gamma = 1/\gamma_a$) for

$$\Sigma_{add}: \begin{cases} \dot{x} = A_m\, x + B_m\, u + 0\, w \\ y = C_m\, x + D_m\, u + I\, w \\ z = \ 0\ \ x + \ I\ \ u \end{cases}$$

Robust Stabilization with Multiplicative Perturbation

Consider an uncertain plant with multiplicative perturbations,



Σ_m has a transfer function $G_m(s) = C_m(sI - A_m)^{-1}B_m + D_m$

Σ_e is an unknown perturbation.

Σ_m and $\Sigma_m \times \Sigma_e$ have same number of unstable poles.

Given a $\gamma_m > 0$, the problem of robust stabilization for plants multiplicative perturbations is to find a proper controller such that when it is applied to the uncertain plant, the resulting closed-loop system is stable for all possible perturbations with their L_∞-norm $\le \gamma_m$. Again, such a problem is equivalent to find an H_∞ γ-suboptimal control law (with $\gamma = 1/\gamma_m$) for the following system,

$$\Sigma_{\text{multi}} : \quad \begin{cases} \dot{x} = A_m\, x + B_m\, u + B_m\, w \\ y = C_m\, x + D_m\, u + D_m\, w \\ z = \ 0 \ \ x + \ I \ \ u \end{cases}$$

Homework Assignment 2 (Hand in your solutions next week)

Rewrite the cruise-control system in Homework Assignment 1 as follows,

$$\Sigma: \quad \begin{cases} \dot{x} = A\,x + B\,u + E\,\tilde{w} \\ y = C_1\,x + D_1\,\tilde{w} \\ z = C_2\,x + D_2\,u \end{cases}$$

where

$$\tilde{w} = \begin{pmatrix} v(t) \\ w(t) \end{pmatrix}, \quad \text{and} \quad z = x_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} x, \text{ i.e., the speed of the car.}$$

- What is the best achievable H_2-norm of the closed-loop system from $\tilde{w}$ to z ? Design an H_2 suboptimal controller such that the H_2-norm of the resulting closed-loop system is reasonably close to the optimal value. Plot the singular values of the closed-loop system.

- What is the best achievable H_∞-norm of the closed-loop system from $\tilde{w}$ to z ? Design an H_∞ suboptimal controller such that the H_∞-norm of the resulting closed-loop system is reasonably close to the optimal value. Plot the singular values of the closed-loop system.

Loop Transfer Recovery Design

Is an LQG Controller Robust?

It is now well-known that the linear quadratic regulator (LQR) has very impressive robustness properties, including guaranteed infinite gain margins and 60 degrees phase margins in all channels. The result is only valid, however, for the full state feedback case. If observers or Kalman filters (i.e., LQG regulators) are used in implementation, no guaranteed robustness properties hold. Still worse, the closed-loop system may become unstable if you do not design the observer of Kalman filter properly. The following example given in Doyle (1978) shows the unrobustness of the LQG regulators.

Example: Consider the following system characterized by

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v, \quad y = [1 \quad 0]\, x + w$$

where x, u and y denote the usual states, control input and measured output, and where w and v are white noises with intensities 1 and $\sigma > 0$, respectively.

The LQG controller consists of an LQR control law + a Kalman filter.

LQR Design: Suppose we wish to minimize the performance index

$$J = \int_0^\infty (x^\mathrm{T} Q x + u^\mathrm{T} R u) dt, \qquad R = 1, \; Q = q \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \quad 1], \; q > 0$$

It is known that the state feedback law $u = -F\,x$ which minimize the performance index J is given by

$$F = R^{-1} B^\mathrm{T} P, \qquad PA + A^\mathrm{T} P - PBR^{-1} B^\mathrm{T} P + Q = 0, \quad P > 0.$$

For this particular example, we can obtain a closed-form solution,

$$F = (2 + \sqrt{4 + q}) [1 \quad 1] = f [1 \quad 1].$$

It can be verified that the open loop of LQ regulator with any $q > 0$ has an infinite gain margin and a phase margin over 105 degrees. Thus, it is very robust.

It can also be shown that the Kalman filter gain for this problem can be expressed as,

$$K = (2 + \sqrt{4 + s}\ ) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which together with the LQR law result an LQG controller,

$$\begin{cases} \dot{\hat{x}} = (A - BF - KC)\,\hat{x} + K\,y \\ u = -F\,\hat{x} \end{cases} \quad \text{or} \quad u = -F(sI - A + BF + KC)^{-1} K\,y$$

Suppose that the resulting closed-loop controller has a scalar gain 1 + *e* (nominally unity) associated with the input matrix, i.e.,

$$\text{the actual input matrix} = (1 + e)B = \begin{bmatrix} 0 \\ 1 + e \end{bmatrix}$$

Tedious manipulations show that the characteristic function of the closed-loop system comprising the given system an the LQG controller is given by

$$\boxed{K(s) = \Pi \cdot s^4 + \Theta \cdot s^3 + \Omega \cdot s^2 + (2e\,k\,f + k + f - 4)s + (1 - e\,k\,f)}$$

A necessary condition for stability is that

$$2e\,kf + k + f - 4 > 0 \quad \text{and} \quad 1 - e\,kf > 0$$

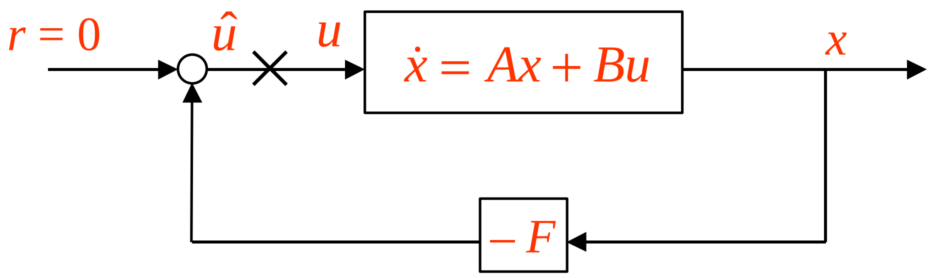
It is easy to see that for sufficient large q and s, the closed-loop could be unstable for a small perturbation in B in either direction. For instance, let us choose $q = s = 60$. Then it is simple to verify the closed-loop system remains stable only when $-0.08 < e < 0.01$.

The above example shows that the LQG controller is not robust at all!

What is wrong?

The answer is that the open-loop transfer function of the LQR design and the open-loop transfer function of the LQG design are totally different and thus, all the nice properties associated with the LQR design vanish in the LQG controller. It can be seen more clearly from the precise mathematical expressions of these two open-loop transfer functions, and this leads to the birth of the so-called Loop Transfer Recovery technique.

Open-Loop Transfer Function of LQR



Open-loop transfer function: When the loop is broken at the input point of the plant, i.e., the point marked ×, we have

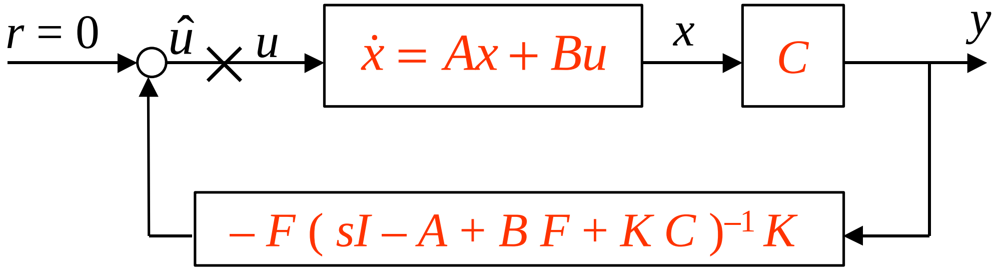
$$\hat{u} = -F(sI - A)^{-1}Bu$$

Thus, the loop transfer matrix from u to $-\hat{u}$ is given by

$$L_t(s) = F(sI - A)^{-1}B$$

We have learnt from our previous lectures that the open loop transfer $L_t(s)$ have very impressive properties if the gain matrix F comes from LQR design.

Open-Loop Transfer Function of LQG



Open-loop transfer function: When the loop is broken at the input point of the plant, i.e., the point marked ×, we have

$$\hat{u} = -F\,(sI - A + BF + KC\,)^{-1} KC\,(sI - A)^{-1} Bu$$

Thus, the loop transfer matrix from u to $-\hat{u}$ is given by

$$L_o(s) = F\,(sI - A + BF + KC\,)^{-1} KC\,(sI - A)^{-1} B$$

Clearly, $L_t(s)$ and $L_o(s)$ are very different and that is why LQG in general does not have nice properties as LQR does.

Loop Transfer Recovery

The above problem can be fixed by choosing an appropriate Kalman filter gain matrix K such that $L_t(s)$ and $L_o(s)$ are exactly identical or almost matched over a certain range of frequencies. Such a technique is called **Loop Transfer Recovery**.

The idea was first pointed out by Doyle and Stein in 1979. They had given a sufficient condition under which $L_o(s) = L_t(s)$. They had also develop a procedure to design the Kalman filter gain matrix K in terms of a tuning parameter q such that the resulting $L_o(s) \to L_t(s)$ as $q \to \infty$, for the invertible and minimum phase systems .

Doyle-Stein Conditions: It can be shown that $L_o(s)$ and $L_t(s)$ are identical if the observer gain K satisfies

$$K(I + C\Phi K)^{-1} = B(C\Phi B)^{-1}, \quad \Phi = (sI - A)^{-1}$$

which is equivalent to $B = 0$ (prove it!). Thus, it is impractical.

Classical LTR Design

The following procedure was proposed by Doyle and Stein in 1979 for left invertible and minimum phase systems: Define

$$Q_q = Q_0 + q^2 BVB^{\mathrm{T}}, \quad R = R_0$$

where Q_0 and R_0 are noise intensities appropriate for the nominal plant (in fact, Q_0 can be chosen as a zero matrix and $R_0 = I$), and V is any positive definite symmetric matrix (V can be chosen as an identity matrix). Then the observer (or Kalman filter) gain is given by

$$K = PC^{\mathrm{T}} R^{-1}$$

where P is the positive definite solution of

$$AP + PA^{\mathrm{T}} + Q_q - PC^{\mathrm{T}} R^{-1} CP = 0$$

It can be shown that the resulting open-loop transfer function $L_o(s)$ from the above observer or Kalman filter has

$$L_o(s) \to L_t(s), \quad as \;\; q \to \infty.$$

Example: Consider a given plant characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 35 \\ -61 \end{bmatrix} v, \quad y = [2 \quad 1]\, x + w$$

with $E[v(t)] = E[w(t)] = 0$ and $E[v(t)v(t\ )] = E[w(t)w(t\ )] = d\,(t - t\ )$.

This system is of minimum phase with one invariant zero at $s = -2$. The LQR control law is given by

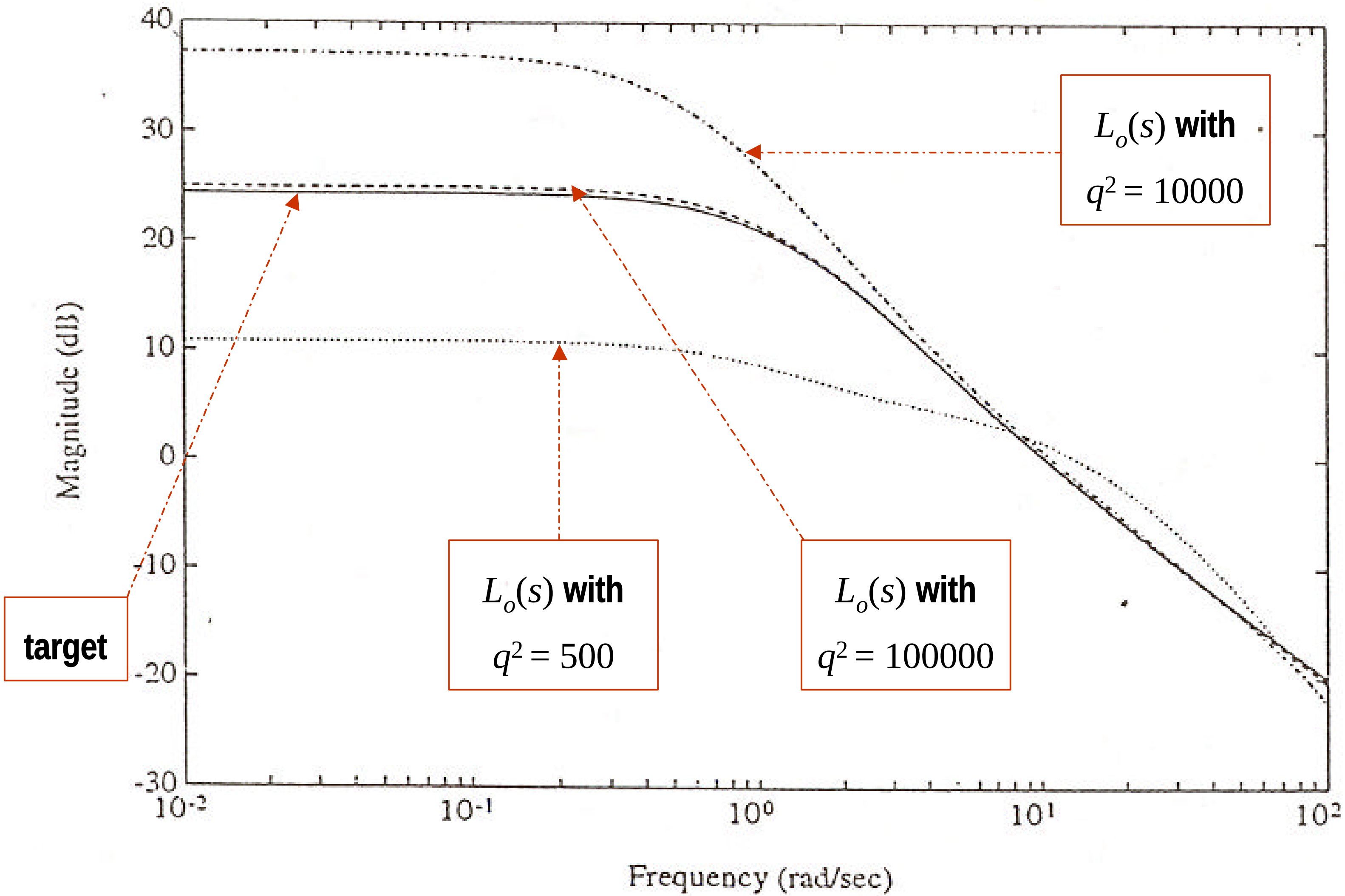
$$u = -F\, x = -[50 \quad 10]\, x$$

The resulting open-loop transfer function $L_t(s)$ has an infinity gain margin and a phase margin over 85°. We apply Doyle-Stein LTR procedure to design an observer based controller, i.e.,

$$u = -F[\Phi^{-1} + BF + KC]^{-1} K\, y$$

where K is computed as on the previous page with

$$Q_q = \begin{bmatrix} 35 \\ -61 \end{bmatrix} [35 \quad -61] + q^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \quad 1] = \begin{bmatrix} 1225 & -2135 \\ -2135 & 3721 + q^2 \end{bmatrix}.$$



40
30
20
10
0
-10
-20
-30
Magnitude (dB)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)
$L_o(s)$ with $q^2 = 10000$
target
$L_o(s)$ with $q^2 = 500$
$L_o(s)$ with $q^2 = 100000$

New Formulation for Loop Transfer Recovery

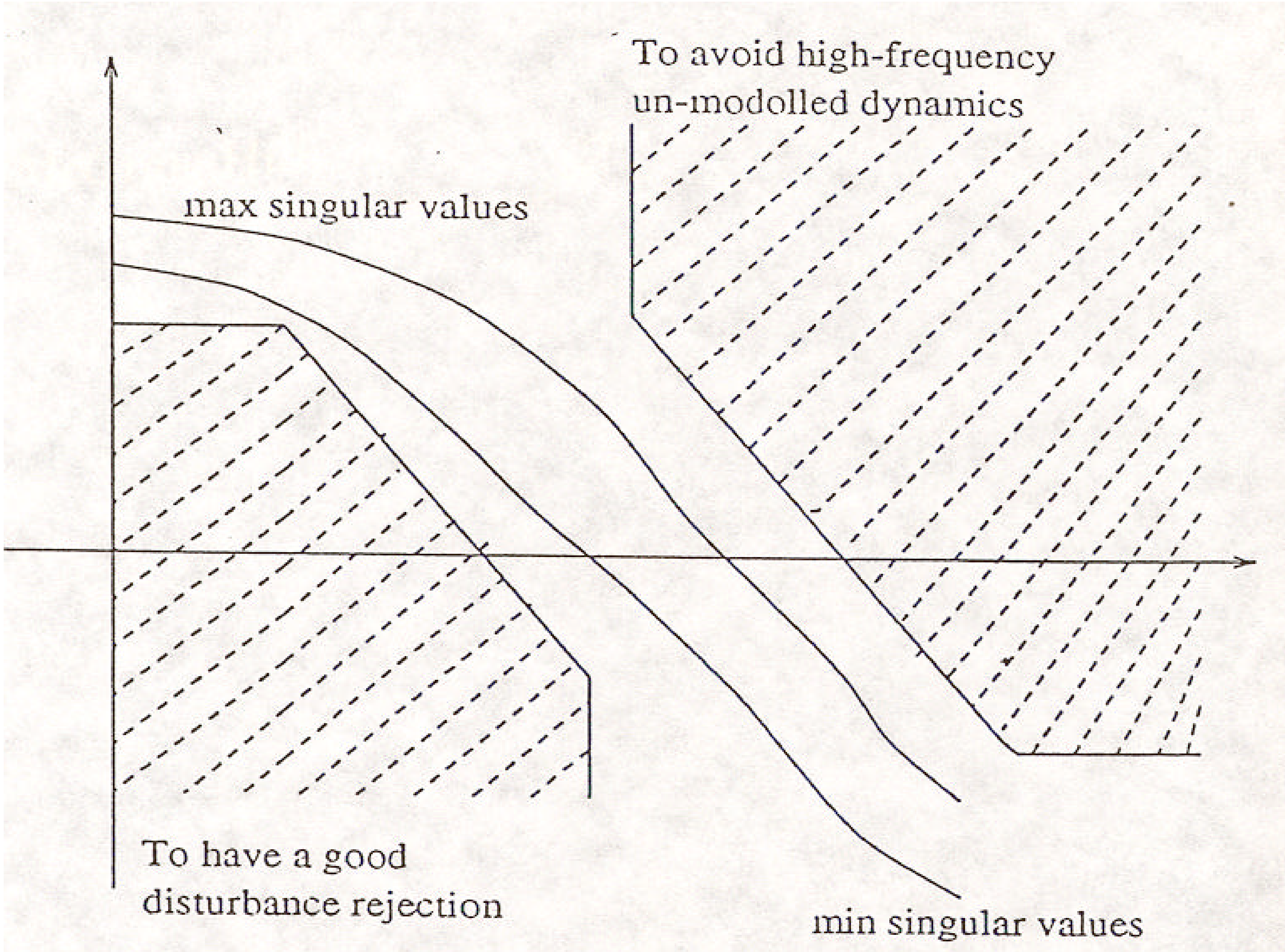
Consider a general stabilizable and detectable plant,

$$\begin{cases} \dot{x} = A\ x + B\ u \\ y = C\ x + D\ u \end{cases}$$

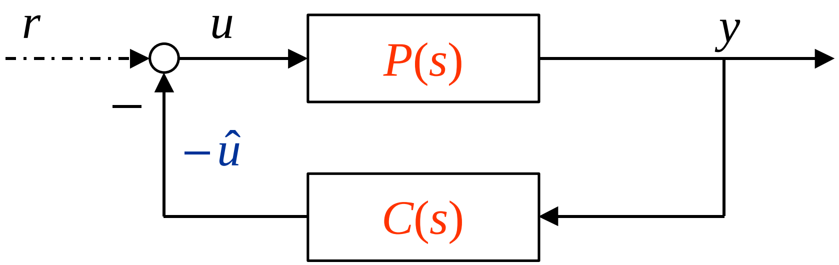
The transfer function is given by $P(s) = C\Phi B + D,\ \ \Phi = (sI - A)^{-1}$. Also, let F be a state feedback gain matrix such that under the state feedback control law $u = -F\,x$ has the following properties:

- the resulting closed-loop system is asymptotically stable; and
- the resulting target loop $L_t(s) = F\Phi B$ meets design specifications (GM, PM, etc).

Such a state feedback can be obtained using LQR design or any other design methods so long as it meets your design specifications. Usually, a desired target loop would have the shape as given in the following figure.


To avoid high-frequency
un-modolled dynamics
max singular values
To have a good
disturbance rejection
min singular values

The problem of loop transfer recovery (LTR) is to find a stabilizing controller $u = -C(s)y$



such that the resulting open-loop transfer function from u to $-\hat{u}$, i.e.,

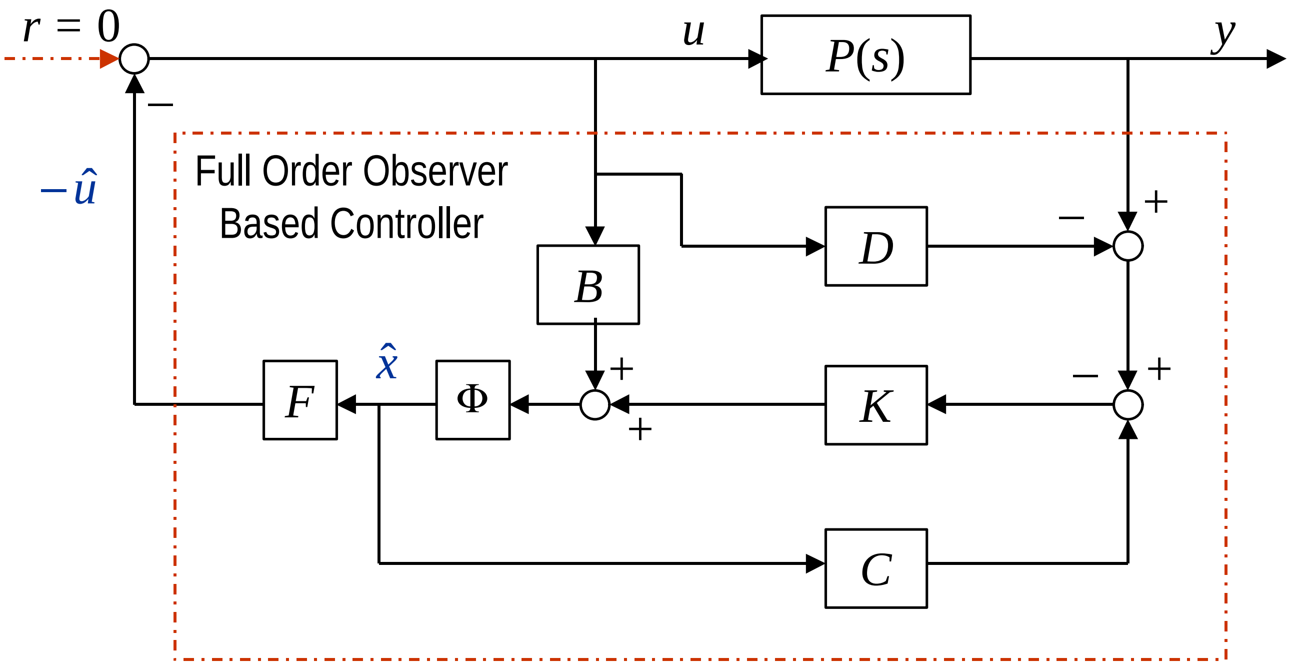
$$L_o(s) = C(s)P(s)$$

is either exactly or approximately equal to the target loop $L_t(s)$. Let us define the recovery error as the difference between the target loop and the achieved loop, i.e.,

$$E(s) = L_t(s) - L_o(s) = F\Phi B - C(s)P(s)$$

Then, we say exact LTR is achievable if $E(s)$ can be made identically zero, or almost LTR is achievable if $E(s)$ can be made arbitrarily small.

Observer Based Structure for C(s)



Dynamic equations of $C(s)$: $$\begin{cases} \dot{\hat{x}} = A\,\hat{x} + B\,u + K\,(y - C\hat{x} - D\,u) \\ \hat{u} = u = -F\,\hat{x} \end{cases}$$

Transfer function of $C(s) = C_o(s) = F(\Phi^{-1} + BF + KC - KDF)^{-1}K$

Achieved open-loop: $L_o(s) = C_o(s)P(s) = F(\Phi^{-1} + BF + KC - KDF)^{-1}K(C\Phi B + D)$

Lemma: Recovery error, $E_o(s)$, i.e., the mismatch between the target loop and the resulting open-loop of the observer based controller is given by

$$E_o(s) = M(s)\,[I + M(s)]^{-1}(I + F\Phi B), \quad M(s) = F(\Phi + KC)^{-1}(B - KD)$$

Proof.

$$\begin{aligned}
L_o(s) = C_o(s)P(s) &= F(\Phi^{-1} + BF + KC - KDF)^{-1}K(C\Phi B + D) \\
&= F[I + (\Phi^{-1} + KC)^{-1}(B - KD)F]^{-1}(\Phi^{-1} + KC)^{-1}K(C\Phi B + D) \\
&= [I + F(\Phi^{-1} + KC)^{-1}(B - KD)]^{-1}F(\Phi^{-1} + KC)^{-1}K(C\Phi B + D) \\
&= [I + M(s)]^{-1}[F(\Phi^{-1} + KC)^{-1}KC\Phi B + F(\Phi^{-1} + KC)^{-1}KD] \\
&= [I + M(s)]^{-1}\{F[I - (\Phi^{-1} + KC)^{-1}\Phi^{-1}]\Phi B + F(\Phi^{-1} + KC)^{-1}KD\} \\
&= [I + M(s)]^{-1}[F\Phi B - F(\Phi^{-1} + KC)^{-1}B + F(\Phi^{-1} + KC)^{-1}KD] \\
&= [I + M(s)]^{-1}[F\Phi B - F(\Phi^{-1} + KC)^{-1}(B - KD)] \\
&= [I + M(s)]^{-1}[F\Phi B - M(s)]
\end{aligned}$$

Note that we have used $(\Phi^{-1} + KC)^{-1}KC = I - (\Phi^{-1} + KC)^{-1}\Phi^{-1}$. Thus,

$$E_o = L_t - L_o = F\Phi B - [I + M]^{-1}[F\Phi B - M] = M[I + M]^{-1}(I + F\Phi B).$$

Loop Transfer Recovery Design

It is simple to observe from the above lemma that the loop transfer recovery is achievable if and only if we can design a gain matrix K such that $M(s)$ can be made either identically zero or arbitrarily small, where $M(s) = F(\Phi + KC)^{-1}(B - KD)$.

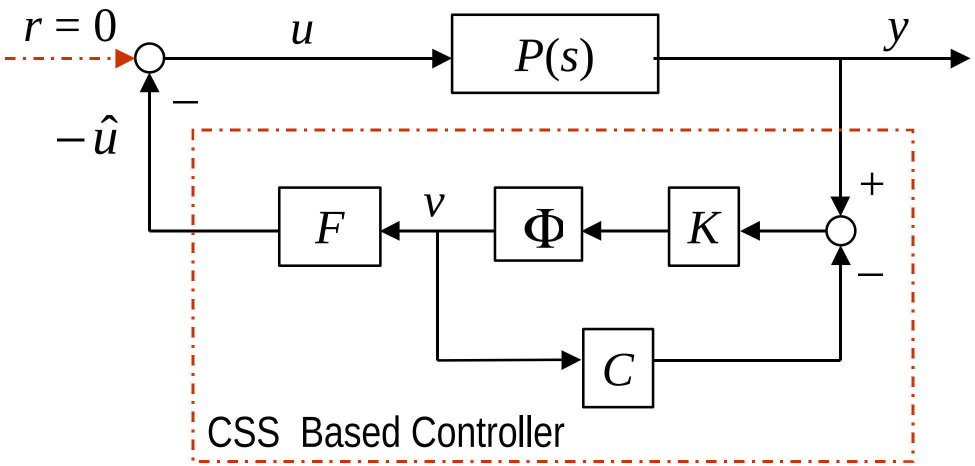
Let us define an auxiliary system

$$\Sigma_{\mathrm{aux}} \; : \; \begin{cases} \dot{x} = A^{\mathrm{T}} x + C^{\mathrm{T}} u + F^{\mathrm{T}} w \\ y = x \\ z = B^{\mathrm{T}} x + D^{\mathrm{T}} u \end{cases} \qquad + \qquad u = -K^{\mathrm{T}} x$$

Closed-loop transfer function from w to z is $(B^{\mathrm{T}} - D^{\mathrm{T}} K^{\mathrm{T}})(sI - A^{\mathrm{T}} + C^{\mathrm{T}} K^{\mathrm{T}})^{-1} F^{\mathrm{T}} = M^{\mathrm{T}}(s)$.

Thus, LTR design is equivalent to design a state feedback law for the above auxiliary system such that certain norm of the resulting closed-loop transfer function is made either identically zero or arbitrarily small. As such, all the design techniques in H_2 and H_∞ can be applied to design such a gain. There is no need to repeat all over again once this is formulated.

LTR Design via CSS Architecture Based Controller



- CSS Structure was proposed by Chen, Saberi and Sannuti in 1992. It has the following:

Dynamic equations of $C(s)$: $\begin{cases} \dot{v} = (A - KC)\, v + Ky \\ \hat{u} = u = -F\, v \end{cases}$

Transfer function of $C(s) = C_c(s) = F(\Phi^{-1} + KC)^{-1} K$

Achieved open-loop: $L_c(s) = C_c(s) P(s) = F(\Phi^{-1} + KC)^{-1} K (C\Phi B + D)$

Lemma: Recovery error, $E_c(s)$, i.e., the mismatch between the target loop and the resulting open-loop of the CSS architecture based controller is given by

$$E_c(s) = M(s) = F(\Phi + KC)^{-1}(B - KD)$$

Proof.

$$\begin{aligned} E_c(s) &= L_t(s) - L_c(s) \\ &= F\Phi B - F(\Phi^{-1} + KC)^{-1}K(C\Phi B + D) \\ &= F(\Phi^{-1} + KC)^{-1}[(\Phi^{-1} + KC)\Phi B - KC\Phi B - KD] \\ &= F(\Phi^{-1} + KC)^{-1}(B + KC\Phi B - KC\Phi B - KD) \\ &= M(s) \end{aligned}$$

It is clear that LTR via the CSS architecture based controller is achievable if and only if one can design a gain matrix K such that the resulting $M(s)$ can be made either identically zero or arbitrarily small. This is identical to the LTR design via the observer based controller. Thus, one can again using the H_2 and H_∞ techniques to carry out the design of such a gain matrix.

What is the Advantage of CSS Structure?

Theorem. Consider a stabilizable and detectable system Σ characterized by (A, B, C, D) and target loop transfer function $L_t(s) = F\Phi B$. Assume that Σ is left invertible and of minimum phase, which implies that the target loop $L_t(s)$ is recoverable by both observer based and CSS architecture based controllers. Also, assume that the same gain K is used for both observer based controller and CSS architecture based controller and is such that for all $w \in \Omega$, where Ω is some frequency region of interest,

$$0 < s_{\max}[M(jw)] << 1, \quad s_{\min}[L_t(jw)] = s_{\min}[F(jwI - A)^{-1}B] >> 1$$

Then, for all $w \in \Omega$,

$$\boxed{s_{\max}[E_c(jw)] << s_{\max}[E_o(jw)].}$$

Proof. See Chen, Saberi and Sannuti, *Automatica*, vol. 27, pp. 257-280, 1991. See also Saberi, Chen and Sannuti, *Loop Transfer Recovery: Analysis and Design*, Springer, 1993.

Remark: In order to have good command following and disturbance rejection properties, the target loop transfer function $L_t(jw)$ has to be large and consequently, the minimum singular value $s_{\min}[L_t(jw)]$ should be relatively large in the appropriate frequency region. Thus, the assumption in the above theorem is very practical.

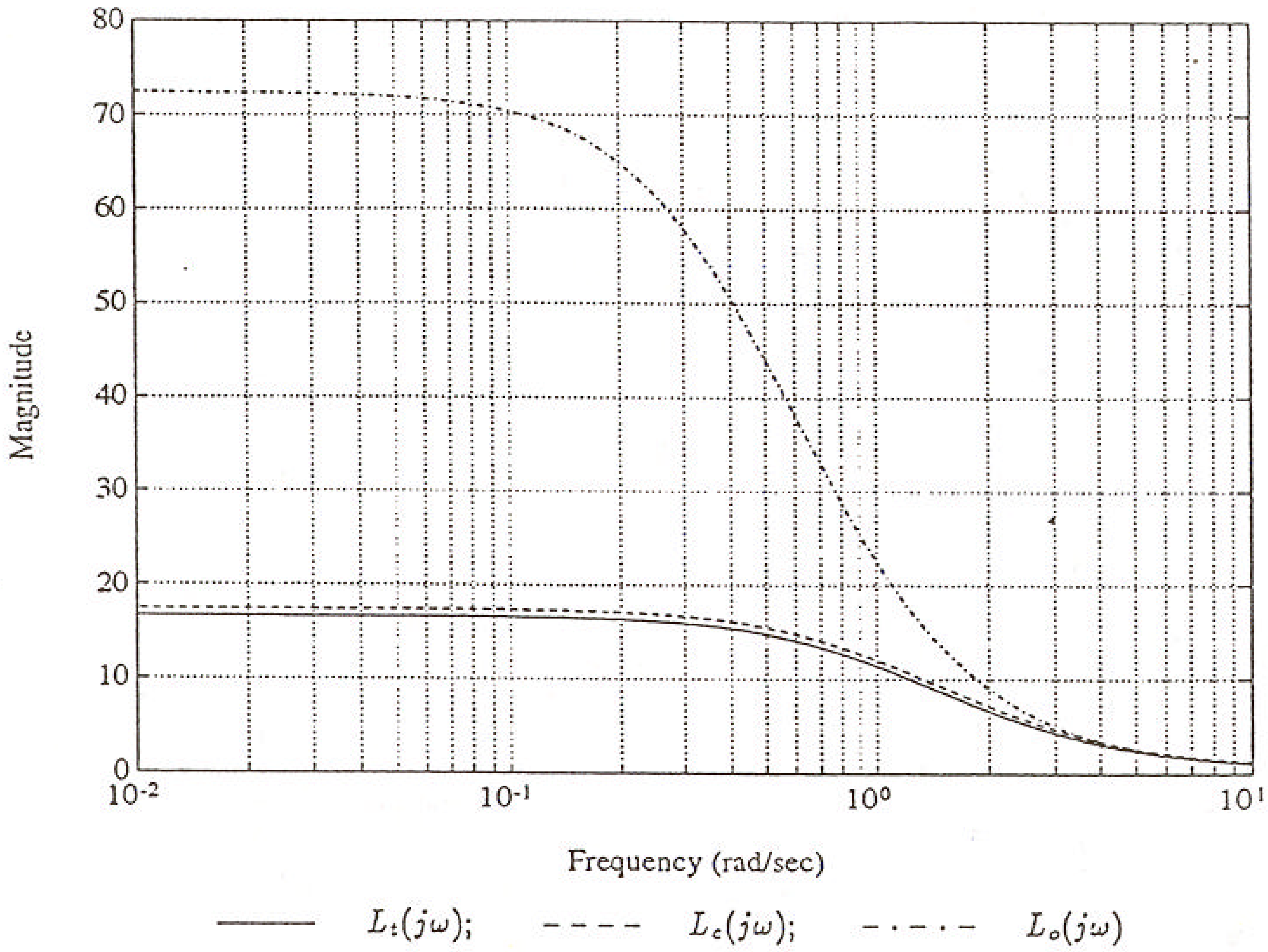
Example: Consider a given plant characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u , \quad y = \begin{bmatrix} 2 & 1 \end{bmatrix} x + 0 \cdot u$$

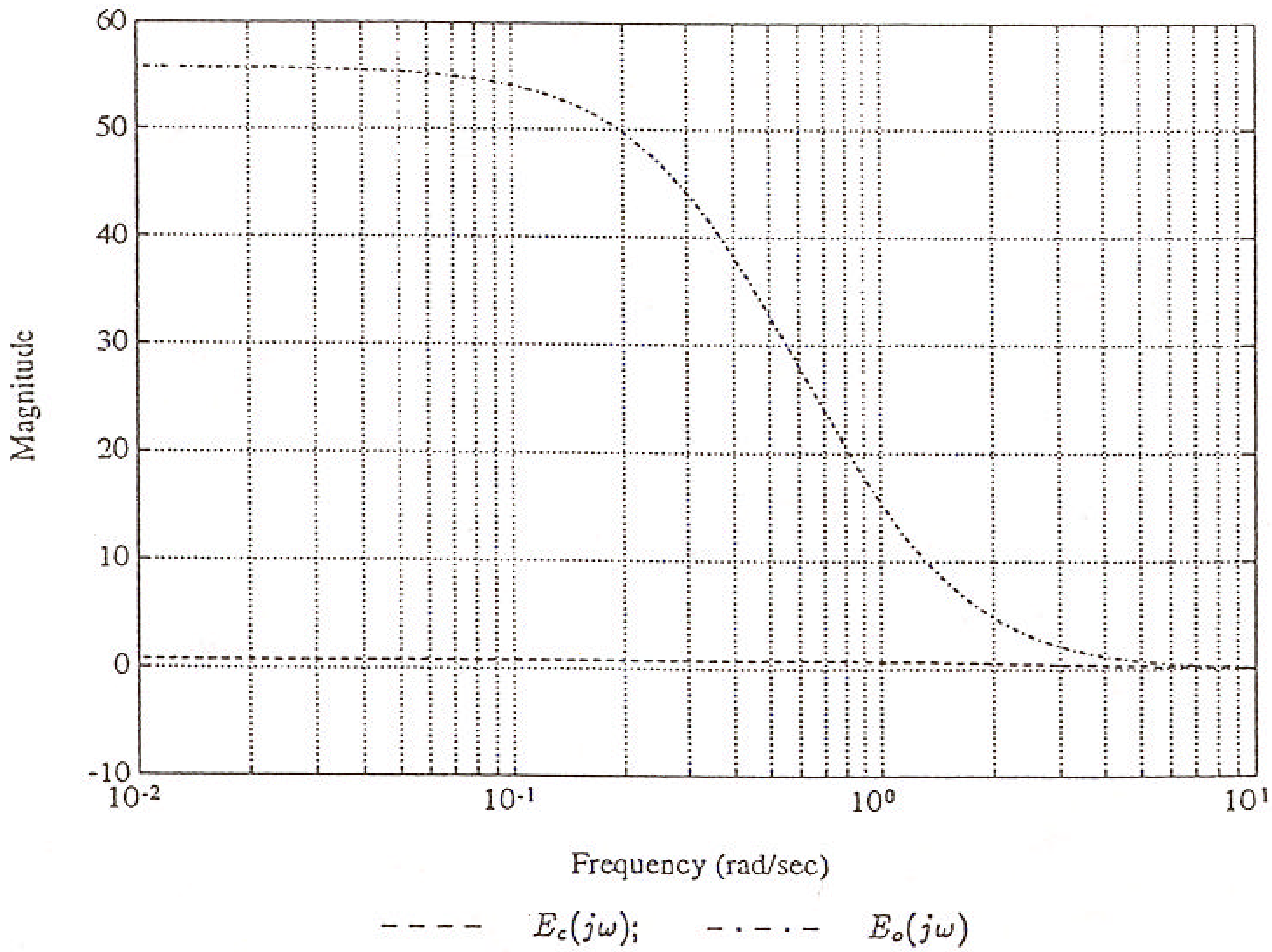
Let the target loop $L_t(s) = F\,\Phi B$ be characterized by a state feedback gain $F = \begin{bmatrix} 50 & 10 \end{bmatrix}$.

Using MATLAB, we know that the above system has an invariant zero at $s = -2$. Hence it is of minimum phase. Also, it is invertible. Thus, the target loop $L_t(s)$ is recoverable by both the observer based and CSS architecture based controllers. The following gain matrix K is obtained by using the H_2 optimization method,

$$K = \begin{bmatrix} 6.9 \\ 84.6 \end{bmatrix}.$$



80
70
60
50
40
30
20
10
0
Magnitude
10^{-2}
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)
$L_t(j\omega)$; $L_c(j\omega)$; $L_o(j\omega)$



60
50
40
30
20
10
0
-10
Magnitude
10-2
10-1
100
101
Frequency (rad/sec)
---- $E_c(j\omega)$; -·-·- $E_o(j\omega)$

Homework Assignment 3 (Hand in your solutions next week)

Recall the cruise-control system in Homework Assignment 1. Assume that the state feedback gain matrix is obtained using the LQR technique as in Item 2 of Homework Assignment 1.

Design an observer and CSS architecture gain matrix K using the classical LTR method with $Q = q^2 BB^{\mathrm{T}}$ and $R = 1$, i.e.,

$$K = PC^{\mathrm{T}}, \quad P > 0 \text{ is the solution of } AP + PA^{\mathrm{T}} + q^2 BB^{\mathrm{T}} - PC^{\mathrm{T}} CP = 0$$

for q^2 = 500, 1000 and 10000, respectively.

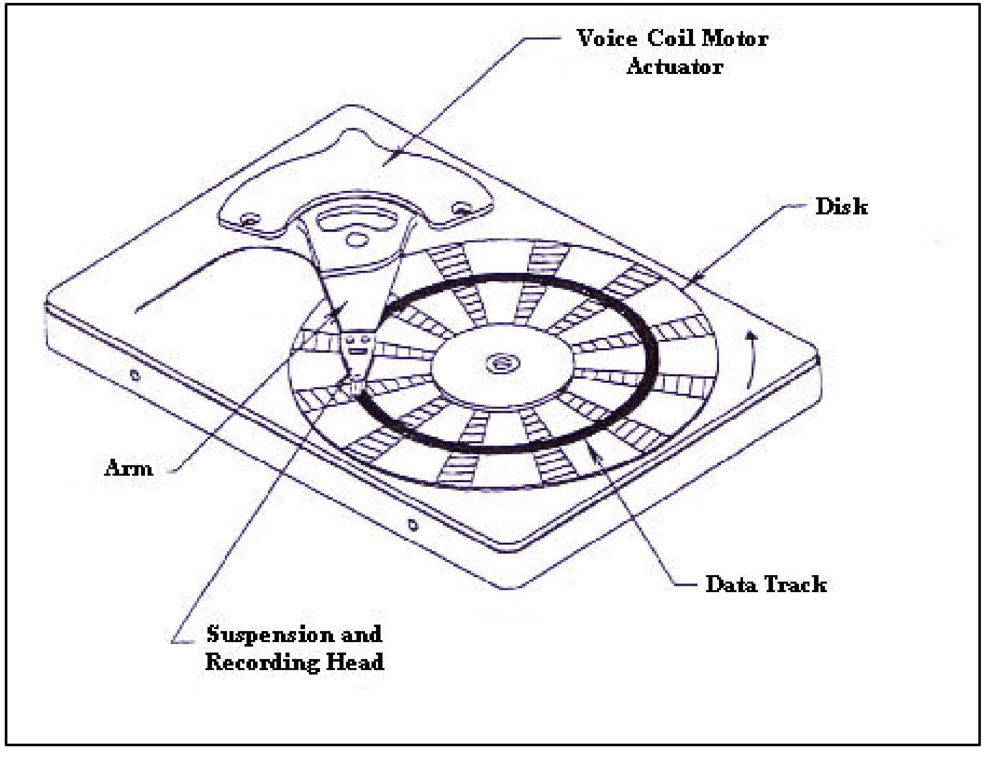
- For each gain matrix K, plot the magnitudes of the target loop $L_t(jw)$, the achieved loop by observer based controller, $L_o(jw)$, and CSS architecture based controller, $L_c(jw)$, over the frequency range $10^{-2} \le w \le 10^{2}$ rad / sec.

- For each gain matrix K, plot the magnitudes of the error functions, $E_o(jw)$ and $E_c(jw)$, over the same range of frequencies. Comment on the outcomes.

Design and Implementation of an HDD Servo System:

— A Robust and Perfect Tracking Approach

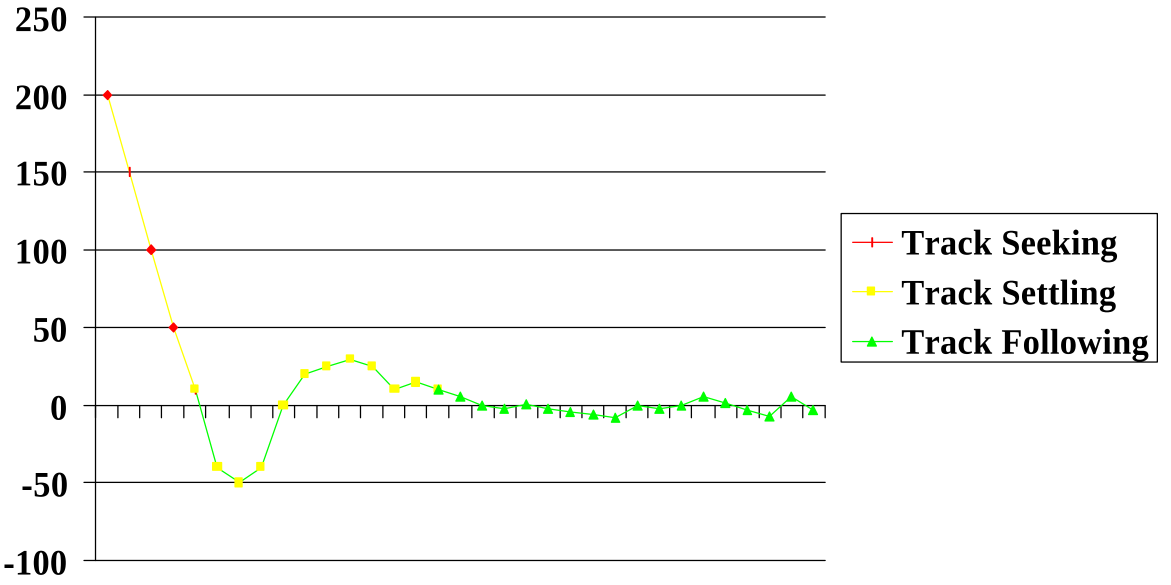
A Typical Hard Disk Drive (HDD)



Hard Disk Drive Servo Illustration

Track Seeking: to move R/W head from the present track to destination track in minimum time using a bounded control effort.

Track Following: to maintain R/W head as close as possible to the center of destination track while data reading and writing.



HDD Trends in Industry

- Smaller in size: Small 3.5” and 2.5” form factor disk drives are in broad use. More compact and efficient disks are expected.
- Higher data capacity: Growth rate is 60% annually.
- Faster access time: High speed actuator motion and shorter data bands of the smaller diameter disks is the reason. Low mass actuator design with highly efficient VCM will lead to faster access time.
- Better reliability: Minimization the parts count, more integrated electronics etc. will increase the reliability.

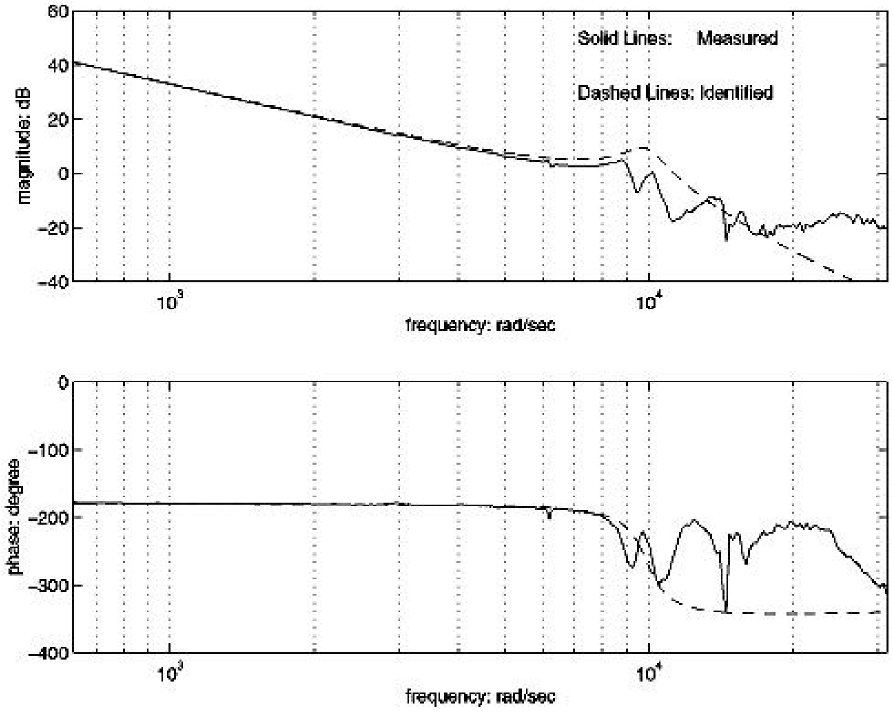
Robust and Perfect Tracking (RPT) Design

- Robust and Perfect Tracking (RPT) Design (Liu, Chen and Lin, International Journal of Control, Jan. 2001. See also Chen, *Robust and H_∞ Control*, Springer, 2000), which is to design a parameterized control law using the framework of robust and H_∞ control such that when it is applied to the given system, the resulting closed-loop system has:
  - Internal Stability;
  - Robustly and perfectly tracks a given reference input, i.e. any L_p-norm of the tracking error can be made zero, in face of external disturbance and any initial condition.

VCM Actuator Modeling

- Voice coil motor as the actuator of R/W head
- Fourth order model identified:

$$\frac{4.3817 \times 10^{10} s + 4.3247 \times 10^{15}}{s^2 (s^2 + 1.5962 \times 10^3 s + 9.6731 \times 10^7)}$$



Bode Plot of VCM Actuator

Design Specifications

- The control input should not exceed ± 2 volts.
- Overshoots and undershoots of step response should be less than 5%.
- The 5% settling time should be less than 2 ms.
- Sampling frequency in implementation is 4k Hz.

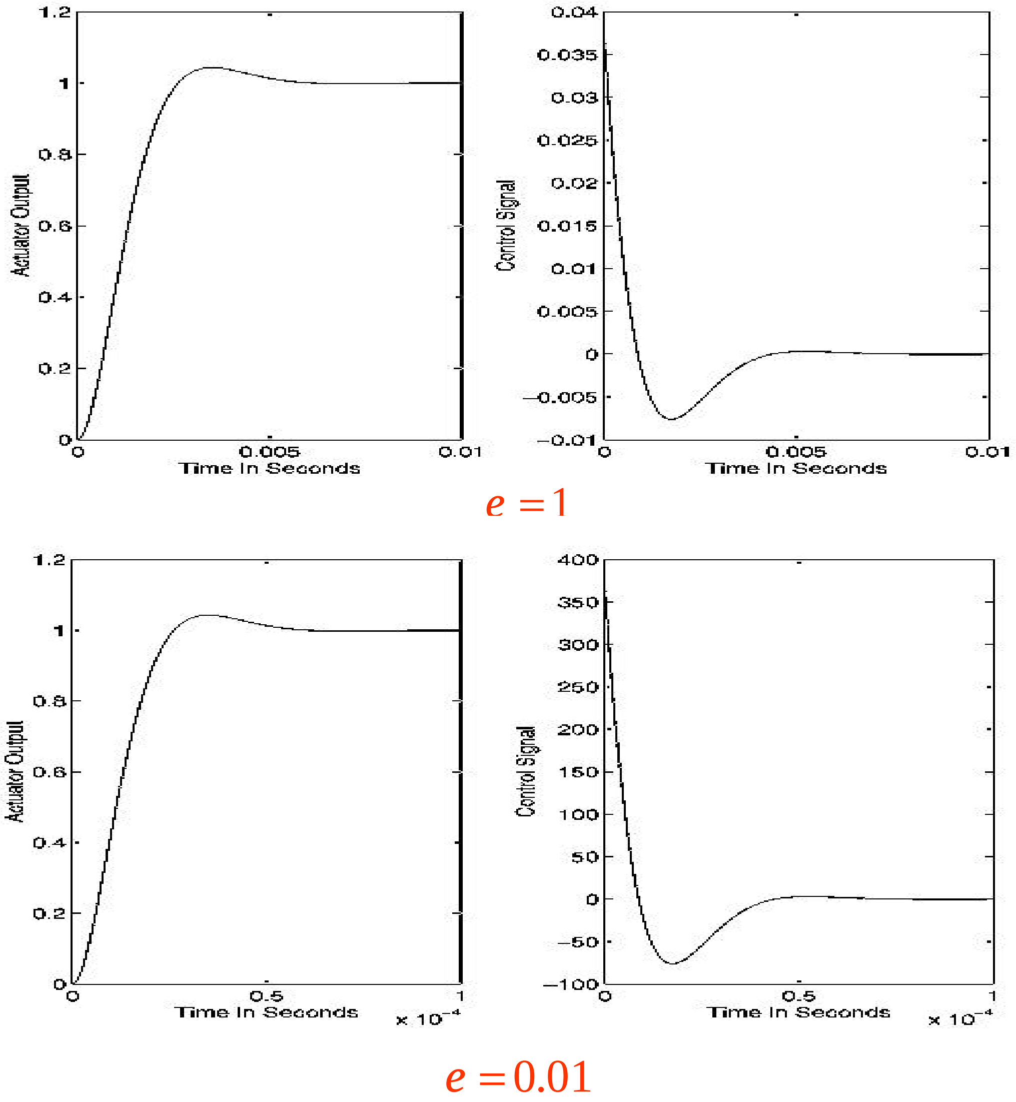
Controller Design via RPT Approach

- Lower order plant consider in control design.
- Then, use the state feedback law to construct a reduced order measurement feedback control law.
- The parameterized first order RPT control law is obtained:

$$\begin{cases} A_{\mathrm{RC}}(e) = -7800/e \\ B_{\mathrm{RC}}(e) = \dfrac{1}{e^2}\left[\, 1.62\mathrm{x}10^{6} \quad -4.842\mathrm{x}10^{7} \,\right] \\ C_{\mathrm{RC}}(e) = -4.063572\,\mathrm{x}10^{-5}/e \\ D_{\mathrm{RC}}(e) = \dfrac{1}{e^2}\left[\, 0.036572 \quad -0.280386 \,\right] \end{cases}$$

e is a tuning parameter, which can be tuned to meet the design specs.

Parameterized RPT Controller Design - Simulation Result



Step response of closed-loop system with parameterized controller.

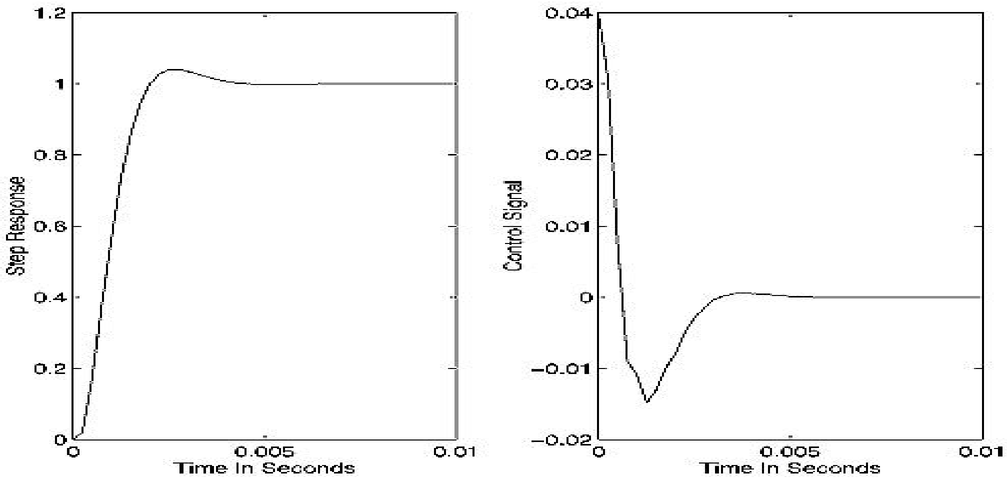
RPT problem is solved as the tuning parameter is pushed to be smaller and smaller.

Discretized RPT Controller

- We found that parameter with 0.9 meets the design specifications.

- The discretized control law with 4kHz sampling rate is given:

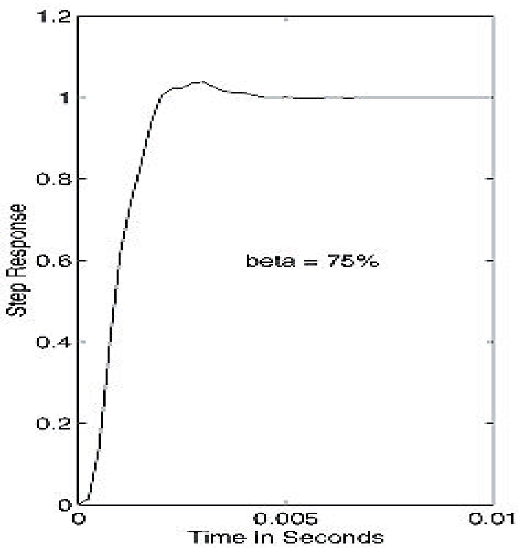
$$\begin{cases} x_v(k+1) = -0.04x_v(k) + 15178.933r(k) - 453681.43y(k) \\ u_v(k) = -3.426708\times10^{-7}x_v(k) + 0.0397325r(k) - 0.1842147y(k) \end{cases}$$

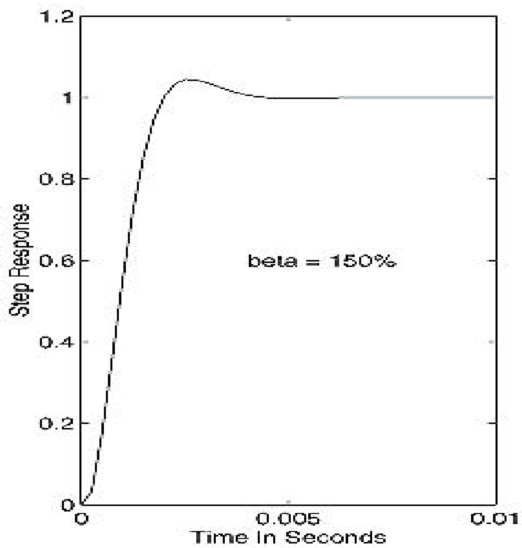


Step Response of the closed-loop System

Step Responses with Different Resonant Frequencies

- In real HDDs, the resonant frequency may vary (= beta $\times$ nominal resonant frequency).
- Common method in practice: adding the notch filters.
- RPT controller can withstand the variation of resonance frequency in the actuator.

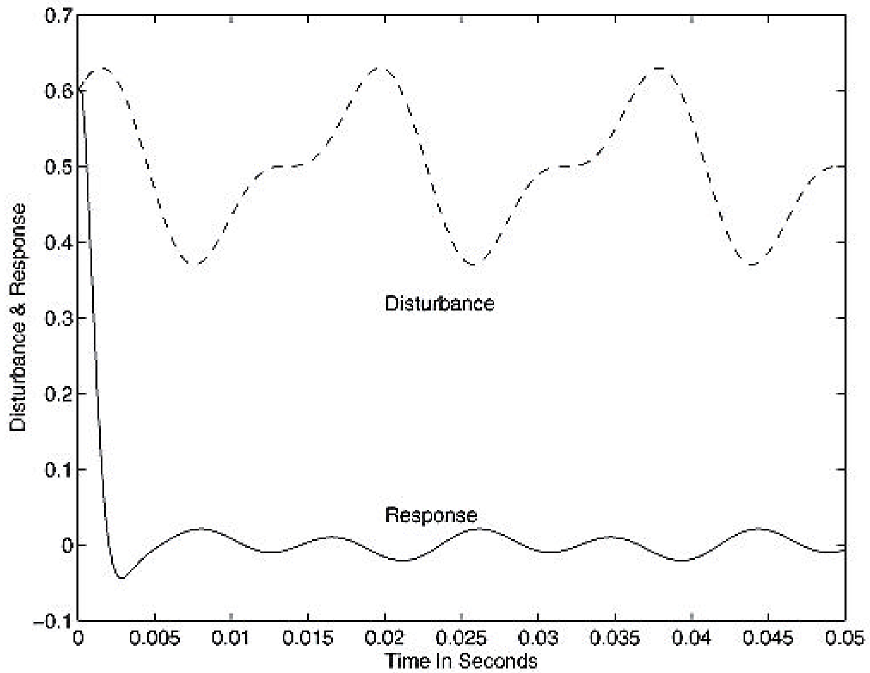




Step Response of the closed-loop System

Test Against Run-out Disturbance

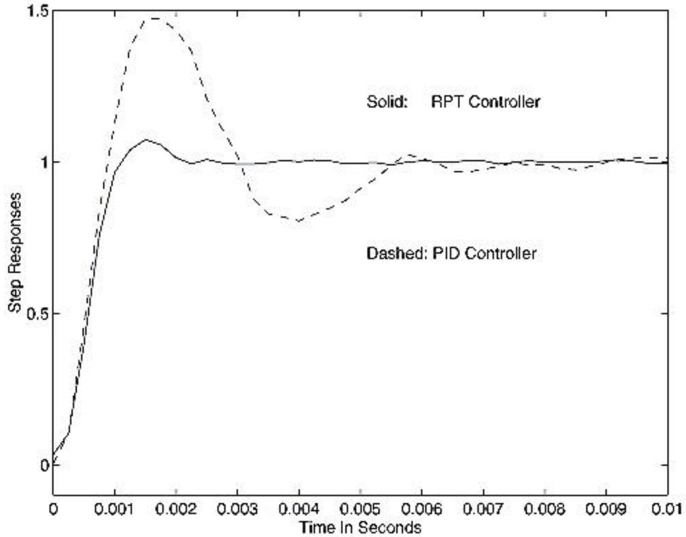
- Run-out: due to imperfectness of the data tracks and the spindle motor speeds.
- Disturbance injected: $0.5 + 0.1\cos(345t) + 0.05\sin(691t)$ and zero reference.
- The effect of this disturbance is minimal.



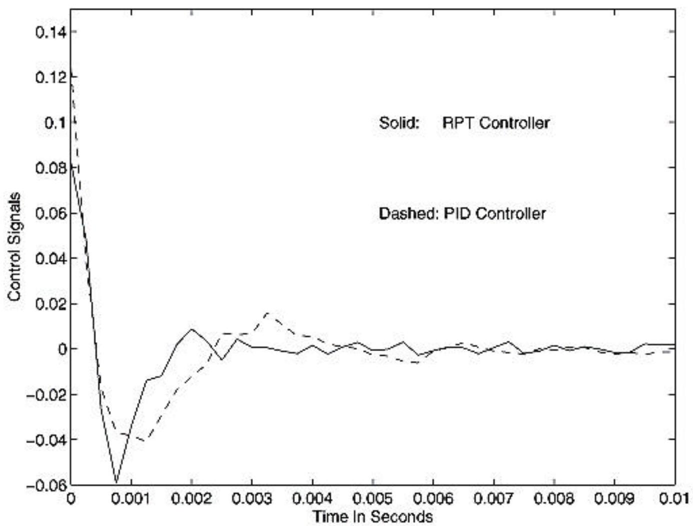
Implementation Result: Track Following Test

- Compared with PID controller below (Done by Goh in his Master of Engineering Thesis). The PID controller:

$$u = \frac{0.13z^2 - 0.23z + 0.10}{z^2 - 1.25z + 0.25}(r - y)$$



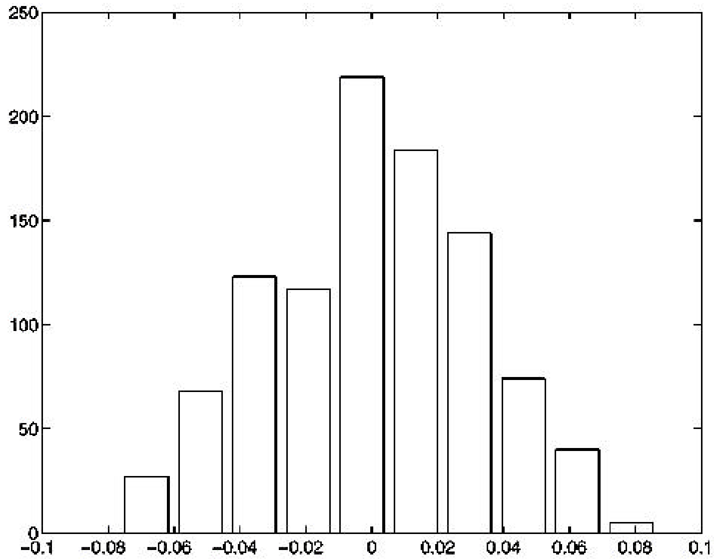
Step Responses



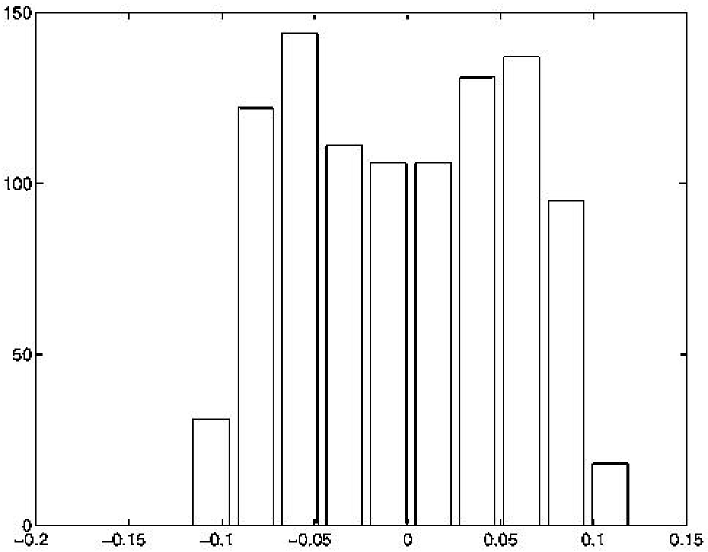
Control Signals

Implementation Result: PES Test

- Position Error Signal (PES) is the measurement output in real HDD servo systems.
- Disturbances: Repeatable Run-outs (RRO) and Non-repeatable Run-Outs (NRRO).
- $3s$ of RPT controller was 0.095μm, while that for PID controller was about 0.175μm.



Histogram of RPT PES test



Histogram of PID PES test

Comparisons Between RPT and PID Designs

	Our Control	**PID Control**
Order of controller	✓ 1^{st} order	2^{nd} order
Closed Loop Resonance Peak	✓ No resonance peak	Resonance peak of 6 dB
Settling Time for Track Seek	Faster than PID method (<2ms)	>5 ms
Overshoot during Track seek	Lower overshoot (6%)	50 %
Closed Loop Bandwidth	Similar bandwidth (500 Hz)	Similar bandwidth (500 Hz)

The conclusion is obvious.

References

For classical optimal control techniques, see

B. D. O. Anderson and J. B. Moore, Optimal Control, Prentice Hall, London, 1989.

F. L. Lewis, Applied Optimal Control and Estimation, Prentice Hall, Englewood Cliffs, 1992.

For more detailed treatment on H_2 optimal control, see

A. Saberi, P. Sannuti and B. M. Chen, H_2 Optimal Control, Prentice Hall, London, 1995.

For robust and H_∞ control, see

B. M. Chen, Robust and H_∞ Control, Springer, London, 2000.

For more on loop transfer recovery techniques, see

A. Saberi, B. M. Chen and P. Sannuti, Loop Transfer Recovery: Analysis and Design, Springer, London, 1993.