EE 2402 Engineering Mathematics III

Tutorial 1

1. Convergence Test of Series:

   (a) Show that the series

   $$\ln(a+h) = \ln a + \frac{h}{a} - \frac{h^2}{2a^2} + \frac{h^3}{3a^3} - \cdots, a > 0,$$

   converges for $|h| < a$.

   (b) Find the open interval of absolute convergence for the power series $\sum_{n=1}^{\infty} u_n(x)$ for

   $$u_n(x) = \frac{(-4)^n}{n(n+1)}(x+2)^{2n}.$$

2. Use the recurrence equations of Bessel functions in the lecture notes and the fact that for any integer n, $J_{-n}(x) = (-1)^n J_n(x)$ to show that:

   (a) $J_0'(x) = -J_1(x)$

   (b) $\int xJ_0(x)dx = xJ_1(x)$

3. The generating function for Bessel function is

   $$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

   By differentiating both sides of the equation with respect to t and equating coefficients of like powers of t, show that

   $$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

4. Use the fact that J_v is a solution of Bessel's equation of order v to show that $x^a J_v(bx^c)$ is a solution of the equation

   $$y'' - \left(\frac{2a-1}{x}\right) y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - v^2c^2}{x^2}\right) y = 0.$$

5. Use the fact that $[x^{-v}J_v(x)]' = -x^{-v}J_{v+1}(x)$ and

   $$J_{3/2} = \sqrt{\frac{2}{\pi x}} \left[\frac{\sin(x)}{x} - \cos(x)\right]$$

   to show that

   $$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3}{x^2} - 1\right)\sin(x) - \frac{3}{x}\cos(x)\right]$$

6. It is proved in the lecture notes that $x^a J_n(bx^c)$ and $x^a Y_n(bx^c)$ are solutions of

$$y'' - \left(\frac{2a-1}{x}\right) y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - n^2c^2}{x^2}\right) y = 0$$

for constants a, b and c and any nonnegative integer n. Use the above fact to write the general solutions to the following differential equations:

(a) $y'' - \dfrac{1}{x}y' + \left(1 - \dfrac{3}{x^2}\right) y = 0$

(b) $y'' - \dfrac{3}{x}y' + \left(\dfrac{1}{4x} + \dfrac{3}{x^2}\right) y = 0$

(c) $y'' + \dfrac{3}{x}y' + \dfrac{1}{16x}y = 0$

(d) $y'' - \dfrac{3}{x}y' + \left(4x^2 - \dfrac{60}{x^2}\right) y = 0$

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Tutorial 2

Interactive Problems

Remark: *This tutorial set is to be conducted in a series of two tutorial sessions on Weeks 5 and 6. Tutors are free to have their own ways to conduct this tutorial set, as long as each student has a chance to solve at least one of the following 4 problems and present it in the tutorial class. Tutors are requested to mark the performance of each student in a scale from 0 to 10, which will consist of 10% of the student's final grade, and submit the completed record to the class lecturer after the completion of the tutorial series.*

1. Show that if a polynomial $P(x) = \sum_{k=0}^{n} a_k x^k \equiv 0$, then $a_k = 0$ for all $k = 0, 1, 2, \cdots, n$. Show also that if a power series $\sum_{k=0}^{\infty} a_k x^k \equiv 0$, then $a_k = 0$ for all $k = 0, 1, 2, \cdots, \infty$. Note that these properties are used in the class without proofs.

2. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and then use it to show that $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin(x)}{x} - \cos(x)\right]$.

3. How does a Bessel's equation arise from real-life problems? Give a practical application of Bessel's equations and/or Bessel functions with details. Note: Do not use those examples in the lecture notes.

4. How does a Legendre's equation arise from real-life problems? Give a practical application of Legendre's equations and/or Legendre functions with details. Note: Do not use those examples in the lecture notes.

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Tutorial 3

1. For $n = 0, 1, 2, 3, 4, 5$ verify that $P_n(x)$ is a solution of Legendre's equation with $\alpha = n$.

2. Expand each of the following in a series of Legendre's polynomials:

   (a) $1 + 2x - x^2$

   (b) $2x + x^2 - 5x^3$

   (c) $2 - x^2 + 4x^4$

3. Let n be a nonnegative integer. Use the fact that $P_n(x)$ is one solution of Legendre's equation with $\alpha = n$ to obtain a second, linearly independent solution

$$Q_n(x) = P_n(x) \int \frac{1}{P_n(x)^2(1-x^2)} dx$$

   Hint: Let $Q_n(x) = P_n(x)z(x)$ and then substitute it to the Legendre's equation to find $z(x)$.

4. Use the result in Problem 3 to obtain

$$\begin{aligned}
Q_0(x) &= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \\
Q_1(x) &= \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) - 1 \\
Q_2(x) &= \frac{1}{4}(3x^2 - 1) \ln\left(\frac{1+x}{1-x}\right) - \frac{3}{2}x
\end{aligned}$$

5. Show the following properties of Legendre polynomials:

   (a) $P_n(-x) = (-1)^n P_n(x)$ for $-1 \le x \le 1$ and $n = 0, 1, 2, \cdots$

   (b) For any integer $n > 0$,

$$\int_{-1}^{1} P_n(x) dx = 0$$

   Hint: $P_n(x) = P_n(x)P_0(x)$.

6. Practice: O'Neil, Sections 6.3 and 6.4.

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Tutorial 4

1. Show that
$$y(x,t) = \sin(x)\cos(at) + \frac{1}{a}\cos(x)\sin(at)$$
is a solution of
$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}, \qquad (0 < x < \pi,\ t > 0)$$
with boundary conditions
$$y(x,0) = \sin(x), \qquad \frac{\partial y}{\partial t}(x,0) = \cos(x) \quad (0 < x < \pi)$$

2. Solve the boundary value problem
$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= 9\frac{\partial^2 y}{\partial x^2} + x^2 && (0 < x < 4,\ \ t > 0) \\ y(0,t) = y(4,t) &= 0 && (t > 0) \\ y(x,0) &= 0 && (0 < x < 4) \\ \frac{\partial y}{\partial t}(x,0) &= 0 && (0 < x < 4) \end{aligned}$$
Hint: Let $z(x,t) = y(x,t) + h(x)$ and substitute into the partial differential equation and boundary conditions to determine $h(x)$ that would wash out the x^2 term in the partial differential equation.

3. Show that the partial differential equation
$$\frac{\partial u}{\partial t} = k\left[\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu\right]$$
can be transformed into a partial differential equation of the form
$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$$
by choosing α and β appropriately and using the change of variables
$$u(x,t) = e^{\alpha x + \beta t}v(x,t)$$
Hint: Substitute $u(x,t) = e^{\alpha x + \beta t}v(x,t)$ into the partial differential equation and determine how to choose α and β.

4. Use the method of Problem 3 to solve the boundary value problem

$$\frac{\partial u}{\partial t} = k\left[\frac{\partial^2 u}{\partial x^2} - \frac{a}{L}\frac{\partial u}{\partial x}\right] \qquad (0 < x < L,\ \ t > 0)$$

$$\begin{aligned} u(0,t) &= u(L,t) = 0 \qquad (t > 0) \\ u(x,0) &= \frac{L}{ak}[1 - e^{-a(1-x/L)}] \qquad (0 < x < L) \end{aligned}$$

Here $u(x,t)$ measures the concentration of positive charge carriers on the base of length L of a transistor, $k > 0$ and $a > 0$ are constants.

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