EE 2402 Engineering Mathematics III

Solutions to Tutorial 4

1. Show that

$$y(x,t) = \sin(x)\cos(at) + \frac{1}{a}\cos(x)\sin(at)$$

is a solution of

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \qquad (0 < x < \pi,\ t > 0)$$

with boundary conditions

$$y(x,0) = \sin(x), \qquad \frac{\partial y}{\partial t}(x,0) = \cos(x) \quad (0 < x < \pi)$$

Solution: Given

$$y(x,t) = \sin(x)\cos(at) + \frac{1}{a}\cos(x)\sin(at)$$

$$\Downarrow$$

$$y(x,0) = \sin(x)$$

$$\frac{\partial y}{\partial t}(x,t) = -a\sin(x)\sin(at) + \cos(x)\cos(at)$$

$$\Downarrow$$

$$\frac{\partial y}{\partial t}(x,0) = \cos(x)$$

The boundary conditions are satisfied.

$$\Downarrow$$

$$\frac{\partial^2 y}{\partial t^2} = -a^2\sin(x)\cos(at) - a\cos(x)\sin(at)$$

$$\frac{\partial y}{\partial x} = \cos(x)\cos(at) - \frac{1}{a}\sin(x)\sin(at)$$

$$\Downarrow$$

$$\frac{\partial^2 y}{\partial x^2} = -\sin(x)\cos(at) - \frac{1}{a}\cos(x)\sin(at)$$

$$\Downarrow$$

$$a^2\frac{\partial^2 y}{\partial x^2} = -a^2\sin(x)\cos(at) - a\cos(x)\sin(at)$$

$$\Downarrow$$

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

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2. Solve the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2} + x^2 \qquad (0 < x < 4,\ \ t > 0)$$
$$y(0,t) = y(4,t) = 0 \qquad (t > 0)$$
$$y(x,0) = 0 \qquad (0 < x < 4)$$
$$\frac{\partial y}{\partial t}(x,0) = 0 \qquad (0 < x < 4)$$

Hint: Let $z(x,t) = y(x,t) + h(x)$ and substitute into the partial differential equation and boundary conditions to determine $h(x)$ that would wash out the x^2 term in the partial differential equation.

Solution: Let

$$z(x,t) = y(x,t) + h(x)$$
$$\Downarrow$$
$$y(x,t) = z(x,t) - h(x)$$
$$\Downarrow$$
$$\frac{\partial y}{\partial t} = \frac{\partial z}{\partial t} \qquad \text{and} \qquad \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 z}{\partial t^2}$$
$$\frac{\partial y}{\partial x} = \frac{\partial z}{\partial x} - h'(x) \qquad \text{and} \qquad \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} - h''(x)$$

Substitute into the given differential equation, i.e.,

$$\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2} + x^2$$
$$\Downarrow$$
$$\frac{\partial^2 z}{\partial t^2} = 9\frac{\partial^2 z}{\partial x^2} - 9h''(x) + x^2$$

We want to select a $h(x)$ such that

$$x^2 - 9h''(x) = 0$$
$$\Downarrow$$
$$h''(x) = \frac{1}{9}x^2$$
$$h'(x) = \frac{1}{27}x^3 + c_1$$
$$h(x) = \frac{1}{108}x^4 + c_1 x + c_2$$
$$\Downarrow$$
$$z(x,t) = y(x,t) + h(x) = y(x,t) + \frac{1}{108}x^4 + c_1 x + c_2$$

$$z(0,t) = y(0,t) + c_2 = 0 + c_2 = 0 \qquad \Longleftarrow \qquad c_2 = 0$$
$$z(4,t) = y(4,t) + \frac{1}{108}(256) + 4c_1 = 0 \qquad \Longleftarrow \qquad c_1 = -\frac{16}{27}$$
$$\Downarrow$$
$$h(x) = \frac{1}{27}\left(\frac{1}{4}x^4 - 16x\right)$$

With this choice of $h(x)$, the original boundary value problem reduces to the following form:

$$\frac{\partial^2 z}{\partial t^2} = 9\frac{\partial^2 z}{\partial x^2} \qquad (0 < x < 4, \;\; t > 0)$$
$$z(0,t) = z(4,t) = 0 \qquad (t > 0)$$
$$z(x,0) = h(x) \qquad (0 < x < 4)$$
$$\frac{\partial z}{\partial t}(x,0) = 0 \qquad (0 < x < 4)$$

Note that this is the wave equation with zero initial velocity, $L = 4$ and $a = 3$. From your text or class notes, we know the general solution to this problem is given by

$$z(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L h(\xi)\sin\left(\frac{n\pi}{L}\xi\right)d\xi\right]\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right)$$

or

$$z(x,t) = \frac{1}{2}\sum_{n=1}^{\infty}\left[\int_0^4 \frac{1}{27}\left(\frac{1}{4}\xi^4 - 16\xi\right)\sin\left(\frac{n\pi}{4}\xi\right)d\xi\right]\sin\left(\frac{n\pi}{4}x\right)\cos\left(\frac{3n\pi}{4}t\right)$$

Let us compute

$$\begin{aligned}
&\int_0^4 \frac{1}{27}\left(\frac{1}{4}\xi^4 - 16\xi\right)\sin\left(\frac{n\pi}{4}\xi\right)d\xi \\
&= -\frac{4}{27n\pi}\int_0^4\left(\frac{1}{4}\xi^4 - 16\xi\right)d\cos\left(\frac{n\pi}{4}\xi\right) \\
&= -\frac{4}{27n\pi}\left[\left(\frac{1}{4}\xi^4 - 16\xi\right)\cos\left(\frac{n\pi}{4}\xi\right)\Big|_0^4 - \int_0^4(\xi^3 - 16)\cos\left(\frac{n\pi}{4}\xi\right)d\xi\right] \\
&= 0 + \frac{4}{27n\pi}\int_0^4(\xi^3 - 16)\cos\left(\frac{n\pi}{4}\xi\right)d\xi \\
&= \cdots \\
&= \frac{1024}{27n^5\pi^5}\left[(3n^2\pi^2 - 6)\cos(n\pi) + 6\right]
\end{aligned}$$
$$\Downarrow$$

$$\begin{aligned}
y(x,t) = -h(x) + z(x,t) &= -\frac{1}{27}\left(\frac{1}{4}x^4 - 16x\right) \\
&+ \frac{512}{27}\sum_{n=1}^{\infty}\frac{1}{n^5\pi^5}\left[(3n^2\pi^2 - 6)\cos(n\pi) + 6\right]\sin\left(\frac{n\pi}{4}x\right)\cos\left(\frac{3n\pi}{4}t\right)
\end{aligned}$$

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3. Show that the partial differential equation

$$\frac{\partial u}{\partial t} = k\left[\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu\right]$$

can be transformed into a partial differential equation of the form

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$$

by choosing α and β appropriately and using the change of variables

$$u(x,t) = e^{\alpha x+\beta t}v(x,t)$$

Hint: Substitute $u(x,t) = e^{\alpha x+\beta t}v(x,t)$ into the partial differential equation and determine how to choose α and β.

Solution: Let

$$u(x,t) = e^{\alpha x+\beta t}v(x,t)$$

$$\Downarrow$$

$$\frac{\partial u}{\partial t} = \beta e^{\alpha x+\beta t}v(x,t) + e^{\alpha x+\beta t}\frac{\partial v}{\partial t}$$

$$\frac{\partial u}{\partial x} = \alpha e^{\alpha x+\beta t}v(x,t) + e^{\alpha x+\beta t}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \alpha^2 e^{\alpha x+\beta t}v(x,t) + 2\alpha e^{\alpha x+\beta t}\frac{\partial v}{\partial x} + e^{\alpha x+\beta t}\frac{\partial^2 v}{\partial x^2}$$

Substituting into the differential equation

$$\frac{\partial u}{\partial t} = k\left[\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu\right]$$

we have

$$e^{\alpha x+\beta t}\left[\beta v + \frac{\partial v}{\partial t}\right] = e^{\alpha x+\beta t}k\left[\alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + A\alpha v + A\frac{\partial v}{\partial x} + Bv\right]$$

$$\Downarrow$$

$$\beta v + \frac{\partial v}{\partial t} = k\left[\alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + A\alpha v + A\frac{\partial v}{\partial x} + Bv\right]$$

$$\Downarrow$$

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \left[k(\alpha^2 + A\alpha + B) - \beta\right]v + k(2\alpha + A)\frac{\partial v}{\partial x}$$

Let us choose

$$\alpha = -\frac{A}{2} \qquad \text{and} \qquad \beta = k(\alpha^2 + A\alpha + B) = k\left(B - \frac{A^2}{4}\right)$$

$$\Downarrow$$

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$$

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4. Use the method of Problem 3 to solve the boundary value problem

$$\frac{\partial u}{\partial t} = k\left[\frac{\partial^2 u}{\partial x^2} - \frac{a}{L}\frac{\partial u}{\partial x}\right] \qquad (0 < x < L,\ \ t > 0)$$

$$u(0,t) = u(L,t) = 0 \qquad (t > 0)$$
$$u(x,0) = \frac{L}{ak}[1 - e^{-a(1-x/L)}] \qquad (0 < x < L)$$

Here $u(x,t)$ measures the concentration of positive charge carriers on the base of length L of a transistor, $k > 0$ and $a > 0$ are constants.

Solution: Comparing with Problem 3, we have

$$A = -\frac{a}{L}, \qquad B = 0$$

So we can choose

$$\alpha = \frac{a}{2L}, \qquad \beta = -\frac{ka^2}{4L^2}$$

Let

$$u(x,t) = e^{\alpha x + \beta t} v(x,t)$$

The p.d.e. becomes

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$$

$$u(0,t) = 0 \quad \Longrightarrow \quad e^{\beta t} v(0,t) = 0 \quad \Longrightarrow \quad v(0,t) = 0$$

$$u(L,t) = 0 \quad \Longrightarrow \quad e^{\alpha L + \beta t} v(L,t) = 0 \quad \Longrightarrow \quad v(L,t) = 0$$

$$u(x,0) = \frac{L}{ak}\left[1 - e^{-a(1-X/L)}\right] = e^{ax/(2L)} v(x,0)$$

$$\Downarrow$$

$$v(x,0) = \frac{L}{ak} e^{-ax/(2L)}\left[1 - e^{-a(1-X/L)}\right] = \frac{L}{ak}\left[e^{-ax/(2L)} - e^{-a} e^{ax/(2L)}\right] := f(x)$$

This is the Heat Equation and the solution of this equation is given by

$$v(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(\xi)\sin\left(\frac{n\pi}{L}\xi\right)d\xi\right]\sin\left(\frac{n\pi}{L}x\right)e^{-n^2\pi^2 kt/L^2}$$

Next, compute

$$\int_0^L f(\xi)\sin\left(\frac{n\pi}{L}\xi\right)d\xi = \int_0^L \frac{L}{ak}\left[e^{-a\xi/(2L)} - e^{-a}e^{a\xi/(2L)}\right]\sin\left(\frac{n\pi}{L}\xi\right)d\xi$$

$$= \frac{L}{ak}\frac{e^{-a\xi/(2L)}}{a^2/(4L^2)+n^2\pi^2/L^2}\left[-\frac{a}{2L}\sin\left(\frac{n\pi}{L}\xi\right) - \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}\xi\right)\right]\Big|_0^L$$

$$- \frac{Le^{-a}}{ak}\frac{e^{a\xi/(2L)}}{a^2/(4L^2)+n^2\pi^2/L^2}\left[\frac{a}{2L}\sin\left(\frac{n\pi}{L}\xi\right) - \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}\xi\right)\right]\Big|_0^L$$

$$= \frac{4L^2 n\pi[1 - e^{-a/2}\cos(n\pi)]}{ak(a^2+4n^2\pi^2} - \frac{4L^2 n\pi[e^{-a} - e^{-a/2}\cos(n\pi)]}{ak(a^2+4n^2\pi^2}$$

$$= \frac{4L^2 n\pi(1-e^{-a})}{ak(a^2+4n^2\pi^2)}$$

$$\Downarrow$$

$$u(x,t) = \frac{8L\pi}{ak}e^{ax/(2L)-ka^2t/(4L^2)}\sum_{n=1}^{\infty}\frac{n(1-e^{-a})}{a^2+4n^2\pi^2}\sin\left(\frac{n\pi}{L}x\right)e^{-n^2\pi^2kt/L^2}$$

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